INVARIANT THEORY

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1. Lecture 1

Prerequisites and literature:

H.P. Kraft [?], S. Lang [?], I.R. Shafarevich [?], T.A. Springer [?], E.B. Vinberg and A.L. Onishchik [?].

1-1. Invariants and a fundamental Lemma.

Definition 1-1.1. Let S be a set and $S = \sqcup_\alpha S_\alpha$ a decomposition of S into disjont subsets S_α. An *invariant* is a map

$$I\colon S \to M$$

to some other set M, such that $I(S_\alpha)$ is a point, for all α. A system of invariants $I_k\colon S \to M$, for $k = 1, \dots$ is *complete* if $I_k(S_\alpha) = I_k(S_\beta)$, for all k, implies that $\alpha = \beta$.

Example 1-1.2. Let S be the class of all oriented closed compact 2 dimensional manifolds and let the S_α's be the classes of homeomorphic surfaces. We have a map $I\colon S \to \mathbf{Z}_+$ from S to the nonnegative integers, which takes a manifold X to its genus $g(X)$. This is a complete system.

Remark 1-1.3. In general we cannot separate the sets that appear in geometric situations. However, it is usually possible to separate them *generically*, in a sence that we shall make more precise later.

Setup 1-1.4. Let G be an algebraic group. Consider the action of G on an affine algebraic variety X. We shall take $X = \sqcup_\alpha S_\alpha$ as being the decomposition of X in orbits under the action of G, and we shall require the invariants to be polynomial functions.

We shall next give some examples where invariants under group actions appear naturally in classification problems from linear algebra.

Example 1-1.5. Examples of linear algebra problems.

- **1:** Given a linear operator A acting on a finite dimensional vector space V. Find a basis for V in which the matrix A has the *simplest possible form.*
- **2:** Given linear operators A and B acting on a finite dimensional vector space V. Find a basis for V in which the matrices A and B have the *simplest possible form.*
- **3:** Let F be bilinear form on a finite dimensional vector space V. Find a basis for V in which F has the *simplest possible form.*
- **4:** Let F be a bilinear form on a finite dimensional Euclidian space V. Find an orthogonal basis for V in which F has the *simplest possible form.*

5: Given symmetric polynomials. Express them in terms of *simpler ones*.

We shall reformulate the above Examples in terms of actions of groups in such a way that the connection with invariants becomes more apparent.

Example 1-1.6. Reformulation of the previous Examples

1: Choose a basis of V. Let X be the matrix of A in this basis. If we choose another basis, then X gets replaced by gXg^{-1} for some g in Gl_n. So the problem is to find the orbits of the action of Gl_n on the space Mat_n of $n \times n$ matrices, or, to find representatives of the orbits. If the basic field is $\mathbf{C}$, then the answer is given by the *Jordan canonical form*. The first step is to consider the map $\mathrm{Mat}_n \to \mathbf{C^n}$ which sends a matrix X to $(a_1(X), \dots, a_n(X))$, where the $a_i(X)$ are the coefficients of the characteristic polynomial $\deg(\lambda I - X) = \lambda^n + a_1(x)\lambda^{n-1} + \cdots + a_n(X)$ of X. The map is invariant under the action of Gl_n.

It is clear that the map separates orbits *generically*. That is, it separates orbits that correspond to matrices with different characteristic polynomials and *most* matrices in Mat_n are diagonizable, with different eigenvalues.

2: An element g of Gl_n acts on pairs of elements of Mat_n by sending the pair (X, Y) to (gXg^{-1}, gYg^{-1}). This gives an action of Gl_n on $\mathrm{Mat}_n \times \mathrm{Mat}_n$. The problem is to classify this action. The answer is unknown and a problem in linear algebra is called *wild* if it contains this problem as a subproblem. It is wild even when X and Y commute.

3: Choose a basis for V. Let X be the matrix of A in this basis. If we choose another basis, then X gets replaced by gX^tg for some g in Gl_n. Over $\mathbf{C}$ and $\mathbf{R}$ it is easy, and classical, to find simple normal forms, but over arbitrary fields it is unknown.

4: Choose an orthogonal basis for V. Let X be the matrix of A in this basis. If we choose another orthogonal basis, then X gets replaced by gX^tg for some g in O_n. Over $\mathbf{C}$ and $\mathbf{R}$ it is easy, and classical, to find simple normal forms, but over arbitrary fields it is unknown.

5: The problem is to describe the invariants of the action of the *symmetric group* $\mathfrak{S}_n$ on the polynomial ring $\mathbf{C}[\mathbf{x_1}, \dots, \mathbf{x_n}]$. The answer is given by the *elementary symmetric functions*, but there are many other natural bases of symmetric functions. See MacDonald [**?**]

The following Lemma is fundamental for finding invariants of group actions. We shall use it, and generalizations, many times in the following.

Lemma 1-1.7. *Let G be a group acting on a variety V and let $P_1, \dots, P_k$ be invariant polynomials. Suppose that $H \subseteq G$ is a subgroup which leaves invariant a subvariety U of V, such that:*

(1) *The functions $P_i|U$ generate the ring of all invariants of H acting on U*
(2) *The set GU is a dense subset of V.*

Then the polynomials $P_1, \dots, P_k$ generate the ring of invariants of the action of G on V.

Proof. Let P be a polynomial which is invariant under the action of G. Consider the restriction $P|U$. Clearly, $P|U$ is H invariant. Hence, by assumption (1), we have that $P|U = F(P_1|U, \dots, P_k|U)$, for some polynomial F. Consider the difference $P' = P - F(P_1, \dots, P_k)$. Then $P'|U \equiv 0$, and P' is G invariant as a polynomial expression in the $P_1, \dots, P_k$. Therefore $P'|GU \equiv 0$. However, GU is dense in V, so $P' \equiv 0$. Hence we have that $P = F(P_1, \dots, P_k)$. □

Problem 1-1.1. The orthogonal group O_n has the quadratic invariant $f(x) = (x, x)$ where $(,)$ is the bilinear form $((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n x_i y_i$ on $\mathbf{R^n}$. Show that the ring of invariants of O_n on $\mathbf{R^n}$ is generated by $f(x)$.

Hint: In the Lemma above we take $U = \mathbf{Re}$ to be a 1 dimensional vector space with $\|e\| = 1$. Moreover, we take $H = \mathbf{Z}/\mathbf{2Z}$, acting on U by $e \to -e$. Then H fixes $U^\perp$. We have that $(,)|U \times U$ is given by $(xe, xe) = x^2$. The invariants of H acting on U is therefore the subring $\mathbf{R}[\mathbf{x^2}]$ of the polynomial ring $\mathbf{R}[\mathbf{x}]$.

We have that $\mathrm{O}_n \mathbf{Re} = \mathbf{R^n}$, because, let v_1 and w_1 be such that $\|v_1\| = \|w_1\|$, and choose orthogonal vectors $v_1, \dots, v_n$ and $w_1, \dots, w_n$ with $\|v_i\| = \|w_i\|$. Then the matrix X defined by $Xv_i = w_i$ is in O_n and sends v_1 to w_1.

2. Lecture 2

2-1. **Group actions and a basic Example.** An *action* of a group G on a set X is a map

$$G \times X \to X,$$

such that $(g_1 g_2)x = g_1(g_2 x)$ and $ex = x$, for all elements g_1, g_2 of G and x of X, where gx denotes the image of the pair (g, x).

The *orbit* of a subset Y of X under the action is the set

$$GY = \{gy\colon g \in G, y \in Y\},$$

and the *stabilizer* of a point x of X is the set

$$G_x = \{g \in G\colon gx = x\}.$$

The stabilizer is a subgroup of G. The set of orbits we denote by $X \backslash G$. Given a subset H of G we let X^H

$$X^H = \{x \in X\colon hx = x, \quad \text{for all } h \in H\}.$$

Definition 2-1.1. The group G acts on the set of functions $\mathcal{F}(\mathcal{X})$ on X with values in any set, by the rule $(gF)(x) = F(g^{-1}x)$, for each g in G, x in X and F in $\mathcal{F}(\mathcal{X})$. An *invariant* is an element of $\mathcal{F}(\mathcal{X})^{\mathcal{G}}$. Sometimes the invariants are called *invariant functions*.

Remark 2-1.2. An element F is in $\mathcal{F}(\mathcal{X})^{\mathcal{G}}$, if and only if $F(gx) = g^{-1}F(x) = F(x)$ for all g in G and x in X. That is, if and only if $F(Gx) = F(x)$. In other words, F is in $\mathcal{F}(\mathcal{X})^{\mathcal{G}}$ if and only if F is constant on the orbits of points in X.

The basic example, that we shall treat below, is when $G = \mathrm{Gl}_n(\mathbf{C})$ and $X = \mathrm{Mat}_n(\mathbf{C})$, and the action of the group is $gx = gxg^{-1}$. One may also use other fields.

Theorem 2-1.3. *Write*

$$\det(tI + x) = t^n + P_1(x)t^{n-1} + \cdots + P_n(x).$$

Then we have that the polynomials $P_1, \dots, P_n$ are algebraically independent and they generate the ring of $\mathrm{Gl}_n(\mathbf{C})$ *invariant polynomials on* $\mathrm{Mat}_n(\mathbf{C})$.

In other words, we have that

$$\mathbf{C}[\mathrm{Mat}_\mathbf{n}(\mathbf{C})]^{\mathrm{Gl}_\mathbf{n}(\mathbf{C})} = \mathbf{C}[\mathbf{P_1}, \dots, \mathbf{P_n}],$$

and the ring on the right hand side is polynomial in $P_1, \dots, P_n$.

Proof. Apply Lemma 1-1.7 with U being the diagonal matrices of $\mathrm{Mat}_n(\mathbf{C})$, and let H be the group of permutations of the basis vectors $e_1, \dots, e_n$, that is $\sigma \in H$ if $\sigma(e_i) = e_{\sigma(i)}$, where $(\sigma(1), \dots, \sigma(n))$ is a permutation of the indices. We have that H operates on U by

$$\sigma \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_{\sigma(1)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{\sigma(n)} \end{pmatrix}$$

According to the Lemma we must check the properties:

(i) $\overline{Gx} = X$.

(ii) $P_1|U_1, \dots, P_n|U_n$ generate the ring of invariants $\mathbf{C[U]^H}$.

Moreover we will check that the polynomials $P_1, \dots, P_n$ are algebraically independent. Assertion (i) follows from Problem 2-1.4, since any matrix with distinct eigenvalues can be diagonalized.

To prove (ii), let x be the matrix with $(\lambda_1, \dots, \lambda_n)$ on the diagonal and zeroes elsewhere. Then we have that

$$\det(tI + x) = \prod_{i=1}^{n}(t + \lambda_i) = t^n + \sigma_1(\lambda)t^{n-1} + \dots + \sigma_n(\lambda),$$

where $\sigma_1 = \sum \lambda_i$, $\sigma_2 = \sum_{i<j} \lambda_i \lambda j$, ... , are the elementary symmetric polynomials. We see that $P_i|U = \sigma_i$. However, the elementary symmetric polynomials generate the ring of all symmetric functions and are algebraically independent. In other words $\mathbf{C[U]}^{\mathfrak{S}_n} = \mathbf{C}[\sigma_\mathbf{1}, \dots, \sigma_\mathbf{n}]$. We have thus verified assertion (ii), and, at the same time, proved the independence of the P_i. Consequently we have proved the Theorem. □

Remark 2-1.4. The proof of Theorem 2-1.3 that we have given holds for all algebraically closed fields. However, the assertion holds for all fields, and the proof can be reduced to that given for algebraically closed fields, using Exercise 2-1.5

The importance of invariants come from the fact that they give rise to the map

$$\pi \colon \operatorname{Mat}_n(\mathbf{C}) \to \mathbf{C^n}.$$

which sends a matrix x to $(P_1(x), \dots, P_n(x))$. This map will be called a *quotient map* and will be the center of study of the remainder of the lectures.

The basic question is to study the fibers of π. Each fiber is a union of orbits. Indeed, if $c = (c_1, \dots, c_n)$ is a point of $\mathbf{C^n}$ we have that

$$\pi^{-1}(c) = \{x \in \operatorname{Mat}_n(\mathbf{C}) \colon \mathbf{P_i(x) = c_i}\}.$$

However, the polynomials P_i are invariant under the action of $\operatorname{Gl}_n(\mathbf{C})$, so the fibers are invariant. The orbits are however, not parametrized by

$\mathbf{C^n}$. Since the coefficients of a polynomial determines its roots we have that the fibers of π are the sets of matrices with prescribed eigenvalues. We know that the *Jordan form* is a normal form for matrices with given eigenvalues. We can therefore determine the orbits in the fiber.

Remark 2-1.5. To determine the orbits, we consider the matrices with given eigenvalues of multiplicities $m_1, \dots, m_s$. There are $p(m_1)\cdots p(m_s)$ Jordan canonical forms with these eigenvalues, where p is the classical *permutation function*, that is, $p(n)$ is the number of permutations of n. Indeed, the Jordan form is

$$\begin{pmatrix} \lambda_1 & 1 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & \lambda_1 & & & & & \\ & & & & \lambda_2 & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & & 1 & \\ & & & & & & & \lambda_2 & \\ & & & & & & & & \ddots \end{pmatrix},$$

where we have indicated the boxes of size m_1 and m_2 belonging to the eigenvalues λ_1 and λ_2. We see that the fiber consists of $p(m_1)\cdots p(m_s)$ orbits. In particular, a fiber consists of a simple orbit, if and only if, all eigenvalues are distinct. Hence, we have that $\mathbf{C^n}$ parametrizes orbits *generically*. We also see that there is only a finite number of orbits.

Problem 2-1.1. Let G be a finite group. Then $|G| = |G_x||Gx|$, for all x in X.

Problem 2-1.2. (Burnsides theorem) Let G and X be finite. Then we have that

$$|X \setminus G| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

Hint: Consider the subset $M \subseteq G\times X$ defined by $M = \{(g,x) \colon gx = x\}$. Calculate $|M|$ in two ways. First $|M| = \sum_{g\in G}|X^g|$, and secondly $|M| = \sum_{x\in X}|G_x|$. Now use previous exercise. We have that X^g is the fiber over g in $M \to G$ and G_x is the fiber over x in $M \to X$. Hence we get $\sum_{x\in X}|G_x| = |G|\sum_{x\in X}\frac{1}{|Gx|} = |G|\sum_{x\in X\setminus G}1$.

Problem 2-1.3. Show that the map $\pi\colon \operatorname{Mat}_n(\mathbf{C}) \to \mathbf{C^n}$ is surjective.

Hint: This map is surjective, for given $c = (c_1, \dots, c_n)$ in $\mathbf{C^n}$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be roots in the characteristic polynomial. Then the matrix with diagonal entries λ and the remaining entries 0 maps to c.

Problem 2-1.4. Show that the matrices with distinct eighenvalues form a dense subset of $\mathrm{Mat}_n(\mathbf{C})$.

Hint: Consider the map $\mathrm{Mat}_n(\mathbf{C}) \to \mathbf{C^n}$, which sends a matrix X to the coefficients $(P_1(X), \dots P_n(X))$ of the characteristic polynomial. It follows from the previous problem that the map is surjective. It is clear that the subset of $\mathbf{C^n}$ consisting of coefficients of monic polynomials that have distinct roots is open. The inverse image of this set is the set of matrices with distinct eigenvalues.

Problem 2-1.5. Let G operate linearly on a vector space V over a field $\mathbf{F}$. Let $\overline{\mathbf{F}}$ be the algebraic closure. Then we have that $(V \otimes_{\mathbf{F}} \overline{\mathbf{F}})^G = V^G \otimes_{\mathbf{F}} \overline{\mathbf{F}}$.

Hint: Choose a basis $\{\epsilon_i\}_{i\in I}$ for $\overline{\mathbf{F}}$ over $\mathbf{F}$ and a basis $\{v_i\}_{j\in J}$ for V over $\mathbf{F}$. Then we have that $V \otimes_{\mathbf{F}} \overline{\mathbf{F}}$ has a basis $\{v_i \otimes \epsilon_j\}_{i\in I, j\in J}$. Let $v = \sum_{ij} a_{ij} v_i \otimes \epsilon_j$. If $gv = v$ we have that $\sum_{ij} a_{ij} g v_i \otimes \epsilon_j = \sum_{i,j} a_{ij} v_i \otimes \epsilon_j$. Consequently, we have that $\sum_i a_{ij} g v_i = \sum_i a_{ij}$, for all j. That is $\sum_i a_{ij} v_i \in V^G$. Hence, if $v \in (V \otimes_{\mathbf{F}} \overline{\mathbf{F}})^G$, we have that $v \in V^G \otimes_{\mathbf{F}} \overline{\mathbf{F}}$.

2-2. **Closedness of orbits.** The first basic problem is to calculte invariants. This is done, for $\mathrm{Gl}_n(\mathbf{C})$ by Theorem 2-1.3. The second basic problem is to decide which of the orbits are closed. We shall next do this for $\mathrm{Gl}_n(\mathbf{C})$.

Theorem 2-2.1. *The orbit* $\mathrm{Gl}_n(\mathbf{C})\mathbf{x}$ *is closed if and only if the matrix* x *is diagonizable. Moreover, every fiber of* π *contains a unique closed orbit, which is the orbit of minimal dimension of the fiber.*

Proof. We may assume that x is in Jordan form. Moreover assume that x has a block of size $m > 1$, that is

$$x = \begin{pmatrix} \lambda & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & \\ & & & \lambda & \\ & & & & B \end{pmatrix}.$$

Then Gx is not closed, that is $\overline{Gx} \not\supseteq Gx$. Indeed, consider the element

$$g_\epsilon = \begin{pmatrix} \epsilon^m & & & & \\ & \epsilon^{m-1} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & I \end{pmatrix}.$$

Then we have that

$$g_\epsilon x = \begin{pmatrix} \lambda & \epsilon & & & & \\ & \lambda & & & & \\ & & \ddots & & & \\ & & & & \epsilon & \\ & & & & \lambda & \\ & & & & & B \end{pmatrix} \tag{2-2.1.1}$$

in Gx, for all $\epsilon \neq 0$. The closure of $g_\epsilon x$ contains the matrix

$$\begin{pmatrix} \lambda & 0 & & \\ & \ddots & & \\ & 0 & \lambda & \\ & & & B \end{pmatrix} \tag{2-2.1.2}$$

in Gx. However, this matrix has different Jordan form from that of x, so it is not in Gx. Hence non diagonizable matrices have non closed orbits.

Conversely, diagonizable elements have closed orbits. Indeed, we have that x is diagonizable if and only if $P(x) = 0$, for some polynomial with distinct roots. Let $x' \in \overline{Gx}$. then $P(gx) = 0$, for $g \in F$, so $P(x') = 0$. Hence, by the same criterion, x' is also diagonizable. But x and x' lie in the same fiber of π, since $\overline{Gx} \in \pi^{-1}(\pi x)$. But, in each fiber there is a unique *semi simple element*, that is diagonizable element, up to conjugacy of diagonizable elements, and the fibers are closed. We conclude that $x' \in Gx$. Hence $\overline{Gx} = Gx$, and we have proved the Theorem. □

Remark 2-2.2. The conclusion of the Theorem we already knew when x has distinct eigenvalues, because the orbits are fibers of points in $\mathbf{C^n}$.

Remark 2-2.3. We have shown that $\mathbf{C^n}$ parametrizes closed orbits. We call π the *quotient map.*

Remark 2-2.4. The fiber $\pi^{-1}(0)$ consists of the matrices with all eigenvalues equal to zero, that is, of the nilpotent elements.

The problem in general is to classify all orbits once the closed orbits are known.

The general picture for a reductive group acting linearly on a vector space is:

(i) Firstly, classify the closed orbits. These are classified by invariants via the quotient map.

(ii) Secondly, classify nilpotent orbits, that is the orbits that satisfy $\overline{Gx} \ni 0$.
(iii) Thirdly, determine the general *Jordan decomposition*, described abstractly by the three conditions of Excercise 2-2.2.

Remark 2-2.5. The general linear group $\mathrm{Gl}_n(\mathbf{C})$ has the properties:
(i) The ring of invariants is a polynomial ring.
(ii) There are finitely many nilpotent orbits.

These two properties do not hold *in general*.

Problem 2-2.1. Show that $\overline{Gx} \ni 0$ if and only if x is a nilpotent matrix.

Hint: We have that $Gx \in \pi^{-1}(\pi(x))$. Since the fibers are closed we have that $\overline{Gx} \ni \pi^{-1}(\pi(x))$. Hence we have that $\overline{Gx} \ni (0)$ if and only if $0 = \pi(0) = \pi(x)$. that is, if and only if x is nilpotent.

Problem 2-2.2. (Jordan decomposition.) Every x can be written $x = x_s + x_n$, where x_s is diagonizable and x_n is nilpotent, and such that
(i) Gx_s is closed.
(ii) $\overline{G_{x_s}x_n} \ni 0$.
(iii) $G_{x_s} \supseteq G_x$.

A decomposition satisfying properties (i), (ii) and (iii) is unique.

Hint: Follows from the Jordan canonical form. We proved property (i) in 2-2.1 and (ii) in the above problem. Property (iii) follows from the argument of the proof of 2-2.1. Multiplying x with g_ϵ of that proof, we get $g_\epsilon x$ of 2-2.1.1, that clearly is in G_{x_s}, and we saw that the closure of the latter matrices contained the matrix 2-2.1.2, that we see is in G_{x_s}, if we do it for all blocks.

We have proved that $x_s \in \overline{G_{x_s}x}$. However, we have that $\overline{G_{x_s}x} = x_s + \overline{G_{x_s}x_n}$, so we obtain that $0 \in \overline{G_{x_s}}x$. The converse is obtained in the same way. To prove that $G_x \subset G_{x_s}$ we observe that $x_s = P(x)$ for some polynomial P. Indeed, we obtain this by taking $P(t) \equiv \lambda_i \pmod{(x-\lambda_i)^{m_i}}$, for $i = 1, \dots, r$, because x_0 and $P(x)$ are the same operators on V.

Problem 2-2.3. Let $G = \mathrm{Sl}_m(\mathbf{C}) \times \mathrm{Sl}_{\mathbf{n}}(\mathbf{C})$ and $V = \mathrm{Mat}_{m,n}(\mathbf{C})$. Then G acts on V by $(a,b)x = axb^{-1}$. Then we have that:
(i) If $m \neq n$, all invariants are constants. There are finitely many orbits. Describe them.
(ii) If $m = n$, we have that $\mathbf{C}[\mathbf{V}]^{\mathbf{G}} = \mathbf{C}[\det \mathbf{x}]$. Consider the map $V \to \mathbf{C}$ that sends x to $\det x$. Then all fibers are closed orbits,

unless $\det x = 0$. The fiber $\{x\colon \det x = 0\}$ consists of finitely many orbits. Describe them.

Hint: When $m \neq n$ we can put all matrices on the form

$$\begin{pmatrix} 1 & & & \dots & & 0 \\ & \ddots & & & & \vdots \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \vdots \\ & & & & 0 & 0 \end{pmatrix}$$

by the action of G. Take

$$U = \begin{pmatrix} 1 & & & \dots & 0 \\ & \ddots & & & \vdots \\ & & 1 & 0 & 0 \end{pmatrix},$$

and $H = (1, 1)$ in Lemma 1-1.7. We see that the invariants are the constants. The orbits consists of the orbits of the above elements.

Let $m = n$. All matrices can be put in the form

$$\begin{pmatrix} 1 & & & \dots & 0 \\ & \ddots & & & \vdots \\ & & 1 & & \\ & & & 0 & \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \lambda \end{pmatrix}$$

by the action of G. Take U to be the matrices of the second form and take $H = (1, 1)$ in Lemma 1-1.7. Let $P(x) = \det x$. Then $P|U = \lambda$ is the only invariant in $\mathbf{C}[\mathbf{t}]$, which is the ring of U, under the trivial action. The fibers when $\det x \neq 0$, are the orbits of the second matrix, and the fibers when $\det x = 0$ consists of all the orbits of all the matrices of the first form.

3. Lecture 3

3-1. **Operations of groups.**

Definition 3-1.1. A *linear action* of a group G on a vector space V is a homomorphism

$$G \to \mathrm{Gl}_n(V).$$

We shall express the action of G on V by $G : V$.

From the action $G : V$ we obtain an action $G : V^*$, where V^* is the *dual space* of V. Indeed, we have seen 2-1.1 that G acts on all functions on V, so it acts on linear functions. This action is called the *contragradient action.*

Given $G : U$ and $G : V$. Then we obtain an action $G : U \oplus V$ by $g(u \oplus v) = gu \oplus gv$, and an action $G : U \otimes V$ by $g(\sum_i u_i \otimes v_i) = \sum_i gu_i \otimes gv_i$.

Finally we have that $G : \mathbf{C}$ by the trivial action $gv = v$.

Definition 3-1.2. A *graded algebra* $R = \oplus_{k=0}^{\infty} R_k$, is an algebra R such that $R_0 = \mathbf{C}$ and $R_m R_n \subseteq R_{m+n}$. By a *graded module* over R we mean an R module $M = \oplus_{k\geq 0} M_k$, such that $R_i M_j \subseteq M_{i+j}$.

We get an action $G : T(V) = \oplus_{k\geq 0} T^k(V)$. Here $T^0(V) = \mathbf{C}$ and $T^k(V) = V \otimes \cdots \otimes V$, the tensor product taken k times, for $k > 0$. Also we get an action $G : S(V)$, where $S(V) = T(V)/(\{(a\otimes b - b\otimes a) \colon a, b \in V\})$ is the *symmetric algebra* of V. Indeed, the ideal generated by the elements $a \otimes b - b \otimes a$, for a and b in V, is G invariant. We have that $S(V)$ is graded, and we write $S(V) = \oplus_{k=0}^{\infty} S^k V$.

Similarly, G acts on the exterior algebra $\bigwedge(V) = T(V)/(\{(a \otimes b + b \otimes a) \colon a, b \in V\})$. Again $\bigwedge(V)$ is graded and we write $\otimes_{k\geq 0} \bigwedge^k V$.

Notation 3-1.3. We let $\mathbf{C[V]}$ be the algebra of polynomial functions on V. That is $\mathbf{C[V]} = \mathbf{S(V^*)}$, and we have a grading $\mathbf{C[V]} = \sum_{\mathbf{k=0}}^{\infty} \mathbf{C[V]_k}$, coming from $S(V^*) = \oplus_{k=0}^{\infty} S^k(V^*)$.

Given a linear action $G : V$. Then G acts on $\mathbf{C[V]}$, preserving the grading. Hence $\mathbf{C[V]^G}$ is again a graded algebra $\mathbf{C[V]^G} = \oplus_{\mathbf{k=0}}^{\infty} \mathbf{C[V]_k^G}$.

