BUT WAIT, WE LIKE FINDING HOW SOMETHING GOES TO 0, BUT HATE INEQUALITIES! ISN'T THERE A BETTER TOOL?

THM: (LIMIT COMPARISON TEST)

LET $\{a_n\}$, $\{b_n\}$ BE TWO SEQUENCES WITH $b_n > 0$ FOR ALL (SUFFICIENTLY LARGE) n.

ASSUME THAT $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ $(\neq \pm\infty)$.

THEN:

(a) IF $\sum_{n=1}^{\infty} b_n$ CONVERGES SO DOES $\sum_{n=1}^{\infty} a_n$.

(b) IF $L \neq 0$ AND $\sum_{n=1}^{\infty} b_n$ DIVERGES, SO DOES $\sum_{n=1}^{\infty} a_n$

IN PARTICULAR, IF $L \neq 0$ THEN $\sum_{n=1}^{\infty} a_n$ CONVERGES IF AND ONLY IF $\sum_{n=1}^{\infty} b_n$.

~~THIS~~ GREATLY SIMPLIFIES OUR COMPARISONS.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{n + 5\sin(n)}{n^2+1}$

IDEA: COMPARE WITH $\frac{1}{n}$.

USING THE TEST:

$$\lim_{m\to\infty} \frac{\frac{m+5\sin(m)}{m^2+1}}{\frac{1}{m}} = \lim_{m\to\infty} \frac{m^2+5m\sin(m)}{m^2+1}$$

$$= \lim_{m\to\infty} \frac{m^2}{m^2+1} + \frac{5m\sin(m)}{m^2+1} = 1+0 = 1$$

SQUEEZE

$$\underset{\downarrow \atop 0}{\frac{-5m}{m^2+1}} \leq \cdot \leq \underset{\downarrow \atop 0}{\frac{5m}{m^2+1}}$$

SO IT DIVERGES BY LIM. COMP WITH $\sum_{m=1}^{\infty} \frac{1}{m}$.

EXAMPLE: $\sum_{m=1}^{\infty} \frac{\log(m)^{50}}{m^2}$

IDEA: $\frac{\log(m)^{50}}{m^2} \leq \frac{1}{m^{2-d}}$ FOR ANY d, EVENTUALLY

USING THE TEST: PICK $d=\frac{1}{2}$ SO WE COMPARE WITH $\frac{1}{m^{\frac{3}{2}}}$. WE HAVE

$$\lim_{m\to\infty} \frac{\frac{\log(m)^{50}}{m^2}}{\frac{1}{m^{\frac{3}{2}}}} = \lim_{m\to\infty} \frac{\log(m)^{50}}{m^{\frac{1}{2}}} = 0$$

ANY POWER OF $\log(x)$ LOSES TO ANY POWER OF x

SO BY LIMIT COMPARISON $\sum_{n=1}^{\infty} \frac{\log(n)^{50}}{n^2}$ MUST CONVERGE AS $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ DOES.

EXAMPLE : $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n \log(n+1)^{\frac{1}{2}}}$

IDEA $\frac{\arctan(n-1)}{n \log(n)^{\frac{1}{2}}} \approx \frac{1}{n \log(n+1)^{\frac{1}{2}}}$ SO IT SHOULD DIVERGE.

USING THE TEST $\lim_{n \to \infty} \frac{\frac{\arctan(n-1)}{n \log(n+1)^{\frac{1}{2}}}}{\frac{1}{n \log(n+1)^{\frac{1}{2}}}} =$

$\lim_{n \to \infty} \arctan(n-1) = \frac{\pi}{2}$ SO BY THE LIMIT COMP. TEST $\sum_{n=1}^{\infty} \frac{\arctan(n-1)}{n \log(n+1)^{\frac{1}{2}}}$ DIVERGES AS $\sum_{n=} \frac{1}{n \log(n+1)}$ DOES.

ALTERNATING SERIES TEST

THERE IS ONE <u>VERY SPECIAL</u> CASE WHEN THE DIVERGENCE TEST ACTUALLY BECOMES A <u>TWO WAYS</u> TEST.

