ALTERNATING SERIES TEST

THERE IS ONE VERY SPECIAL CASE WHEN THE DIVERGENCE TEST ACTUALLY BECOMES A TWO WAYS TEST.

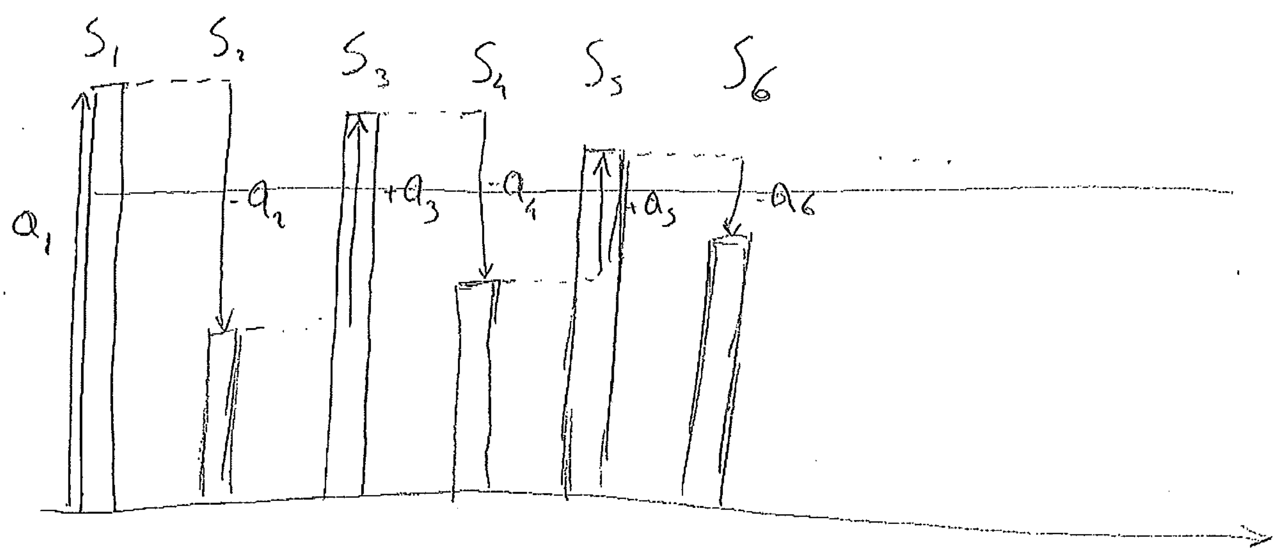
DEF: A SERIES IS CALLED ALTERNATING IF THE SIGNS OF THE SUCCESSIVE TERMS ALTERNATE BETWEEN + AND −. WE WILL WRITE IT AS

$$a_1 - a_2 + a_3 - a_4 + a_5 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

$$\left(\text{OR } -a_1 + a_2 - a_3 + \dots = \sum_{n=1}^{\infty} (-1)^n a_n \right)$$

WHERE $\{a_n\}$ IS A SEQUENCE OF POSITIVE NUMBERS

LET'S SAY THAT a_n IS DECREASING AND IT GOES TO 0. WHAT HAPPENS THEN?



AS THE INCREASES / DECREASES GET SMALLER AND SMALLER AND BALANCE OUT, ONCE WE ARE OVER N THE FUTURE PARTIAL SUMS WILL ALL BE WITHIN S_{N-1} AND S_N!

WITH SOME WORK WE OBTAIN:

THM: (ALTERNATING SERIES TEST)
SUPPOSE THAT $a_n \geq 0$ FOR ALL n AND a_n IS DECREASING, I.E. $a_n \geq a_{n+1}$ FOR ALL n. THEN IF $\lim_{n \to \infty} a_n = 0$ THE SERIES $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ CONVERGES.