The *Poicaré series* of an action $G : V$ is the formal power series

$$P_{G:V}(t) = \sum_{k\geq 0} (\dim \mathbf{C[V]_k^G})\mathbf{t^k},$$

in the variable t. The spectacular thing about the Poincaré series is that it often can be computed.

Example 3-1.4. Let $G = \mathbf{Z_2}$ and let G act on V by $-1v = -v$. Moreover, let $\dim V = n$. Then, by Moliens formula in Exercise 3-1.1, we have that $P_{G:V}(t) = \frac{1}{2}(\frac{1}{(1-t)^n} + \frac{1}{(1+t)^n}) = \frac{1}{2}\frac{(1+t)^n+(1-t)^n}{(1-t^2)^n}$. This is a symmetric expression in t. When $n = 1$ we get $P_{G:V}(t) = \frac{1}{1-t^2}$. In this case we have that $\mathbf{C[V]^G} = \mathbf{C[x^2]}$, so the ring of invariants is a polynomial ring. For $n = 2$ we have $P_{G:V} = \frac{1+t^2}{(1-t^2)^2}$. Hence, by Exercise 3-1.2 we have that the ring of invariants is not polynomial. We have that $\mathbf{C[V]^G} = \mathbf{C[x^2, y^2, xy]}$.

Remark 3-1.5. Even when $P_{G:V}(t) = 1/\prod_i(1-t^{m_i})$, we do not necessarily have that $\mathbf{C[V]^G}$ is a polynomial ring. Indeed, let

$$G =< \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} > .$$

We get a group of order 8 generated by 2 elements. It follows from Exercise 3-1.4 that the ring of invariants is not polynomial.

Theorem 3-1.6. (Hilbert-Samuel-Serre.) *Let $R = \oplus_{k\geq 0} R_i$ be a graded algebra generated by homogeneous elements $r_1, \dots, r_s$ of degrees $m_1, \dots, m_s$, all strictly greater than zero. Moreover, let $M = \oplus_{k\geq 0} M_k$ be a finitely generated graded module over R. Then we have that*

$$P_M(t) = \sum_{k\geq 0} (\dim M_k)t^k = \frac{f(t)}{\prod_{i=1}^g (1-t^{m_i})}, \quad \text{where } f(t) \in \mathbf{Z[t]}.$$

$P_M(t)$*In particular $P_R(t)$ has the above form.*

Proof. We use induction on the number s of generators. If $s = 0$ we have that $R = \mathbf{C}$, and consequently that $\dim M < \infty$. Hence we have that $P_M(t) \in \mathbf{Z[t]}$. Assume that $s > 0$ and consider the exact sequence

$$0 \to K_n \to M_n \xrightarrow{r_s} M_{n+m_s} \to L_{n+m_s} \to 0.$$

Let $K = \oplus_{n\geq 0} K_n$ and $L = \oplus_{n\geq 0} L_n$. Then K and L are graded $R/(r_s)$ modules. We have that L is finitely generated as a quotient of M and K is finitely generated by the Hilbert basis theorem 6-1.2. We have that

$$\sum_{n\geq 0} (\dim M_n)t^{n+m_s} + \sum_{n\geq 0} (\dim L_{n+m_s})t^{n+m_s} = \sum_{n\geq n} \dim(M_{n+m_s})t^{n+m_s} + \sum_{n\geq 0} (\dim K_n)t^{n+m_s}.$$

Consequently we have that

$$t^{m_s}P_M(t) + P_L(t) = P_M(t) + t^{m_s}P_K(t) + \sum_{i=0}^{m_s-1} (\dim M_i - \dim L_i)t^i.$$

Apply the inductive assumption to L and K. We obtain that

$$P_M(t) = \frac{f_1(t)}{\prod_{i=1}^{s}(1-t^{m_s})} - \frac{f_2(t)}{\prod_{i=1}^{s}(1-t^{m_s})} + \frac{f_3(t)}{\prod_{i=1}^{s}(1-t^{m_s})} = \frac{f_1(t)-f_2(t)+f_3(t)}{\prod_{i=0}^{s}(1-t^{m_s})}.$$

□

Remark 3-1.7. A large part of invariant theory consist of (i), to show that there is a finite number of generators of the ring of invariants, and (ii), to study the function $f(t)$ appearing in the Theorem.

Problem 3-1.1. Let G be a finite group acting linearly on a vector space V. Then:

(a) We have that $\frac{1}{|G|}\sum_{g\in G} g$ is a projection $V \to V^G$, which commutes with the action of G. A *projection* is a linear map that maps a space to a subspace in such a way that it is the identity on the subspace.

Assume that V is finite dimensional.

(b) We have that $\dim V^G = \frac{1}{|G|}\sum_{g\in G} \operatorname{tr} g$.

(c) (Moliens formula) We have that $P_{G:V}(t) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-tg)}$.

Hint:(a) is clear. For (b) we have that $\frac{1}{|G|}\sum_g \operatorname{tr} g = \frac{1}{|G|}\operatorname{tr}\sum_g g = \operatorname{tr}\frac{1}{|G|}\sum_g g = \dim V^G$, where the last equality follows from (a). To prove (c) we have that $P_{G:V}(t) = \sum_i \dim S^i(V^*)t^i = \sum_i\sum_g \frac{1}{|G|}\operatorname{tr}_{S^i(V^*)} gt^i = \frac{1}{|G|}\sum_g\sum_i \operatorname{tr}_{S^i(V^*)} gt^i$. Hence it suffices to show that $1/\det(a-gt) = \sum_i \operatorname{tr}_{S^i(V^*)} gt^i$. We have that $g^n = 1$ for some n since G is finite. Hence g is diagonalizable. Let g be the diagonal matrix with diagonal entries $(\lambda_1,\dots,\lambda_1,\dots,\lambda_r,\dots,\lambda_r)$ in some basis, where $\det(1-gt) = \prod_{i=1}^{r}(1-g\lambda_i)^{n_i}$, and $n_1+\cdots+n_r = \dim V$. We use that $1/(1-tx_1)\cdots(1-tx_n) = \sum_i\sum_{i_1+\cdots+i_n=i} x_1^{i_1}\cdots x_n^{i_n}t^i$. Then we have that $\dim S^i(V^*) = \sum_{i_1+\cdots+i_n=i} x_1^{i_1}\cdots x_n^{i_n}$ and when we let $x_1 = \cdots = x_{n_1} = \lambda_1$, $x_{n_1+1} + \cdots x_{n_1+n_2} = \lambda_2, \dots$, we obtain an expression for $\operatorname{tr}_{S^i(V^*)} g$, which gives the formula.

Remark 3-1.8. If G is compact group with invariant measure $d\mu$, then (a), (b) and (c) hold, with $|G| = \int_G d\mu$.

Problem 3-1.2. If $\mathbf{C[V]^G}$ is equal to the polynomial ring $\mathbf{C[P_1,\dots,P_k]}$ in polynomials P_i, where the polynomials P_i are homogeneous of degree m_i, then $P_{G:V}(t) = 1/\prod_i(1-t^{m_i})$.

Problem 3-1.3. Let $G_1 : V_1$ and $G_2 : V_2$. We may form the *direct product action*, $G_1\times G_2 : V_1\oplus V_2$, by $(g_1,g_2)(v_1+v_2) = g_1v_1+g_2v_2$. Show that $\mathbf{C[V_1\oplus V_2]^{G_1\times G_2} = C[V_1]^{G_1}\otimes C[V_2]^{G_2}}$. Hence, we have that $P_{G_1\times G_2:V_1\otimes V_2}(t) = P_{G_1:V_1}(t)_{G_2:V_2}(t)$.

Hint: We clearly have homomorphisms $\mathbf{C}[\mathbf{V_i}] \to \mathbf{C}[\mathbf{V_1} \oplus \mathbf{V_2}]$, for $i = 1, 2$, and hence we get a homomorphism $\mathbf{C}[\mathbf{V_1}] \otimes \mathbf{C}[\mathbf{V_2}] \to \mathbf{C}[\mathbf{V_1} \oplus \mathbf{V_2}]$. Chosing bases for the rings we easily see that this is an isomorphism. It is clear that the latter map induces a surjection $\mathbf{C}[\mathbf{V_1}]^{\mathbf{G_1}} \otimes \mathbf{C}[\mathbf{V_1}]^{\mathbf{G_2}} \to \mathbf{C}[\mathbf{V_1} \oplus \mathbf{V_2}]^{\mathbf{G_1} \times \mathbf{G_2}}$. Let f be contained in the right hand side and write $f = \sum_\beta f_\beta(x) y^\beta$, for some $f_\beta \in \mathbf{C}[\mathbf{V_1}]$. Let G_1 act. We see that $f_\beta \in \mathbf{C}[\mathbf{V_1}]^{\mathbf{G_1}}$. Let $f_1, \dots, f_r$ be a basis for the vector space spanned by the f_β. We get that $f = \sum_{i=1}^n \sum_\beta a_{\beta i} f_i(x) y^\beta$. Let G_2 act. We see that $g_i(y) = \sum_\beta a_{\beta i} y^\beta \in \mathbf{C}[\mathbf{V_2}]^{\mathbf{G_2}}$, for all i. Thus we have that $f = \sum_{i=1}^r f_i g_i \in \mathbf{C}[\mathbf{V_1}]^{\mathbf{G_1}} \otimes \mathbf{C}[\mathbf{V_2}]^{\mathbf{G_2}}$. We have proved the first part of the exercise. For the second part we note that, for graded algebras $A = \sum_{i \geq 0} A_i$ and $B = \sum_{i \geq 0} B_i$, we have that $P_{A \otimes B}(t) = P_A(t) P_B(t)$, since $\dim(A \otimes B)_l = \dim \sum_{i+j=l} A_k \otimes B_j = \sum_{i+j=l} \dim A_i \dim B_j$.

Problem 3-1.4. Let G be the group of Remark 3-1.5

(a) We have that $\mathbf{C}[\mathbf{V}]^{\mathbf{G}}$ is generated by x^2, y^2, xy, z^4. This is not a polynomial ring.
(b) We have that $P_{G:V}(t) = \frac{1}{(1-t^2)^3}$.

Hint: The group takes monomials to themselves with changed coefficients. We therefore only have to consider the monomials. If $x^i y^j z^k$ is invariant we must have that $4|k$ and $2|i + j$. We have that (b) follows from the previous Exercise.

4. Lecture 4

4-1. **Finite generation of invariants.** Given a group G operating linearly on a vector space V. The problem of deciding when the algebra $\mathbf{C[V]^G}$ is finitely generated is called *Hilbert's 14th problem.* We shall show that the ring is finitely generated when G is a reductive group. In particular it is true for finite groups, and, more generally, for compact groups. Nagata gave an example of a group and an action where the ring of invariants is not finitely generated.

Definition 4-1.1. Let G be a group acting by automorphism on an algebra R. A linear map

$$\sharp \colon R \to R^G$$

$\sharp$ is called a *Reynolds operator* if

(i) The map $\sharp$ is a projection onto R^G, that is we have $\sharp|R^G = I_{R^G}$.
(ii) We have that $(ab)^\sharp = ab^\sharp$ if $a \in R^G$ and $b \in R$.

Remark 4-1.2. Note that condition (i) implies that $\sharp$ is surjective and that condition (i) holds, if and only if $(\sharp|R^G)\sharp = \sharp$.

With an *operation* of a group G on a graded algebra $R = \oplus_{m\geq 0} R_i$, we shall always mean an operation that preserves the grading, that is $gR_m \subseteq R_m$, for all g and m. Then we clearly have that R^G is a graded subalgebra of R.

Theorem 4-1.3. *Let G act by automorphisms on a graded algebra $R = \oplus_{m\geq 0} R_m$. Suppose that R is a finitely generated $\mathbf{C} = \mathbf{R_0}$ algebra and that there is a Reynolds operator $\sharp \colon R \to R^G$ on R. Then R^G is finitely generated.*

Proof. We have that R^G is a graded algebra. Let I be the ideal of R generated by the elements $\oplus_{m>0} R_m^G$. By the Hilbert basis theorem we have that I is generated by a finite subset of a system of generators for I. We thus have that I can be generated by a finite number of homogeneous elements $P_1, \dots, P_n$ of positive degrees $k_1, \dots, k_n$.

We claim that the polynomials $P_1, \dots, P_n$ generate R^G. To prove this we let $P \in R^G$ be a homogeneous invariant of positive degree k. We shall prove the claim by induction on k. Write $P = \sum_{i=1}^n Q_i P_i$, where $Q_i \in R$. Apply the Reynolds operator to this expression. We get that $P = \sum_{i=0}^n Q_i^\sharp P_i$. Replace $Q_i^\sharp$ by its homogeneous component of degree $k - k_i$ for all i. Then the equation $P = \sum_{i=1}^n Q_i^\sharp P_i$ still holds. Since all k_i are positive we have that $k - k_i < k$. Consequently $\deg Q_I^\sharp < \deg P = k$. By the induction assumption we have that $Q_i^\sharp \in \mathbf{C}[\mathbf{P_1}, \dots, \mathbf{P_n}]$. Consequently we have that $P \in \mathbf{C}[\mathbf{P_1}, \dots, \mathbf{P_n}]$ and we have proved the Theorem. ☐

Problem 4-1.1. Take a line in the plane through origin and with irrational angle, so that it does not pass through any of the lattice points $\mathbf{Z^2}$. Take $R = \oplus_{(\alpha,\beta)\in S\cap \mathbf{Z}^2_+}\mathbf{C}\mathbf{x}^\alpha\mathbf{y}^\beta$. Then R is not finitely generated.

Hint: Let the line have the equation $y = \tau x$, with τ irrational. The monomials $x^\alpha y^\beta$ that lie below the line $\beta \le \gamma\alpha$, for some γ, generate the algebra of monomials under the line. That is, we have that $x^\delta y^\epsilon$ is in the algebra when $\epsilon \le \gamma\delta$. If R were finitely generated, we could take a maximal rational γ. We then must have that $\gamma < \tau$. Take a rational $\gamma < \frac{b}{a} < \tau$. Then we have that $x^a y^b$ lies in R, but is not under $y \le \gamma x$.

Remark 4-1.4. It is an open problem to decide whether R, in the previous Exercise, is not the ring of invariants of some group G.

4-2. **Construction of Reynolds operators.** Let G be a group acting linearly on a finite dimensional vector space V. We want to construct a Reynolds operator

$$\sharp\colon \mathbf{C[V]} \to \mathbf{C[V]^G}.$$

When G is finite we let $\sharp = \frac{1}{|G|}\sum_{g\in G} g$, where each element g of G is considered as a linear map for V to itself, sending v to gv. Then $\sharp$ is a Reynolds operator by Exercise 4-2.1. Similarly, when G is compact we let $\sharp = \frac{1}{\mu(G)}\int_{g\in G} g\, d\mu$. For finite groups it was E. Noether who first proved that $\mathbf{C[V]^G}$ is finitely generated.

Definition 4-2.1. A *linear action*, or *representation*, of a group G in a vector space V is called *completely reducible* if it is a direct sum of irreducible representation.

Remark 4-2.2. We have not assumed that V is a finite dimensional vector space in the above definition. However, if $R = \oplus_{m\ge 0} R_m$ is a graded ring and each R_m is a finite dimensional vector space, we only need to verifiy that G is completely reducible on finite dimensional spaces.

Proposition 4-2.3. *Let G act by automorphisms on an algebra R, and suppose that this representation is completely reducible. Then there exists a Reynolds operator. Moreover, the Reynolds operator has the property that:*

If $U \subseteq R$ is a G invariant subspace, then $U^\sharp \subseteq U^G$.

Proof. Since the action of G is completely reducible we have that there is a decomposition $R = R^G \oplus R''$, where R'' is the canonical invariant complement guaranteed by Exercise 4-2.2. Then R'' is the sum of all nontrivial irreducible representations of G in R. Let $\sharp\colon R \to R^G$ be the projection of R on R^G.

We shall check that $\sharp$ is a Reynolds operator. Let $a \in R^G$ and $b \in R$. We have that $b = b' + b''$, where $b' \in R^G$ and $b'' \in R''$. Then we have that $b' = b^\sharp$. We also have that $aR'' \subseteq R''$, for $a \in R^G$, because let $V_\alpha \subseteq R''$. Then the map $V_\alpha \xrightarrow{a} V$ given by multiplication with a is either zero, in which case $aV_\alpha \subseteq R''$, or the image is isomorphic to V_α, in which case aV_α is one of the subspaces of R''. Hence we have that $ab = ab^\sharp + ab''$. However $ab^\sharp \in R^G$ and $ab'' \in R''$. Consequently we have that $(ab)^\sharp = ab^\sharp$. Hence $\sharp$ is a Reynolds operator.

To prove the last part we note that, since U is completely reducible, we can, by Exercise 4-2.2, write $U = U^G \oplus U''$, where U'' is the sum of all non trivial G invariant submodules of U. Hence we have that $U'' \subseteq R''$, where R'' is the module of the proof of the previous Proposition. Moreover we have that $U^G \subseteq R^G$. Hence we obtain that $U^\sharp \subseteq U^G$. $\square$

Problem 4-2.1. Let G be finite group. Show that the map $\sharp = \frac{1}{|G|}\sum_{g\in G} g$ is a Reynolds operator.

Hint:It is clear that $v^\sharp \in R^G$ for all v and that $v^\sharp = v$ when $v \in R^G$.

Problem 4-2.2. Given a completely reducible representation of G in V. Then we have that:

(i) Every invariant subspace has an invariant complement.
(ii) The space V decomposes uniquely into a direct sum of *isotypic* components. By a isotypic component we mean the sum of all equivalent irreducible representations.
(iii) When V is finitely dimensional, then V is completely reducible if and only if every G invariant subspace is completely reducible.

Indeed we have that, if every G invariant vector space $W \subseteq V$ has a G invariant submodule and we have that $V = W \oplus W'$ for some invariant subspace W', then V is completely reducible.

Hint: (i)Assume that V is a sum of irreducible G modules $\{V_\alpha\}_{\alpha\in I}$. for every G invariant module $W \subseteq V$ we have that, either $V_\alpha \cap X = 0$, or $V_\alpha \subseteq W$, because $V_\alpha \cap X \subseteq V_\alpha$ is invariant. A totally ordered family $\{I_\beta\}_{\beta\in J}$ of indices, that has the property that $\oplus_{\beta\in I_\beta} \oplus W$ is a direct sum, has a maximal element because a relation $\sum_{\beta\in I_\beta} r_\beta + w$ must lie in one of the subsets. Let I' be a maximal element. Then we have that $\oplus_{\beta\in I'} \oplus W = V$ because we can use the above observations to a V_α that is not in the sum.

(ii) Let $V = \sum_{\alpha\in I} W_\alpha$ be the sum of the isotypic components. Such a decomposition exists by the assumption. From (i) we conclude that $W_\alpha = \oplus_{\beta\in I_\alpha} V_\beta$, where all the spaces V_β are isomorphic irreducible. Let $\oplus_{\alpha\in J} W_\alpha$ be a maximal direct sum decomposition. If it is not the whole of V there is an V_β outside and consequently a W_β that is outside.

Hence $W_\beta \cap \oplus_{\alpha\in J} W_\alpha \neq \emptyset$. Then there is a map $V_\beta \subseteq W_\beta \subseteq V \to V_\alpha$ that is not zero for any α or β. This is impossible.

(iii) Let W be a G invariant subspace. Since V is completely reducible we have that $V = W \oplus W'$ where W' is G invariant. However, then W has a G invariant irreducible subspace. Indeed, we have a map $\sum_\alpha V_\alpha = V = W \oplus W' \to V/W = W$ and all the subspaces V_α of V can not be mapped to zero by this surjection. We have proved that all G invariant subspaces have G invariant irreducible subspaces. Let W_0 be the sum of all irreducible G invariant subspaces of W. Then, by assumption, we have that $V = W_0 \oplus W_0'$ for some G invariant subspace W_0' of V. We get that $W = W_0 \oplus (W_0' \cap W)$, because, if $w \in W$, we have that $w = w_0 + w_0'$, with $w_o \in W_0$ and $w_0' \in W_0'$, and $w_0' = w - w_0 \in W$. However, $W_0' \cap W$ contains irreducible G invariant subspaces, which is impossible by the definition of W_0.

5. Lecture 5

5-1. **Reductive actions.**

Remark 5-1.1. Proposition 4-2.3 makes it interesting to determine which groups have reductive action.

We have the following picture of groups:

{groups}⊃ {topological groups}⊃ {real Lie groups} ⊃ {complex Lie groups} ⊃ {algebraic groups}.

As for group actions $G \times X \to X$ we have the picture:

{group action} ⊃ {continous action} *supset* {smooth action} ⊃ {complex analytic action} ⊃ { algebraic action}.

When we have toplogical groups, smooth groups, analytic groups, . . . , we shall always assume that the action is continous, smooth, analytic, ... In particular, when the action of G on a space V is linear, we always have that it is analytic.

Lemma 5-1.2. *Let K be compact group acting linearly on a real finite dimensional vector space V. Let Ω be a convex K invariant subset of V. Then there exists a point p in Ω which is fixed by K.*

Proof. Take a point x in Ω and take the convex hull $[Kx]$ of the orbit of x by K. Then $[Kx]$ is compact convex K invariant subset of Ω. Let p be the center of gravity of $[Kx]$. Then $Kp = p$ because the center of gravity is unique. □

Proposition 5-1.3. *Let K be a compact group acting linearly on a real, or complex, finite dimensional vector space V. Then there exists a positive definite, respectively Hermitian, form on V, which is K invariant.*

Proof. We apply Lemma 5-1.2 to the real vector space of all symmetric, respectively Hermition, forms on V, with Ω the subset of all positive definite forms. Then Ω is convex because sums with positive coefficients of positive definite forms are again positive definite forms. □

Remark 5-1.4. We can obtain an alternative proof of the previous result by taking an arbitrary positive form F and forming $F_0 \in_{g\in K} gF\, d\mu(g)$. Then F_0 is a positive definit form which is K invariant.

Theorem 5-1.5. *Let G be a group acting linearly on a finite dimensional vector space V. Then we have that:*

(a) *When G is a compact topological group and V is a real, or complex, vector space V. Then the action is completely reducible.*
(b) *When G be a complex Lie group acting on a complex vector space V. Assume that*

(i) *G has finitely many connected components.*
(ii) *G contains a compact real Lie subgroup K such that* $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{T_e K} = \mathbf{T_e G}$.

Then the action of G on V is completely reducible.

Proof. (a) By Proposition 5-1.3 there is a positive definite, or Hermitian, bilinear G invariant form F on V. If U is a G invariant real, respectively complex, subspace of V, we have that $U^\perp = \{v \in V : F(u, U) = 0\}$ is a complementary real, respectively complex, subspace to U, which is G invariant. Hence we have that $V = U \oplus U^\perp$. If one of the two factors are not irreducible we can, by Exercise 4-2.2 decompose that factor further. Since V is finitely dimensional we get a decomposition of V into irreducible subspaces after fintely many steps.

(b) It follows from assumption (i) that the unity component has finite index. Hence it follows from Exercise 5-1.1 that we can assume that G is connected.

We must show that if $U \subseteq V$ is a complex G invariant subspace, then there exists a complementary complex G invariant subspace. Clearly U is K invariant. Hence, it follows from (a), that there is a K invariant complement $U^\perp$. We choose the latter notation because we may construct the complement from a Hermitian form. We must now show that $U^\perp$ is G invariant. Since K operates linearly on V we have a map $\varphi\colon K \to \mathrm{Gl}(V)$. We obtain a map of tangent spaces $d\varphi\colon T_e K \to T_e \mathrm{Gl}(V) = \mathrm{Mat}(\mathbf{C}, \mathbf{V})$. From $d\varphi$ we obtain an action of $T_e K$ on V. Let $X \in T_e K$ and let $\gamma(t)\colon \mathbf{R} \to \mathbf{G}$ be a one parameter group corresponding to X. We get that $\varphi\gamma\colon \mathbf{R} \to \mathbf{G}$ corresponds to $d\varphi X$. Hence we have that $\frac{\varphi\gamma(t) - I}{t}$ acts on V as matrices and we have that $\lim_{t\to 0} \frac{\varphi\gamma(t) - I}{t} v = d\varphi X v$. Since $U^\perp$ is K invariant we have that $\frac{\gamma(t) - I}{t} v = \frac{\varphi\gamma(t) - I}{t} v$ is in $U^\perp$, for all t and all $v \in U^\perp$. Consequently, the limit $d\varphi X v = X v$ will be in $U^\perp$, where we consider the action of $T_e K$ via $d\varphi$ and the action of G via φ. However, since G is analytic, we have that $T_e G \to \mathrm{Mat}(\mathbf{C}, \mathbf{V})$ is $\mathbf{C}$ linear, and we have that $T_e G$ acts $\mathbf{C}$ linearly on V. However, by assumption, we have that $T_e G = T_e K \otimes_{\mathbf{R}} \mathbf{C}$, so that $U^\perp$ is invariant under $T_e G$. We have that, if $X \in T_e G$ correponds to the one parameter subgroup $\gamma(t)$, then $\gamma(t)$ is the image of X by the exponential map $\exp\colon T_e G \to G$. Consequently we have that $X v = d\varphi X v$. Moreover we have that $\varphi \exp X = \exp d\varphi X$, so that $\exp X v = \varphi \exp X v = \exp d\varphi X v = e^M v$, where $M = d\varphi X \in \mathrm{Mat}(\mathbf{C}, \mathbf{V})$. However, if $M v \in U^\perp$, we have that $e^M v \in U^\perp$, so $\exp X v \in U^\perp$ for all $X \in T_e G$. Consequently $U^\perp$ is invariant under all elements in an open subset S of e in G. It thus follows from Exercise 5-1.2 that $U^\perp$ is invariant under G, and part (b) of the Theorem follows. $\square$

Remark 5-1.6. The proof of part (b) of the Theorem is calles *Weyl's unitary trick.*

We use that the action is analytic because we must have that the action on the tangent space $T_e\mathbf{C}$ is $\mathbf{C}$ linear.

Definition 5-1.7. A complex Lie group satisfying (i) and (ii) of the Theorem is called a *reductive* Lie group, and the compact subgroup is called its *compact form.*

Remark 5-1.8. We notice that from the standard linear action of a group, it is easy to construct an abundance of other linear actions. Indeed, the contragredient action, which is given by the inverse matrices, the action on the tensor products, symmetric products, exterior products, as well as all subrepresentations are linear actions. So is the direct sum of linear actions.

