

WARM UP :

i) WE KNOW THAT $\sum_{m=2}^{\infty} \frac{(-1)^m}{m-\sqrt{m}}$ CONVERGES,

$$\text{SAY TO } \sum_{m=2}^{\infty} \frac{(-1)^m}{m-\sqrt{m}} = S.$$

FIND N SUCH THAT THE ERROR

$$|R_N| = |S - S_N| \text{ IS BELOW } \frac{1}{1000}.$$

$$\text{SOL: WE HAVE } |R_N| \leq a_{N+1} = \frac{1}{N+1-\sqrt{N+1}}$$

OPTION 1): WE JUST BRUTALLY SAY

$$N+1 - \sqrt{N+1} \geq \frac{N+1}{2} \text{ FOR } N \geq 3 \text{ AND THUS}$$

$$\frac{2}{N+1} \leq \frac{1}{1000} \sim N+1 \geq 2000 \sim N \geq 1999$$

WORKS

$$\text{OPTION 2) } \frac{1}{N+1-\sqrt{N+1}} \leq \frac{1}{1000} \sim N+1-\sqrt{N+1}-1000 \geq 0$$

$$\text{SOLVE FOR } \sqrt{N+1}: \sqrt{N+1} \geq \frac{1+\sqrt{4001}}{2}$$

$$\frac{1+\sqrt{4001}}{2} \approx \frac{64.3}{2} = 32.15 \text{ SO } N+1 \geq (32.15)^2$$

$$= 1033,6225 \text{ SO } N+1 \geq 1034 \text{ } N \geq 1033$$

(ACTUALLY THE BEST SOLUTION IS 1032 AS WE ROUNDED UP)

ABSOLUTE AND CONDITIONAL CONVERGENCE

OUR INTUITIVE IDEA THAT

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots = S$$

WHEN THE SERIES CONVERGES SEEMS TO SUGGEST THAT CONVERGENT SERIES WOULD FOLLOW THE MAIN PROPERTIES OF USUAL ADDITION, IN PARTICULAR THAT THE ORDER OF SUMMANDS DOES NOT AFFECT THE END RESULT.

UNFORTUNATELY, THIS TURNS OUT TO BE HORRIBLY WRONG IN GENERAL.

EXAMPLE: WE KNOW THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGES (TO $\log(2)$). WE CAN DIVIDE THE TERMS IN TWO TYPES:

- ODD TERMS $\frac{1}{n}$ WITH n ODD
- EVEN TERMS $-\frac{1}{n}$ WITH n EVEN

IN THE ORIGINAL SERIES THEY ARE ARRANGED AS

ODD + EVEN + ODD + EVEN + ODD + ...

LET'S ARRANGE THEM, KEEPING THE RELATIVE ORDER, AS

$$\text{ODD} + \text{EVEN} + \text{EVEN} + \text{ODD} + \text{EVEN} + \text{EVEN} + \dots =$$

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots =$$

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \dots =$$

$$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \dots =$$

$$\frac{1}{2} \left(1 - \frac{1}{2}\right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2} \left(\frac{1}{5} - \frac{1}{6}\right) + \dots =$$

$$\frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right) = \frac{\log(2)}{2} !!$$

WE JUST GOT HALF THE PREVIOUS VALUE!

IN FACT, WE CAN REARRANGE THE TERMS TO GET ANY NUMBER OR EVEN TO HAVE IT DIVERGE TO $\pm\infty$!!

SO WE CAN SEE THAT THIS KIND OF CONVERGENCE IS RATHER FLIMSY...
BUT! IF WE TRIED TO REARRANGE

FOR EXAMPLE $\sum_{n=1}^{\infty} \frac{1}{n^2}$ WE WOULD ALWAYS GET THE SAME NUMBER, AND THE SAME WOULD HAPPEN FOR $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$. SO WHICH ARE THE "BETTER" SERIES?