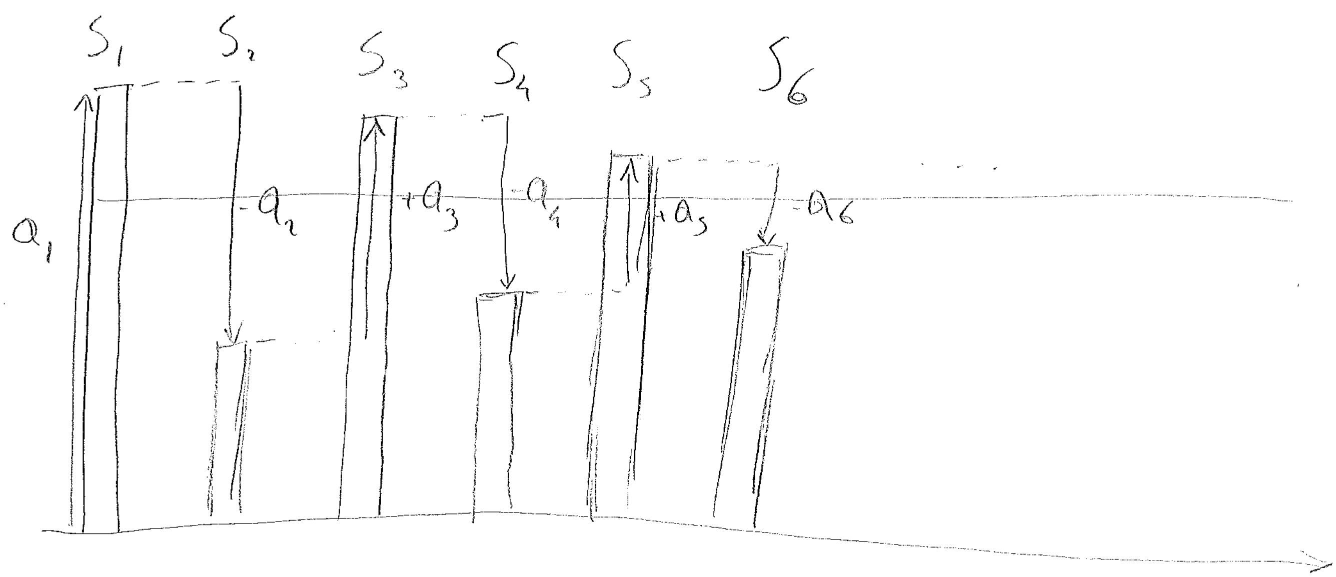
DEF: A SERIES IS CALLED <u>ALTERNATING</u> IF THE SIGNS OF THE SUCCESSIVE TERMS ALTERNATE BETWEEN + AND −. WE WILL WRITE IT AS

$$a_1 - a_2 + a_3 - a_4 + a_5 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

$$\left(\text{OR } -a_1 + a_2 - a_3 + \dots = \sum_{n=1}^{\infty} (-1)^n a_n \right)$$

WHERE $\{a_n\}$ IS A SEQUENCE OF <u>POSITIVE NUMBERS</u>

LET'S SAY THAT a_n IS <u>DECREASING</u> AND IT GOES TO 0. WHAT HAPPENS THEN?



AS THE INCREASES / DECREASES GET SMALLER AND SMALLER AND BALANCE OUT, ONCE WE ARE OVER N THE FUTURE PARTIAL SUMS WILL ALL BE WITHIN S_{N-1} AND S_N!

WITH SOME WORK WE OBTAIN:

THM: (ALTERNATING SERIES TEST)
SUPPOSE THAT $a_m \geq 0$ FOR ALL m AND a_m IS DECREASING, I.E. $a_m \geq a_{m+1}$ FOR ALL m. THEN IF $\lim_{m\to\infty} a_m = 0$ THE SERIES $\sum_{m=1}^{\infty} (-1)^{m-1} a_m$ CONVERGES.