Example 5-1.9. (Examples of reductive groups and their compact form)

1 $G = \mathbf{C}^* \supset \mathbf{K} = \mathbf{S^1} = \{\mathbf{z}\colon |\mathbf{z}| = \mathbf{1}\}$.

Hint: We have that $T_e\mathbf{C}^* = \mathbf{C}\frac{\partial}{\partial \mathbf{z}} = \mathbf{C}\frac{\partial}{\partial \mathbf{x}} = \mathbf{C}\frac{\partial}{\partial \mathbf{y}}$, since $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{i\partial f}{\partial y}$, which is a consequence of the definition of derivation of analytic functions. Moreover we have that $T_eS^1 = \mathbf{R}\frac{\partial}{\partial \mathbf{y}}$ since $(1+\epsilon(a+ib))(1+\epsilon(a-ib)) = 1$ gives $a = 0$.

2 $G = \mathrm{Gl}_n(\mathbf{C}) \supset \mathbf{K} = \mathrm{U}_\mathbf{n} = \{\text{unitary matrices}\} = \{\mathbf{A} \in \mathrm{Gl}_\mathbf{n}(\mathbf{C})\colon \mathbf{A^t\bar{A}} = \mathbf{I}\}$. It is clear that K is compact. Moreover we have that

(a) G is *arcwise connected*, hence *connected.*

(b) $T_e\,\mathrm{Gl}_n(\mathbf{C})\,\mathrm{Mat}_\mathbf{n}(\mathbf{C}) = \mathbf{C} \otimes_\mathbf{R} \mathbf{T_eU_n}$.

Proof. To prove (a) we give $g \in \mathrm{Gl}_n(\mathbf{C})$. We will find a curve that passes e and g. Write $g = AJA^{-1}$, where J is the Jordan form of A. If we can find a path $J(t)$ from J to e, then $g(t)AJ(t)A^{-1}$ is a path joining g to e. But we can find the curve separately for each block of the Jordan form. Hence we can assume that the matrix has the form

$$\begin{pmatrix} \lambda & \epsilon & & \\ & \ddots & \ddots & \\ & & & \epsilon \\ & & & \lambda \end{pmatrix},$$

with $\epsilon = 1$. Letting ϵ go to zero we can connect J to a diagonal matrix, with λ on the diagonal. Now we can let λ go to 1 and connect J to e, and we have proved (a).

To prove (b) we use that $T_e \operatorname{U}_n$ is the space of *skew Hermitian* matrices

$$\begin{pmatrix} i\lambda_1 & & & \\ & \ddots & a_{ij} & \\ & -\bar{a}_{ij} & & \\ & & & i\lambda_n \end{pmatrix},$$

with the λ_i real numbers. Indeed, we have that $T_e \operatorname{U}_n = \{X\colon {}^t(I+\epsilon X)^t(I+\epsilon \bar{X}) = I\} = \{X + {}^t\bar{X} = 0$, so each X in $T_e \operatorname{U}_n$ is skew Hermitian. Then we have that iX is *Hermitian*, that is of the form

$$\begin{pmatrix} \mu_1 & & & \\ & \ddots & b_{ij} & \\ & \bar{b}_{ij} & & \\ & & & i\mu_n \end{pmatrix}.$$

However, every matrix A can be written as $A = \frac{1}{2}(A + {}^tA) + \frac{1}{2}(A - {}^tA)$, where the matrix in the the first parenthesis is Hermitian and the one in the second skew Hermitian. Hence we have proved (b) □

3 $G = \operatorname{Sl}_n(\mathbf{C}) \subset \mathbf{K} = \mathrm{SU}_\mathbf{n}$. Again we have that G is connected. The proof is the same is in Example (**2**).

Hint: We can, like in the previous example connect the element to a diagonal matrix. In this case the product of the diagonal elements is 1. We can replace the diagonal elements $\lambda_1, \dots, \lambda_n$ by $\lambda_1/\sqrt[n]{\lambda}, \dots, \lambda_n\sqrt[n]{\lambda}$, and let λ tend to 1 to get the unit matrix. Moreover, we have that $T_e \operatorname{Sl}_n(\mathbf{C}) = \{\mathbf{X}\colon \det(\mathbf{I} + \epsilon\mathbf{X}) = \mathbf{1}\} = \{\mathbf{X}\colon \mathbf{1} + \epsilon \operatorname{tr} \mathbf{X} = \mathbf{1}\} = \{\mathbf{X}\colon \operatorname{tr} \mathbf{X} = \mathbf{0}\}$. Hence we get those of the matrices of the previous example with trace equal to zero.

4 $G = \mathrm{O}_n = \{A \in \operatorname{Gl}_n(\mathbf{C})\colon \mathbf{A}^\mathbf{t}\mathbf{A} = \mathbf{I}\} \supset \mathbf{K} = \mathrm{O}_\mathbf{n}(\mathbf{R})) = \{\mathbf{A} \in \operatorname{Gl}_\mathbf{n}(\mathbf{R})\colon \mathbf{A}^\mathbf{t}\mathbf{A} = \mathbf{I}\}$. It follows from Exercise 5-1.3 that $\mathrm{O}_n(\mathbf{C})$ and $\mathrm{O}_n(\mathbf{R})$ have two connected components, and that the conditions of the Theorem are satisfied.

5 $G = \mathrm{SO}_n(\mathbf{C}) \supset \mathbf{K} = \mathrm{SO}_\mathbf{n}(\mathbf{R})$. It follows from Exercise 5-1.3 that G is connected and satifies the conditions of the Theorem.

6 Let n be even. $G = \mathrm{Sp}_n(\mathbf{C}) = \{\mathbf{A} \in \mathrm{Gl}_\mathbf{n}(\mathbf{C}) \colon \mathbf{A}\mathbf{J}{}^\mathbf{t}\mathbf{A} = \mathbf{J}\} \supseteq \mathbf{K} = \mathrm{Sp}_\mathbf{n}(\mathbf{C}) \cap \mathrm{U}_\mathbf{n}$, where

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

It follows from Exercise 5-1.4 that $\mathrm{Sp}_n(\mathbf{C})$ is archwise connected and K its connected compact form.

Remark 5-1.10. Considering the tremendous efforts made during the 19'th century to prove that the ring of invariants of $\mathrm{Sl}_2(\mathbf{C})$ on $S^k\mathbf{C}^\mathbf{2}$ was finitely generated, the following result by Hilbert was amazing.

Theorem 5-1.11. (Hilberts theorem.) *Let G be a reductive complex Lie group acting linearly on a finite dimensional complex vector space V. Then $\mathbf{C}[\mathbf{V}]^\mathbf{G}$ is finitely generated.*

Proof. We have that $\mathbf{C}[\mathbf{V}] = \mathbf{S}(\mathbf{V}^*)$. Since $G : V$ is linear, and thus analytic, the same is true for $G : V^*$, since the contragredient action is given by the inverse matrices. Hence the same is true for $G : S^k(\mathbf{V}^*)$. It follows from Theorem 5-1.5 that $G : S^k(V^*)$ is completely reducible. Hence, it follows from Proposition 4-2.3 that there exists a Reynolds operator $\sharp$ for $G : \mathbf{C}[\mathbf{V}]$. Hilbert's theorem now follows from Theorem 4-1.3. □

Remark 5-1.12. We note the in the proof of Hilberts theorem we constructed a Reynolds operator $\sharp$ for $G : \mathbf{C}[\mathbf{V}]$ such that, for all G invariant subspaces U of $\mathbf{C}[\mathbf{V}]$, we have that $U^\sharp \subseteq U$.

Problem 5-1.1. Let G be a group acting linearly on a finite dimensional vector space V over $\mathbf{R}$ or $\mathbf{C}$, and let G_0 be a subgroup of finite index in G. Then G is completely recucible if and only if G_0 is.

Hint: Assume that V is G reducible. Writing V as a sum of irreducible invariant subspaces, we see that we can assume that V is G irreducible. Since V is finite dimensional there is a smallest G_0 invariant subspace $W \subseteq V$. Then W must be G_0 irrducible. Choose the one with lowest dimension among the G_0 irreducible subspaces. Let $g_1, \dots, g_r$ be right representatives for the residual set G/G_0. The module spanned by $g_1W, \dots, g_rW$ will then be V. We have that g_iW is G_0 invariant, because W is G_0 invariant. The space g_iW is G_0 irreducible, because, if not, it would include a G_0 irreducible subspace of lower dimension. We have thus seen that V is the sum of the G_0 invariant subspaces. Hence V is G_0 reducible.

Assume that V is G_0 reducible. Choose a G invariant subspace $W \subseteq V$. We have a G_0 linear map $V \xrightarrow{\varphi} W$ which is the identity on W. Choose $g_1, \dots, g_r$ to be right representatives for G/G_0. Let $V \xrightarrow{\psi} W$ be the map defined by $\psi(v) = \frac{1}{|G/G_0|}\sum_i g_i^{-1}\varphi(g_i v)$. For $v \in W$ we have that $g_i v \in W$ so that $\varphi(g_i v) = g_i v$ and $\psi(v) = v$. Take $g \in G$ and set $g_i g = h_i g_{j_i}$, with $h_i \in G_0$. Then we have that $\psi(gv) = \frac{1}{|G/G_0|}\sum_i g_i^{-1}\varphi(g_i g v) = \frac{1}{|G/G_0|}\sum_i g_i^{-1}h_i\varphi(g_{j_i}v) = \frac{1}{|G/G_0|}\sum_i g g_{j_i}^{-1}\varphi(g_{j_i}v)$. Now we have that $g_{j_1}, \dots, g_{j_r}$ are all different because, if not, we have that $g_{j_r}g_{j_s}^{-1} = h_r^{-1}g_r g g^{-1} g_s^{-1} h_s \in G_0$. However, then we have that $g_{j_r}g_{j_s}^{-1} \in G_0$.

Problem 5-1.2. Let G be a connected topological group. Then G is generated by the elements of any neighbourhood of e.

Hint: Let U be a neighbourhood of e and let H be the subgroup generated by U. Then we have that H is open because if $h \in H$, then $hU \subseteq H$. However, H is also closed since the cosets of H, different from H, are also open. Since G is connected we have that $H = G$.

Problem 5-1.3. Show that $\mathrm{O}_n(\mathbf{C})$ and $\mathrm{O}_n(\mathbf{R})$ have two connected components, and that the conditions of Theorem 5-1.5 are satisfied.

Hint: We can find eigenvectors for all matrices A in $\mathrm{O}_n(\mathbf{C})$, and, since we have that $< Ax, Ay >=< x, y >$, all the eigenvaules are ± 1. We can take the space W generated by an eigenvector. Then $W^\perp$ is also invariant. By induction we can find an orthogonal basis such that A is diagonal, with ± 1 on the diagonal. If we have two -1's on the diagonal, say in the coordinates i and j, we can multiply by the matrix that has ones on the diagonal, except in the coordinates i and j and that looks like

$$\begin{pmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

in rows and columns i and j and has zeroes elsewhere. We see that we can get the matrix into a form where either all the elements on the diagonal are 1, or they are all 1 except for a 0 in the n the coordinate. This shows that $\mathrm{O}_n(\mathbf{C})$ has two connected components. Now, give $A \in \mathrm{O}_n(\mathbf{R})$. Write $S = A + {}^tA$. Then S is symmetric and consequently has a real eigenvalue λ, because $< Ax, x >=< x, {}^tAx >=< x, Ax >$, so $< \lambda x, x >= \lambda < x, x >=< x, \lambda x >= \bar{\lambda} < x, x >$. Let v be an eigenvector. We have that $\lambda Av = ASv = AAv + v$, so that $AAv = -v + \lambda Av$. Consequently the space $\{v, Av\}$ is stable under A. Choose an orthogonal basis for $\{v, Av\}$. Then we have that A acts like

$$\pm\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We can again transform the matrix to one with 1 on the diagonal, except possibly one, which is -1. It follows that $\mathrm{O}_n(\mathbf{R})$ has two components.

The tangent space to $\mathrm{O}_n(\mathbf{C})$ is $\{X\colon {}^t(I+\epsilon X)(1+\epsilon X) = I\} = \{X\colon X + {}^tX = 0\}$, that is the *antisymmetric* complex matrices. In the same way we obtain that $T_e\,\mathrm{O}_n(\mathbf{R})$ are the antisymmetric real matrices. It follows that $T_e\,\mathrm{O}_n(\mathbf{C}) = \mathbf{T_e}\,\mathrm{O}_\mathbf{n}(\mathbf{R}) \otimes_\mathbf{R} \mathbf{C}$.

Problem 5-1.4. Show that that $\mathrm{Sp}_n(\mathbf{C})$ is archwise connected and K its connected compact form.

Hint: We have a bilinear form $<,>$ on V such that $< x.x >= 0$ and $< x,y >= 0$ for all y implies that $x = 0$. A *transvection* on V is a linear map $\tau\colon V \to V$ defined by $\tau(u) = u + \lambda < x,u > x$ for all $u \in V$, where $\lambda \in \mathbf{C}$ and x a nonzero vector in V. It is clear that each transvection is a symplectic transformation.

For every pair x, y of elements of V such that $< x,y >\neq 0$ the transvection associated to $X - y$ and $\lambda < x,y >= 1$ will satisfy $\tau(x) = y$. Indeed, $\tau(x) = x + \lambda < x-y, x > (x-y) = x - \lambda < x,y > x + \lambda < x,y > y$.

For every pair x, y of nonzero vectors of V there is a product of at most 2 transvections that send x to y. Indeed, assume that $x \neq y$. By what we just saw it suffices to find an element z such that $< x,z >\neq 0$ and $< y,z >\neq 0$. If $< x >^\perp = < y >^\perp$ we can take z to be any element outside $< x >^\perp$ and if $< x >^\perp \neq < y >^\perp$ we take $u \in < x >^\perp \setminus < y >^\perp$ and $u' \in < y >^\perp \setminus < x >^\perp$ and $z = u + u'$.

Let a, b, a', b' be vectors in V such that $< a,b >= 1$ and $< a',b' >= 1$. Then there is a product of at most 4 transvections that sends a to a' and b to b'. Indeed, we have seen that we can find two transvections, whose product σ sends a to a'. Let $\sigma(b) = b''$. Then $1 =< a',b' >=< a,b >=< a',b'' >$. Consequently it suffices to find two more transvections that send b'' to b' and that fix a'. If $< b',b'' >$, we let $\tau(u) = u + \lambda < b'' - b', u > (b'' - b')$. Then we have that $\tau(b'') = b'$, by the same calculations as above, and we have that $\tau(a') = a''$ because $< b'' - b', a' >= 1 - 1 = 0$. On the other hand, when $< b',b'' >= 0$, we have that $1 =< a',b'' >=< a', a' + b'' >=< a',b' >$ and $< b'', a' + b'' >\neq 0 \neq < b', a', a' + b'' >$, so we can first tranform (a', b'') to $(a', a' + b'')$ and then the latter pair to (a', b').

We can now show that the symplectic group is generated by transvections. Choose a basis $e_1, e_1', \dots, e_m, e_m'$ of V such that $< e_i, e_i' >= 1$, for $i = 1, \dots, m$, and all other products of basis elements are 0. Let σ be an element in the symplectic group and write $\sigma(e_i) = \bar{e}_i$ and $\sigma(e_i') = \bar{e}_i'$. We have seen above that we can find a product τ of transvections that sends the pair (e_1, e_1') to $(\bar{e}_1, \bar{e}_1')$. Then $\tau^{-1}\sigma$ is the identity on the space generated by (e_1, e_1'). Thus $\tau^{-1}\sigma$ acts on the orthogonal complement of (e_1, e_1'), which is generated by the remaining basis vectors. Hence we can use induction on the dimension of V to conclude that σ can be written as a product of transvections.

Problem 5-1.5. Take $\mathbf{C}^* : \mathbf{C^3}$, via the operation

$$t \to \begin{pmatrix} t^k & & \\ & t^m & \\ & & t^{-n} \end{pmatrix},$$

where k, m, n are positive integers.

(a) Find a finite system of generators of the algebra of invariants.
(b) Show that the algebra of invariants is a polynomial algebra, if and only if, $n = \gcd(n, k)\gcd(n, m)$.

Hint: Note that, if all exponents were positive, there would be no invariants because the orbit contains 0.

Assume that k, m, n do not have a common divisor. The invariants are spanned by all $x^a y^b z^c$ such that $ak + bm - cn = 0$. Hence the generators are given by all points (a, b, c) with integer coefficients in the plane $kx + my - nz$. Let $d = (n, k)$ and $e = (n, m)$. Then we have that $(e, d) = 1$, and $de|n$. We get the points $(\frac{n}{d}, 0, \frac{k}{d})$ and $(0, \frac{n}{e}, \frac{m}{e})$. The ring is polynomial exactly if it is generated by these elements. To see this we assume that $de < n$. Solve the congruence $\frac{k}{d}a \equiv -m \pmod{\frac{n}{d}}$ for a Then we get an $a \geq 0$ such that $\frac{k}{d}a + m = c\frac{n}{d}$ for some integer c, and we must have $c > 0$ since the left hand side of the latter equation is positive. Then we have that $ka + md = cn$, so (a, d, c) is a solution. However, $d < \frac{n}{e}$, and consequently it is not in the space spanned by the other. Hence we do not have a polynomial ring.

6. Lecture 6

6-1. **Fundamental results from algebraic geometry.**

Definition 6-1.1. A commutative associative ring R is called *Noetherian* if one of the following three equivalent conditions hold, see Exercise 6-1.1:

(i) Any ideal I of A is finitely generated.
(ii) Any set of generators of an ideal I contains a finite subset of generators.
(iii) Any increasing sequence of ideals stablizes.

We have the following three fundamental results:

6-1.2. Hilbert's basis theorem *If A is Noetherian, then $A[t]$ is Noetherian.*

Corollary 6-1.3. *The polynomial ring $\mathbf{C}[\mathbf{t_1}, \dots, \mathbf{t_n}]$ and all its quotient rings are Noetherian.*

Definition 6-1.4. The set $\sqrt{I} = \{a \in R \colon a^k \in I, \text{for some integer } k\}$ is an ideal, which is called the *radical* of the ideal I.

6-1.5. Radical. *We have that $\sqrt{0}$ is the intersection of all prime ideals of R.*

6-1.6. (Extension of maps.) *Let B be a domain over $\mathbf{C}$. Let A be a subalgebra, and $t \in B$ such that A and t generate B. Fix a nonzero element $b \in B$. Then there exists an element a of A such that any homomorphism $\varphi \colon A \to \mathbf{C}$ with $\varphi(a) \neq 0$ extends to a homomorphism $\tilde{\varphi} \colon B \to \mathbf{C}$, such that $\tilde{\varphi}(b) \neq 0$.*

By induction we get:

Corollary 6-1.7. *The assertion of 6-1.5 holds for all finitely generated algebras B over A.*

6-1.8. Hilbert's Nullstellensatz. *Let I be an ideal in $\mathbf{C}[\mathbf{V}]$, where V is finitely dimensional, and let $M = \{x \in V : I(x) = 0\}$. Let $F \in \mathbf{C}[\mathbf{V}]$ be such that $F|M \equiv 0$. Then $F^k \in I$, for some positive integer k.*

Proof. We shall show that Hilbert's Nullstellensatz, follows from the assertions 6-1.5 and 6-1.6. Suppose, to the contrary, that $F \notin \sqrt{I}$. Then, by the assertion 6-1.5 applied to A/I, we have that there is a prime $J \supseteq I$ such that $F \notin J$. Consider its image b in $B = [\mathbf{V}]/\mathbf{J}$. Let $A = \mathbf{C}$ and $a = 1$. By assertion 6-1.6 we have that there exists a homomorphism $\tilde{\varphi} \colon B \to \mathbf{C}$ such that $\tilde{\varphi}(b) \neq 0$. Hence we have a map $\tilde{\varphi} \colon \mathbf{C}[\mathbf{V}] \to \mathbf{C}$ so that $\tilde{\varphi}(F) \neq 0$ and $\tilde{\varphi}(I) = 0$. But $\tilde{\varphi}$ defines a point p of V such that $I(p) = 0$ and $F(p) \neq 0$. A contradiction. □

Corollary 6-1.9. *If $f_1, \dots, f_k$ in $\mathbf{C}[\mathbf{V}]$ have no common zeroes, then there exists $g_1, \dots, g_k$ in $\mathbf{C}[\mathbf{V}]$ such that $1 = f_1 g_1 + \dots + f_k g_k$.*

Proof. We apply the Hilbert Nullstellensatz 6-1.8 to $I = (f_1, \dots, f_k)$ and choose $F = 1$. □

Problem 6-1.1. Show that the three conditions of Definition 6-1.1 are equivalent.

Problem 6-1.2. Show that the assertion 6-1.5 holds.

Problem 6-1.3. Prove the Corollary to assertion 6-1.6.

6-2. **Separation of invariants.**

Theorem 6-2.1. (Separation of invariants.) *Let G be a complex reductive group acting linearly on a finite dimensional vector space. Let I_1 and I_2 be G invariant ideals in $\mathbf{C}[\mathbf{V}]$, and let $M_i = \{x \in V \colon I_i(x) = 0\}$, for $i = 1, 2$. Assume that $M_1 \cap M_2 = \emptyset$. Then there exists a polynomial F in $\mathbf{C}[\mathbf{V}]^{\mathbf{G}}$, such that $F|M_1 = 0$ and $F|M_2 = 1$.*

Proof. Let $P_1, \dots, P_k$ be generators of I_1 and let $Q_1, \dots, Q_s$ be generators of I_2. Then $P_1, \dots, P_k, Q_1, \dots, Q_s$ have no common zeroes, by the assumption that $M_1 \cap M_2 = \emptyset$. Consequently, by the Corollary of Hilbert's Nullstellensatz 6-1.8 we have that there exists polynomials $f_1, \dots, f_k$ and $g_1, \dots, g_s$, such that $\sum_i P_i f_i + \sum_j Q_j g_j = 1$. Let $\varphi_1 = \sum_i P_i f_i$ and $\varphi_2 = \sum_j Q_j g_j$. Then we have that $\varphi_i \in I_i$, for $i = 1, 2$ and $\varphi_1 + \varphi_2 = 1$. Since G is reductive, it follows from Theorem 5-1.5 and Proposition 4-2.3 that there is a Reynolds operator $\sharp \colon \mathbf{C}[\mathbf{V}] \to \mathbf{C}[\mathbf{V}]^{\mathbf{G}}$, such that $I_i^\sharp \subseteq I_i$, for $i = 1, 2$. Apply the Reynolds operator to $\varphi_1 + \varphi_2 = 1$. We obtain $\varphi_1^\sharp + \varphi_2^\sharp = 1$, where $\varphi_i^\sharp \in I_i \cap \mathbf{C}[\mathbf{V}]^{\mathbf{G}}$. Put $F = \varphi_1^\sharp$. Then we have that $F \in \mathbf{C}[\mathbf{V}]^{\mathbf{G}}$. Moreover, we have that $F|M_1 = 0$ because $F \in I_1$. However, we have that $F|M_2 = (1 - \varphi_2^\sharp)|M_2 = 1$. □

There is a compact counterpart of the above result.

6-2.2. *Let K be a compact group that acts linearly and continously on a real vector space V. Let F_1 and F_2 be disjont K invariant compact subsets of V, with disjoint orbits. Then there exists a polynomial $P \in \mathbf{R}[\mathbf{V}]^{\mathbf{K}}$, such that $P|F_1 > 0$ and $P|F_2 < 0$.*

Proof. There exists a continous function P_1 on V such that $P|F_1 > 0$ and $P|F_2 < 0$. Indeed, fix a point $x \in F_1$ and take $P_1(x) = \epsilon - d(x_1 F_1)$. Here $d(x_1 F_1)$ is zero on F_1. Let P_2 be a polynomial which approximates P_1 up to δ. Such a polynomial exists by the Stone Weierstrass theorem, that asserts that on a compact set a continous function can be approximated by a polynomial. For small δ we have that $P_2|F_1 > 0$ and $P_2|F_2 < 0$. Let $\deg P_2 \leq N$, and denote by Ω the set of all polynomial functions of degree at most equal to N, that are positive on F_1 and negative on F_2. Then Ω is nonempty. It is clearly also convex. Hence, by the fix point theorem, there is a fixed point $P \in \Omega$. This polynomial has the desired properties. $\square$

7. Lecture 7

7-1. **Linear algebraic groups.** The *Zariski toplogy* on $V = \mathbf{C}$ is the topology whose closed subsets are the sets $F = \{x \in \mathbf{C^n} \colon \mathbf{P_1(x)} = \cdots = \mathbf{P_k(x)}$, for all collections of polynomials $P_1, \dots, P_k \in \mathbf{C[V]}\}$, see Exercise 7-1.1.

Example 7-1.1. On $\mathbf{C^1}$ the closed subsets are the finite subsets together with the whole space and the empty set.

Remark 7-1.2. We have that V is *quasi compact*, that is, every decreasing sequence of closed sets $F_1 \supseteq F_2 \subseteq \cdots$ stabilizes. Indeed, let $I_k = \{f \in \mathbf{C[V]} \colon \mathbf{f|F_k = 0}\}$. Then, by the Hilbert basis theorem 6-1.2, the sequence $I_1 \subseteq I_2 \subseteq \cdots$ stabilizes. Consequently the sequence $F_1 \supseteq F_2 \supseteq \cdots$ stabilizes, by the Hilbert Nullstellensatz.

Note that the ideals I_k are radical. Moreover, there is a bijection between closed subsets of $\mathbf{C^n}$ and radical ideals of I. Indeed, we have seen above how a closed set F gives rise to a radical ideal $\{f \in \mathbf{C[V]} \colon \mathbf{f|F = 0}\}$. Conversely, given an ideal I we get a closed set $\{x \in \mathbf{C^n} \colon \mathbf{I(x) = 0}, \text{for all } \mathbf{F \in I}\}$. By Hilbert Nullstellensatz these are inverses.

The topology induced by the Zariske topology of V on subset M, is called the *Zariski* topology on M.

Definition 7-1.3. A set is *irreducible* if it can not be decomposed into a union of two nonempty closed subset.

Proposition 7-1.4. *Let F be a closed subset of V. Then $F = \cup_{j=1}^N F_j$, where the F_j are all maximal irreducible subsets of F.*

Proof. Exercise 7-1.2 □

Remark 7-1.5. If F is a closed irreducible subset of V, then it is archwise connected. This is not trivial, see [**?**].

Definition 7-1.6. A *linear algebraic group G* in $\mathrm{Gl}(V)$, is a subgroup which is given as the zeroes of polynomials. That is, there exists polynomials $P_1, \dots, P_k$ in $\mathbf{C}(\mathrm{End}(\mathbf{V}))$, for which $G = \{g \in \mathrm{Gl}(V) \colon P_i(g) = 0, \text{ for } i = 1, \dots, n\}$.