DEF: (ABSOLUTE/CONDITIONAL CONVERGENCE)

• A SERIES $\sum_{n=1}^{\infty} a_n$ IS SAID TO ABSOLUTELY CONVERGE IF $\sum_{n=1}^{\infty} |a_n|$ CONVERGES

• A SERIES $\sum_{n=1}^{\infty} a_n$ IS SAID TO CONDITIONALLY CONVERGE IF $\sum_{n=1}^{\infty} a_n$ CONVERGES BUT $\sum_{n=1}^{\infty} |a_n|$ DIVERGES.

THM: AN ABSOLUTELY CONVERGENT SERIES IS CONVERGENT, I.E. IF

$\sum_{n=1}^{\infty} |a_n|$ CONVERGES THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES

FACT (HOPEFULLY MIND-BLOWING): REARRANGING AN ABSOLUTELY CONVERGENT SERIES WILL ALWAYS YIELD THE SAME VALUE. A COND. CONV. SERIES CAN BE REARRANGE TO YIELD ANY VALUE: $\rightarrow \infty$ OR $-\infty$

EXAMPLE:

• $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ IS CONDITIONALLY CONVERGENT.

• $\sum_{n=1}^{\infty} \frac{\sqrt{n} \cos(n)}{n^2-1}$ IS ABSOLUTELY CONVERGENT

AS $\left| \frac{\sqrt{n} \cos(n)}{n^2-1} \right|$ CAN BE SHOWN TO CONVERGE

BY EITHER NORMAL OR LIMIT COMPARISON
WITH $\frac{1}{n^{\frac{3}{2}}}$.

• $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ IS CONDITIONALLY CONVERGENT,

BUT PROVING IT REQUIRES FINER TOOLS THAN
THE ONES AVAILABLE IN THIS COURSE.

• $\sum_{n=1}^{\infty} \log\left(1 + \frac{(-1)^{n-1}}{n}\right)$ IS CONDITIONALLY CONV.

AS $\log\left(1 + \frac{1}{n-1}\right) + \log\left(1 - \frac{1}{n}\right) =$

$\log\left(\frac{n}{n-1}\right) + \log\left(\frac{n-1}{n}\right) = 0$ SO IT SUMS TO 0.

BUT $\sum_{n=1}^{\infty} \left| \log\left(1 + \frac{(-1)^{n-1}}{n}\right) \right| \geq \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right)$

SO IT DIVERGES BY COMPARISON.

• $\sum_{n=1}^{\infty} \frac{(-1)^{\pi_n}}{n^2}$ WHERE $\pi_n = n$ -TH DIGIT OF π
CONVERGES ABSOLUTELY.

NOTE (IMPORTANT!):

OUR MAIN TOOLS, THAT IS THE INTEGRAL TEST, COMPARISON TEST AND LIMIT COMPARISON TEST ALL CHECK FOR ABSOLUTE CONVERGENCE AS THEY ALL REQUIRE POSITIVITY OF THEIR COMPARISON TERMS.

THE RATIO TEST

THE FINAL TEST WE'LL USE IS A RATHER "ROUGH" TEST (IT ONLY SELECTS SERIES THAT HAVE A BEHAVIOUR SIMILAR TO A GEOMETRIC SERIES) BUT IT HAS THE RELEVANT ADVANTAGE THAT WE WON'T HAVE TO CHOOSE A COMPARISON TERM. MOREOVER, IT WILL BE OUR PRIME TOOL WHEN DISCUSSING POWER SERIES.

THM (RATIO TEST):

LET a_m BE A SEQUENCE, AND SUPPOSE THAT $a_m \neq 0$ FOR ALL (LARGE ENOUGH) m . THEN:

• IF $\lim_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right| = L < 1$ THEN $\sum_{m=1}^{\infty} a_m$ CONVERGES ABSOLUTELY.

• IF $\lim_{m \rightarrow \infty} \left| \frac{a_{m+1}}{a_m} \right| = L > 1$ THEN $\sum_{m=1}^{\infty} a_m$ DIVERGES.

WARNING: THE RATIO TEST TELLS
YOU NOTHING IF $L = 1$.

EXAMPLE: $\sum_{m=1}^{\infty} \frac{(-1)^m (x+2)^m}{\sqrt{m}}$ (FUNCTION OF x !)