EXAMPLE: WE KNOW THAT $\sum_{m=1}^{\infty} \frac{1}{m}$ DIVERGES. BUT WHAT ABOUT $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}$? WELL THE SERIES IS ALTERNATING, $\frac{1}{m}$ IS DECREASING, AND $\lim_{m\to\infty} \frac{1}{m} = 0$ SO BY THE ALTERNATING TEST $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}$ CONVERGES. WE'LL SEE LATER IN THE COURSE THAT $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} = \log(2)$

EXAMPLE: $\sum_{m=1}^{\infty} \frac{(-1)^m}{m - \sqrt{m}}$

THE SERIES IS ALTERNATING AND CLEARLY $\lim a_m = 0$, SO WE JUST HAVE TO PROVE THAT

$a_n = \frac{1}{n-\sqrt{n}}$ IS DECREASING. WE HAVE

$a_n = f(n)$ WHERE $f(x) = \frac{1}{x-\sqrt{x}}$, SO WE CAN JUST SHOW THAT $f(x)$ IS DECREASING, WHICH IS EQUIVALENT TO $x-\sqrt{x}$ BEING INCREASING. $(x-\sqrt{x})' = 1 - \frac{1}{2\sqrt{x}} > 0$ SO $f(x)$ IS DECREASING, WHICH SHOWS THAT

$a_{n+1} = f(n+1) \leq f(n) = a_n$. THUS WE HAVE ALL WE NEED TO USE THE ALTERNATING TEST AND CONCLUDE THAT $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$ CONVERGES.

<u>EXAMPLE</u> $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$.

FIRST WE NOTICE THAT $\cos(n\pi) = (-1)^n$, SO THE SERIES IS ALTERNATING.

NOW, $a_n = \frac{n}{2^n}$ CLEARLY GOES TO 0, SO WE HAVE TO SHOW IT'S DECREASING.

$$a_n - a_{n+1} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}} = \frac{2n-n+1}{2^{n+1}} = \frac{n-1}{2^n} \geq 0$$

SO $a_n \geq a_{n+1}$ AND BY THE ALTERNATING TEST THE SERIES CONVERGES.

EXAMPLE $\displaystyle\sum_{n=1}^{\infty} \log\left(1+\frac{(-1)^{n-1}}{n^2}\right)$

$\log\left(1-\frac{1}{n^2}\right) < 0$ SO THE SERIES IS ALTERNATING,

ALSO $\displaystyle\lim_{n\to\infty} \left|\log\left(1+\frac{(-1)^{n-1}}{n^2}\right)\right| = |\log(1)| = 0$

WE HAVE TO CHECK THAT

$$\left|\log\left(1+\frac{(-1)^{n-1}}{n^2}\right)\right| = \begin{cases} \log\left(1+\frac{1}{n^2}\right) & n \text{ ODD} \\ -\log\left(1-\frac{1}{n^2}\right) & n \text{ EVEN} \end{cases}$$

IS DECREASING

n ODD: $\log\left(1+\frac{1}{n^2}\right) - \left(-\log\left(1-\frac{1}{(n+1)^2}\right)\right) =$

$\log\left(1+\frac{1}{n^2}\right) + \log\left(1-\frac{1}{(n+1)^2}\right) = \log\left(\frac{n^2+1}{n^2}\right) +$

$\log\left(\frac{(n+1)^2-1}{(n+1)^2}\right) = \log\left(\frac{n^2+1}{n^2}\cdot\frac{(n+1)^2-1}{(n+1)^2}\right) =$

$$\log\Bigg(\underbrace{\frac{n^4+2n^3+n^2+2n}{n^4+2n^3+n^2}}_{\to 1}\Bigg) > 0$$

THE CASE n EVEN IS SIMILAR. THIS SHOWS THAT THE SERIES $\displaystyle\sum_{n=1}^{\infty} \log\left(1+\frac{(-1)^{n-1}}{n^2}\right)$ CONVERGES.