EXAMPLE : $\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n^2+1)(n!)^2}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(2n+2)!}{((n+1)^2+1)(n+1!)^2}}{\frac{(-1)^n(2n)!}{(n^2+1)(n!)^2}} \right| = \lim_{n\to\infty} \overbrace{\frac{(2n+2)(2n+1)}{(n+1)^2}}^{\to 2} \cdot \overbrace{\frac{(n^2+1)}{((n+1)^2+1)}}^{\to 1}$$

$= 2$ SO THE SERIES DIVERGES

NOTE:

IDEA 1 - IF YOU SEE FACTORIALS, A RATIO TEST CAN PROBABLY HELP YOU

IDEA 2 - $\frac{(2n)!}{(n!)^2} = \frac{2n \cdot \ldots \cdot n+1 \cdot \cancel{n \cdot \ldots \cdot 1}}{n \cdot \ldots \cdot 1 \cdot \cancel{n \cdot \ldots \cdot 1}} =$

$$\frac{2n \cdot \ldots \cdot n+1}{n \cdot \ldots \cdot 1} = \underbrace{\frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdot \ldots \cdot \frac{n+1}{1}}_{\text{ALL} \geq 2} \geq 2^n$$

EXAMPLE : $\sum_{n=1}^{\infty} \frac{2^n + 3^n \cdot n^2}{5^n - n^3}$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1} + 3^{n+1} \cdot (n+1)^2}{5^{n+1} - (n+1)^3}}{\frac{2^n + 3^n \cdot n^2}{5^n - n^3}} \right| = \lim_{n\to\infty} \frac{2^{n+1} + 3^{n+1}(n+1)^2}{2^n + 3^n n^2} \cdot \frac{5^n - n^3}{5^{n+1} - (n+1)^3}$$

$$= \lim_{n\to\infty} 3 \underbrace{\left(\frac{\frac{2}{3}^{n+1} + (n+1)^2}{\frac{2}{3}^n + n^2} \right)}_{\to 1} \cdot \frac{1}{5} \underbrace{\left(\frac{1 - \frac{n^3}{5^n}}{1 - \frac{(n+1)^3}{5^{n+1}}} \right)}_{\to 1}$$

$= \frac{3}{5}$ SO THE SERIES CONVERGES.

EXAMPLE: $\sum_{n=1}^{\infty} n^{1000}$

THE SERIES OBVIOUSLY DIVERGES, BUT

$$\lim_{n\to\infty}\left|\frac{(n+1)^{1000}}{n^{1000}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right|^{1000} = \left(\lim_{n\to\infty}\left|\frac{n+1}{n}\right|\right)^{1000}$$

$= 1^{1000} = 1$ SO THE RATIO TEST IS USELESS IN THIS CASE!

EXAMPLE: $\sum_{n=1}^{\infty} \frac{1}{n^{10^{10^{10}}}}$

SAME THING! $\lim_{n\to\infty}\left|\frac{\left(\frac{1}{n+1}\right)^{10^{10^{10}}}}{\left(\frac{1}{n}\right)^{10^{10^{10}}}}\right| = \left(\lim\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right|\right)^{10^{10^{10}}}$

$= 1$ THE RATIO TEST IS UTTERLY USELESS ON ANYTHING NON-EXPONENTIAL.

POWER SERIES

LET X BE A NUMBER. WE KNOW THAT IF $|X|<1$ THEN

$$\sum_{m=0}^{\infty} X^m = \frac{1}{1-X} \quad \text{(GEOMETRIC RATIO X STARTING 1)}$$

THIS IS TRUE FOR ALL $-1<X<1$, SO WE CAN THINK OF THIS AS AN EQUALITY OF FUNCTIONS!

$$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \dots$$

BUT WHY SHOULD THIS BE USEFUL?

IDEA: MAYBE WE CAN APPLY OPERATIONS BY DOING THEM ON EACH TERM? SO THAT

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}1 + \underbrace{\frac{d}{dx}x}_{=1} + \underbrace{\frac{d}{dx}x^2}_{=2x} + \dots = \sum_{m=0}^{\infty}(m+1)X^m$$

AND

$$(-\log(1-x)+C)= \int \frac{1}{1-x}\,dx = C + \int 1\,dx + \int x\,dx + \dots = C + \sum_{m=1}^{\infty}\frac{X^m}{m} \; !$$

WAIT, IF WE PLUG IN $C=0$, $X=-1$

$$-\log(2) = 0 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m} \quad \text{SO} \quad \log(2) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}$$

SO ALTOUGH WE HAVENT ACTUALLY STATED ANY THM YET, WE CAN SEE HOW POWER SERIES, I.E. SERIES IN THE FORM

$$\sum_{n=0}^{\infty} A_n (x-c)^n = A_0 + A_1(x-a) + A_2(x-a)^2 + \dots$$

CAN BE USEFUL (ASSUMING THE PROPERTIES WE JUST GUESSED WORK!). WE'LL BEGIN BY STUDYING THEM AS JUST SERIES DEPENDING ON X, THEN WE'LL SEE THEM AS FUNCTIONS, AND FINALLY GIVEN A FUNCTION WE'LL BE ABLE TO PRODUCE POWER SERIES EQUAL TO IT (AROUND SOME POINT).