Example 7-1.7. The groups $\mathrm{Gl}_n(\mathbf{C})$, the diagonal matrices $(\mathbf{C}^*)^\mathbf{n}$, $\mathrm{Sl}_n(\mathbf{C})$, $\mathrm{O}_n(\mathbf{C})$, $\mathrm{SO}_n(\mathbf{C})$, $\mathrm{Sp}_n(\mathbf{C})$ are all algebraic groups. The unitary group U_n is not a complex algebraic group, but it is a real algebraic group.

Remark 7-1.8. The general linear group $\mathrm{Gl}_n(\mathbf{C})$ is not naturally a closed set in an affine space. However, it can be made a closed subgroup of

$\mathrm{Sl}_{n+1}(\mathbf{C})$, by the map

$$g \to \begin{pmatrix} g & 0 \\ 0 & (\det g)^{-1} \end{pmatrix}.$$

Remark 7-1.9. Any linear algebraic group is a complex Lie group. This follows from the implicit function theorem, at every smooth point, but since we are on a group all points are smooth.

Proposition 7-1.10. *Let G be a linear algebraic group. Then the irreducible components are disjoint, the unity component G_0 is a normal subroup, and all other components are cosets of G_0 in G.*

Proof. First we have that G_0 is a group because the image of $G_0 \times G_o$ in G, by multiplication, contains G_0 and is irreducible by Exercise 7-1.3. Write $G = \cup_{j=0}^{N} G_j$, where G_0 is the unity component. Suppose that $h \in G_0 \cap G_j$, for some $j \neq 0$. Then gG_j is an irreducible component by Exercise 7-1.3. Through any other point $h_1 \in G_0$ we have an irreducible component $h_1 h^{-1} G_j$, and it is not equal to G_0 because $h_1 h^{-1} G_j = G_0$ implies $G_j = G_0$, and h and h_1 are in G. Hence, each point of G_0 is contained in another irreducible component and consequently we have that $G_0 = \cup_{j \neq 0}(G_0 \cap G_j)$. But this is a decomposition which contradicts the irreducibility of G_0. Consequently $G = \cup_{j=0}^{n} G_j$, where $G_0 \cap G_j = \emptyset$.

We have that G_0 is normal since gG_0g^{-1} contains e and this is an irreducible component. Hence $gG_0g^{-1} = G_0$ for all g in G. Finally gG_0 is an irreducible component. Consequently cosets of G_0 in G are irreducible components. □

Corollary 7-1.11. *A linear algebraic group is irreducible if and only if it is connected in the Zariski topology.*

Remark 7-1.12. A linear algebraic group is connected in the Zariski topology, if and only if, it is archwise connected in the *metric topology*, see Remark 7-1.5.

Problem 7-1.1. Show that the sets $F = \{x \in \mathbf{C^n} \colon \mathbf{P_1(x)} = \cdots = \mathbf{P_k(x)}\}$, for all collections of polynomials $P_1, \dots, P_k \in \mathbf{C}[\mathbf{V}]$, are the closed sets of a topology on $\mathbf{C^n}$. Show that the topology is T_1, that is, given points x and y of $\mathbf{C^n}$, then there is an open set containing one, but not the other.

Problem 7-1.2. Prove Proposition 7-1.4.

Problem 7-1.3. (a) Given an increasing sequence of irreducible sets $M_1 \subset M_2 \subset \cdots$. Then the union $\cup_i M_i$ is irreducible.
(b) Given $M_1 \subseteq V_1$, and $M_2 \subseteq V_2$, both irreductible. Then $M_1 \times M_2$ is an irreducible subset of $V_1 \times V_2$.

(c) Suppose $M \subset \mathbf{C^n}$ is irreducible and let $\varphi\colon \mathbf{C^m} \to \mathbf{C^n}$ be given by rational functions defined on M. Then $\varphi(M)$ is irreducible.

7-2. **The Lie-Kolchin theorem.** Given a group G and elements a, b. We call the element $[a, b] = aba^{-1}b^{-1}$ the *commutator* of a and b and we denote by $G^{(1)}$ the subgroup og G generated by all the commutators. Moreover, we denote by $G^{(i+1)}$ the subgroups of $G^{(i)}$ generated by all the commutators of $G^{(i)}$. A group G is *solvable* if the chain $G \supseteq G(1) \supseteq G^{(2)} \supseteq \cdots$ ends in e. This is equivalent to having a chain of groups $G \supset G_1 \supset G_2 \supset \cdots \supset e$, such that G_i/G_{i+1} is abelian for all i. Indeed, if we have a chain of the latter type, then $G^{(i)} \subseteq G_i$, for all i.

Lemma 7-2.1. *The derived group $G^{(1)}$ is connected if G is connected.*

Proof. Let $G^{(1)}_k = \{[a_1, b_1] \cdots [a_k, b_k]\}$ consist of the products of k commutators. Then $G^{(1)}_1 \subseteq G^{(1)}_2 \subseteq \cdots$ and $\cup_k G^{(1)}_k = G^{(1)}$, and each $G^{(1)}_k$ is irreducible by Excercise 7-1.3 (c), because $G^{(1)}_k$ is the image of the product of G with itself $2k$ times. It follows from Exercise 7-1.3 (a) that $G^{(1)}$ is irreducible, and consequently connected. □

Theorem 7-2.2. (The Lie-Kolchin theorem.) *Let G be a solvable, but not necessarily algebraic, subgroup of $\mathrm{Gl}_n(\mathbf{C})$ that acts on $V = \mathbf{C^n}$, via the inclusion. Then there exists a non zero vector v in V such that $gv = \lambda v$, where $\lambda(g) \in \mathbf{C}^*$, for all g in G.*

Proof. Since G is solvable, we have a series $G \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots \supseteq G^{(k)} = e$. We prove the Theorem by induction on k. If $k = 1$, the Theorem holds since $G^{(1)} = e$. Thus G is commutative, and, over the complex numbers, any commutative set has common eigenvectors. Indeed, if S is a commuting set of matrices, then they have a common eigenvector. To see this choose an A in S and let $V_\lambda = \{v \in V\colon Av = \lambda v\}$. This vector space is not empty for some λ. Since the matrices commute, we have that each matrix of S sends V_λ to itself. Hence we can continue with the next element of S. Since V has finite dimension this process will stop and give a common eigenvector for the matrices in S, see also 13-1.10.

Assume that $k > 1$ and apply the inductive assumption to $G^{(1)}$. Then there exists a nonzero vector v in V such that $gv = \lambda(g)v$ for all $g \in G^{(1)}$. Note that $\lambda\colon G^{(1)} \to \mathbf{C}^*$ is a character, that is $\lambda(g_1g_2) = \lambda(g_1)\lambda(g_2)$. Given a character λ of $G^{(1)}$, we let V_λ be the space of all vectors v satisfying $gv = \lambda(g)v$, for all $g \in G^{(1)}$. We know that $V_\lambda \neq 0$ for some character λ. We have that $gV_\lambda \subseteq V_{\lambda^g}$, for any $g \in G$, where $\lambda^g(h) = \lambda(g^{-1}hg)$, for $h \in G^{(1)}$ is again a character of $G^{(1)}$. Indeed, take $v \in V_\lambda$ and $h \in G^{(1)}$. Then $h(gv) = g(g^{-1}hg)v = \lambda(g^{-1}hg)gv$, because $g^{-1}hg \in G^{(1)}$.

By Exercise 7-2.1 the V_λ for different characters λ of $G^{(1)}$ form a direct sum. In particular there is only a finite number of characters λ of $G^{(1)}$, such that $V_\lambda \neq 0$.

Let $G' = \{g \in G \colon gV_\lambda \subseteq V_\lambda\}$. Then G' is a subgroup of G. We have that G acts as a permutation on the finite set of nonzero V_λ's. Consequently G' has finite index in G. Moreover, it is closed in G since, if $v_1, \dots, v_n$ is a basis for V whose first m elements form a basis for V_λ. Indeed, the condition that $gV_\lambda \subseteq V_\lambda$ is expressed as the condition that the coefficients for the last $n-m$ vectors in the expression for gv_i in the basis is zero, for $i = 1, \dots, m$. Since G is connected we must have that $G = G'$, that is $GV_\lambda \subseteq V_\lambda$. But the elements of $G^{(1)}$ considered as linear maps on V_λ have determinant 1, since they are products of commutators, and they operate as scalars $\lambda(g)$ on V_λ. However, there is only a finite number of scalar matrices with determinant 1. Thus the image of $G^{(1)}$ in $\mathrm{Gl}(V_\lambda)$ is finite. Since $G^{(1)}$ is connected, the image must be 1. So $G^{(1)}$ acts trivially on V_λ, and thus the elements of G, considered as linear maps on V_λ, commute. However, as we saw above, commuting matrices have a common eigenvector. □

Corollary 7-2.3. *Under the assumptions of the Lie-Kolchin theorem, there exists a basis of V, on which all elements $g \in G$ are upper triangular matrices. Hence, elements of $G^{(1)}$ are matrices with units on the diagonal.*

Proof. We prove the Corollary by induction on the dimension of V. Let V_1 be generated by a common eigenvector for the elements of G. Use the Theorem on V/V_1. □

Remark 7-2.4. Every finite sovable group G, with a non trivial 1 dimensional representation, gives an example where the Lie-Kolchin theorem fails when G is not connected.

Hint: Take for example the subroup of $\mathrm{Gl}_2(\mathbf{C})$ spanned by

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then G has order 8 and consists of all the matrices

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

We have the relation $(\sigma\tau)^4 = 1$. It is clear that σ and τ do not have common eigenvectors. In fact, we have $\sigma v_1 = v_2$, $\sigma v_2 = v_1$, $\tau v_1 = v_1$ and $\tau v_2 = -v_2$.

Problem 7-2.1. Let G be a group. Then the nonzero spaces V_λ, for different characters λ of G, form a direct sum.

8. Lecture 8

8-1. Algebraic groups.

Definition 8-1.1. An *affine algebraic variety* is a Zariski closed subset of $\mathbf{C^n} = \mathbf{V}$. To an affine algebraic variety M there corresponds an ideal $I = \{f\colon f|M = 0\}$ of $\mathbf{C}[\mathbf{V}] = \mathbf{C}[\mathbf{x_1}, \dots, \mathbf{x_n}]$. We have that I is a radical ideal. To I we associate the ring $\mathbf{C}[\mathbf{M}] = \mathbf{C}[\mathbf{V}]/\mathbf{I}$.

To an affine algebraic set M we have associated an algebra $\mathbf{C}[\mathbf{M}]$ which is a finitely generated algebra which has no nilpotent elements, and with a fixed system of generators. This defines is a bijective correspondence between the algebraic sets of $\mathbf{C^n}$ and the finitely generated algebras with no nilpotent elements, and with a fixed system of n generators, see Exercise 8-1.1.

Definition 8-1.2. Given closed subsets M_1 and M_2 of $\mathbf{C^n}$. A *regular map*, or *morphism*, from M_1 to M_2 is a map $\varphi\colon M_1 \to M_2$, which is the restriction to M_1 of a polynomial map $\varphi\colon \mathbf{C^m} \to \mathbf{C^n}$. We say that φ is an *isomorphism* between M_1 and M_2 if there exists a regular map $\psi\colon M_1 \to M_2$, such that $\varphi\psi$ and $\psi\varphi$ are identity maps.

To a regular map $\varphi\colon M_1 \to M_2$ there corresponds a homomorphism $\varphi^*\colon \mathbf{C}[\mathbf{M_1}] \to \mathbf{C}[\mathbf{M_2}]$. If follows from Exercise 8-1.2 that every homomorphism is obtained in this way. It follows that the correspondence that sends an affine algebraic set M to the algebra $\mathbf{C}[\mathbf{M}]$ gives a contravariant functor between the category of affine algebraic varieties with regular maps, and the category of finitely generated algebras without nilpotent elements, with algebra homomorphisms.

Remark 8-1.3. Let M be an affine algebraic variety, and let $P_1, \dots, P_k$ be a system of generators of $\mathbf{C}[\mathbf{M}]$. Consider the map $\varphi\colon M \to \mathbf{C^k}$ defined by $\varphi(x) = (P_1(x), \dots, P_k(x))$. Then $\varphi(M)$ is closed in $\mathbf{C^k}$ and $\varphi\colon M \to \varphi(M)$ is an isomorphism. This follows from Exercise 8-1.1.

Given closed subsets M_1 and M_2 of $\mathbf{C^m}$ respectively $\mathbf{C^n}$. Then $M_1 \times M_2$ is a closed subset of $\mathbf{C^m} \times \mathbf{C^n}$.

Proposition 8-1.4. *We have that* $\mathbf{C}[\mathbf{M_1} \times \mathbf{M_2}]$ *is canonically isomorphic to* $\mathbf{C}[\mathbf{M_1}] \otimes \mathbf{C}[\mathbf{M_2}]$.

Proof. Define the map $\varphi\colon \mathbf{C}[\mathbf{M_1}] \otimes \mathbf{C}[\mathbf{M_2}] \to \mathbf{C}[\mathbf{M_1} \times \mathbf{M_2}]$ by $\varphi(f \otimes g)(x_1, x_2) = f(x_1)g(x_2)$. Any element of $\mathbf{C}[\mathbf{M_1} \times \mathbf{M_2}]$ comes from a polynomial in $\mathbf{C}[\mathbf{C^m} \times \mathbf{C^n}]$ and thus are sums of products of polynomials in $\mathbf{C}[\mathbf{C^m}]$ and $\mathbf{C}[\mathbf{C^n}]$. Hence the map is surjective. To show that it is injective, take an element of $\mathbf{C}[\mathbf{M_1}] \otimes \mathbf{C}[\mathbf{M_2}]$ of the form $\sum_i f_i \otimes g_i$, with the f_i linearly independent over $\mathbf{C}$. If $\sum_i f_i \otimes g_i \neq 0$, then at least one of the g_i must be a nonzero function. Assume that $\sum_i f_i \otimes g_i$ is in the kernel of φ. We have that $\sum_i f_i(x)g_i(y) = 0$, for all $x \in M_1$ and $y \in M_2$. Fix a y_0 at which one of the g_i is not zero. Then we obtain a nontrivial linear dependence between the f_i. □

Definition 8-1.5. An *affine algebraic group* G is an affine algebraic variety, together with regular maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$, and an element $e \in G$, wuch that the group axioms hold for the product μ and the inverse ι.

A *regular action* of an affine algebraic group G on an affine algebraic variety is a map $G \times M \to M$, that satisfies the axioms of an action.

Example 8-1.6. Examples of linear groups

(a) Let G be a closed subset of $\mathrm{Mat}_n(\mathbf{C})$, which is a closed subvariety and which is a group with respect to matrix multiplication. Then G is an affine algebraic group. Indeed, we have that $\mathbf{C}[\mathrm{Gl}_\mathbf{n}(\mathbf{C})] = \mathbf{C}[\mathbf{x_{ij}}, \mathbf{d}]/(\mathbf{d}\det(\mathbf{x_{ij}}) - \mathbf{1})$, where the x_{ij} and d are indeterminates. Hence we have that $\mathrm{Gl}_n(\mathbf{C})$ is an affine algebraic variety. In the example it is clear that the multiplication is given by polynomials. However, also the inverse is a polynomial in x_{ij} and d. Hence $\mathrm{Gl}_n(\mathbf{C})$, is an affine algebraic group and thus all the closed subgroups are.

(b) Let M be closed subset in $\mathbf{C^n}$ which is G invariant, where $G \subseteq \mathrm{Gl}_n(\mathbf{C})$ is as in (a). Then the restiction of the action of G to M is a regular action. This action is clearly algebraic.

Remark 8-1.7. We shall show that all algebraic groups are linear and that all actions are obtained as in (b) above.

Lemma 8-1.8. *Let $G : M$, where G is an affine algebraic group and M an affine algebraic variety, and let $F \in \mathbf{C}[\mathbf{M}]$. Then the subspace of $\mathbf{C}[\mathbf{M}]$ spanned by the elements $\{gF\colon g \in G\}$ has finite dimension.*

Proof. We have a map $\alpha\colon G \times M \to M$, with corresponding map of coordinate rings $\alpha^*\colon \mathbf{C}[\mathbf{M}] \to \mathbf{C}[\mathbf{G}] \otimes [\mathbf{M}]$, where $\alpha^* F(g, m) = F(gm)$. We have that $\alpha^* F = \sum_{i=1}^N \varphi_i \otimes \psi_i$. We obtain that, if $g \in G$ and $m \in M$, then $(\alpha^* F)((g^{-1}, m)) = \sum_{i=1}^N \varphi_i(g^{-1})\psi_i(m)$, by the identification $\mathbf{C}[\mathbf{G}] \otimes \mathbf{C}[\mathbf{M}] \cong \mathbf{C}[\mathbf{G} \times \mathbf{M}]$. Moreover, we obtain that $F(g^{-1}m) = \sum_{i=1}^N \varphi_i(g^{-1})\psi_i(m)$, for all $g \in G$ and $m \in M$. This means that $gF = \sum_{i=1}^N \varphi(g^{-1})\psi_i$. Consequently the space generated by the elements gF is contained in the space generated by $\psi_1, \dots, \psi_N$. □

Remark 8-1.9. The Lemma is false if we instead of polynomials take rational functions. For example take $G = \mathbf{C}^*$ acting on $\mathbf{C}$ by multiplication. Then G acts on fuctions by $g\varphi(R) = \varphi(g^{-1}R)$. Consequently, an element $\lambda \in \mathbf{C}^*$ sends the polynomial x to $\lambda^{-1}x$. If $F = a_0 + a_1x + \cdots + a_nx^n$ we get that $\lambda F = a_0 + a_1\frac{1}{\lambda}x + \cdots + a_n\frac{1}{\lambda^n}x^n$. Consequently, we have that $\{\lambda F \colon \lambda \in \mathbf{C}^*\} \subseteq \{\mathbf{1}, \mathbf{x}, \ldots, \mathbf{x^n}\}$. On the other hand we have that λ sends $\frac{1}{1+x}$ to $\frac{1}{1+\lambda^{-1}x}$, which has a pole in $-\lambda$. Consequently the space generated by the elements gF can not be in a finite dimensional space of rational functions.

Theorem 8-1.10. (Linearization theorem.) *Let $G : M$ be a regular action of an affine algebraic group G on an affine algebraic variety M. Then there exists a linear regular action $G : V$, a closed G invariant subset $\tilde{M}$ of V, and an isomorphism $\psi\colon M \to \tilde{M}$, such that the following diagram is commutative:*

$$\begin{array}{ccc} G \times M & \xrightarrow{1\times\psi} & G \times \tilde{M} \\ \alpha\big\downarrow & & \big\downarrow\tilde{\alpha} \\ M & \xrightarrow{\psi} & \tilde{M} \end{array},$$

where $\tilde{\alpha}\colon G \times \tilde{M} \to \tilde{M}$ is the action $G : \tilde{M}$ induced by $G : V$.

Proof. Let $Q_1, \ldots, Q_n$ be a system of generators of $\mathbf{C[M]}$. Morever, let U be the space generated by $\{gQ_i \colon g \in G, i = 1, \ldots, n\}$. We have that U generates $\mathbf{C[M]}$, since it contains the Q_i, and it is G invariant. It follows from Lemma 8-1.8 that U is finite dimensional. Let $P_1, \ldots, P_N$ be a basis for U. Then $P_1, \ldots, P_N$ is also a system of generators of $\mathbf{C[M]}$.

Let $\psi\colon M \to V = \mathbf{C^N}$ be defined by $\psi(x) = (P_1, (x), \ldots, P_N(x))$, and let $\tilde{M}$ be the image of M. By Remark 8-1.3 we have that $\tilde{M}$ is a closed subset of V and that $\psi\colon M \to \tilde{M}$ is an isomorphism. The action of G on V is defined to be induced from that of G on U. We have that $gP_i(m) = P_i(g^{-1}m) = \alpha^*P_i((g^{-1}, m)) = \sum_{j=1}^N (a_{ij} \otimes P_J)(g^{-1}, m) = \sum_{j=1}^N a_{ij}(g^{-1}P_j(m)$. Consequently, we have that $gP_i = \sum_{j=1}^N \varphi_{ij}(g^{-1})P_j$, where $\varphi_{ij} \in \mathbf{C[G]}$. We obtain an *anticommutativ* linear algebraic action of G on M via the map that sends g to $(\varphi_{ij}(g^{-1}))$ in $\mathrm{Mat}_N(\mathbf{C})$. This is *anticommutative* because we have $g(hP_i(m)) = g(P_i(h^{-1}m)) = P_i(g^{-1}h^{-1}m) = P_i((hg)^{-1}m)$. We obtain an action of G on M by sending g to $(\varphi_{ij}(g))$, and we get an action of G on V by $g(a_1, \ldots, a_N) = (a_{ij}(g))^t(a_1, \ldots, a_N)$. By this action we have that $g(P_1(m), \ldots, P_N)) = (P_1(gm), \ldots, P_N(gm))$.

For $(P_1(x), \ldots, P_N(x))$ in $\tilde{M}$, we have that $g\psi(x) = (P_1(g^{-1}x), \ldots, P_N(g^{-1}x))$ and, since $g^{-1}x \in M$, we have that $g\psi(x) \in \tilde{M}$. Consequently $\tilde{M}$ is invariant under the action of G.

That the diagram is commutative follows from the following equalities, obtained from the above action:

$$\psi\alpha(g,x) = \psi(gx)$$
$$= (P_1(gx), \ldots, P_N(gx)) = (g^{-1}P_1(x), \ldots, g^{-1}P_N(x)) = g(P_1(x), \ldots, P_N(x)),$$

and

$$\tilde{\alpha}(1 \times \psi)(g,x) = \tilde{\alpha}(g, P_1(x), \ldots, P_N(x))$$
$$= g(P_1(x), \ldots, P_N(x)) = (P_1(gx), \ldots, P_N(gx)).$$

□

Corollary 8-1.11. *Any affine algebraic group is isomorphic to a linear algebraic group.*

Remark 8-1.12. Let N be a closed subset of M and let φ be the inclusion. The the restriction map $\varphi^*\colon \mathbf{C[M]} \to \mathbf{C[N]}$ is surjective. Indeed, $N \subseteq M \subseteq V$, where V is a vector space, and consequently $I(M) \subseteq I(N)$. Hence $\varphi^*\colon \mathbf{C[M]} = \mathbf{C[V]/I(M)} \to \mathbf{C[N]} = \mathbf{C[V]/I(N)}$ is surjective.

Problem 8-1.1. Show that the correspondence that sends an algebraic subset M of $\mathbf{C^n}$ to the algebra $\mathbf{C[M]}$ defines bijective correspondence between the algebraic sets of $\mathbf{C^n}$ and the finitely generated algebras with no nilpotent elements, and with a fixed system of n generators.

Problem 8-1.2. Show that every homomorphism $\mathbf{C[M_2]} \to \mathbf{C[M_1]}$ is obtained from a regular map $M_1 \to M_2$.

Problem 8-1.3. Prove that an affine algebraic variety is irreducible if and only if $\mathbf{C[M]}$ is a domain.

Problem 8-1.4. Fix a point x in an algebraic variety M. We define a homomorphism $\mathbf{C[M]} \to \mathbf{C}$ by sending P to $P(x)$. We also define an ideal $\mathfrak{M}_x = \{P \in \mathbf{C[M]}\colon \mathbf{P(x) = 0}\}$. Then $\mathfrak{M}_\mathfrak{x}$ is a maximal ideal, which is the kernel of the above homomorphism. Show that this gives a bijection between points, maximal ideals, and homomorphisms from $\mathbf{C[M]}$ to $\mathbf{C}$.

9. Lecture 9

9-1. **Reductive algebraic groups.**

Definition 9-1.1. A *reductive algebraic group* is an affine algebraic group which is a reductive a complex Lie group. Any algebraic group has finitely many components, so the condition of 5-1.7 is that G_0 contains a compact real Lie group K, such that $T_eK \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{T_e G_0}$.

Remark 9-1.2. Let G be a reductive group operating on an affine algebraic subset M of $\mathbf{C^n}$. Then G operates completely reductively on $\mathbf{C[M]}$. Indeed, by the linearization theorem 8-1.10 we have a linear action $G : V$, where M is a closed G invariant subspace of V. We have seen in Remark 8-1.12 that we have a surjective map $\mathbf{C[V]} \to \mathbf{C[M]}$ of G modules. It follows from Theorem 5-1.5 that the representation of G on $\mathbf{C[V]}$ is completely reducible. Hence the same is true for the action of G on $\mathbf{C[M]}$ by Excercise 9-1.1.

Lemma 9-1.3. *Let G be a reductive algebraic group acting regularly on an affine algebraic variety M, and let N be a closed G invariant subvariety of M. Then the restriction map $\mathbf{C[M]^G} \to \mathbf{C[N]^G}$ is surjective.*

Proof. We know, by Remark 8-1.12, that the restriction map $\mathbf{C[M]} \to \mathbf{C[N]}$ is surjective with kernel I, which consists of all functions on M that vanish on N. Since G is reductive we have, by Theorem 5-1.5 and Remark 9-1.2, that the action on all subspaces of $\mathbf{C[M]}$ is completely reducible. Since N is invariant we have that I is invariant. Consequently we have that $\mathbf{C[M]} = \mathbf{I} \oplus \mathbf{A}$, where A is G invariant, and the map $r\colon \mathbf{C[M]} \to \mathbf{C[N]}$ induces an isomorphism $A \to r(A) = \mathbf{C[N]}$. Taking invariants under G we obtain an isomorphism $A^G \to r(A)^G = \mathbf{C[N]^G}$, induced by $\mathbf{C[M]^G} = \mathbf{I^G} \oplus \mathbf{A^G} \to \mathbf{C[N]^G}$. Consequently the map r induces a surjection. □

Problem 9-1.1. Let G be a group and V a vector space on which G acts completely reducibly. For any G invariant subspace of V we have that G acts completely reducibly on U and V/U.

Example 9-1.4. The group $\mathbf{C_a} = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \mathbf{a} \in \mathbf{C}\}$ is not reductive, because it has no compact subroups. We have that $\mathbf{C_a}$ acts on $V = \mathbf{C^2}$ by $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ay \\ y \end{pmatrix}$, so $\mathbf{C[V]^{C_a}} = \mathbf{C[y]}$. Take $M = \mathbf{C^2}$ and $N = \{(x,0)\colon x \in \mathbf{C}\}$. Then N is invariant and $\mathbf{C[V]^{C_a}} = \mathbf{C[y]}$. However, $\mathbf{C_a}$ acts trivially on N. Consequently $\mathbf{C[N]^{C_a}} = \mathbf{C[x]}$, and the restriction map is not surjective.

