EXAMPLE: $\sum_{m=0}^{\infty} (\pi_m+1)(x-c)^m$ $\pi_m = m$-th DIGIT OF π

THE LIMIT $\lim_{m\to\infty} \left|\frac{\pi_{m+1}+1}{\pi_m+1}\right|$ DOES NOT EXIST

(BECAUSE π IS IRRATIONAL, ... DON'T WORRY ABOUT EXACTLY WHY)

BUT $\pi_m+1 \leq 10$ FOR ALL m SO

$\sum_{m=0}^{\infty} (\pi_m+1)(x-c)^m$ CONVERGES ABSOLUTELY

BY COMPARISON WITH $\sum_{n=0}^{\infty} 10(x-c)^n$ FOR

$|x-c| < 1$

AND $\pi_m+1 \geq 1$ FOR ALL m SO

$\sum_{m=0}^{\infty} (\pi_m+1)(x-c)^m$ DIVERGES BY COMPARISON

WITH $\sum_{m=0}^{\infty} (x-c)^m$ FOR $|x-c| > 1$

THM: LET $\sum_{m=0}^{\infty} A_m(x-c)^m$ BE A POWER SERIES. ONE OF THE FOLLOWING MUST HOLD, EITHER:

i) THRE IS $R > 0$ S.T. $\left|\sum_{m=0}^{\infty} A_m(x-c)^m\right.$ CONVERGES FOR $|x-c| < R$ AND DIVERGES FOR $|x-c| > R$.

ii) $\sum_{n=0}^{\infty} A_n(x-c)^n$ CONVERGES FOR ALL x ($R=\infty$).

iii) $\sum_{n=0}^{\infty} A_n(x-c)^n$ CONVERGES FOR $x=c$ AND DIVERGES FOR ANY $x \neq c$ ($R=0$).

EXAMPLE:

Q: WE KNOW THAT A POWER SERIES CENTERED AT $c=10$ CONVERGES AT $x=7$ BUT DIVERGES AT $x=15$. WHAT CAN WE CONCLUDE?

A: $R \geq |10-7| = 3$ AND $R \leq |10-15| = 5$

Q: WHAT IF WE ALSO KNEW THAT IT CONVERGES AT $x=5$?

$R \geq |10-5| = 5$ AND $R \leq |10-15| = 5$ SO $R = 5$.

DEF: THE SET OF POINTS WHERE $\sum_{n=0}^{\infty} A_n(x-c)^n$ CONVERGES IS CALLED THE INTERVAL OF CONVERGENCE OF $\sum_{n=0}^{\infty} A_n(x-c)^n$.

IT ALWAYS CONTAINS $(c-R, c+R)$, AND MAY OR MAY NOT CONTAIN THE ENDPOINTS

NOTE: ALWAYS REMBER CHECKING THE ENDPOINTS!

EXAMPLE: $\sum_{m=0}^{\infty} \log(m+1)x^m$, $\sum_{m=0}^{\infty} \frac{x^m}{\sqrt{m+1}}$, $\sum_{m=0}^{\infty} \frac{x^m}{(m+1)^2}$

ALL THREE SERIES HAVE $R=1$, $C=0$, BUT

- $\sum_{m=0}^{\infty} \log(m+1)x^m$ DIVERGES AT BOTH ENDPOINTS AS $\sum_{m=0}^{\infty} (-1)^m \log(m+1)$ AND $\sum_{m=0}^{\infty} \log(m+1)$ BOTH DIVERGE BY THE DIVERGENCE TEST
- $\sum_{m=0}^{\infty} \frac{x^m}{\sqrt{m+1}}$ CONVERGES AT $X=-1$ AS $\sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{m+1}}$ CONV. BY ALTERNATING TEST, AND DIVERGES AT $X=1$ AS $\sum_{m=0}^{\infty} \frac{1}{\sqrt{m+1}}$ DIVERGES BY THE P-TEST
- $\sum_{m=0}^{\infty} \frac{x^m}{(m+1)^2}$ CONVERGES AT BOTH ENDPOINTS AS $\sum_{m=0}^{\infty} \frac{1}{(m+1)^2}$ AND $\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^2}$ BOTH CONVERGE ABSOLUTELY.