<u>EXAMPLE</u>: WE KNOW THAT $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES. BUT WHAT ABOUT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$? WELL THE SERIES IS ALTERNATING, $\frac{1}{n}$ IS DECREASING, AND $\lim_{n \to \infty} \frac{1}{n} = 0$ SO BY THE ALTERNATING TEST $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ CONVERGES. WE'LL SEE LATER IN THE COURSE THAT $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log(2)$

<u>EXAMPLE</u>: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \sqrt{n}}$

THE SERIES IS ALTERNATING AND CLEARLY $\lim a_n = 0$, SO WE JUST HAVE TO PROVE THAT

$a_n = \frac{1}{n - \sqrt{n}}$ IS DECREASING. WE HAVE

$a_n = f(n)$ WHERE $f(x) = \frac{1}{x - \sqrt{x}}$, SO WE CAN JUST SHOW THAT $f(x)$ IS DECREASING, WHICH IS EQUIVALENT TO $x - \sqrt{x}$ BEING INCREASING. $(x - \sqrt{x})' = 1 - \frac{1}{2\sqrt{x}} > 0$ SO $f(x)$ IS DECREASING, WHICH SHOWS THAT

$a_{n+1} = f(n+1) \leq f(n) = a_n$. THUS WE HAVE ALL WE NEED TO USE THE ALTERNATING TEST AND CONCLUDE THAT $\sum_{n=1}^{\infty} \frac{(-1)^n}{n - \sqrt{n}}$ CONVERGES.

<u>EXAMPLE</u> $\sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n}$

FIRST WE NOTICE THAT $\cos(n\pi) = (-1)^n$, SO THE SERIES IS ALTERNATING.

NOW, $a_n = \frac{n}{2^n}$ CLEARLY GOES TO 0, SO WE HAVE TO SHOW IT'S DECREASING.

$$a_n - a_{n+1} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}} = \frac{2n - n + 1}{2^{n+1}} = \frac{n-1}{2^n} \geq 0$$

SO $a_n \geq a_{n+1}$ AND BY THE ALTERNATING TEST THE SERIES CONVERGES.

THE REMAINDER

THERE IS ANOTHER GREAT PROPERTY OF ALTERNATING SERIES THAT WE CAN UNDERSTAND JUST BY LOOKING AT THE PICTURE OF THE PARTIAL SUMS S_N!
WE CAN ESTIMATE HOW FAR S_N IS FROM THE VALUE $S = \sum_{m=1}^{\infty} (-1)^{m-1} a_m$ VERY EASILY!

THM: (ALTERNATING SERIES TEST REMAINDER)
SUPPOSE A SERIES $\sum_{m=1}^{\infty} (-1)^{m-1} a_m$ PASSES THE ALTERNATING SERIES TEST, AND THUS CONVERGES TO $\sum_{m=1}^{\infty} (-1)^{m-1} a_m = S$.
LET $R_N = S - S_N$ BE THE DIFFERENCE BETWEEN THE N-TH PARTIAL SUM AND THE VALUE OF THE SERIES. THEN:

R_N IS BETWEEN 0 AND $(-1)^N a_{N+1}$

IN PARTICULAR $|R_N| \leq a_{N+1}$.

<u>EXAMPLE</u>: $\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}$

THE REMAINDER R_N IS BETWEEN 0 AND $\frac{(-1)^N}{N+1}$

Q: IS S_N BIGGER OR SMALLER THAN S FOR $N=351$
A: $(-1)^{351} = -1$ SO $S - S_{351}$ IS IN $\left[-\frac{1}{351}, 0\right]$; THEN THE VALUE OF S_{351} MUST BE BIGGER THAN S.

EXAMPLE $\sum_{n=1}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$

THE REMAINDER R_N IS BETWEEN 0 AND $\frac{(-1)^{N+1}}{N+1-\sqrt{N+1}}$

(NOTE THAT WE HAVE $(-1)^n$, NOT $(-1)^{n-1}$ IN THE SERIES)

Q: HOW MANY STEPS DO WE NEED TO BE AT A DISTANCE LOWER THAN $\frac{1}{1000}$ (IN MAGNITUDE) FROM S?