Remark 9-1.5. It follows from Nagatas conterexample 4-1 to Hilbert's 14'th problem, that there exists a regular action of $\mathbf{C_a}$ on an affine variety M such that $\mathbf{C[M]^{C_a}}$ is not finitely generated. Indeed, Nagata gave an example of a unitary group U acting on an affine space V, such that the ring of invariants is not finitely generated. Choose vectors $v_r, \dots, v_n$ in V such that $gv_i = v_i$, for $i = r+1, \dots, n$ and $gv_r = v_r + a_{r+1}(g)v_{r+1} + \cdots + a_n(g)v_n$, for all $g \in U$. We have that $gg'v_r = g''(v_r + a_{r+1}(g)v_r + \cdots) = v_r + (a_{r+1}(g) + a_{r+1}(g'))v_{r+1} + \cdots$. Let $U_0 = \{g \in V \colon a_n(g) = 0\}$. Then we have that $e \in U_0$, and U_0 is a subgroup, by the above equations. We obtain an injetion $U/U_0 \to \mathbf{C_a}$, which sends g to $a_n(g)$, and this must be a surjection, so $U/U_0 \cong \mathbf{C_a}$. We have that $\mathbf{C_a} : \mathbf{C[V]^{U_0}}$.

Assume that $\mathbf{C[V]^U}$ is not finitely generated. If $\mathbf{C[V]^{U_0}}$, is finitely generated, we have that $\mathbf{C_a}$ acts on $M = \operatorname{Spec} \mathbf{C[V]^{U_0}}$ and $\mathbf{C[M]^{C_a}} = \mathbf{C[V]^U}$ is not finitely generated. Hence, we have finished the construction. On the other hand, if $\mathbf{C[V]^{U_0}}$ is not finitely generated, we continue, like above, with U_0, and, by induction, we end up with an U_i such that $\mathbf{C[V]^{U_i}}$ is finitely generated. **Problem:** Find an explicit simple construction of such an action.

We now state the main result on actions of reductive groups on affine varieties. The result has been proved earlier in Theorems 5-1.11 and 6-2.1 in the linear case, and we use the linearization theorem 8-1.10 to obtain the general case.

Theorem 9-1.6. (Main theorem on actions of reductive groups on affine varieties.) *Let G be a reductive algebraic group acting on an affine variety M. Then we have that:*

(a) $\mathbf{C[M]^G}$ *is a finitely generated* $\mathbf{C}$ *algebra.*
(b) *Given two disjoint G invariant subspaces F_1 and F_2. Then there is a $P \in \mathbf{C[M]^G}$ such that $P|F_1 = 1$ and $P|F_2 = 0$.*

Proof. By the linearization theorem we can find a G equivariant inclusion $\alpha \colon M \to V$ of affine varieties. By Lemma 9-1.3 we have that the map $\alpha^* \colon \mathbf{C[V]^G} \to \mathbf{C[M]^G}$ is surjective. That is, any G invariant polynomial on M extends to a G invariant polynomial on V. But $\mathbf{C[V]^G}$ is finitely generated by Hilbert's theorem 5-1.11. Consequently we have that $\mathbf{C[M]^G}$ is finitely generated. Moreover, by Lemma 6-2.1, there is a polynomial $P_1 \in \mathbf{C[V]^G}$ such that $P_1|F_1 = 1$ and $P_1|F_2 = 0$. We can now take $P = P_1|M$, to obtain an element of $\mathbf{C[M]^G}$ that separates F_1 and F_2. $\square$

Let G be an affine group acting regularly on an affine algebraic variety M. Then we have an inclusion $\mathbf{C[M]^G} \to \mathbf{C[M]}$ of $\mathbf{C}$ algebras.

Assume that $\mathbf{C}[\mathbf{M}]^{\mathbf{G}}$ is a finitely generated algebra. Then we shall denote by $\mathrm{M} /\!/ \mathrm{G}$ the space $\operatorname{Spec} \mathbf{C}[\mathbf{M}]^{\mathbf{G}}$, that is, the variety which has $\mathbf{C}[\mathbf{M}]^{\mathbf{G}}$ as its ring of functions. The map $\mathbf{C}[\mathbf{M}]^{\mathbf{G}} \to \mathbf{C}[\mathbf{M}]$ gives us a map of the corresponding varieties $M \to \mathrm{M} /\!/ \mathrm{G}$, which is called the *quotient map*. In the following we shall study the fibers of this map.

Remark 9-1.7. Let $\varphi\colon N \to M$ be a regular map of affine algebraic varieties. We have that $\varphi^*\colon \mathbf{C}[\mathbf{M}] \to \mathbf{C}[\mathbf{N}]$ is injective if and only if φ is *dominant*, that is $\overline{\varphi(N)} = M$, where $\overline{\varphi(N)}$ is the Zariski closure of $\varphi(N)$.

It follows from the above remark that the quotient map $M \to \mathrm{M} /\!/ \mathrm{G}$ is dominant.

Definition 9-1.8. Let G be an algebraic group acting on a variety M, and let $\alpha\colon \mathbf{C}[\mathbf{M}]^{\mathbf{G}} \to \mathbf{C}$ be a homomorphism of algebras. The set $M_\alpha = \{x \in M\colon P(x) = \alpha(P)$, for all $P \in \mathbf{C}[\mathbf{M}]^{\mathbf{G}}$ is called the *level variety* associated to α.

The level varieties are G invariant, because if $x \in M_\alpha$, then $\alpha(P) = P(x) = P(gx)$, for all $g \in G$ and $P \in \mathbf{C}[\mathbf{M}]^{\mathbf{G}}$. We shall show that the fibers of $M \to \mathrm{M} /\!/ \mathrm{G}$ are exactly the level varieties.

Remark 9-1.9. We have seen in Exercise 8-1.4 that there is a one to one correspondence between homomorphisms $\alpha\colon \mathbf{C}[\mathbf{M}]^{\mathbf{G}} \to \mathbf{C}$, maximal ideals in $\mathbf{C}[\mathbf{M}]^{\mathbf{G}}$, and points on $\mathrm{M} /\!/ \mathrm{G}$. Let $\varphi\colon M \to N$ be a regular map of affine algebraic varieties. Take $a \in N$, and let $\mathfrak{M}_\mathfrak{a} \subseteq \mathbf{C}[\mathbf{N}]$ be the corresponding maximal ideal. Then the fiber $\varphi^{-1}(a)$ consists of the points of M, which correspond to the maximal ideals of $\mathbf{C}[\mathbf{M}]$ containing $\mathfrak{M}_\mathfrak{a}$. Equivalently, the fiber consists of all the homomorphisms $\mathbf{C}[\mathbf{M}] \to \mathbf{C}$, whose kernels contain $\mathfrak{M}_\mathfrak{a}$.

9-1.10. Assume that $\mathbf{C}[\mathbf{M}]^{\mathbf{G}}$ is finitely generated. Then we have a quotient map $M \to \mathrm{M} /\!/ \mathrm{G}$ of algebraic varieties. The fibers of this map are exactly the level varieties corresponding to homomorphisms $\alpha\colon \mathbf{C}[\mathbf{M}]^{\mathbf{G}} \to \mathbf{C}$. Indeed, to α there corresponds a point a of $\mathrm{M} /\!/ \mathrm{G}$. We have that $M_\alpha = \pi^{-1}(a)$. Indeed, the points of $\pi^{-1}(a)$ correspond to the homomorphism $\beta\colon \mathbf{C}[\mathbf{M}] \to \mathbf{C}$, that extend α, so that $\alpha(P) = \beta_x(P) = P(x)$, for all $P \in \mathbf{C}[\mathbf{M}]^{\mathbf{G}}$. Conversely, if $x \in M_\alpha$, that is $\alpha(P) = P(x)$, for all $P \in \mathbf{C}[\mathbf{M}]^{\mathbf{G}}$, we have that the map $\beta_x\colon \mathbf{C}[\mathbf{M}] \to \mathbf{C}$ defined by $\beta_x(P) = P(x)$, will extend α.

We have that a function on M is G invariant, if and only if it is constant on all level varieties. Indeed, we have seen that an invariant function P on M take the same value $\alpha(P)$ on all the points of M_α.

Conversely, we have that a function in $\mathbf{C[M]}$ that take the same value on all level varieties is G in variant, because the level varieties are G invariant and cover M.

Example 9-1.11. (Example when the quotient map is not surjective.)

Let $G = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : \quad a \in \mathbf{C}\}$, and let G act on $V = \mathrm{Mat}_2(\mathbf{C})$ by left multiplication $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} x+au & y+av \\ u & v \end{pmatrix}$. Clearly, we have that $\mathbf{C[V]} = \mathbf{C[x,y,u,v]}$. The map $\pi\colon V \to \mathrm{V} /\!\!/ \mathbf{C_a} = \mathbf{C^3}$ is not surjective. Indeed, it follows from Excercise 9-1.2 that $\{(0,0,d \neq 0)\} \notin \pi(V)$.

Problem 9-1.2. Show that $\mathbf{C[V]^{C_a}} = \mathbf{C[u,v,d]} = \mathbf{C}[\det \begin{pmatrix} x & y \\ u & v \end{pmatrix}]$, where $\mathbf{C_a}$ and V are as in Example 9-1.11. Moreover, show that $\pi(V) = \mathrm{V} /\!\!/ \mathbf{C_a} \setminus \{0,0,d \neq 0\}$.

Hint: We have that $\mathbf{C[u,v,d]} \subseteq \mathbf{C[V]^{C_a}}$. Every polynomial in $\mathbf{C[V]^{C_a}}$ can, after possible multiplication by a high power of v, be written $\sum_{1\leq i,j} a_{ij} y^i u^j v^{d-i-j}$, modulo an element in $\mathbf{C[u,v,d]}$, beacuse $xv = yu + d$. However, this sum is not invariant since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ applied to the polynomial gives $\sum_{ij} a_{ij}(y+av)^i u^j v^{d-i-j}$, which contains a to the same power as y.

Problem 9-1.3. Let $G = \mathrm{Gl}_n(\mathbf{C})$, or $G = \mathrm{Sl}_n(\mathbf{C})$ act on $V = \mathrm{Mat}_n(\mathbf{C})$ by conjugation. Show that the map $X \to \det(\lambda I - X)$ is a quotient map.

Hint: Let $a_1(x), \dots, a_n(x)$ be the coefficients of the characteristic polynomial. We saw in Exercise 1-1 that the inclusion $\mathbf{C[y_1,\dots,y_n]} \to \mathbf{C[V]}$, from the polynomial ring that sends y_i to $a_i(x)$ gives an isomorphism onto $\mathbf{C[V]^G}$. We get a map $V \to \mathbf{C^n}$, which sends x to $(a_1(x), \dots, a_n(x))$.

Problem 9-1.4. Let $G = \mathrm{Gl}_n(\mathbf{C})$, or $G = \mathrm{Sl}_n(\mathbf{C})$ and let $V = \mathbf{C^n} + (\mathbf{C^n})^*$. Show that the map $V \to \mathbf{C}$ which sends (v,f) to $f(v)$ is a quotient map. Find the orbits of G on V.

Hint: We first describe the orbits. Given (v,f) and (v',f'), such that $f(v) = f'(v')$. Choose bases $v_2, \dots, v_n$ and $v_2', \dots, v_n'$ for $\ker f$, respectively $\ker g$, and choose $g \in \mathrm{Gl}_n(\mathbf{C})$ such that $gv = v'$ and gv_iv_i', for $i = 2, \dots, n$. Then g is uniquely determined by the choise of bases. We have that $(gf)(v_i') = f(g^{-1}v_i') = f(v_i) = 0$. Consequently, we have that $gf = \beta f'$, for some $\beta \in \mathbf{C}$. However, we have that $f(v) =$

$gf(gv) = \beta f'(v') = \beta f'(gv) = \beta f'(v')$, so we must have $\beta = 1$. It follows that $g(v, g) = (gv, gf) = (v', f')$. If $f(v) = f'(v') = 0$ and v, v', f, f' are not all zero, we choose $g \in \mathrm{Gl}_n(\mathbf{C})$ such that $gv_i v_i'$, for all i, and choose v_2 and v_2' by $v = v_2$ and $v' = v_2'$. Choose w and w' such that $f(w) = f'(w') \neq 0$, and proceed as above, with w, w' instead of v, v'. We obtain that $g(v, f) = (gv, gf) = (v', f')$. Hence we have that $\{(v, f) \colon f(v) = 0, f \neq 0, v \neq 0\}$ is an orbit. We have also an orbit $\{(v, 0) \colon v \in V\}$ and $\{(0, f) \colon f \in V^*\}$.

As for invariants, we fix (v, f) such that $f(v) = 1$. Let $U \subseteq V$ be the vector space generated by the elements $\{v, f\}$. Moreover, let $H \subseteq G$ be the set $\{g \in G \colon gv = \lambda^{-1} v \text{ and } gf = \lambda f, \text{ for some } \lambda\}$. We have that H is a subgroup which leaves U invariant, and we have seen that GU is dense in V. Moreover, the functions $(v, f)|U$ generate the ring of invariants because we have $\mathbf{C}[\mathbf{U}] = \mathbf{C}[\mathbf{x}, \mathbf{y}]$, with action $hx = \lambda^{-1} x$, $hy = \lambda y$, and the only invariant function is $(v, f) = xy$.

10. Lecture 10

10-1. Quotients.

Definition 10-1.1. Let G be an affine algebraic group acting on an affine varitey M. The *categorical quotient* of the action is a pair $(X, \pi\colon M \to X)$, where X is an affine variety, such that:

($*$) The map π maps orbits of G to points in X.

($**$) The map π is universal with respect to the above property. That is, if $(X', \pi'\colon M \to X')$ is another pair satisfying property ($*$), then there exists a unique regular map $\varphi\colon X \to X'$, such that $\varphi\pi = \pi'$.

Property ($**$) implies that a categorical quotient is unique. The following result proves the existence of a categorical quotient when the ring of invariants is finitely generated.

Proposition 10-1.2. (Existence of categorical quotients.) *Assume that $\mathbf{C[M]^G}$ is finitely generated. Then the pair $(M /\!\!/ G, \pi)$, where π is the canonical quotient map, is a categorical quotient.*

Proof. Let $\pi'\colon M \to X'$ be a map such that the orbits of G maps to points. We want so show there exists a unique map $\varphi : X \to X'$ such that $\varphi\pi = \pi'$. Since π' maps orbits to points we have that $(\pi')^*\colon \mathbf{C[X']} \to \mathbf{C[M]}$, has image contained in $\mathbf{C[M]^G}$. Indeed, the pullback of a function on X' by π' is invariant under G. The resulting inclusion $\mathbf{C[X']} \to \mathbf{C[M]^G}$ defines a map $\varphi\colon \mathrm{M} /\!\!/ \mathrm{G}$, such that φ^* is the inclusion. It follows from the functorial properties of the map between algebras without nilpotent elements and algebraic varieties, see Problem 8-1.1, that the map φ has the desired properties. □

The opposite of the above Proposition also holds, see Exercise 10-1.1.

Definition 10-1.3. A *geometrical quotient* of $G : M$ is a pair $(M/G, \pi\colon M \to M/G)$, such that all fibers of π are orbits.

Example 10-1.4. Let $G = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t \in \mathbf{C}^*\} : \mathbf{C^2} = \mathbf{V}$. Then there are no invariants. That is $\mathrm{V} /\!\!/ \mathrm{G}$ is a point. Indeed, we have that $\mathbf{C[v]^G} = \mathbf{C}$, so the quotient $\mathrm{M} /\!\!/ \mathrm{G}$ is a point. The orbits consist of the lines through the origin, with the origin removed, that are not closed, and the origin, which is closed. If $y \neq 0$ we have that $\mathbf{C[x, y, \frac{1}{y}]^G} = \mathbf{C[\frac{x}{y}]}$, and every line $(\frac{b}{a})$ corresponds to a point of $\mathbf{C[\frac{x}{y}]}$. Hence if we remove the x axis, the geometric quotient exists.

Theorem 10-1.5. *Let G be an affine reductive group acting on an affine variety M. Then the quotient map $\pi\colon M \to N /\!\!/ G$ is surjective.*

Proof. Since G is reductive we have that $\mathbf{C[M]^G}$ is finitely generated by Theorem 9-1.6, and there is a Reynolds operator $\sharp$ on $\mathbf{C[M]}$ by Theorem 5-1.5 and Proposition 4-2.3. We have an injection $\pi^*\colon \mathbf{C[M]^G} \to \mathbf{C[M]}$. A point a in $\mathrm{M}/\!\!/\mathrm{G}$ corresponds to a maximal ideal $\mathfrak{M}_\mathfrak{a} \in \mathbf{C[M]^G}$. The points of the fiber $\pi^{-1}(a)$ correspond to the maximal ideals of $\mathbf{C[M]}$ that contain $\mathfrak{M}_\mathfrak{a}$. The fiber is therefore empty exactly when $\mathfrak{M}_\mathfrak{a}\mathbf{C[M]} = \mathbf{C[M]}$, or equivalently, when $1 = \sum_i f_i P_i$, where $f_i \in \mathbf{C[M]}$ and $P_i \in \mathfrak{M}_\mathfrak{a} \subseteq \mathbf{C[M]^G}$. Apply the Reynolds operator to both sides of the last equation. We obtain that $1 = \sum_i f_i^\sharp P_i \in \mathfrak{M}_\mathfrak{a}$, which is a contradiction since 1 does not vanish at a. □

Corollary 10-1.6. *Let G be a reductive algebraic group acting on an affine variety M, and let N be a G invariant closed subset of M. Let $\pi_N\colon N \to \pi(N)$, be the map induced by the map $\pi\colon M \to M/\!\!/G$. Then $\overline{\pi(N)} = \pi(N)$ in $M/\!\!/G$ and $\pi_N\colon N \to \pi(N)$ is a quotient map for the action of G on N.*

Proof. We have a surjective map $\mathbf{C[M]} \to \mathbf{C[N]}$, by Remark 8-1.12, which iduces a surjection $\mathbf{C[M]^G} \to \mathbf{C[N]^G}$, by 9-1.3. We obtain a commutative diagram

$$\begin{array}{ccc} N & \longrightarrow & M \\ {\scriptstyle \pi_N}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathrm{N}/\!\!/\mathrm{G} & \longrightarrow & \mathrm{M}/\!\!/\mathrm{G}, \end{array}$$

where the horizontal maps define N and $\mathrm{N}/\!\!/\mathrm{G}$ as closed subvarieties of M respectively $\mathrm{M}/\!\!/\mathrm{G}$. The vertical maps are surjective by the Theorem. Hence we have that $\mathrm{N}/\!\!/\mathrm{G} = \pi(N)$, and that $\pi(N) = \overline{\pi(N)}$. □

Problem 10-1.1. Show that, if the categorical quotient exists, then $\mathbf{C[M]^G}$ is finitely generated, and hence is the categorical quotient $(\mathrm{M}/\!\!/\mathrm{G}, \pi)$.

Hint: If $\pi\colon M \to \mathrm{M}/\!\!/\mathrm{G}$ exists, we have that π must be dominating. Otherwise we would have that $\overline{\pi(M)}$ would give a categorical quotient. Hence we have an injection $\pi^*\colon \mathbf{C}[\mathrm{M}/\!\!/\mathrm{G}] \to \mathbf{C[M]}$, and since the functions on $\mathrm{M}/\!\!/\mathrm{G}$ give invariant functions on M, via π, we have that the image of π^* lies in $\mathbf{C[M]^G}$. Conversely, let $f \in \mathbf{C[M]^G}$. We get a map $f\colon M \to \mathbf{C}$ which sends orbits to points. Consequently the map f factors via $M \to \mathrm{M}/\!\!/\mathrm{G}$. In other words we have that $\mathbf{C[f]} \subseteq \mathbf{C}[\mathrm{M}/\!\!/\mathrm{G}]$. Consequently we have that $\mathbf{C}[\mathrm{M}/\!\!/\mathrm{G}] = \mathbf{C[M]^G}$.

Problem 10-1.2. If G is a finite group, then the categorical quotient is automatically the geometric quotient.

Hint: We have that $\pi\colon M \to \mathrm{M} /\!/ \mathrm{G}$ has finite fibers. Indeed, the coefficients of the points are roots of a finite number of polynomials in one variable. However, if we had two orbits in the same fiber, they would both be closed, and hence there would be an invariant function on M which takes 0 on one of the orbits and 1 on the other. On the other hand, all invariant fuctions take the fibers to the same point. We obtain a contradiction.

Problem 10-1.3. Given a linear action $G : V$. If the categorical quotient is geometric, then G is finite.

11. Lecture 11

11-1. **Closed orbits.**

Proposition 11-1.1. *Let $\varphi\colon M \to N$ be a regular map of affine algebraic varieties. Then $\varphi(M)$ contains a dense open subset of $\overline{\varphi(M)}$.*

Proof. We may assume that M is irreducible, since it suffices to prove the Proposition for each irreducible component of M. When M is irreducible we have that $\overline{\varphi(M)}$ is an irreducible subset of N. Consequently we may assume that $N = \overline{\varphi(M)}$. When the latter equality holds we have an injection $\varphi^*\colon \mathbf{C}[\mathbf{N}] \to \mathbf{C}[\mathbf{M}]$, of one finitely generated domain into another. Hence, by the result 6-1.6, there exists an element $f \in \mathbf{C}[\mathbf{M}]$ such that any homomorphism $\psi\colon \mathbf{C}[\mathbf{N}] \to \mathbf{C}$, with $\psi(f) \neq 0$, can be extended to a homomorphism $\tilde{\psi}\colon \mathbf{C}[\mathbf{M}] \to \mathbf{C}$. This means that the fibers of φ over the subset $U = \{x \in N\colon f(x) \neq 0\}$ of N are all nonempty. Consequently $U \subseteq \varphi(M)$. It is clear that U is open in N and it follows from Excercise 11-1.1 that it is nonempty. ☐

Theorem 11-1.2. *Let G be an affine algebraic group acting on an affine algebraic variety. Then each orbit Gx is open in $\overline{Gx}$.*

Proof. Consider the map $\varphi\colon G \to M$ given by $\varphi(g) = gx$. By Proposition 11-1.1 we have that $\varphi(G) = Gx$ contains a set U which is open in $\overline{Gx}$. Choose an element $y_0 = g_0x$ in U. For any $y \in Gx$ we have that $y = g_1x$ and thus $y = g_1{g_0}^{-1}y_0$. Consequently, y is contained in the open subset $g_1{g_0}^{-1}U$ of Gx. ☐

Lemma 11-1.3. *Let G be an affine algebraic group acting on an algebraic variety M, and let N be a nonempty closed G invariant subset of M. Then N contains a closed G orbit.*

Proof. If N contains a closed orbit we are done. So assume that N contains properly a non closed orbit Gx. Then $N_1 = \overline{Gx} \setminus Gx \neq \emptyset$. The set N_1 is G invariant, and, by Theorem 11-1.2 we have that Gx is open in $\overline{Gx}$. Hence N_1 is closed. If N_1 does not contain a closed orbit, we can repeat the argument for N_1 and get a sequence $N \not\supseteq N_1 \not\supseteq N_2 \not\supseteq \cdots$ of closed G invariant subsets. Since M is quasi compact, by Remark 11-1.2, we have that the sequence stops, and the interesection must contain a closed orbit. ☐

Theorem 11-1.4. *Let G be a reductive affine algebraic group acting on an affine variety M, and let $\pi\colon M \to M /\!\!/ G$ be the categorical quotient. Then the correspondence between orbits of G and points in $M /\!\!/ G$, wich, to an orbit F associates the point $\pi(F)$, is bijective.*

Proof. Since all fibers of π are closed and G invariant, it follows from Lemma 11-1.3 that, in every fiber of π, there is a closed orbit. Moreover, all fibers are nonempty by Theorem 10-1.5. Hence, for any point of $M /\!/ G$ there exists a closed orbit mapping to the point.

It remains to prove that we have exactly one closed orbit in each fiber. Assume, that we had two closed G invariant subsets F_1 and F_2 in the same orbit. It follows from Theorem 6-2.1, that there is a G invariant function that takes the value 0 on F_1 and the value 1 on F_2. However, an invariant function takes the same value on all point of a fiber by 9-1.10. Hence there is only one G invariant closed set in each fiber. □

Example 11-1.5. Let $\mathbf{C_a} = \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : \mathbf{a} \in \mathbf{C}\} : \mathbf{C^2}$. We have that $\mathbf{C}[\mathbf{V}]^{\mathbf{C_a}} = \mathbf{C}[\mathbf{y}]$ and $V /\!/ \mathbf{C_a} = \mathbf{C^1}$.

Hint: The orbits are horizontal lines with nonzero first coordinate, together with all the points on the x axis. We have that $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + a\beta \\ \beta \end{pmatrix}$. The fiber of $V \to V /\!/ \mathbf{C_a} = \mathbf{C^1}$ over $\alpha \neq 0$ is the line with heigh α. The fiber over 0 is the x axis with each point as an orbit.

Example 11-1.6. Let $\mathbf{C}^* = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \mathbf{a} \in \mathbf{C}\} : \mathbf{C^2}$. We have that $V /\!/ \mathbf{C}^*$ is a point. In this case the Theorem holds.

Hint: We saw in Example 10-1.4 that the fiber has infinitely many orbits, but only $(0, 0)$ is closed.

Example 11-1.7. Let $\mathbf{C}^* = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\} : \mathbf{C^2}$. We have that $\mathbf{C}[\mathbf{V}]^{\mathbf{C}^*} = \mathbf{C}[\mathbf{x}, \mathbf{y}]$ and $V /\!/ \mathbf{C}^* = \mathbf{C^1}$. Indeed, the orbits consist of hyperbolas $(a\alpha, a^{-1}\beta)$, where $\alpha\beta \neq 0$. When $\alpha\beta = 0$ we have orbits $X \setminus 0$, $Y \setminus 0$ and 0. Moreover, we have that $\mathbf{C}[\mathbf{V}]^{\mathbf{C}^*} = \mathbf{C}[\mathbf{x}, \mathbf{y}]$. Consequently, over a point where $\alpha\beta \neq 0$, we get the entire hyperbola, and $\pi^{-1}(0) = \{X \setminus 0, Y \setminus 0, 0\}$.

Remark 11-1.8. In all the above Examples we may remove a closed set, in fact the x axis, in such a way that in the remaining variety all the fibers are orbits.

