TAYLOR SERIES

TWO QUESTIONS:

i) IF WE KNOW THAT $f(x) = \sum_{m=0}^{\infty} A_m x^m$

AROUND $x = 0$, CAN WE FIND $\frac{d^m}{dx^m} f(0)$?

ii) IF WE KNOW $f(0)$ AND $\frac{d^m}{dx^m} f(0)$ FOR ALL m, CAN WE FIND THE COEFFICIENTS A_m?

WELL,

i) IF $f(x) = A_0 + A_1 x + A_2 x^2 + \ldots$ THEN

$\frac{d}{dx} f(x) = A_1 + 2A_2 x + 3A_3 x^3 + \ldots$ AND

$\frac{d^m}{dx} f(x) = 2 \cdot 3 \cdot \ldots \cdot m A_m + 2 \cdot 3 \cdot \ldots \cdot m{+}1 \, A_{m+1} x + \ldots$

SO $f(0) = 2 \cdot 3 \cdot \ldots \cdot m A_m + 0 + 0 + \ldots$

$= 2 \cdot 3 \cdot \ldots m A_m = m! A_m$

ii) CONVERSELY $A_m = \frac{d^m}{dx} f(0) \cdot \frac{1}{m!}$

THIS CLEARLY WORKS FOR ANY CENTER c, GIVING US

$$A_m = \frac{d^m}{dx} f(c) \cdot \frac{1}{m!}$$

SO THAT AROUND c WE HAVE

$$f(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(c)\,(x-c)^n =$$

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3$$

$$+ \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

WAIT, THIS LOOKS FAMILIAR!

RECALL:

THE TAYLOR POLYNOMIAL OF DEGREE n CENTERED AT c FOR $f(x)$ IS

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

IT APPROXIMATES $f(x)$ WITH AN ERROR

$$|E_n(x)| = |f(x) - T(x)| \le \left| \frac{M}{(n+1)!}(x-c)^{n+1} \right|$$

WHERE $M = \mathrm{MAX}_{[c,x]} f^{(n+1)}$

WE CAN NOW EXTEND THIS POWERFUL IDEA!

THEOREM: (TAYLOR SERIES)

THE TAYLOR SERIES (EXPANSION) FOR $f(x)$

WITH CENTER c IS

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

$$= \sum_{n=0}^{\infty} \frac{d^n}{dx^n} f(c) \cdot \frac{(x-c)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

- FOR ALL THE FUNCTIONS $f(x)$ WE'LL BE CONCERNED ABOUT WE HAVE

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(x-c)^n$$

WHEREVER THE SERIES CONVERGES

- MOREOVER, IF WE KNOW ALREADY THAT

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n$$

THEN THIS IS ITS TAYLOR SERIES AND

$$A_n = \frac{f^{(n)}(c)}{n!}$$

EXAMPLE: WE HAVE ALREADY DERIVED

THE FOLLOWING EQUALITIES:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0} x^n$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^m}{m}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} x^{2m-1}$$

BY THE SECOND PART OF THE THM

THESE ARE THE TAYLOR EXPANSIONS OF

$\frac{1}{1-x}$, e^x, $\log(1+x)$, $\arctan(x)$ AT $c=0$

NOTE THAT ALSO (BY $x \to x-1$)

$$\log(x) = x-1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

EXAMPLE: EXPANSION OF $\sin(x)$ AT $c=0$

WE HAVE $(\sin(x))' = \cos(x)$, $(\cos(x))' = -\sin(x)$

SO THE SEQUENCE OF $f^{(n)}(c)$ AT $c=0$

GOES $0, 1, 0, -1, 0, 1, 0, -1$ SO THE TAYLOR

SERIES IS

$$0 + x + 0 - \frac{x^3}{6} + 0 + \frac{x^5}{120} + 0 - \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)!} x^{2m-1}$$

THIS CONVERGES EVERYWHERE. WHY?

IDEA 1: COMPARE WITH $\sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$

IDEA 2: $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^{m-1} x^{2m-1}}{(2m-1)!} = x \sum_{m=0}^{\infty} \frac{(-1)^m (x^2)^m}{(2m+1)!}$

NOW BY RATIO TEST (ON COEFFICIENTS) $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^m}{(2m+1)!}$ HAS

$R = \infty$, AND WE CAN APPLY THE SUBSTITUTION THM

IDEA 3: DIRECTLY APPLY RATIO TEST ON WHOLE

$\displaystyle\sum_{m=0}^{\infty} \frac{(-1) x^{2m-1}}{(2m-1)!}$! $\displaystyle\lim_{m\to\infty} \left| \frac{\frac{x^{2m+1}}{2m+1!}}{\frac{x^{2m-1}}{2m-1!}} \right| = \lim_{m\to\infty} \frac{x^2}{(2m+1)(2m)}$

$= 0$

EXAMPLE: EXPANSION OF $\cos(x)$ AT $c=0$

THE VALUES $f^{(m)}(0)$ GO $1, 0, -1, 0, 1, 0, -1, \ldots$

SO $\displaystyle\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2m!}$

EXERCISE: USE THE IDEAS ABOVE TO SHOW THAT THIS POW. SER. HAS $R = \infty$.