A: FOR SURE $N+1-\sqrt{N+1} \geq \frac{N+1}{2}$ SO $N \geq 1999$ WILL SUFFICE... TO BE MORE PRECISE LET $u = \sqrt{N+1}$. $u^2 - u - 1000 \geq 0$ GIVES US $u \geq \frac{1+\sqrt{1+4000}}{2} \approx 33$ SO

$N+1 \geq 33^2 = 1089$

EXAMPLE $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{2^n}$

R_N IS BETWEEN 0 AND $(-1)^{N+1} \frac{N+1}{2^{N+1}}$

Q: FIND N SUCH THAT $0 \geq R_N \geq \frac{-1}{10{,}000}$

A_1: WE CAN DO THIS BY HAND! $2^{10} = 1024$, $2^{20} = (1024)^2 > 10^6$, $0 \leq R_{20} \leq \frac{21}{10^6}$, SO

$0 \geq R_{19} \geq \frac{-38}{10^6} > \frac{-1}{10.000}$

A_2: USING $N+1 \leq 2^{\frac{N+1}{12}}$ WE GET THAT $N=27$ IS ENOUGH.

EXAMPLE : $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

WE KNOW (OR WILL KNOW BY THE END OF THE COURSE) THAT

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} \quad \text{SO} \quad \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} = e^{-1}$$

(NOTE THAT :

- $\begin{cases} 0! = 1 \\ m! = 1 \cdot 2 \cdot \ldots \cdot m \end{cases}$, SO $\frac{1}{m!}$ IS DECREASING

AND $\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}$ CONV BY ALTERNATING TEST)

Q: HOW FAR IS $S_4 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$ TO $\frac{1}{e}$?

A: $|R_4| \leq a_5 = \frac{1}{5!} = \frac{1}{120}$

Q: HOW MANY TERMS DO WE NEED TO APPROXIMATE $\frac{1}{e}$ TO WITHIN $\frac{1}{1000}$?

A: WE NEED $a_{N+1} \leq \frac{1}{1000}$ SO $\frac{1}{(N+1)!} \leq \frac{1}{1000}$

SO $(N+1)! \geq 1000$. NOW, $5! = 120$, $6! = 720$, $7! > 1000$ SO $\underline{N = 6}$.

EX: $\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$ IT'S KNOWN THAT $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

Q: HOW MANY TERMS ARE NEEDED TO APPROXIMATE $\frac{\pi}{4}$ WITH AN ERROR OF LESS THAN 0.0000001?

A: $0.0000001 = \frac{1}{10^7}$ SO WE NEED $|R_N| \leq \frac{1}{10^7}$; THUS WE WANT

$$a_{N+1} = \frac{1}{2N+3} \leq \frac{1}{10^7} \sim 2N+3 \geq 10^7$$

$$N \geq \frac{10^7-3}{2} \quad \text{SO} \quad N > 4999998$$

EX: CONSIDER $\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m}}(x+2)^m$ (FUNCTION OF X!)

DOES IT CONVERGE AT $x=-1$, $x=-3$?

$x=-1$ SO $x+2=1$ AND SERIES BECOMES

$\sum_{m=1}^{\infty} \frac{(-1)^m}{\sqrt{m}}$, CONVERGES BY ALTERNATING TEST

$x=-3$ SO $x+2=-1$ SERIES BECOMES

$\sum_{m=1}^{\infty} \frac{(-1)^m(-1)^m}{\sqrt{m}} = \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}}$ <u>DIVERGES</u> BY INTEGRAL TEST!

EXAMPLE: $\sum_{n=5}^{\infty} (-1)^{n+1} \frac{3n^2+n}{2n^2}$

WE HAVE $a_n = \frac{3n^2+n}{2n^2}$ SO $\lim_{n\to\infty} a_n = \frac{3}{2}$

AND THE SERIES DIVERGES BY DIV. TEST

EXAMPLE: $\sum_{n=3}^{\infty} \sin\left(\frac{(-1)^n}{n}\right)$

- WE HAVE $\sin(-x) = -\sin(x)$ SO $\sin\left(\frac{(-1)^n}{n}\right) = (-1)^n \sin\left(\frac{1}{n}\right)$ IS ALTERNATING
- $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$
- $\sin\left(\frac{1}{n}\right)$ IS DECREASING AS $\sin\left(\frac{1}{x}\right)' = \frac{-1}{x^2}\cos\left(\frac{1}{x}\right)$ WHICH IS NEGATIVE.