Hint: If we remove $y = 0$ in the first of the examples we have that $\mathbf{C}[\mathbf{x}, \mathbf{y}, \frac{\mathbf{1}}{\mathbf{y}}]^{\mathbf{G}} = \mathbf{C}[\mathbf{y}, \frac{\mathbf{1}}{\mathbf{y}}]$ and we get all the orbits with $y = \beta \neq 0$, that correspond to the nonzero points on the quotient $V /\!/ G$. Take $y \neq 0$ in the second Example and we get $\mathbf{C}[\mathbf{x}, \mathbf{y}, \frac{\mathbf{1}}{\mathbf{y}}]^{\mathbf{G}} = \mathbf{C}[\frac{\mathbf{x}}{\mathbf{y}}]$ and all the directions $\frac{\alpha}{\beta}$, with $\beta \neq 0$ correspond to the nonzero points on $V /\!/ G$.

Finally, take $y \neq 0$ in the third Example. We have that $\mathbf{C}[\mathbf{x}, \mathbf{y}, \frac{\mathbf{1}}{\mathbf{y}}]^{\mathbf{G}} = \mathbf{C}[\mathbf{x}, \mathbf{y}]$ and all the orbits (α, β), with $\alpha\beta \neq 0$ correspond to $\alpha\beta$ on $\mathrm{V} /\!/ \mathrm{G}$, and the orbit with $\alpha = 0$ and $\beta \neq 0$ correspond to $Y \setminus 0$.

Problem 11-1.1. If M is an irreducible affine variety and $f \in \mathbf{C}[\mathbf{M}]$. Then $U = \{x \in M \colon f(x) \neq 0\}$ is dense in M.

Hint: Use Excercise 6-1.6.

Problem 11-1.2. (a) Let $G \subseteq \mathrm{Gl}(V)$ be an arbitrary subroup. Then the Zariski closure $\overline{G}$ is an affine algebraic group, and $\mathbf{C}[\mathbf{V}]^{\mathbf{G}} = \mathbf{C}[\mathbf{V}]^{\overline{\mathbf{G}}}$.

(b) If G contains a subset U which is open and dense in $\overline{G}$, then $G = \overline{G}$.

(c) The image by a a regular map of an affine algebraic group G to another affine algebraic group is closed.

(d) We have that the commutator group $G^{(1)}$ is an affine algebraic subgroup of G.

Hint: (a) The restriction, $G \to \overline{G}$, of the inverse map on $\mathrm{Gl}(V)$ is continous, so it sends $\overline{G}$ to $\overline{G}$. Hence the inverse of elements in $\overline{G}$ are in $\overline{G}$. For every g_0 in G, we have that the map $G \to \overline{G}$ which sends g to $g_0 g$ is continous, so $g_o\overline{G} \subseteq \overline{G}$. Let $g \in \overline{G}$. By what we just saw we have that $Gg \subseteq \overline{G}$. Consequently $\overline{G}g \subseteq \overline{G}$, so $\overline{G}$ is closed under products.

(b) Translating the open subset we see that G is open in $\overline{G}$. However, then every coset is open, so G is also closed. Thus $G = \overline{G}$.

(c) It follows from Proposition 11-1.1 that $\operatorname{im} G$ contains a set which is open in $\overline{\operatorname{im} G}$. It thus follows from (b) that $\operatorname{im} G = \overline{\operatorname{im} G}$.

(d) Let $G_k^{(1)}$ be the set of elements in $G^{(1)}$ that can be written as a product of k commutators. This set is the image of the product of G with itself $2k$ times and is therefore irredusible. We have that the commuator group is the union of these sets. Moreover, we have that $\overline{G_1^{(1)}} \subseteq \overline{G_2^{(1)}} \subseteq \cdots \subseteq \overline{G^{(1)}}$, and all the sets in the sequence are irreducible. Hence the sequence stabilizes. However, it is clear that $\overline{G^{(1)}}$ is the union of the sets in the sequence. Hence $\overline{G_n^{(1)}} = \overline{G^{(1)}}$, for big n. We have, in particular that $\overline{G_n^{(1)}}$ is a group. However, we have that the product of G by itself $2n$ times maps surjectively onto $G_n^{(1)}$ Consequently this set contains an open subset of its closure. As we just saw, the closure is irreducible, hence the open set is dense. We thus obtain that $\overline{G_n^{(1)}} = G_n^{(1)}$ by part (b), and thus $G^{(1)} = \overline{G^{(1)}}$.

11-2. **Unipotent groups.**

Definition 11-2.1. A closed subgroup of the group $\mathrm{U}_n = \{ \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \}$ is called a *unipotent* algebraic group. A *unipotent element* u of an algebraic group G is an element with all eigenvalues equal to 1 in some representation of G on a finite dimensional vector space, or equivalently, an element that can be written in the form $\begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$, in some representation.

Theorem 11-2.2. *As an algebraic variety, a unipotent group is isomorphic to* $\mathbf{C}^{\mathbf{k}}$*, for some integer* k*.*

Proof. Let U be a unipotent group. The map $\varphi_t \colon U \to \mathrm{Gl}(\mathbf{C})$ which send u to $\exp(t \log u)$, is regular. Indeed, we have that $u = 1 + v$, where $v^n = 0$, for some n which is independent of u. We also have that $(\log u)^n = v - \frac{v^e}{2} + \cdots)^n = 0$, for all u in U_n.

When t is an integer it follows from the properties of exp and log that $\varphi_t(u) = u^t$. Moreover, for fixed u, we obtain a map $\psi \colon \mathbf{C} \to \mathrm{Gl}_{\mathbf{n}}(\mathbf{C})$, given by $\psi(t) = \varphi_t(u)$. We have seen that ψ send $\mathbf{Z}$ to U. Since U is closed in U_n it follows that that ψ sends the closure $\overline{\mathbf{Z}}$ to U. However, $\overline{\mathbf{Z}} = \mathbf{C}$, since $\mathbf{Z}$ is infinite. Consequently we have a regular map $\varphi_t \colon U \to U$.

We have that log is defined on all of U. Consequently we have that log induces a map $U \to \log U \subseteq T_e U$. Since we have a map $\exp \colon \mathrm{Mat}_n(\mathbf{C}) \to \mathrm{Gl}_{\mathbf{n}}(\mathbf{C})$, we get that $\log U = T_e(U)$, and log and exp are inverses on U and $\log U$. Thus U and $T_e U$ are isomorphic. □

Corollary 11-2.3. *All nonconstant regular functions on a unipotent group* U *have a zero. In particular, there is no regular homomorphism, that is, there is no character,* $U \to \mathbf{C}^*$*.*

Proof. It follows from the Theorem that U is isomorphic to an affine space. However, on an affine space all polynomials have a zero. □

Corollary 11-2.4. *The image of a unipotent group* U *under a regular map is a unipotent group. In particular, in any finite dimensional representation, there is a nonzero vector which is fixed by* U*.*

Proof. By Theorem 8-1.10, we can consider U as a subvariety of $\mathrm{Gl}_n(\mathbf{C})$ via a map $\varphi \colon U \to \mathrm{Gl}_n(\mathbf{C})$. The image of φ is a connected solvable group, since the image of any solvable group is solvable. It follows from the Lie-Kolchin theorem 7-2.2, that $\varphi(U)$ has an eigenvector v. Consequently $uv = \lambda(u)v$, for some character $\lambda \colon U \to \mathbf{C}^*$. However, it follows from the above Corollary that $\lambda = 1$. $\square$

Corollary 11-2.5. *Let u be a unipotent element of an affine algebraic group G, and let $\psi \colon G \to G$ be a regular homorphism. Then $\varphi(u)$ is a unipotent element.*

Proof. Take $U = \overline{\{u^k\}}_{k \in \overline{\mathbf{Z}}}$. Then U is unipotent because, if u is triangular with 1's on the diagonal, then the same is true for all powers, and for the closure of the set of all powers. The Corollary consequently follows from the above Corollary. $\square$

Theorem 11-2.6. *The orbits for a regular action of a unipotent group U on an affine variety M are closed.*

Proof. Let Ω be an orbit under the action of U on M. If Ω is not closed, then $F = \overline{\Omega} \setminus \Omega$ is a nonempty subset of M, which is invariant under U. It follows from Theorem 11-1.2 that F is closed. Since F is closed and Ω is open in F there is a nonzero function $h \in \mathbf{C}[\overline{\Omega}]$, such that $h|\overline{\Omega} \neq 0$. Let $V =< gh >_{g \in U}$ be the linear span in $[\overline{\Omega}]$ of the elements gh. It follows from Lemma 8-1.8 that V is finite dimensional, and it is invariant under U. By Corollary 11-2.4 we have that there exists a nonzero function h_1 in V, such that $Uh_1 = h_1$. For all $h \in V$ we have that $h_1|F = 0$, since $h|F = 0$ and F is U invariant. Consequently $h_1|F = 0$. However, $h_1 \neq 0$ so $h_1(p) = c \neq 0$ for some $p \in \Omega$. Since h_1 is U invariant, we have that $h_1|Up = c \neq 0$, and thus $h_1|\overline{\Omega} = c \neq 0$, which contradicts that $h_1|F = 0$. $\square$

Problem 11-2.1. Let U be a unipotent group acting on an affine algebraic space. If any two closed disjont orbits can be separated by invariants, then U acts trivially.

Hint: Since U can be represented by upper triangular matrices with 1's on the diagonal, there is an r such that $Uv_i = v_i$ for $i = r+1, \dots, n$ and $uv_r = v_r + a_{r+1} + \cdots$, for some $u \in U$ and some $a_{r+1}, \dots, a_n$, not all zero. It suffices to consider $U : \{v_{r+}, \dots, v_n\}$. Let $h = (x_r, \dots, x_n)$ be invariant. We get that $uh(x_r, \dots, x_n)h(x_r + a_{r+1}x_{r+1} + \cdots a_n x_n, x_{r+1}, \dots, x_n)$. Consider the highest and next highest powers of x_r in $h(x_r, \dots, x_n)$. We see that h can not depend on x_r. However, then $v_{r+1}, \dots, v_n$ are different orbits in the same fiber.

Problem 11-2.2. Show that on $\mathrm{Sl}_n(\mathbf{C})$ any regular function vanishes at some point. In particular, it has no *characters*, that is, there are no homomorphisms $\mathrm{Sl}_n(\mathbf{C}) \to \mathbf{C}^*$.

Hint: Let V be the open subset of Sl_n consisting of matrices whose 1×1, 2×2, $\dots$, minors in the upper left corner are nonzero. Then, for every m in V there are elements u and v in U_n such that um^tv is diagonal. It follows from Corollary 11-2.3 that a polynomial h either has a zero on Sl_n or is constant on all sets Um^tU. Hence, the values of h are the same vaues it takes on the diagonal of $\mathrm{Sl}_n(\mathbf{C})$. However, the diagonal is isomorphic to the open subset of $\mathbf{C^n}$, where the product of the coordinaes is zero. On this set any nonconstant polynomial has a zero.

12. Lecture 13

12-1. **Classical invariant theory.** More information on the material covered in this section can be found in Weyl's book [**?**].

12-1.1. We shall consider a closed subgroup G of $\mathrm{Gl}_n(\mathbf{C})$. The group G operates $V = \mathbf{C^n}$ via the action of $\mathrm{Gl}_n(\mathbf{C})$ on V. Let $V_m = \mathbf{C^n} \oplus \cdots \oplus \mathbf{C^n}$, where the sum is taken m times. An element in V_m we shall write $(v_1 \oplus \cdots \oplus v_m)$, and we shall consider the v_i as column vectors. Hence, we shall consider the element $(v_1 \oplus \cdots \oplus v_m)$ of V_m as the $n \times m$ matrix:

$$\begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix}. \tag{12-1.1.1}$$

The determinant of the matrix taken from rows $j_1, \dots, j_n$, where $n \leq m$, we shall denote by $d_{j_1,\dots,j_n}(v_1 \oplus \cdots \oplus v_m)$.

Remark 12-1.2. It is clear that the functions $d_{j_1,\dots,j_n}$ are invariant under the action of $\mathrm{Gl}_n(\mathbf{C})$.

We let G operate on V_m from the left, and $\mathrm{Gl}_m(\mathbf{C})$ from the right. Hence the group $G \times \mathrm{Gl}_m(\mathbf{C})$ operates on V_m by:

$$\begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix} \begin{pmatrix} c_{11} & \dots & c_{1m} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mm} \end{pmatrix}^{-1}.$$

The action of the two groups commute. Given a polynomial P in $\mathbf{C}[\mathbf{V_n}]$ and an $m \times n$ matrix $A = (a_{ij})$, we let

$$P^A(v_1 \oplus \cdots \oplus v_m) = P(\sum_{i=1}^m a_{i1} v_i \oplus \cdots \oplus \sum_{i=1}^m a_{in} v_i).$$

In other words, P^A is a polynomial in $\mathbf{C}[\mathbf{V_m}]$ and its value at the element 12-1.1.1 is the value of P at the $n \times n$ matrix

$$\begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix} \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \tag{12-1.2.1}$$

of V_n.

The following is the main tool of classical invariant theory, for finding invariants. We shall give a reinterpretation of the result below and also show how the result is the basis of the *symbolic method*.

Theorem 12-1.3. *(Basic theorem of classical invariant theory.) Let $m \geq n$. We have that $\mathbf{C[V_m]^G}$ is generated by the polynomials*

$$\{P^A\colon P \in \mathbf{C[V_n]^G}, \mathbf{A} \in \mathrm{Mat}_{\mathbf{m,n}}(\mathbf{C})\}.$$

Proof. Since multiplication with the $m \times n$ matrix (a_{ij}) in equation 12-1.2.1 commutes with the action of G, it is clear that, if $P \in \mathbf{C[V_n]^G}$, then $P^A \in \mathbf{C[V_m]^G}$.

Conversely, since the actions of G and $\mathrm{Gl}_m(\mathbf{C})$ on V_m commute, we have that $\mathbf{C[V_m]^G}$ is $\mathrm{Gl}_m(\mathbf{C})$ invariant. Moreover, since $\mathrm{Gl}_m(\mathbf{C})$ is reductive, it follows from Theorem 5-1.5 applied to each of the homogeneous components of $\mathbf{C[V_m]^G}$, that $\mathbf{C[V_m]^G}$ decomposes into a direct sum of finite dimensional irreducible $\mathrm{Gl}_m(\mathbf{C})$ invariant subspaces $\mathbf{C[V_m]^G} = \oplus_\mathbf{i} \mathbf{U}_\alpha$.

Consider the subgroup $N_m = \{\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}\}$ of $\mathrm{Gl}_m(\mathbf{C})$ consisting of upper diagonal matrices wiht 1's on the diagonal. It follows from Corollary 11-2.4 that there is a nonzero element P_α in U_α for each α, which is fixed under the action of N_m. We have that $< \mathrm{Gl}_m(\mathbf{C})\mathbf{P}_\alpha >= \mathbf{U}_\alpha$, since $< \mathrm{Gl}_m(\mathbf{C})\mathbf{P}_\alpha >$ is invariant under $\mathrm{Gl}_m(\mathbf{C})$ and U_α is irreducible under $\mathrm{Gl}_m(\mathbf{C})$. Consequently we have that $\mathbf{C[V_m]^G}$ is generated by the elements of the set $S = \{gP\colon g \in \mathrm{Gl}_m(\mathbf{C}), \mathbf{P} \in \mathbf{C[V_m]^{G\times N_m}}\}$.

The subset Ω of V_m consisting of the elements $(v_1 \oplus \cdots \oplus v_m)$, such that $v_1, \dots v_n$ are linearly independent, is dense in V_m. Take $v \in \Omega$. Then there is an $A \in N_m$ such that $(v_1 \oplus \cdots \oplus v_m)A = (v_1 \oplus \cdots \oplus v_n \oplus 0 \oplus \cdots \oplus 0)$. (In fact we can choose A to have the unit $(m-n) \times (m-n)$ matrix in the lower right corner.) For every polynomial P in $\mathbf{C[V_m]^{G\times N_m}}$ we get that $P(v_1 \oplus \cdots \oplus v_m) = A^{-1}P(v_1 \oplus \cdots \oplus v_m) = P(v_1 \oplus \cdots \oplus v_n \oplus 0 \oplus \cdots \oplus 0)$. Consequently the latter equality holds for all $(v_1 \oplus \cdots \oplus v_m)$ in V_m. That is, we have that $P(v_1 \oplus \cdots \oplus v_m) = P|V_n(v_1 \oplus \cdots \oplus v_n)$. Let $a = (a_{ij})$ be an element in $\mathrm{Gl}_m(\mathbf{C})$, and let A be the $m \times n$ matrix taken from the first n columns of a. We obtain that $(aP)(v_1 \oplus \cdots \oplus v_m) = P(\sum_{i=1}^m a_{i1}v_i \oplus \cdots \oplus \sum_{i=1}^m a_{in}v_i) = P(\sum_{i=1}^m a_{i1}v_i \oplus \cdots \oplus \sum_{i=1}^m a_{in}v_i \oplus 0 \oplus \cdots \oplus 0) = (P|V_n)(\sum_{i=1}^m a_{i1}v_i \oplus \cdots \oplus \sum_{i=1}^m a_{in}v_i) = (P|V_n)^A(v_1 \oplus \cdots \oplus v_m)$. Hence every element gP in S is of the form $(P|V_n)^A$ for some $m \times n$ matrix A. However, when $P \in \mathbf{C[V_m]^G}$ we have that $(P|V_n) \in \mathbf{C[V_n]}$, so we have proved that every element of S is of the form P^A for some $P \in \mathbf{C[V_n]}$ and some $m \times n$ matrix A. □

Remark 12-1.4. We may reformulate the basic theorem of classical invariant theory as follows:

We have that G acts homogeneously on every column $v_1, \dots, v_n$ of $v_1 \oplus \cdots \oplus v_n$. Hence, every polynomial $P \in \mathbf{C}[\mathbf{V_n}]^{\mathbf{G}}$ is homogeneous in each of the coordinates $v_1, \dots, v_n$, say of degrees $k_1, \dots, k_n$. Then we can consider P as a k_i multilinear function in v_i, for $i = 1, \dots, n$. Consequently we can write P in the form $P(v_1 \oplus \cdots \oplus v_n) = \tilde{P}(v_1, \dots, v_1, v_2, \dots, v_2, \dots)$, where v_i appears k_i times on the right hand side, and $\tilde{P}$ is a multilinear function in each of the variables $v_1, \dots, v_n$. We have that

$$\begin{aligned} P^A(v_1 \oplus \cdots \oplus v_m) &= P(\sum_{i=1}^m a_{i1}v_i \oplus \cdots \oplus \sum_{i=1}^m a_{in}v_i) \\ &= \tilde{P}(\sum_{i-1}^m a_{i1}v_i, \dots, \sum_{i=1}^m a_{i1}v_i, \sum_{i-1}^m a_{i2}v_i, \dots, \sum_{i=1}^m a_{i2}v_i, \dots). \end{aligned}$$

Consequently we have that $P^A(v_1 \oplus \cdots \oplus v_m)$ is a linear combination of $\tilde{P}(v_{j_1}, \dots, v_{j_k})$ for $1 \leq j_1, \dots, j_k \leq m$. Let

$$\tilde{P}_{j_1,\dots,j_k}(v_1 \oplus \cdots \oplus v_m) = \tilde{P}(v_{j_1}, \dots, v_{j_k}).$$

Then we have that the algebra $\mathbf{C}[\mathbf{V_m}]^{\mathbf{G}}$ is generated by the invariants $\tilde{P}_{j_1,\dots,j_k}$ for $1 \leq j_1, \dots, j_k \leq m$.

Proposition 12-1.5. *Let $G = \mathrm{Sl}_n(\mathbf{C})$. Then the following two assertions hold:*

(a) *If $m < n$, we have that G has an open orbit. In particular $\mathbf{C}[\mathbf{V_m}]^{\mathbf{G}} = \mathbf{C}$ and 0 is the only closed orbit.*
(b) *If $m \geq n$, we have that $\mathbf{C}[\mathbf{V_m}]^{\mathbf{G}}$ is generated by the polynomials $d_{j_1,\dots,j_n}(v_1 \oplus \cdots \oplus v_m)$, for all $1 \leq j_1 < \cdots < j_n \leq m$. Moreover, all the fibers of the quotient map $\pi \colon V_m \to V_m /\!\!/ G$ are closed orbits, except for the fiber over $\pi(0)$. In other words, the orbit of a nonzero point v in V_m is closed if and only if $d(j_1, \dots, j_n)(v) \neq 0$, for at least one set $j_1 < \cdots < j_n$.*

Proof. Assertion (a) is obvious since, if $m < n$, then $\mathrm{Sl}_n(\mathbf{C})$ operates transitively on the sets $v = v_1 \oplus \cdots \oplus v_m$ of linearly independent elements.

To prove (b) , it follows from 12-1.3 that it is advantageous to consider first the case $m = n$. Then, choose linearly independent vectors $v_1, \dots, v_n$ such that the matrix $v_1 \oplus \cdots \oplus v_n$ has determinant 1. Then $\mathrm{Sl}_n(\mathbf{C})\{\mathbf{v_1} \oplus \cdots \oplus \mathbf{v_{n-1}} \oplus \mathbf{cv_n} \colon \mathbf{c} \in \mathbf{C}\}$, consists of all elements $w_1 \oplus \cdots \oplus w_n$, such that $w_1, \dots, w_n$ are linearly independent. In particular, this set is dense in V_n. Moreover we have that $d_{1,\dots,n}|\{v_1 \oplus \cdots \oplus v_{n-1} \oplus cv_n\}$ is the coordinate function c, that is

$d_{1,\dots,n}(v_1 \oplus \cdots \oplus v_{n-1} \oplus cv_n) = c$, and consequently $\mathbf{C}[\mathbf{x}]^{\{\mathbf{e}\}} = \mathbf{C}[\mathbf{x}] = \mathbf{C}[\mathbf{d}_{1,\dots,\mathbf{n}}|\{\mathbf{v_1} \oplus \cdots \oplus \mathbf{cv_n}\}]$. We now apply Lemma 1-1.7 to $G \supseteq \{e\}$ and $V_n \supseteq \{v_1 \oplus \cdots \oplus v_{n-1} \oplus cv_n\}$, and conclude that $\mathbf{C}[\mathbf{V_n}]^{\mathbf{G}} = \mathbf{C}[\mathbf{x}]$.

For $m > n$, it follows from Theorem 12-1.3, that the invariants are given by the expression $d_{1,\dots,n}(\sum_{i=1}^m a_{i1}v_i, \dots, \sum_{i=1}^m a_{in}v_i)$. These expressions are linear combination of the invariants $d_{j_1,\dots k j_n}$.

For the last statement of the Proposition, we may assume that $d_{1,\dots,n}(v_1 \oplus \cdots \oplus v_m) = c \neq 0$. We want to show that the set of all such $v_1 \oplus \cdots \oplus v_m$ is closed. The condition that $c \neq 0$ implies that the vectors $v_1, \dots v_n$ are linearly independent. Consequently, for $j > n$, we have that $v_j = \sum_{i=1}^n c_{ji}v_i$. The set $\{w = w_1 \oplus \cdots \oplus w_m \in V_m \colon d_{1,\dots,n}(w) = c$, and $w_j = \sum_{i=1}^n c_{ji}w_i\}$ is a closed subset of V_m containing v. There exists a, unique, $g_w \in \mathrm{Sl}_n(\mathbf{C})$, such that $g_w v = w$. Hence the orbit $\mathrm{Sl}_n(\mathbf{C})\mathbf{v}$ is closed, and, as an algebraic variety, it is isomophic to $\mathrm{Sl}_n(\mathbf{C})$.

We have seen that the fibers of the quotient map $\pi \colon V_m \to V_m /\!/ \mathrm{G}$ over points different from $\pi(0)$ consists of closed orbits. However, every fiber of π contains a unique closed orbit by Theorem 11-1.4. Hence every fiber over a point different from $\pi(0)$ cosists of a closed orbit. □

Problem 12-1.1. Consider the quotient map $\pi \colon V_m \to V_m /\!/ \mathrm{Sl}_n(\mathbf{C})$ with $m \geq n$. Show that the dimension of $\pi^{-1}(0)$ is $(n-1)(m+1)$. Consequently $\dim \pi^{-1}(0) > \dim \mathrm{Sl}_n(\mathbf{C})$ if $m > n$.

Hint: The kernel consists of $n \times m$ matrices of the form 12-1.1.1 of rank $n-1$. To find those with the first $n-1$ columns linearly independent we choose the $n(n-1)$ entries in the first $n-1$ columns general. Then we choose an arbitrary $(m-n+1)(n-1)$ matrix (a_{ij}) and let $v_i = \sum_{j=1}^{n-1} a_{ij}v_j$, for $i = n, \dots, m$. In this way we determine the remaining rows. This gives dimension $n(n-1) + (m-n+1)(n-1) = (n-1)(m+1)$.

Problem 12-1.2. Describe all the orbits of the zero fiber for the case $m = n$ of Proposition 12-1.5.

Hint: The kernel of $d_{1,\dots,n}$ consists of $n \times n$ matrices of rank at most $n-1$, and the action of $\mathrm{Sl}_n(\mathbf{C})$ is multiplication on the left. The orbits consists of matrices of the same rank. Let $v = v_1 \oplus \cdots \oplus v_n$ be in kernel of $d_{1,\dots,n}$ and have rank r, and such that $v_1, \dots, v_r$ are linearly independent. Let $v_i = \sum_{j=1}^r a_{ij}v_j$, for $i = r+1, \dots, n$. Then the orbit consists of $w = w_1 \oplus \cdots \oplus w_n$, where $w_1, \dots, w_r$ are linearly independent and $w_i = \sum_{j=1}^r a_{ij}v_j$ for $i = r+1, \dots, n$. Consequently the orbit is of dimension $r - n$, and there is an $r(n-r)$ dimensional family of such orbits. We get similar orbits when $v_{i_1}, \dots, v_{i_r}$ are linearly independent.

Problem 12-1.3. Consider $\mathbf{Z_2} = \{\pm\mathbf{1}\}$ acting on $\mathbf{C^2}$. Describe the invariants and quotient map for $\mathbf{Z_2} : \mathbf{V_m}$, when $m \geq 2$.

Hint: We do not need the above results. It is clear that $\mathbf{C[V_m]} = \mathbf{C[x_{ij}]_{i=1,\ldots,n,j=1,\ldots,m}}$ and $\mathbf{C[V_m]^{Z_2}} = \mathbf{C[x_{ij}x_{kl}]_{i,k=1,\ldots,n,j,l=1,\ldots,m}}$. The fiber consist of (x_{ij}) and $(-x_{ij})$, if some coordinate is nonzero, and the fiber over zero is zero.