WE HAVE SEEN ALREADY THAT FOR $x = -1$ THE SERIES CONVERGES AND FOR $x = -3$ IT DIVERGES. HOW ABOUT ALL THE OTHER VALUES? LET'S USE THE RATIO TEST

$$\lim_{m \rightarrow \infty} \left| \frac{(-1)^{m+1} (x+2)^{m+1}}{\sqrt{m+1}} \cdot \frac{\sqrt{m}}{(-1)^m (x+2)^m} \right| = \lim_{m \rightarrow \infty} \left| (x+2) \cdot \frac{\sqrt{m}}{\sqrt{m+1}} \right|$$

$$= |x+2|$$

SO FOR $|x+2| < 1$, THAT IS $-3 < x < -1$ THE SERIES CONVERGES ABSOLUTELY, FOR $|x+2| > 1$, THAT IS $x \leq -3$ OR $x > -1$ THE SERIES DIVERGES, AND FOR $x = -1$ IT CONV. CONDITIONALLY.

EXAMPLE: $\sum_{m=1}^{\infty} \frac{a^m}{m!}$

$$\lim_{m \rightarrow \infty} \left| \frac{\frac{a^{m+1}}{(m+1)!}}{\frac{a^m}{m!}} \right| = \lim_{m \rightarrow \infty} \frac{a}{m+1} = 0 \quad \text{SO THE SERIES}$$

CONVERGES ABSOLUTELY FOR ANY a .

EXAMPLE: $\sum_{m=0}^{\infty} \frac{(-1)^m (2m)!}{(m^2+1)(m!)^2}$

$$\lim_{m \rightarrow \infty} \left| \frac{(-1)^{m+1} (2m+2)!}{((m^2+1)+1)(m+1!)^2} \cdot \frac{(-1)^m (2m)!}{(m^2+1)(m!)^2} \right| = \lim_{m \rightarrow \infty} \underbrace{\frac{(2m+2)(2m+1)}{(m+1)^2}}_{\rightarrow 2} \cdot \underbrace{\frac{(m^2+1)}{(m+1)^2+1}}_{\rightarrow 1}$$

= 2 SO THE SERIES DIVERGES

NOTE:

IDEA 1 - IF YOU SEE FACTORIALS, A RATIO TEST CAN PROBABLY HELP YOU

IDEA 2 - $\frac{(2m)!}{(m!)^2} = \frac{2m \cdot \dots \cdot m+1 \cdot m \cdot \dots \cdot 1}{m \cdot \dots \cdot 1 \cdot m \cdot \dots \cdot 1} =$

$$\frac{2m \cdot \dots \cdot m+1}{m \cdot \dots \cdot 1} = \frac{2m}{m} \cdot \frac{2m-1}{m-1} \cdot \dots \cdot \frac{m+1}{1} \geq 2^m$$

ALL ≥ 2

EXAMPLE: $\sum_{m=1}^{\infty} \frac{2^m + 3^m \cdot m^2}{5^m - m^3}$

$$\lim_{m \rightarrow \infty} \left| \frac{2^{m+1} + 3^{m+1} \cdot (m+1)^2}{5^{m+1} - (m+1)^3} \cdot \frac{2^m + 3^m \cdot m^2}{5^m - m^3} \right| = \lim_{m \rightarrow \infty} \frac{2^{m+1} + 3^{m+1} (m+1)^2}{2^m + 3^m m^2} \cdot \frac{5^m - m^3}{5^{m+1} - (m+1)^3}$$

$$= \lim_{m \rightarrow \infty} 3 \left(\frac{\frac{2^{m+1}}{3} + (m+1)^2}{\frac{2^m}{3} + m^2} \right) \cdot \frac{1}{5} \left(\frac{1 - \frac{m^3}{5^m}}{1 - \frac{(m+1)^3}{5^{m+1}}} \right)$$

$\rightarrow 1$ $\rightarrow 1$

= $\frac{3}{5}$ SO THE SERIES CONVERGES.