EXAMPLE: EXPANSION OF e^{x^3} AT $c = 0$

DO WE HAVE TO TAKE A LOT OF UGLY DERIVATIVES?

NO! WE HAVE $\displaystyle e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \ldots$

$\displaystyle = \sum_{m=0}^{\infty} \frac{x^{3m}}{m!}$ BY SUBSTITUTION, AND BY THM THIS IS THE TAYLOR SERIES!

EXAMPLE: FIND $\frac{d^9}{dx^9} e^{x^3}(0)$

WE KNOW THAT $e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \dots$

$$= 1 + 0 + 0 + x^3 + 0 + 0 + \frac{x^6}{2} + 0 + 0 + \frac{x^9}{6}$$

SO IN THE TAYLOR EXPANSION $A_9 = \frac{1}{6}$

BUT $A_9 = \frac{d^9}{dx^9} e^{x^3}(0) \cdot \frac{1}{9!}$ SO

$$\frac{1}{6} = \frac{(e^{x^3})^{(9)}(0)}{9!} \sim (e^{x^3})^{(9)}(0) = \frac{9!}{6}$$

EXAMPLE: FIND THE COEFFICIENT OF x^4 IN THE TAYLOR SERIES OF

$$\frac{e^x - \cos(x) - \sin(x)}{x^2} \quad \text{AT} \quad c = 0$$

LET'S TRY TO REWRITE THIS A SERIES DIVIDED BY x^2

$$\frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) - \left(x - \frac{x^3}{6} + \dots\right)}{x^2}$$

$$= \frac{0 + 0 + x^2 + \frac{x^3}{3} + 0 + 0 + \frac{x^6}{360} + \dots}{x^2} = 1 + \frac{x}{3} + \frac{x^4}{360} + \dots$$

SO $A_4 = \frac{1}{360}$

EXAMPLE: SOLVE $\lim_{x \to 0} \frac{\cos(x)-1}{x^2}$ WITHOUT USING L'HOPITAL.

WE CAN REWRITE $\frac{\cos(x)-1}{x^2} = \frac{\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\dots\right)-1}{x^2}$

$$= \frac{-\frac{x^2}{2}+\frac{x^4}{24}-\dots}{x^2}$$

SO $\lim_{x \to 0} \frac{\cos(x)-1}{x^2} = \lim_{x \to 0} \frac{-\frac{x^2}{2}+\frac{x^4}{24}-\dots}{x^2}$

$$= \lim_{x \to 0} -\frac{1}{2} + \underbrace{x^2}_{\to 0} \underbrace{\left(\frac{1}{24} - \dots\right)}_{\text{CONV. TO SOME VALUE}} = -\frac{1}{2}$$

EXAMPLE: TAYLOR EXPANSION WITH $c \neq 0$

- $\frac{1}{x}$ AT $c=3$ $\quad \frac{d^m}{dx^m} \frac{1}{x} = \frac{(-1)^m \cdot m!}{x^{m+1}}$

AT $c=3$ $\frac{(-1)^m \cdot m!}{3^{m+1}}$ SO $A_m = \frac{(-1)^m \cdot m!}{3^{m+1} \cdot m!} = \frac{(-1)^m}{3^{m+1}}$

$\frac{1}{x} = \sum_{m=0}^{\infty} \frac{(-1)^m}{3^{m+1}} (x-3)^m$ ON $(0,6)$ (CHECK!)

- $\log(x)$ AT $c=10$ $\quad \frac{d^m}{dx^m} \log(x) = \begin{cases} \log(x) & m=0 \\ \frac{(-1)^{m-1} (m-1)!}{x^m} & m \neq 0 \end{cases}$

SO $A_0 = \log(10)$; $A_m = \dfrac{(-1)^{m-1}(m-1)!}{10^m \cdot m!} = \dfrac{(-1)^{m-1}}{10^m \cdot m}$

$$\log(x) = \log(10) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{10^m \cdot m}(x-10)^m$$

ON $(0, 20]$ (CHECK!)

EXAMPLE: AREA-SO-FAR FUNCTIONS

IF $F(x) = \int_0^x f(\tau)\,d\tau$ THEN INTEGRATING $f(x)$

WE GET AROUND $c = 0$ $F(x) = \int f(x)\,dx$

- $F(x) = 0 + f(0)x + \frac{f'(0)}{2}x^2 + \frac{f''(0)}{6}x^3 + \dots$

$C = 0$

- $F(x) = \int_0^x e^{-\tau^2}\,d\tau$ AT $c = 0$

WHAT IS THE TAYLOR EXPANSION OF e^{-x^2}? $f(x) = e^x \Rightarrow f(-x^2) = e^{-x^2}$

SO AS $f(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \Rightarrow f(-x^2) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^{2m}$

SO $\int_0^x f(-\tau^2)\,d = \int_0^x 1 - \tau^2 + \frac{\tau^4}{2} - \frac{\tau^6}{6} + \frac{\tau^8}{24} - \dots\,d\tau$

$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^8}{216} - \dots = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\, x^{2m-1}}{(2m-1)(m-1)!}$ ON $(-\infty, +\infty)$