13. Lecture 14

13-1. **First fundamental theorem for the general and special linear group.**

13-1.1. We shall use the same setup as in 12-1.1. Let $V_{m,k}$ denote the space $\mathbf{C^n} \oplus \cdots \oplus \mathbf{C^n} \oplus \mathbf{C^{n*}} \oplus \cdots \oplus \mathbf{C^{n*}}$, where the sum of $V = \mathbf{C^n}$ is taken m times and that of $V^* = \mathbf{C^{n*}}$ is taken k times.

Given a polynomial P in $\mathbf{C}[\mathbf{V_{n,n}}]$ and an $m \times n$ matrix $A = (a_{ij})$, and an $n \times k$ matrix $B = (b_{ij})$. We define $P^{A,B}$ in $\mathbf{C}[\mathbf{V_{m,k}}]$ by

$$P^{A,B}(v_1 \oplus \cdots \oplus v_m \oplus \alpha_1 \oplus \cdots \oplus \alpha_k)$$
$$= P(\sum_{i=1}^{m} a_{i1} v_i \oplus \cdots \oplus \sum_{i=1}^{m} a_{in} v_i \oplus \sum_{j=1}^{k} b_{1j}\alpha_j \oplus \cdots \oplus \sum_{j=1}^{k} b_{nj}\alpha_j).$$

We denote by b_{ij} the polynomial *coordinate* function on $V_{m,k}$ which is defined by $b_{ij}(v_1 \oplus \cdots \oplus v_m \oplus \alpha_1 \oplus \cdots \oplus \alpha_k) = \alpha_j(v_i)$.

Remark 13-1.2. It is clear that the functions b_{ij} are invariant under the action of $\mathrm{Gl}_n(\mathbf{C})$.

Theorem 13-1.3. (Generalization of the basic theorem of classical invariant theory.) *Let $m, k \geq n$. We have that $\mathbf{C}[\mathbf{V_{m,k}}]^{\mathbf{G}}$ is generated by the polynomials*

$$\{P^{A,B} \colon P \in \mathbf{C}[\mathbf{V_{n,n}}]^{\mathbf{G}}, \mathbf{A} \in \mathrm{Mat}_{\mathbf{m,n}}(\mathbf{C}) \text{ and } \mathbf{B} \in \mathrm{Mat}_{\mathbf{n,k}}(\mathbf{C}).\}$$

Proof. The proof is similar to that in the special case Theorem 12-1.3, see Exercise 13-1.1 □

Remark 13-1.4. When G is a reductive group acting on an affine algebraic set and N is an invariant closed subset we have that the resulting map $\mathbf{C}[\mathbf{M}] \to \mathbf{C}[\mathbf{N}]$ induces a surjection $\mathbf{C}[\mathbf{M}]^{\mathbf{G}} \to \mathbf{C}[\mathbf{N}]^{\mathbf{G}}$, by Lemma 9-1.3. Since the inclusion $V_n \subseteq V_m$, and the similar surjection $V_k^* \to V_n^*$, are both G invariant, we may assume that $m, k \geq n$, when we study the classical invariant theory of reductive groups G.

Theorem 13-1.5. (First fundamental theorem of classical invariant theory for the special linear group. First version.) *Let $\mathrm{Sl}_n(\mathbf{C}) : \mathbf{V_m}$. Then the algebra $\mathbf{C}[\mathbf{V_m}]^{\mathrm{Sl}_\mathbf{n}(\mathbf{C})}$ is generated by the polynomials $d_{j_1,\dots,j_n}$, where $1 \leq j_1 < \cdots < j_n \leq m$.*

Proof. The assertion is part of Proposition 12-1.5. □

Theorem 13-1.6. (First fundamental theorem of classical invariant theory for the general linear group.) *Let $\mathrm{Gl}_n(\mathbf{C}) : \mathbf{V_{m,k}}$. Then the algebra $\mathbf{C}[\mathbf{V_{m,k}}]^{\mathrm{Gl}_\mathbf{n}(\mathbf{C})}$ is generated by b_{ij}, for $i = 1, \dots, m$ and $j = 1, \dots, k$.*

Proof. It follows from Remark 13-1.4 and the main theorem 13-1.3, that it suffices to prove the Theorem when $m = k = n$.

We apply the basic Lemma 1-1.7 to the groups $\mathrm{Gl}_n(\mathbf{C})$ and $\{e\}$ and the action of the first on $V_{n,n}$ and the second on the closed set $N = \{e_1 \oplus \cdots \oplus e_n \oplus \alpha_1 \oplus \cdots \oplus \alpha_n \colon \alpha_1, \dots, \alpha_n,$ arbitary linear functions$\}$, where $e_1, \dots, e_n$ is the standard basis for V. Moreover, we let the polynomials of $\mathbf{C}[\mathbf{V_{n,n}}]^{\mathrm{Gl_n(\mathbf{C})}}$ used in the basic Lemma consist of all the b_{ij}. We have that $b_{kj}|N = \alpha_j(e_i)$, so they give all the coordinate functions $\alpha_1, \dots, \alpha_n$ of $\mathbf{C}[\mathbf{N}]$. Moreover, we have that $\mathrm{Gl}_n(\mathbf{C})\mathbf{N}$ consists of all $v_1 \oplus \cdots \oplus v_n \oplus \alpha_1 \oplus \cdots \oplus \alpha_n$, where $v_1, \dots, v_1$ are linearly independent. Consequently $\mathrm{Gl}_n(\mathbf{C})\mathbf{N}$ is dense in $V_{n,n}$. The conditions of Lemma 1-1.7 are thus satisfied, and the conclusion of the Lemma, in this case, is the assertion of the Theorem. □

Remark 13-1.7. For a geometric interpretation of the above result see Exercise 13-1.2. We note that if we use that $M^n_{k,m}$ has the structure of a *normal* variety, one can prove that the map φ of that Exercise is, in fact, bijective.

Theorem 13-1.8. (First fundamental theorem for the special linear group.) *Let* $\mathrm{Sl}_n(\mathbf{C}) : \mathbf{V_{m,k}}$. *Then the algebra* $\mathbf{C}[\mathbf{V_{m,k}}]^{\mathrm{Sl_n(\mathbf{C})}}$ *is generated by the polynomials* b_{ij}, $d_{i_1,\dots,i_n}$ *and* $d^*_{j_1,\dots,j_n}$, *for* $i = 1, \dots, m$ *and* $j = 1, \dots, k$, *for* $1 \le i_1 < \cdots < i_n \le m$, *and for* $1 \le j_1 < \cdots < j_n \le k$.

Proof. To compute $\mathbf{C}[\mathbf{V_{m,k}}]^{\mathrm{Sl_n(\mathbf{C})}}$ we note that $\mathrm{Gl}_n(\mathbf{C}) =< \mathrm{Sl}_\mathbf{n}(\mathbf{C}), \begin{pmatrix} \lambda & \dots & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} >$, and the scalar matrices commute with $\mathrm{Sl}_n(\mathbf{C})$. Hence $\mathrm{Gl}_n(\mathbf{C})$ acts on $\mathbf{C}[\mathbf{V_{m,k}}]$ leaving $A = \mathbf{C}[\mathbf{V_{m,k}}]^{\mathrm{Sl_n(\mathbf{C})}}$ invariant. Since $\mathrm{Sl}_n(\mathbf{C})$ acts trivially on A, it follows from the exact sequence

$$0 \to \mathrm{Sl}_n(\mathbf{C}) \to \mathrm{Gl}_\mathbf{n}(\mathbf{C}) \to \mathbf{C}^* \xrightarrow{\det} \mathbf{0},$$

that $\mathbf{C}^*$ acts on A. It follows from Exercise 13-1.3 that $A = \oplus_{k \in \mathbf{Z}} A_k$, where $A_k = \{P \in A \colon gP = (\det g)^k P\}$. Take $P \in A_k$, with $k \ge 0$. We must show that P can be expressed in terms of the invariants given in the Theorem. The rational function $P/d^k_{1,\dots,n}$ is a $\mathrm{Gl}_n(\mathbf{C})$ invariant rational function because $g d^k_{1,\dots,n} = (\det g)^k d_{1,\dots,n}$. The restriction of this rational function to the closed set$N = \{(e_1 \oplus \cdots \oplus e_n \oplus \alpha_1 \oplus \cdots \oplus \alpha_n) \colon \alpha_1, \dots, \alpha_n$ are linear forms$\}$, where $e_1, \dots, e_n$ is the standard basis for V, is a polynomial in the coordinates of the α_i's, which is the restriction to N of a polynomial of the form $P_1(b_{kj})$, because $b_{ij}|N$ is the j'th coordinate of α_i.

The rational function $P/d^k_{1,\dots,n} - P_1(b_{ij})$ is $\mathrm{Gl}_n(\mathbf{C})$ invariant, and it is zero on N. Moreover we have that $\mathrm{Gl}_n(\mathbf{C})\mathbf{N}$ consists of all $v_1 \oplus \cdots \oplus v_n \oplus \alpha_1 \oplus \cdots \oplus \alpha_n$, where $v_1, \dots, v_1$ are linearly independent, and hence is dense in $V_{n,n}$. Consequently, we have that $P/d^k_{1,\dots,n} - P_1(b_{ij}) \equiv 0$. Similarly we show that $P = (d^*_{1,\dots,n})^{-k} P_1(b_{kj})$. □

Remark 13-1.9. Note that we did not use the fundamental Lemma 1-1.7 in the previous proof because we had rational functions, rather than polynomials. However, one can easily generalize the fundamental result to the case of rational functions.

Remark 13-1.10. In order to assert that $A = \oplus_k A_k$ we use the following two results:

> The elements of a commutative group, whose elements all have finite order, that acts on a vectore space of finite dimensions, can be simultaneously diagonalized.
>
> Let $\mathbf{C}^* : \mathbf{V}$, where V has finite dimension. Then we have that $V = \oplus_k V_k$, where each V_k is $\mathbf{C}^*$ invariant and irreducible, that is $V_k = \mathbf{C}^* \mathbf{v_k}$, for some vector v_k.

To prove the first assertion, we note that each element has a minimal polynomial with distinct roots, and therefore can be diagonalized. We start by diagonalizing one element. Since the group is commutative the eigenspaces of this element are invariant under the group. Hence we can consider the restriction of a second element to each of the eigenspaces, and diagonalize it on each space. Then we continue with the next element. Since the space is finitely dimensional we arrive at a simultaneous diagonalization for all the elements.

To prove the second assertion we use the first assertion to the subgroup H of $\mathbf{C}^*$ consisting of roots of unity. We can thus write $V = \oplus_k \mathbf{C}\mathbf{v_k}$ where $Hv_k = \mathbf{C}\mathbf{v_k}$. The map $\mathbf{C}^* \to \mathbf{V}$ which sends g to gv_k is continous and sends H to V_k. However, H is dense in $\mathbf{C}^*$. Consequently the image $\mathbf{C}\mathbf{v_k}$ of $\mathbf{C}$ is also in V_k. Thus the spaces $\mathbf{C}\mathbf{v_k}$ are $\mathbf{C}^*$ invariant.

Problem 13-1.1. Prove Theorem 13-1.3 by looking at $V_{m,k}$ as a pair (A, B), where $A \in \mathrm{Mat}_{n,m}(\mathbf{C})$ and $B \in \mathrm{Mat}_{k,n}(\mathbf{C})$, and the action is $g(A, B) = (gA, Bg^{-1})$. Now introduce the action of $\mathrm{Gl}_m(\mathbf{C}) \times \mathrm{Gl}_{\mathbf{k}}(\mathbf{C})$ by $(g_1, g_2)(A, B) = Ag_1^{-1}, g_2 B)$. This action commutes with the action of G. Now continue the argument og the proof of Theorem 12-1.3.

Problem 13-1.2. Let $m, k \geq n$. Show that the map $\psi\colon V_{m,k} = \mathrm{Mat}_{n,m} \times \mathrm{Mat}_{k,n} \to \mathrm{Mat}_{k,m}$ which send (A, B) to BA has image equal to the set $M^n_{k,m}$ of all matrices of rank at most n. Moreover show that there is a map $\varphi\colon V_{m,k} /\!\!/ \mathrm{Gl}_n(\mathbf{C}) \to M^n_{k,m}$ such that $\psi = \varphi\pi$, and that this map is bijective regular.

Hint: It is clear that the image is $M^n_{k,m}$ since the map gives a composition. $V^m \to V^n \to V^k$. Moreover, it is clear that every element in $M^n_{k,m}$ can be obtained by using the matrix $\begin{pmatrix} 1 & & & \dots & 0 \\ & \ddots & & & \vdots \\ & & 1 & \dots & 0 \end{pmatrix}$ to the left. We have that $g(A,B) = (gA, Bg^{-1})$ is mapped to BA, so the orbits are contained in the same fiber. Hence the map φ exists. The fiber over C is $\{(A,B)\colon BA = C\}$. It follows that we have equalities

$$b_{ij}\pi(A,B) = b_{ij}\pi(\begin{pmatrix} v_{11} & \dots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nm} \end{pmatrix}, \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{k1} & \dots & \alpha_{kn} \end{pmatrix})$$

$$= b_{ij}(w_1, \dots, w_m)^t(\alpha_1, \dots, \alpha_k) = \alpha_j(w_i) = \sum_{k=1}^{n} \alpha_{jk}(v_{ki}),$$

which is the (i,j)'th coordinate of BA. Thus we obtain that $(b_{ij}\pi(A,B)) = C$ holds, if and only if $\pi(B,A) = C$, that is, if and only if (A,B) is in the fiber. Thus φ is bijective.

Problem 13-1.3. Show that the only characters of $\mathbf{C}^*$ are those that send z to z^k for some integer k.

Hint: A character is the same as a map $\mathbf{C}[\mathbf{x}, \frac{\mathbf{1}}{\mathbf{x}}] \to \mathbf{C}[\mathbf{s}, \mathbf{t}, \frac{\mathbf{1}}{\mathbf{s}}, \frac{\mathbf{1}}{\mathbf{t}}]$, that is having polynomials $g(s,t) = f(s,t)s^m t^n$ such that $f(x,t)^{-1}s^{-m}t^{-n}$ is of the same form. But then $f(s,t)s^m t^n = s^{m'}t^{n'}$. However, $g(s,t) = g(t,s)$ so that $m' = n'$.

14. Lecture 15

14-1. **First fundamental theorem for the orthogonal group.**

14-1.1. Let $V = \mathbf{C^n}$. On V we have a bilinear form defined by $(u,v) = \sum_i u^i v^i$, for all vector $u = {}^t(u^1,\dots,u^n)$ and $v = {}^t(v^1,\dots,v^n)$ of V. Recall that $\mathrm{O}_n(\mathbf{C}) = \{\mathbf{g} \in \mathrm{Gl}_\mathbf{n}(\mathbf{C})\colon (\mathbf{gu},\mathbf{gv}) = (\mathbf{u},\mathbf{v})\}$ We define polynomials b_{ij} on $V_m = \mathbf{C^n} \oplus \cdots \oplus \mathbf{C^n}$, where the sum is taken m times, by

$$b_{ij}(v_1 \oplus \cdots \oplus v_m) = (v_i, v_j), \qquad \text{for } i,j = 1,\dots,m.$$

Theorem 14-1.2. *(First fundamental theorem for the orthogonal group.) The algebra $\mathbf{C}[\mathbf{V_m}]^{\mathrm{O}_\mathbf{m}(\mathbf{C})}$, is generated by the polynomials b_{ij} for $i,j = 1,\dots,m$.*

Proof. We shall use the fundamental Lemma 1-1.7 for rational invariants, see Remark 13-1.9. First, by the basic theorem 12-1.3, we may take $m = n$, see 13-1.4, and, when $m < n$ we have that all invariants come from $\mathbf{C}[\mathbf{V_n}]^\mathbf{G}$ by the map $\mathbf{C}[\mathbf{V_n}]^\mathbf{G} \to \mathbf{C}[\mathbf{V_m}]^\mathbf{G}$.

Consider V_n as $\mathrm{Mat}_n(\mathbf{C})$, on which $\mathrm{O}_n(\mathbf{C})$ acts by left multiplication. Let B_m be the subset of $\mathrm{Mat}_n(\mathbf{C})$ consisting of all upper triangular matrices. Then $\mathrm{O}_n(\mathbf{C})\mathbf{B_n}$ is dense in $\mathrm{Mat}_n(\mathbf{C})$. Indeed, consider the map $\varphi\colon\ \mathrm{O}_n(\mathbf{C})\times\mathbf{B_n} \to \mathrm{Mat}_\mathbf{n}(\mathbf{C})$, given by multiplication. Then $\varphi(1,0) = 0$. Let $a = (1,0)$. Then the map $d\varphi_a$ is given by the sum of tangent vectors. But $T_e\,\mathrm{O}_n(\mathbf{C})$ consists of all skew symmetric matrices, and T_eB_n consists of all upper triangular matrices. However, every matrix is the sum of an upper triangular and a skew symmetric matrix. That is, $T_e(\mathrm{O}_n(\mathbf{C})) + \mathbf{T_eB_n} = \mathbf{T_e}\,\mathrm{Mat}_\mathbf{n}(\mathbf{C})$. Hence the map $d\varphi_a$ is surjective. By the implicit function theorem we have that the set $\varphi(\mathrm{O}_n(\mathbf{C}) \times \mathbf{B_n})$ contains an open neighbourhood of a. However, the Zariski closure of a ball in an affine space is the whole space, since a function that vanishes on a ball vanishes everywhere.

We now apply the fundamental Lemma 1-1.7, for rational functions, see Remark 13-1.9, to the subgroup H of $\mathrm{O}_n(\mathbf{C})$ consisting of diagonal matrices with ± 1 on the diagonal, where H acts on B_n and $\mathrm{O}_n(\mathbf{C})$ on $\mathrm{Mat}_n(\mathbf{C})$, and with the polynomials b_{ij} of $\mathbf{C}[\mathrm{Mat}_\mathbf{n}(\mathbf{C})]$. We have already seen that $\mathrm{O}_n(\mathbf{C})\mathbf{B_n}$ is dense in $\mathrm{Mat}_n(\mathbf{C})$. We check that the b_{ij}'s generate the field $\mathbf{C}(\mathbf{B_n})^\mathbf{H}$. Since B_n consists of the upper diagonal matrices, we have that $\mathbf{C}(\mathbf{B_n})$ conists of the rational functions in the indeterminates a_{ij}, for $1 \le i \le j \le n$. We have that $b_{11}|B_n = ({}^t(a_{11},0,\dots,0), {}^t(a_{11},0\dots,0)) = a_{11}^2$, $b_{12}|B_n = ({}^t(a_{11},0,\dots,0), {}^t(a_{11},a_{12},0\dots,0)) = a_{11}a_{12}$, $b_{22}|B_n = a_{12}^2 + a_{22}^2$, $\dots$. All of these elements are in $\mathbf{C}[\mathbf{B_n}]^\mathbf{H}$. However, the action of H on B_n changes the signs of the rows of the matrices. Hence the ring

$\mathbf{C(B_n)^H}$ consists of the rational functions generated by the products of any two elements in the same row, that is, by the elements $a_{ij}a_{il}$ for $i, j, l = 1, \dots, n$. From Exercise 14-1.1 it follows that this ring is generated by the $b_{ij}|B_n$. It now follows from the fundamental Lemma for rational functions that $\mathbf{C}(\mathrm{Mat}_\mathbf{n}(\mathbf{C}))^{\mathrm{O}_\mathbf{n}(\mathbf{C})} = \mathbf{C}(\mathbf{b_{ij}})$.

We must deduce that $\mathbf{C}[\mathrm{Mat}_\mathbf{n}(\mathbf{C})]^{\mathrm{O}_\mathbf{n}(\mathbf{C})} = \mathbf{C}[\mathbf{b_{ij}}]$. To this end, we consider $\mathbf{C}^{\frac{\mathbf{n(n+1)}}{\mathbf{2}}}$ as the space of upper symmetric matrices. Let $\pi' \colon V_n \to \mathbf{C}^{\frac{\mathbf{n(n+1)}}{\mathbf{2}}}$ be the map defined by $\pi'(v_1 \oplus \cdots \oplus v_n) = ((v_i, v_j))$. Since the orthogonal group $\mathrm{O}_n(\mathbf{C})$ preserves the form $(,)$ we have that the orbits of V_n under $\mathrm{O}_n(\mathbf{C})$ are mapped to points by π'. Consequently there is a unique map $\psi \colon V_n /\!/ \mathrm{O}_n(\mathbf{C}) \to \mathbf{C}^{\frac{\mathbf{n(n+1)}}{\mathbf{2}}}$, such that $\pi' = \psi\pi$, where π is the quotient map $V_n \to V_n /\!/ \mathrm{O}_n(\mathbf{C})$. It follows from Excercise 14-1.2 that π' is surjective.

We have an inclusion $\mathbf{C}[\mathbf{b_{ij}}] \subseteq \mathbf{C}[\mathbf{V_n}]^{\mathrm{O}_\mathbf{n}(\mathbf{C})}$. To prove the opposite inclusion we choose a polynomial $P \in \mathbf{C}[\mathbf{V_n}]^{\mathrm{O}_\mathbf{n}(\mathbf{C})}$. We have, by the first part of the proof, that $P = Q_1(b_{ij})/Q_2(b_{ij})$, where Q_1 and Q_2 are in $\mathbf{C}[\mathbf{b_{ij}}]$. We can assume that Q_1 and Q_2 do not have a common divisor. If Q_2 is not a constant we can find a point $p \in \mathbf{C}^{\frac{\mathbf{n(n+1)}}{\mathbf{2}}}$ such that $q_2(p) = 0$, but $Q_1(p) \neq 0$, as is seen by using the Hilbert Nullstellensatz 6-1.6. However, the equation $P = Q_1(b_{kj})/Q_2(b_{ij})$ means that $P = \psi^*Q_1/\psi^*Q_2$. Since π' is surjective we can choose $v_1, \dots, v_n$ such that $\pi'(v_1 \oplus \cdots \oplus v_n) = p$. Then we have that $\psi^*(Q_1)(\pi(v_1 \oplus \cdots \oplus v_n)) = Q_1(\psi\pi(v_1 \oplus \cdots \oplus v_n)) = Q_1(\pi'(v_1 \oplus \cdots \oplus v_n)) = Q_1(p) \neq 0$ and that $(P\psi^*(Q_2))(\pi(v_1 \oplus \cdots \oplus v_n)) = P(\pi)(v_1 \oplus \cdots \oplus v_n)Q_2\psi\pi(v_1 \oplus \cdots \oplus v_n)P(\pi(v_1 \oplus \cdots \oplus v_n))Q_2(p) = 0$, and we have a contradiction. Thus Q_2 must be a constant, and P must be a polynomial. □

Remark 14-1.3. In the proof of the Theorem we could have used Grahm-Schmidt orthogonalization instead of reasoning on tangen spaces. Indeed, we had to show that there, for every set $v_1, \dots, v_n$ of linearly independent vectors of V there is an A in $\mathrm{O}_n(\mathbf{C})$, such that $Av_1b_{11}i_1$, $Av_2 = b_{21}e_1 + b_{22}e_2$, $Av_3 = b_{31}e_1 + b_{32}e_2 + b_{33}e_3$, $\dots$. To find such an element A we get equations $< v_1, v_1 >=< Av_1, Av_1 >= b_{11}^2$, $< v_1, v_2 >=< Av_1, Av_2 > -b_{11}b_{21}$, $< v_2, v_2 >= b_{21}^2 + b_{22}^2$, $< v_1, v_3 >= b_{11}b_{31}$, $< v_2, v_3 >= b_{21}b_{31} + b_{22}b_{32}$, $< v_3, v_3 >= b_{31}^2 + b_{32}^2 + b_{33}^2$, $\dots$. We see that these equations can be solved when $b_{11}, b_{22}, \dots$ all are nonzero. However, they are zero on a closed set, and for $v_1 = e_1, v_2 = e_2, \dots$, they are nonzero. Hence the complement is an open dense subset where we can solve the equations.

Problem 14-1.1. Show that the ring $\mathbf{C(B_n)^H}$ of rational functions in the expressions $a_{ij}a_{il}$ for $i, j, l = 1, \dots, n$ is generated by the elements

$b_{ij}|B_n$ for $i, j = 1, \dots, n$. Here a_{ij} and B_n are as in the proof of Theorem 14-1.2.

Hint: We have that $b_{11}|B_n = a_{11}^2$, $b_{12}|B_n = a_{11}a_{12}$, $b_{13}|B_n = a_{11}a_{13}, \dots, b_{1n}|B_n = a_{11}a_{1n}$. We multiply two such equation and use that a_{11}^2 is expressed in terms of the b_{ij}. Consequently all $a_{1j}a_{1l}$ can be expressed in this way. Then we use the equation $b_{22}|B_n = a_{12}^2 + a_{22}^2$, $b_{23}|B_n = a_{12}a_{13} + a_{22}a_{23}$, $\dots$, $b_{2n}|B_n = a_{12}a_{1n} + a_{22}a_{2n}$. We consequently get that a_{22}^2, $a_{22}a_{23}$, $\dots$, $a_{22}a_{2n}$ are rational in the b_{ij}. By multiplication of the equations and using what we have proved, we get that the expressions $a_{2j}a_{2l}$ are rational. Then we use that $b_{33} = a_{13}^2 + a_{23}^2 + a_{33}^2$, $b_{34}|B_n = a_{13}a_{14} + a_{23}a_{24} + a_{33}a_{34}$, $\dots$, $b_{3n}|B_n = a_{13}a_{1n} + a_{23}a_{24} + a_{33}a_{34}$, and obtain $a_{23}^2, a_{23}a_{24}, \dots, a_{23}a_{2n}$ are rational. Multiplication gives that all $a_{3j}a_{3l}$ are rational, and we can continue.

Problem 14-1.2. The map $\pi' \colon V_n \to \mathbf{C}^{\frac{\mathbf{n(n+1)}}{\mathbf{2}}}$ of the proof of Theorem 14-1.2, is surjective.

Hint: Try to solve the problem with matrices $v_1 \oplus \dots \oplus v_n = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ 0 & v_{22} & & \\ 0 & 0 & & \\ \vdots & & & \vdots \\ 0 & & \dots & v_{nn} \end{pmatrix}$.