POWER SERIES AS FUNCTIONS

NOW THAT WE KNOW A SERIES WILL ALWAYS HAVE AN INTERVAL OF CONVERGENCE, WE CAN THINK OF A SERIES WHICH CONVERGES ON SOME NON-ZERO RADIUS R AS A FUNCTION $f(x) = \sum_{n=0}^{\infty} A_n (x-c)$ DEFINED ON THE INTERVAL I, WHICH CONTAINS $(c-R, c+R)$ AND MAY OR MAY NOT CONTAIN THE ENDPOINTS.

<u>THM</u>: LET $f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n$, $g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$ POWER SERIES WITH $R_f, R_g \neq 0$. THEN

- THE RADIUS OF CONVERGENCE OF $f(x)+g(x) = \sum_{n=0}^{\infty} (A_n + B_n)(x-c)^n$ IS AT LEAST $\text{MIN}(R_f, R_g)$
- THE RADIUS OF CONV. OF $(x-c)^m f(x)$, $Kf(x)$ IS R_f.

<u>EXAMPLE</u>: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $g(x) = \sum_{n=0}^{\infty} \frac{x^n}{10^n}$

THEN R.O.C. OF $x^{10}(f(x) + x^{90} g(x))$ IS $\geq$ $\text{MIN}(R_f, R_g) = \text{MIN}(\infty, 10) = 10$

BUT THAT'S NOT ALL; A LITTLE MIRACLE HAPPENS WITH POWER SERIES, THEY CONVERGE TO <u>DIFFERENTIABLE</u> FUNCTIONS!

<u>THM:</u> $f(x) = \sum_{m=0}^{\infty} A_m (x-c)$, $R_f \neq 0$.

- $f(x)$ IS <u>DIFFERENTIABLE</u> ON ITS DOMAIN OF CONVERGENCE, <u>EXCEPT POSSIBLY AT $c \pm R$</u>
- $$\frac{d}{dx} f(x) = \sum_{m=0}^{\infty} (m+1) A_{m+1} (x-c)^m = A_1 + 2A_2(x-c) + 3A_3(x-c)^2 + \dots$$
- $$\int f(x)\,dx = \sum_{m=1}^{\infty} \frac{A_{m-1}}{m} (x-c)^m + C = C + A_0(x-c) + \frac{A_1}{2}(x-c)^2 + \frac{A_2}{3}(x-c)^3 + \dots$$

<u>EXAMPLE:</u> $\log(1+x)$ AS A POW. SER.

WE KNOW THAT $\sum_{m=0}^{\infty} (-1)^m x^m = \frac{1}{1+x}$

THUS BY THE THM

$$\log(|1+x|) + C = \int \frac{1}{1+x}\,dx = C + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m}$$

$$= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{2} + \dots \quad \text{ON } \underline{\underline{(-1,1]}}$$

<u>EXAMPLE:</u> $\frac{1}{(1-x)^2}$ AS A POW. SER.

WE KNOW THAT $\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m$

SO HOW DO WE GET TO $\frac{1}{(1-x)^2}$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) \; !$$

SO $$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{m=0}^{\infty}(m+1)x^m$$

$$= 1+2x+3x^2+4x^3+\dots \quad \underline{\text{ON } (-1,1)} \quad (\text{CHECK!})$$

<u>EXAMPLE</u> $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

HOW ABOUT e^x?

LET'S LOOK AT $\sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)$

WE HAVE $$\frac{d}{dx}f(x) = \sum_{m=0}^{\infty} \frac{m+1}{m+1!}\cdot x^m = \sum_{m=0}^{\infty} \frac{x^m}{m!} \; !!$$

SO $\frac{d}{dx}f(x) = f(x)$, AND $f(0)=1$... THEN

$f(x) = e^x$!! <u>EVERYWHERE.</u>

<u>THEOREM</u> (SUBSTITUTION IN POWER SERIES)

$f(x) = \sum_{m=0}^{\infty} A_m x^m$ HAS RADIUS OF CONV. R, THEN

- $f(Kx) = \sum_{m=0}^{\infty} A_m K^m x^m$ HAS R.O.C. $\frac{R}{K}$
- $f(x^m) = \sum_{m=0}^{\infty} A_m x^{m\cdot m}$ HAS R.O.C. $\sqrt[m]{R}$

$$= \sum_{k=0}^{\infty}\left(\begin{cases} 0 & \text{IF } k \neq m\cdot m \\ A_m & \text{IF } k = m\cdot m \end{cases}\right)\cdot x^k$$