DEF:

A SERIES IN THE FORM

$$A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 = \sum_{n=0}^{\infty} A_n (x-c)^n$$

IS CALLED A POWER SERIES WITH CENTER c.

FIRST, SOME BASIC ARITHMETIC PROPERTIES.

THM: WE HAVE

- $\displaystyle\sum_{n=0}^{\infty} A_n (x-c)^n + \sum_{n=0}^{\infty} B_n (x-c)^n = \sum_{n=0}^{\infty} (A_n+B_n)(x-c)^n$

- $\displaystyle d \cdot \sum_{n=0}^{\infty} A_n (x-c)^n = \sum_{n=0}^{\infty} d \cdot A_n (x-c)^n$ FOR ALL d

- $\displaystyle (x-c)^m \sum_{n=0}^{\infty} A_n (x-c)^n =$

$$A_0(x-c)^m + A_1(x-c)^{m+1} + A_2(x-c)^{m+2} + \dots$$

$$= \sum_{n=m}^{\infty} A_{n-m}(x-c)^n = \sum_{n=0}^{\infty} \tilde{A}_n (x-c)$$

WHERE $\tilde{A}_n = \begin{cases} 0 & \text{IF } n < m \\ A_{n-m} & \text{IF } n \geq m \end{cases}$

WE ALWAYS WANT TO REINDEX SO THAT THE n-Th COEFFICIENT MULTIPLIES $(x-c)^n$. ALSO NOTE THAT WE ARE ALWAYS COMBINING POWER SERIES WITH THE SAME CENTER c.

EXAMPLE: $\displaystyle (x^4-1) \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} = x^4 \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^4\left(1 + \frac{1}{24}\right) + x^5\left(1 + \frac{1}{120}\right) + x^6\left(\frac{1}{2} + \frac{1}{720}\right) + \dots$$

$$= \sum_{n=0}^{\infty} A_n x^n \qquad A_n = \begin{cases} \frac{1}{n!} & \text{IF } n < 4 \\ \frac{1}{n!} + \frac{1}{(n-4)!} & \text{IF } n \geq 4 \end{cases}$$

WE WANT TO STUDY CONVERGENCE OF $\sum A_m(x-c)^m$. NOTE FIRST THAT THE SERIES WILL ALWAYS CONVERGE IN AT LEAST ONE POINT, AS WHEN $x=c$

$$\sum_{m=0}^{\infty} A_m(c-c)^m = A_0 + 0 + 0 + \dots + = A_0$$

<u>NOTE</u>: THE NOTATION $A_0(x-c)^0$ IS INTENDED TO ALWAYS MEN $A_0 \cdot 1$, WATHEVER THE VALUE OF x. IT'S MEANT TO SHORTEN OUR EXPRESSIONS. A MORE PRECISE NOTATION WOULD BE:

$$A_0 + \sum_{m=1}^{\infty} A_m(x-c)^m$$

OUR TOOL OF CHOICE FOR FINDING OUT WHERE $\sum_{m=0}^{\infty} A_m(x-c)^m$ WILL BE THE RATIO TEST. ASSUME $A_m \neq 0$ FOR ALL LARGE ENOUGH m.

$$\lim_{m\to\infty} \left| \frac{A_{m+1}(x-c)^{m+1}}{A_m(x-c)^m} \right| = \lim_{m\to\infty} |x-c| \left| \frac{A_{m+1}}{A_m} \right| =$$

$$|x-c| \lim_{m\to\infty} \left| \frac{A_{m+1}}{A_m} \right|$$ SO IF

$$\lim_{m\to\infty} \left| \frac{A_{m+1}}{A_m} \right| = L$$ EXISTS BY THE RATIO TEST THE SERIES WILL CONVERGE

ABSOLUTELY FOR $|x-c|L < 1$, THAT IS, $|x-c| < \frac{1}{L}$. IT WILL DIVERGE IF $|x-c|L > 1$, THAT IS $|x-c| > \frac{1}{L}$.

WE JUST PROVED:

<u>THM-DEF:</u> IF $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = L$ EXISTS AND IS NOT 0, WE DEFINE $R = \frac{1}{L}$ TO BE THE <u>RADIUS OF CONVERGENCE</u> OF $\sum_{n=0}^{\infty} A_n(x-c)^n$.

THE SERIES $\sum_{n=0}^{\infty} A_n(x-c)^n$

- <u>CONVERGES ABSOLUTELY</u> IF $|x-c| < R$

I.E. $c-R < x < c+R$.

- <u>DIVERGES</u> IF $|x-c| > R$, I.E.