We obtain equations $a_{ij} = \sum_{l=1}^{j} v_{li}v_{lj}$, for $1 \le i \le j \le n$. That is equations $a_{1j} = f_{11}f_{1j}$, for $j = 1, \dots, n$, $a_{2j} = v_{12}v_{1j} + v_{22}v_{2j}$, for $j = 2, \dots, n$, $a_{3j} = v_{13}v_{1j} + v_{23}v_{2j} + v_{33}v_{3j}$, and for $j = 3, \dots, n$, $\dots$. We obtain $v_{11} = \sqrt{a}_{11}$. From the second set of equations we obtain that $a_{2j} - r(a_{1j}) = v_{23}v_{2j}$, where r is a rational function. We can solve if $a_{2j} - r(a_{1j})$ is not zero. From the third set of equations we obtain that $a_{3j} - s(a_{1j}a_{2j}) = v_{33}v_{3j}$, where s is a rational function. We can solve the equations if $a_{3j} - s(a_{1j}a_{2j})$ is not zero. We continue and see that, for all a_{ij} in an open set, we have solutions.

Problem 14-1.3. (First fundamental theorem for the special orthogonal group.) Show that the first fundamental theorem holds for $\mathrm{SO}_n(\mathbf{C})$.

Hint: We proceed as we did for $\mathrm{O}_n(\mathbf{C})$, however we choose H to be the set of diagonal matrices with ± 1 on the diagonal, but with an even number of -1's. We then get the invariants $a_{ij}a_{il}$ and the invariants $a_{1i_1}a_{2i_2}\cdots a_{ni_n}$, that is, we also get the products of the elements where each factor is taken from a different row. In the expansion of $a_{1i_1}a_{2i_2}\cdots a_{ni_n}\det(a_{ij})$ we have that every term is in $\mathbf{C}(\mathbf{b_{ij}})$. Indeed, it follows from Theorem 14-1.2 that $a_{ij}a_{il} \in (\mathbf{b_{ij}})$. It follows that $a_{1i_1}a_{2i_2}\cdots a_{ni_n}$ is in $\mathbf{C}(\mathbf{b_{ij}}, \det(\mathbf{a_{ij}}))$. We now proceed as in the proof of 14-1.2.

Problem 14-1.4. (First fundamental theorem for the symplectic group.) Show that the first fundamental theorem holds for $\mathrm{Sp}_n(\mathbf{C})$.

Hint: Let $B = \{\begin{pmatrix} a & 0 & * & * & \cdots \\ 0 & a & * & * & \\ & & b & 0 & \\ & & 0 & b & \\ & & & & \vdots \end{pmatrix}\}$. If $v_1, \dots, v_n$ are such that $(v_1, v_2) \neq 0$, we can find an A in $\mathrm{Sp}_n(\mathbf{C})$, such that $Av_i = a_{i1}e_1 + \cdots + a_{(i-1)i}e_{i-1} + a_ie_i$ and $Av_{i+1} = a_{1(i+1)}e_1 + \cdots + a_{(i-1)(i+1)}e_{i-1} + a_ie_{i+1}$, for i odd. Indeed, this follows by induction. Consequently we have that the set $\mathrm{Sp}_n(\mathbf{C})\mathbf{B_n}$ is dense in $\mathrm{Mat}_n(\mathbf{C})$. Let H be the subgroup of B where row i and $i+1$ have the same sign, for all odd i. We then have that $b_{11} = 0$, $b_{12} = a_1a_{22}$, $b_{13} = a_1a_{23}$, $\dots$, $b_{1n} = a_1a_{2n}$, $b_{22} = 0$, $b_{23} = -a_1a_{13}$, $b_{24} = -a_{23}a_{14}$, $\dots$, $b_{n4} = -a_1a_{12}$, $b_{33} = 0$, $b_{34} = a_{24}a_{13} - a_{14}a_{23} + a_3a_{44}$, $\dots$, $b_{3n} = a_{13}a_{2n} - a_{23}a_{1n}$, $b_{44} = 0$, $b_{45} = (a_{14}a_{25} = a_{24}a_{15}) - a_{33} - a_3$, $b_{4n} = (a_{14}a_{2n} = a_{24}a_{2n}) - a_{3n}a_3$. From the first set of equations we get that $a_1a_{2j} \in \mathbf{C}[\mathbf{b_{ij}}]$. From the second set we get $a_1a_{1j} \in \mathbf{C}[\mathbf{b_{ij}}]$ since multiplication by $a_1^2 \in \mathbf{C}[\mathbf{b_{ij}}]$ gives $a_{1j}a_{2l} \in \mathbf{C}(\mathbf{b_{ij}})$. From the third set we now get that $a_3a_{4j} \in \mathbf{C}(\mathbf{b_{ij}})$, and the forth $a_3a_{3j} \in \mathbf{C}(\mathbf{b_{ij}})$. Again we get $a_3^2 \in \mathbf{C}(\mathbf{b_{ij}})$, so multiplication gives $a_{3j}a_{4l} \in \mathbf{C}(\mathbf{b_{ij}})$. It is clear how to continue.

Problem 14-1.5. Find the expression of $d_{1,\dots,n} \in \mathbf{C}[\mathbf{V_n}]^{\mathrm{Sp}_\mathbf{n}(\mathbf{C})}$ in terms of the functions b_{ij}.

Hint: We have that $\det(a_{ij}) = \mathrm{Pf}(b_{ij})$. The latter formula follows from the formula $\mathrm{Pf}(AJ^tA) = \det A \,\mathrm{Pf}\, J$, which holds for all A. The latter equation we obtain by taking the square, which gives $\mathrm{Pf}(AJ^tA)^2 = \det(AJ^tA) = \det A^2 \det J = \det A^2 \,\mathrm{Pf}\, J^2$, which shows that the formula holds up to sign. To verify that the sign is $+1$, we choose $A = 1$. However we have that $AJ^tA = ((v_i, v_j)) = b_{ij}$, and we have finished.

14-2. **The second fundamental theorem of classical invariant theory.**

Theorem 14-2.1. *(The second fundamental theorem of classical invariant theory.) The ideal of relations between the generating invariants of the groups considered in Subsections 14-1 and 13-1 are:*
$\mathrm{Gl}_n(\mathbf{C}) : \mathbf{V_{m,k}}$. *Form* $(b_{ij})_{i\in\{i_1,\dots,i_{n+1}\}, j\in\{j_1,\dots,j_{n+1}\}}$, *where* $1 \le i_1, \dots, i_{n+1} \le m$ *and* $1 \le j_1, \dots, j_{n+1} \le k$, *and* $m, k \ge n$. *The relations are* $\det(b_{ij}) = 0$.

Notice that the determinant is zero because we have $n+1$ vectors from $\mathbf{C^n}$.

$\mathrm{Sl}_n(\mathbf{C}) : \mathbf{V_{m,k}}$. *The relations are*

$$\sum_{\text{cyclic permutation of } \{i_1,\dots,i_{n+1}\}} \pm d_{i_1,\dots,i_n} d_{i_{n+1}j_1,\dots,j_{n-1}} = 0,$$

and the same for $d^*_{i_1,\dots,i_n}$.

Moreover, we have the relation $d_{i_1,\dots,i_n} d^*_{j_1,\dots,j_n} = \det(b_{ij})_{i\in\{i_1,\dots,i_n\}, j\in\{j_1,\dots,j_n\}}$.

$\mathrm{O}_n(\mathbf{C}) : \mathbf{V_{m,k}}$ *and* $\mathrm{Sp}_n(\mathbf{C}) : \mathbf{V_{m,k}}$. *The relations are the same as for* $\mathrm{Gl}_n(\mathbf{C})$ *together with the relations* $\det(b_{ij})_{i,j\in\{k_1,\dots,i_{n+1}\}} = 0$.

$\mathrm{SO}_n(\mathbf{C}) : \mathbf{V_{m,k}}$. *The relations are the same as for* $\mathrm{O}_n(\mathbf{C})$ *and the first relation for* $\mathrm{Sl}_n(\mathbf{C})$.

Proof. (Sketch of proof.) If m or k are at most equal to n there are clearly no relations. We have a map $\pi' \colon V_{m,k} \to \mathrm{Mat}_{m,k}$ given by multiplication $\mathrm{Mat}_{m,n} \times \mathrm{Mat}_{n,k} \to \mathrm{Mat}_{m,k}$. Denote the image by $\mathcal{M}$. In Excercise 13-1.2 we proved that $\mathcal{M}$ consists of matrices of rank at most equal to n. In the same Exercise we saw there is a map $\varphi \colon V_{m,k} /\!\!/ \mathrm{Gl}_n(\mathbf{C}) \to \mathrm{Mat}_{m,k}$, such that $\pi' = \varphi\pi$ and which, as mentioned in Remark 13-1.7, induces an isomorphism from $V_{m,k} /\!\!/ \mathrm{Gl}_n(\mathbf{C})$ to $\mathcal{M}$. Consequently, it suffices to prove the second main theorem for the subvariety $\mathcal{M}$ of $\mathrm{Mat}_{m,k}$. However, a matrix has rank at most n if all the submatrices of rank $n+1$ have determinant zero. This gives all the relations. $\square$

15. Lecture 16

15-1. **Symbolic method of invariant theory.**

15-1.1. The first theorem of invariant theory gives us a tool to compute a vector space basis for the invariants, and the second fundamental theorem to find a finite set of generators. By classical invariant teory we can calculate all the invariants of the *classical groups* $\mathrm{Gl}_n(\mathbf{C})$, $\mathrm{Sl}_n(\mathbf{C})$, $\mathrm{O}_n(\mathbf{C})$, $\mathrm{SO}_n(\mathbf{C})$, $\mathrm{Sp}_n(\mathbf{C})$, acting linearly on a subspace V of the vector space $(\otimes^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}})$, that is for *tensor representations*. For the group $\mathrm{Gl}_n(\mathbf{C})$, $\mathrm{Sl}_n(\mathbf{C})$, $\mathrm{Sp}_n(\mathbf{C})$, we get all the irreducible representations this way, and for the groups $\mathrm{O}_n(\mathbf{C})$, $\mathrm{SO}_n(\mathbf{C})$, we get half of them, see Humphreys [**?**]. In the following we shall only look at tensor representation. It follows, as in Remark 13-1.4, that it suffices to consider the whole space $V = (\otimes^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}})$ of (k, s) tensors, for the linear, and special linear group, and $\otimes^k \mathbf{C^n}$, for the other groups.

Given a polynomial F in $\mathbf{C[V]^G}$ of degree p. We can consider F as an element in $S^p((\otimes^k \mathbf{C^{n*}}) \otimes (\otimes^s \mathbf{C^n}))^{\mathbf{G}} \subseteq \otimes^{\mathbf{p}}((\otimes^{\mathbf{k}} \mathbf{C^{n*}} \otimes (\otimes^{\mathbf{s}} \mathbf{C^n}))^{\mathbf{G}}$. Thus we can consider F as a G invariant tensor in $((\otimes^{kp} \mathbf{C^{n*}}) \otimes (\otimes^{sp} \mathbf{C^n}))^{\mathbf{G}}$, or, equivalently, as a G invariant multilinear function on $(\otimes^{kp} \mathbf{C^n}) \otimes (\otimes^{\mathbf{sp}} \mathbf{C^{n*}})$. We associate to F, considered as a multilinear function, a G invariant polynomial $\tilde{F}$ on $V_{kp,sp}$, defined by

$$\tilde{F}(v_1 \oplus \cdots \oplus v_{kp} \oplus \alpha_1 \oplus \cdots \oplus \alpha_{sp}) = F(v_1 \otimes \cdots \otimes v_{kp} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{sp}).$$

In order to obtain a description of $\tilde{F}$ by classical invariant theory, we introduce the two following G invariant operations:

Alternation: For $1 \leq i_1, \dots, i_n \leq kp$ we define operations:

$$A_{i_1,\dots,i_n} \colon (\otimes^{kp} \mathbf{C^n}) \otimes (\otimes^{\mathbf{sp}} \mathbf{C^{n*}}) \to (\otimes^{\mathbf{kp-n}} \mathbf{C^n}) \otimes (\otimes^{\mathbf{sp}} \mathbf{C^{n*}}),$$

by

$$\begin{aligned} &A_{i_1,\dots,i_n}(v_1 \otimes \cdots \otimes v_{sp} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{sp}) \\ &\qquad = \det(v_{i_1}, \dots, v_{i_n}) v_1 \otimes \cdots \otimes \hat{v}_{i_1} \otimes \cdots \otimes \hat{v}_{i_n} \otimes \cdots \otimes v_{kp} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{ps}. \end{aligned}$$

Similarly, we define:

$$A^*_{i_1,\dots,i_n} \colon (\otimes^{kp} \mathbf{C^n}) \otimes (\otimes^{\mathbf{kp}} \mathbf{C^{n*}}) \to (\otimes^{\mathbf{kp}} \mathbf{C^n}) \otimes (\otimes^{\mathbf{kp-n}} \mathbf{C^{n*}}),$$

by

$$\begin{aligned} &A^*_{i_1,\dots,i_n}(v_1 \otimes \cdots \otimes v_{kp} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{sp}) \\ &\qquad = \det(\alpha_{i_1}, \dots, \alpha_{i_n}) \alpha_1 \otimes \cdots \otimes \hat{\alpha}_{i_1} \otimes \cdots \otimes \hat{\alpha}_{i_n} \otimes \cdots \otimes \alpha_{kp}. \end{aligned}$$

Contraction: For $1 \leq i \leq k$ and $1 \leq j \leq s$, we define operations:

$$C_{ij} \colon (\otimes^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}}) \to (\otimes^{\mathbf{k-1}} \mathbf{C^n}) \otimes (\otimes^{\mathbf{s-1}} \mathbf{C^{n*}})$$

by

$$C_{ij}(v_1 \otimes \cdots v_k \otimes \alpha_1 \otimes \cdots \otimes \alpha_s) = \alpha_j(v_i) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_k \otimes \alpha_1 \otimes \cdots \otimes \hat{\alpha}_j \otimes \cdots \otimes \alpha_s.$$

Definition 15-1.2. We define the product of alternations and contractions $A_{i_1,\dots,i_n}$, ..., $A^*_{j_1,\dots,j_n}$, ..., and C_{ij}, ..., with disjoint sets of indices, by taking the value at $v_1 \otimes \cdots \otimes v_{kp} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{sp}$ to be the product of the corresponding $d_{i_1,\dots,i_n}$, ..., $d^*_{j_1,\dots,j_n}$, ..., and $\alpha_j(v_i)$, ..., with the element obtained from $v_1 \otimes \cdots \otimes v_{kp} \otimes \alpha_1 \otimes \cdots \otimes \alpha_{sp}$, by deleting all the v_i's and α_j's that have coordinates in any of the sets $\{i_1, \dots, i_n\}$, ..., $\{j_1, \dots, j_n\}$, ..., and $\{i, j\}$, ... A *complete* alternation and contraction is a sequence of consecutive alternations and contractions which maps to $\mathbf{C}$.

Note that any contraction and alternation reduces the number of tensor products.

Example 15-1.3. We have that $C_{13}C_{24} \colon (\otimes^2 \mathbf{C^2}) \otimes (\otimes^{\mathbf{2}} \mathbf{C^{2*}}) \to \mathbf{C}$.

Theorem 15-1.4. *Let G be a classical group acting on a subspace V of $(\otimes^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}})$. Any invariant F of degree p in $\mathbf{C[V]^G}$ can be written as a linear combination of invariants otained from a complete alternation and contraction on $V^{\otimes p}$.*

Proof. As remarked in 15-1.1 we can assume that $V = (\otimes^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}})$, and we associated to F a G invariant multilinear function $\tilde{F}$ on $V_{kp,sp}$. It follows from the first fundamental theorems of classical invariant theory, 13-1.6, 13-1.8, 14-1.2, 14-1.3, 14-1.4, that we can write $\tilde{F}(v_1, \oplus \cdots \oplus v_{kp} \oplus \alpha_1 \oplus \cdots \oplus \alpha_{sp}) = P(d_{i_1,\dots,i_n}, d^*_{j_1,\dots,j_n}, b_{ij})$, where P is a homogeneous polynomial. However, $\tilde{F}$ is linear in each of the variables v_i and α_j. It follows that each of the monomials in the $d_{i_1,\dots,i_n}$, $d^*_{j_1,\dots,j_n}$ and b_{ij} that appear in $P(d_{i_1,\dots,i_n}, d^*_{j_1,\dots,j_n}, b_{ij})$ must be linear in the v_i and α_j. Consequently the indices $i_1, \dots, i_n$, ..., $j_1, \dots, j_n$, ..., and ij, ..., must be disjoint. Thus, $P(d_{i_1,\dots,i_n}, d^*_{j_1,\dots,j_n}, b_{ij})$ can be expressed as a sum of complete alternations and contractions. □

Corollary 15-1.5. (a) *Let $G = \mathrm{Gl}_n(\mathbf{C})$ operate on $(\oplus^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}})$. If there exists nonconstant invariant functions, we have that $k = s$.*

(b) *Let $\mathrm{Sl}_n(\mathbf{C})$ act on $V = \otimes^k \mathbf{C^n}$. If there exists invariants of degree p, we have that $n|kp$.*

Proof. In case (a), it follows from Theorem 13-1.6 that we have $\mathbf{C[V_{m,k}]^G} = \mathbf{C[b_{ij}]}$. Consequently we only have to use contractions. However, it is impossible to find a complete product of contractions $(\otimes^k \mathbf{C^n}) \otimes (\otimes^{\mathbf{s}} \mathbf{C^{n*}}) \to \mathbf{C}$ unless $s = k$. Thus we have proved assertion (a).

In case (b) we have that there are no contractions since $V = \otimes^k \mathbf{C^n}$. Hence V does not involve any of the dual spaces $\mathbf{C^{n*}}$. In order to have have a complete product of alternations $\otimes^{kp}\mathbf{C^n} \to \mathbf{C}$, we must therefore have that $n|kp$. □

Theorem 15-1.6. *Let* $G = \mathrm{Gl}_n(\mathbf{C})$ *act by conjugation on* $V = \mathrm{Mat}_n(\mathbf{C}) \oplus \cdots \oplus \mathrm{Mat}_\mathbf{n}(\mathbf{C})$*, where the sum is taken* m *times. Then* $\mathbf{C[V]^G}$ *is generated by the following polynomials:*

$$P_{j_1,\dots,j_k}, \text{ where } 1 \le j_1, \dots, j_k \le m,$$

and

$$P_{j_1,\dots,j_k}(A_1 \oplus \cdots \oplus A_m) = \mathrm{Tr}(A_{j_1}, \dots, A_{j_k})$$

Proof. We have that $\mathrm{Mat}_n(\mathbf{C}) = \mathbf{C^n} \otimes \mathbf{C^{n*}}$, and consequently, that V is contained in $(\otimes^m \mathbf{C^n})(\otimes^\mathbf{m}\mathbf{C^n})^* = (\mathbf{C^n} \times \mathbf{C^{n*}})^{\otimes \mathbf{m}}$.

In case (a), it follows from Theorem 13-1.6 and Theorem 15-1.4 that we have $\mathbf{C[V_{m,k}]^G} = \mathbf{C[b_{ij}]}$. Hence we only have to use contractions. Let F be an element in $\mathbf{C[V]^G}$. We associate to F the G invariant multilinear form $\tilde{F}$ of 15-1.1. Then, by the first main theorems of the classical groups, 13-1.6, 13-1.8, 14-1.2, 14-1.3, 14-1.4, we have that $\tilde{F}$ can be expressed as $P(v)$, where P is a homogeneous polynomial. Let k be the degree of P. Write $v = (\sum_{i_1 j_1} a_{i_1}^{j_1} e^{i_1} \otimes e_{j_1}^*) \otimes \cdots \otimes (\sum_{i_m j_m} a_{i_m}^{j_m} e^{i_m} \otimes e_{j_m}^*)$, where $e^1, \dots, e^n$ is a standard basis for $\mathbf{C^n}$ and $e_1^*, \dots, e_n^*$ is the dual basis for $\mathbf{C^{n*}}$. We have that $v^k = (\sum_{i_1 j_1} a_{i_1}^{j_1} e^{i_1} \otimes e_{j_1}^*) \otimes \cdots \otimes (\sum_{ij} a_{\substack{j\\i}} e^i \otimes e_j^*) \otimes \cdots \otimes (\sum_{i_m j_m} a_{i_m}^{j_m} e^{i_m} \otimes e_{j_m}^*)$, with mk factors. The invariant $P(v)$ is obtained by taking sums of complete contraction of this element. Since the contractions commute, so we can take them in any order. Thus we can write the product of contractions in the form $C_{1\sigma(1)} \cdots C_{mk,\sigma(mk)}$, where σ is a permutation of $1, \dots, mk$. We use that every permutation can be written as a product of cycles. Consequently, the contraction can be written as a product of contractions of the form $C_{i_1 i_2} C_{i_2 i_3} \cdots C_{i_s i_1}$, corresponding to the cycle $(i_1, i_2, \dots, i_s)$. The action of a contraction of the latter form produces a sum $\sum_{i_1 i_2 \dots i_s} a_{i_1}^{i_2} a_{i_2}^{i_3} \cdots a_{i_s}^{i_1}$, multiplied by a product of e_i's and e_j^*'s, where the factors e^{i_1}, $e_{i_1}^*$, … , e^{i_s}, $e_{i_s}^*$ are removed. However, we have that $\mathrm{Tr}\, A_{i_1} \cdots A_{i_s} = \sum_{i_1 \dots i_s} a_{i_1}^{i_2} a_{i_2}^{i_3} \cdots a_{i_s}^{i_1}$. Consequently, any invariant is a linear combination of products of elements of the form $P_{i_1 \dots, i_s}$. □

Remark 15-1.7. Note that $g(A_1 \oplus \cdots \oplus A_n) = (gA_1 g^{-1} \oplus \cdots \oplus gA_n g^{-1})$, so we have $\mathrm{Tr}(A_{j_1} \cdots A_{j_k})$ is invariant. Moreover, we note that k in the Theorem is arbitrary so that there are infinitely many invariants.

Theorem 15-1.8. (Procesi.) *It suffices to take* $k \le 2^n - 1$ *in Theorem 15-1.6.*

Example 15-1.9. In the case $n = 2$ of the Procesi theorem we use Cayley-Hamilton, and get $A^2 - \mathrm{tr}(A)A = \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2)) = 0$. Multiplying to the right by $BC \dots D$ and taking the trace, we get $\mathrm{tr}(A^2BC \dots D) = \mathrm{tr}(A)\,\mathrm{tr}(ABC \dots D) = \frac{1}{2}((\mathrm{tr}(A))^2 - \mathrm{tr}\,A^2)\,\mathrm{tr}(BC \dots D)$. Hence, if a matrix appear more than twice, we can reduce the number of times it appear. It follows from Theorem 15-1.6, for $n = 2$, that $\mathbf{C}[\mathbf{V}]^{\mathbf{G}}$ is generated by $F_{j_1 \dots j_k}$ and P_{11}, when $j_1, \dots, j_k$ are distinct. The last assertion holds because $\mathrm{tr}(A^2)$ can not be expressed by a lower number of matrices. Using Cayley-Hamilton again one can contine and reduce the number of matrices k to 3, or less.

Problem 15-1.1. Show that for $G = \mathrm{Sl}_2(\mathbf{C}) : \mathrm{Mat}_{\mathbf{2}}(\mathbf{C}) \oplus \mathrm{Mat}_{\mathbf{2}}(\mathbf{C})$, we have that the invariants P_1, P_2, P_{11}, P_{22} and P_{12} generate $\mathbf{C}[\mathbf{V}]^{\mathbf{G}}$.

Hint: It follows from Theorem 13-1.8 and Theorem 15-1.4, that the invariants are generated by $A_{i_1 \dots i_n}$, $A^*_{j_1 \dots j_n}$ and C_{ij}. We saw, in the proof of Theorem 15-1.6, that the product of contractions give $\mathrm{Tr}\,A_{i_1} \cdots A_{i_k}$. Moreover, the use of a $A_{i_1 \dots i_n}$ must be compensated by the use of an $A^*_{j_1 \dots j_n}$, in order to end up with a map to $\mathbf{C}$. Use A_{12} and A^*_{12} to the element $(a^1_1 e^1 \otimes e^*_1 + a^2_1 e^1 \otimes e^*_2 + a^1_2 e^2 \otimes e^*_1 + a^2_2 e^2 \otimes e^*_2)(b^1_1 e^1 \otimes e^*_1 + b^2_1 e^1 \otimes e^*_2 + b^1_2 e^2 \otimes e^*_1 + b^2_2 e^2 \otimes e^*_2)$, and we get that $a^1_1 b^2_2 + a^2_2 b^1_1 = a^2_1 b^1_2 - a^1_2 b^2_1 = (a^1_1 + a^2_2)(b^1_1 + b^2_2) - (a^1_1 b^1_1 + a^2_1 b^1_2 + a^2_2 b^2_2) = \mathrm{Tr}\,A\,\mathrm{Tr}\,B - \mathrm{Tr}\,AB$. We now use the formula $\mathrm{Tr}\,AB = \mathrm{Tr}\,A\,\mathrm{Tr}\,B - A_{12}A^*_{12}(a \otimes b)$, to the reduce the number of factors in $\mathrm{Tr}(A_{i_1} \cdots A_{i_k})$ to 2.

Problem 15-1.2. Let $G = \mathrm{Sl}_n(\mathbf{C})$ act on $V = \mathbf{C^n} \oplus \cdots \oplus \mathbf{C^n}$, where the sum is taken m times. Show that $\mathbf{C}[\mathbf{V}]^{\mathbf{G}}$ is a polynomial ring, if and only if $m \le n - 1$. Find the generating invariants.

Hint: It follows from 13-1.8 that $\mathbf{C}[\mathbf{V}^{\mathbf{G}}]$ is generated by $a_{i1}a_{1j} + \cdots + a_{in}a_{nj}$, for $i, j = 1, \dots, n$, and, when $m \ge n$, also by $\det(a_{ij})$. The first n^2 elements clearly are transcendentally independent. Consequently $\mathbf{C}[\mathbf{V}]^{\mathbf{G}}$ is polynomial when $m \le n - 1$. When $m \ge n$ the element $\det(a_{ij})$ is algebraically dependent on the n^2 first. However, it does not lie in the ring genrated by these beacause, choose $a_{ij} = 0$ for $i \ne j$. If $\det(a_{ij})$ was in the ring we then see that the element $a_{11} \cdots a_{nn}$ would lie in the ring generated by $a^2_{11}, \dots, a^2_{nn}$, which is impossible.

Problem 15-1.3. Let $\mathrm{Sl}_n(\mathbf{C}) : \mathbf{V} = \mathbf{S^2C^n} \oplus \mathbf{S^2C^n}$, that is the special linear group operating on pairs of quadratic forms. Show that $\mathbf{C}[\mathbf{V}]^{\mathrm{Sl}_{\mathbf{n}}(\mathbf{C})}$ is a polynomial ring generated by $n + 1$ polynomials of degree n.

Hint: Two of them are discriminants of degree n, but there are $n - 1$ more.

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