SO $\sum_{n=3}^{\infty} \sin\left(\frac{(-1)^n}{n}\right)$ CONVERGES BY THE ALTERNATING TEST.

EXAMPLE $\sum_{n=1}^{\infty} \log\left(1+\frac{(-1)^{n-1}}{n^2}\right)$

$\log\left(1-\frac{1}{n^2}\right) < 0$ SO THE SERIES IS ALTERNATING,

ALSO $\lim_{n\to\infty} \left|\log\left(1+\frac{(-1)^{n-1}}{n^2}\right)\right| = |\log(1)| = 0$

WE HAVE TO CHECK THAT

$$\left|\log\left(1+\frac{(-1)^{n-1}}{n^2}\right)\right| = \begin{cases} \log\left(1+\frac{1}{n^2}\right) & n \text{ ODD} \\ -\log\left(1-\frac{1}{n^2}\right) & n \text{ EVEN} \end{cases}$$

IS DECREASING

n ODD: $\log\left(1+\frac{1}{n^2}\right) - \left(-\log\left(1-\frac{1}{(n+1)^2}\right)\right) =$

$\log\left(1+\frac{1}{n^2}\right) + \log\left(1-\frac{1}{(n+1)^2}\right) = \log\left(\frac{n^2+1}{n^2}\right) +$

$\log\left(\frac{(n+1)^2-1}{(n+1)^2}\right) = \log\left(\frac{n^2+1}{n^2} \cdot \frac{(n+1)^2-1}{(n+1)^2}\right) =$

$\log\left(\underbrace{\frac{n^4+2n^3+n^2+2n}{n^4+2n^3+n^2}}_{>1}\right) > 0$

THE CASE n EVEN IS SIMILAR. THIS SHOWS THAT THE SERIES $\sum_{n=1}^{\infty} \log\left(1+\frac{(-1)^{n-1}}{n^2}\right)$ CONVERGES.

EXAMPLE $\displaystyle\sum_{m=1}^{\infty} \log\left(1+\frac{(-1)^{m-1}}{m^2}\right)$

THE REMAINDER R_N IS BETWEEN 0 AND

$$\log\left(1+\frac{(-1)^N}{(N+1)^2}\right)$$

Q: FIND N SUCH THAT $|R_N| < \frac{1}{1000}$ (HARD)

A: $$\left|\log\left(1+\frac{(-1)^N}{(N+1)^2}\right)\right| < \frac{1}{1000}$$

LET'S PICK N EVEN FOR SIMPLICITY

$$\log\left(1+\frac{1}{(N+1)^2}\right) < \frac{1}{1000}$$

OPTION 1: $$1+\frac{1}{(N+1)^2} < e^{\frac{1}{1000}}$$

SO $$(N+1)^2 > \frac{1}{1-e^{\frac{1}{1000}}} \sim N+1 > \sqrt{\frac{1}{1-e^{\frac{1}{1000}}}}$$

OPTION 2: LINEAR APPROXIMATION TELLS US THAT

$$\log\left(1+\frac{1}{(N+1)^2}\right) - \frac{1}{(N+1)^2} \leq \frac{1}{2(N+1)^4}$$

SO $$\log\left(1+\frac{1}{(N+1)^2}\right) \leq \frac{1}{2(N+1)^4} + \frac{1}{(N+1)^2} \leq \frac{2}{(N+1)^2}$$

SO $$\frac{2}{(N+1)^2} \leq \frac{1}{1000} \qquad (N+1)^2 \geq 2000$$

$$N+1 \geq \sqrt{2000}.$$