$c-R > x$ OR $x > c+R$

IF $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = 0$ THEN $\sum_{n=0}^{\infty} A_n(x-c)^n$ <u>CONVERGES ABSOLUTELY</u> FOR ALL x! (R "=" ∞)

NOTE: WE STILL NEED TO CHECK THE ENDPOINTS $x = c-R$, $x = c+R$ BY HAND.

EXAMPLE: WE HAVE SEEN THAT

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} (x+2)^n = 0 - (x+2) + \frac{(x+2)^2}{\sqrt{2}} - \dots$$

CONVERGES ABS. FOR $-3 < x < -1$, CONDIT. FOR $x = -1$ AND DIVERGES FOR $x \leq -3$ OR $x > -1$.

WE CAN THINK OF IT AS A POWER SERIES WITH $A_0 = 0$, $A_n = \frac{(-1)^n}{\sqrt{n}}$ FOR $n \geq 1$

THEN $\lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1$

SO $R = \frac{1}{1} = 1$, WHICH CONFIRMS WHAT WE ALREADY FOUND OUT $(c = -2)$.

EXAMPLE: WE HAVE SEEN THAT $\sum_{n=0}^{\infty} \frac{a^n}{n!}$

CONVERGES FOR ALL a. WE CAN SEE IT AS THE POWER SERIES $\sum_{n=0}^{\infty} \frac{x^n}{n!}$;

THEN $\lim_{n \to \infty} \left| \frac{A_{n+1}}{A_n} \right| = \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n} = 0$

THIS TELLS US THAT $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ CONVERGES EVERYWHERE.

SOMETIMES THE TEST MIGHT FAIL, I.E. $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ DOES NOT EXIST. WE MAY STILL GET SOME INFORMATION, THOUGH

<u>EXAMPLE</u>: $\sum_{n=0}^{\infty} n!\,x^n$

WE HAVE $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty} n = \infty$

SO THE THM TELLS US NOTHING. BUT IF WE APPLY THE DIVERGENCE TEST WE SEE THAT $\lim_{n\to\infty} n!\,x^n = \infty$ FOR ALL $x \neq 0$, SO THE SERIES ONLY CONVERGES AT THE CENTER $c=0$.

IN FACT, THIS ALWAYS HAPPENS:

<u>THM</u>: IF $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \infty$, THE SERIES $\sum_{n=0}^{\infty} A_n(x-c)^n$ HAS RADIUS OF CONVERGENCE 0, THAT IS IT ONLY CONVERGES AT $x=c$.

RMK: IT CAN BE SHOWN THAT THE RADIUS OF CONVERGENCE EXISTS FOR ANY POWER SERIES, EVEN IF THE RATIO TEST FAILS

EXAMPLE: $\sum_{m=0}^{\infty}(\pi_m+1)(x-c)^m$ $\pi_m = m$-Th DIGIT OF π

THE LIMIT $\lim_{m\to\infty}\left|\frac{\pi_{m+1}+1}{\pi_m+1}\right|$ DOES NOT EXIST

(BECAUSE π IS IRRATIONAL, ∴ DON'T WORRY ABOUT EXACTLY WHY)

BUT $\pi_m+1 \leq 10$ FOR ALL m SO

$\sum_{m=0}^{\infty}(\pi_m+1)(x-c)^m$ CONVERGES ABSOLUTELY

BY COMPARISON WITH $\sum_{n=0}^{\infty}10(x-c)^n$ FOR $|x-c|<1$

AND $\pi_m+1 \geq 1$ FOR ALL m SO

$\sum_{m=0}^{\infty}(\pi_m+1)(x-c)^m$ DIVERGES BY COMPARISON WITH $\sum_{m=0}^{\infty}(x-c)^m$ FOR $|x-c|>1$

THM: LET $\sum_{m=0}^{\infty}A_m(x-c)^m$ BE A POWER SERIES. ONE OF THE FOLLOWING MUST HOLD, EITHER:

i) THRE IS $R>0$ S.T. $\sum_{m=0}^{\infty}A_m(x-c)^m$ CONVERGES FOR $|x-c|<R$ AND DIVERGES FOR $|x-c|>R$.

ii) $\sum_{n=0}^{\infty} A_n(x-c)^n$ CONVERGES FOR ALL X ($R=\infty$).

iii) $\sum_{n=0}^{\infty} A_n(x-c)^n$ CONVERGES FOR $X=C$ AND DIVERGES FOR ANY $X \neq C$ ($R=0$).

EXAMPLE:

Q: WE KNOW THAT A POWER SERIES CENTERED AT $C=10$ CONVERGES AT $X=7$ BUT DIVERGES AT $X=15$. WHAT CAN WE CONCLUDE?

A: $R \geq |10-7| = 3$ AND $R \leq |10-15| = 5$

Q: WHAT IF WE ALSO KNEW THAT IT CONVERGES AT $X=5$?

$R \geq |10-5| = 5$ AND $R \leq |10-15| = 5$ SO $R = 5$.