

WARM-UP

USE THE APPROPRIATE TEST TO DECIDE CONVERGENCE OF THE FOLLOWING SERIES:

i) $\sum_{m=1}^{\infty} \frac{4m-8}{m^2-4m+20}$

ii) $\sum_{m=1}^{\infty} \arctan\left(\sin\left(\frac{\pi m}{4}\right)\right)$

iii) $\sum_{m=1}^{\infty} \frac{3m - \sin(m^3)}{5m+1}$

iv) $\sum_{m=1}^{\infty} e^{-m}(m^2-5m+6)$

SOL: i) FOR $x > 6$ $f(x) = \frac{4x-8}{x^2-4x+20}$ IS DECREASING

AND POSITIVE ($f(x)' = \frac{4(x^2-4x-12)}{(x^2-4x+20)^2}$) SO COMPARE

$\sum_{m=6}^{\infty} \frac{4m-8}{m^2-4m+20}$ WITH $\int_6^{\infty} \frac{4x-8}{x^2-4x+20} dx = \dots =$

$u = (x-2)$

$\lim_{R \rightarrow \infty} 2 \log(x^2-4x+20) \Big|_6^R = \infty$

SO $\sum_{m=1}^{\infty} \frac{4}{m^2-3m+12}$ DIVERGES BY

THE INTEGRAL TEST.

ii) $\sin\left(\frac{\pi m}{4}\right)$ CYCLES BETWEEN

$$\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0,$$

THUS $\arcsin\left(\sin\left(\frac{\pi m}{4}\right)\right)$ CYCLES

BETWEEN $\arctan(b_1), \dots, \arctan(b_8)$

WHERE b_1, \dots, b_8 ARE THE NUMBERS

ABOVE. BUT $\arctan(1) = \frac{\pi}{4}$, $\arctan(-1)$

$= -\frac{1}{4}\pi$ SO THE SEQUENCE $\arctan\left(\sin\left(\frac{\pi}{4}m\right)\right)$

CANNOT HAVE A LIMIT. THEN $\sum_{m=1}^{\infty} \arctan\left(\sin\left(\frac{\pi}{4}m\right)\right)$

DIVERGES BY THE DIVERGENCE TEST.

$$\text{iii) } \lim_{m \rightarrow \infty} \frac{3m - \sin(m^3)}{5m+1} = \lim_{m \rightarrow \infty} \underbrace{\frac{3m}{5m+1}}_{\rightarrow \frac{3}{5}} - \underbrace{\frac{\sin(m^3)}{5m+1}}_{\text{SQUEEZE!}}$$

$$= \frac{3}{5} + 0 \quad \text{SO}$$

$\rightarrow \frac{3}{5}$ SQUEEZE!

$$\sum_{m=1}^{\infty} \frac{3m - \sin(m^3)}{5m+1} \quad \text{DIVERGES}$$

$$\frac{-1}{5m+1} \leq \cdot \leq \frac{1}{5m+1}$$

$\searrow 0 \swarrow$

BY THE DIVERGENCE TEST.

$$\text{iv) } (e^{-x}(x^2 - 5x + 6))' = -e^{-x}(x^2 - 8x + 11)$$

WHICH IS < 0 FOR $x > 4 + \sqrt{5}$

SO $e^{-x}(x^2 - 5x + 6)$ IS DECREASING

FOR $x > 4 + \sqrt{5}$. $4 + \sqrt{5} < 7$ SO

WE COMPARE

$$\sum_{m=7}^{\infty} e^{-m}(m^2 - 5m + 6) \quad \text{WITH} \quad \int_7^{\infty} e^{-x}(x^2 - 5x + 6) dx =$$

$$\dots = \left[-e^{-x}(x^2 - 3x + 3) \right]_7^{\infty} = e^{-7}(31) \quad \text{SO}$$

↑
BY PARTS

$\sum_{m=1}^{\infty} e^{-m}(m^2 - 5m + 6)$ CONVERGES BY THE

INTEGRAL TEST

THE COMPARISON TEST

WE WANT TO BE ABLE TO CHECK CONVERGENCE WITHOUT GOING THROUGH AN INTEGRAL

(WHICH WE MIGHT NOT BE ABLE TO EVALUATE),

THIS IS SIMILAR TO WHAT WE WANTED FOR IMPROPER INTEGRALS, AND WE'LL USE THE SAME IDEA:

THM: (COMPARISON TEST) LET $K > 0$.

a) IF $|a_m| \leq Kc_m$ FOR ALL (SUFFICIENTLY LARGE) m AND $\sum_{m=1}^{\infty} c_m$ CONVERGES, THEN $\sum_{m=1}^{\infty} a_m$ CONVERGES.

b) IF $a_m \geq Kd_m \geq 0$ FOR ALL (SUFFICIENTLY LARGE) m AND $\sum_{m=1}^{\infty} d_m$ DIVERGES THEN $\sum_{m=1}^{\infty} a_m$ DIVERGES

NOTE: STATEMENT (a) WORKS FOR A SEQUENCE $\{a_m\}$ WHOSE SIGN ALTERNATES, BUT STATEMENT (b) DOES NOT.

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{n^{0.1}}{n^{0.99} + n^{1.1} + 1}$$

IDEA:

$$\frac{n^{0.1}}{n^{0.99} + n^{1.1} + 1} \approx \frac{1}{n} \text{ SO IT SHOULD DIVERGE}$$

USING THE TEST: WE COMPARE WITH $\frac{1}{n}$.

$$\text{WE HAVE } n^{0.99} + n^{1.1} + 1 \leq 3n^{1.1} \text{ SO}$$

$$\frac{n^{0.1}}{n^{0.99} + n^{1.1} + 1} \geq \frac{n^{0.1}}{3n^{1.1}} = \frac{1}{3n^{1.1-0.1}} = \frac{1}{3n} \text{ THEN BY}$$

THE COMPARISON TEST $\sum_{n=1}^{\infty} \frac{n^{0.1}}{n^{0.99} + n^{1.1} + 1}$

DIVERGES AS $\sum_{n=1}^{\infty} \frac{1}{n}$ DOES.

EXAMPLE: $\sum_{n=1}^{\infty} \frac{1 + \sin(n)}{n^2}$

IDEA: $\frac{1 + \sin(n)}{n^2} \approx \frac{1}{n^2}$ SO IT SHOULD CONVERGE

USING THE TEST: $1 + \sin(n) \leq 2$ SO

$$\frac{1 + \sin(n)}{n^2} \leq \frac{2}{n^2} \text{ THEN } \sum_{n=1}^{\infty} \frac{1 + \sin(n)}{n^2}$$

CONVERGES AS $\sum_{n=1}^{\infty} \frac{1}{n^2}$ DOES.

EXAMPLE :
$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+3n+2}$$

IDEA : $\frac{n-1}{n^3+3n+2} \approx \frac{1}{n^2}$ so it should converge

USING THE TEST : WE WANT TO COMPARE

WITH
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
.

WE HAVE
$$\frac{n-1}{n^3+3n+2} \leq \frac{n}{n^3+3n+2} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

SO BY THE COMP. TEST
$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+3n+2}$$
 CONVERGES

AS
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
 CONVERGES

EXAMPLE :
$$\sum_{n=1}^{\infty} \frac{n+5\sin(n)}{n^2+1}$$
 (* HARDER. DO IT LAST)

IDEA : $\frac{n+5\sin(n)}{n^2+1} \approx \frac{1}{n}$ so it should diverge

USING THE TEST : WE COMPARE WITH $\frac{1}{n}$.

WE HAVE
$$\frac{n+5\sin(n)}{n^2+1} \geq 0 \text{ FOR } n \geq 5$$

AND
$$n+5\sin(n) \geq n-5 \geq \frac{n}{2} \text{ FOR } n \geq 10$$
 so:

WE START AT $n=10$.

NOW
$$n^2+1 \leq 2n^2$$
 so for $n \geq 10$ we have

$$\frac{m+5 \sin(m)}{m^2+1} \geq \frac{m}{2} \cdot \frac{1}{2m^2} = \frac{m}{4m^2} = \frac{1}{4} \cdot \frac{1}{m}$$

SO BY THE COMPARISON TEST WE CAN CONCLUDE THAT $\frac{m+5 \sin(m)}{m^2+1}$ DIVERGES

EXAMPLE: $\sum_{m=1}^{\infty} \frac{\sqrt{m}}{m^2+m}$

IDEA: $\frac{\sqrt{m}}{m^2+m} \approx \frac{\sqrt{m}}{m^2} = \frac{1}{m^{\frac{3}{2}}}$ SO IT SHOULD CONVERGE

USING THE TEST: WE COMPARE WITH $\frac{1}{m^{\frac{3}{2}}}$

WE HAVE $m^2+m \leq m^2$ SO $\frac{\sqrt{m}}{m^2+m} \leq \frac{\sqrt{m}}{m^2} = \frac{1}{m^{\frac{3}{2}}}$

SO $\sum_{m=1}^{\infty} \frac{\sqrt{m}}{m^2+m}$ CONVERGES AS $\sum_{m=1}^{\infty} \frac{1}{m^{\frac{3}{2}}}$ DOES.

EXAMPLE: $\sum_{m=1}^{\infty} \frac{e^{-m}}{2m-5}$

IDEA: $\frac{e^{-m}}{2m-5} \approx \frac{e^{-m}}{m} \leq e^{-m}$ SO IT SHOULD CONVERGE

USING THE TEST: WE START AT $m=5$ SO OUR FUNCTION IS POSITIVE, AND $2m-5 \geq m$, THUS

$\frac{e^{-m}}{2m-5} \leq \frac{e^{-m}}{m} \leq e^{-m}$; SO BY COMPARISON

$\sum_{m=1}^{\infty} \frac{e^{-m}}{2m-5}$ CONVERGES AS $\sum_{m=1}^{\infty} e^{-m}$ DOES.