INTERACTION BOUNDS IN MULTIVARIABLE CONTROL SYSTEMS

Karl Henrik Johansson

Department of Electrical Engineering & Computer Sciences,
UC Berkeley, 333 Cory Hall # 1770, Berkeley, CA 94720-1770,
johans@eecs.berkeley.edu

Abstract: Time-domain limitations due to right-half plane zeros and poles in linear multivariable control systems are studied. Lower bounds on the interaction are derived. They show not only how the location of zeros and poles are critical in multivariable systems, but also how the zero and pole directions influence the performance. The results are illustrated on the Quadruple-Tank Process, which is a new multivariable laboratory process.

Keywords: Performance limits; Multivariable control systems; Linear systems; Process control

1. INTRODUCTION

When designing technical systems, it is useful to know what characteristics that limit the performance. In many situations this is a non-trivial task. Recently there has been increased interest in fundamental limits for the achievable performance in feedback systems [16,1,6]. One reason for this is new possibilities for integrated process and control design in many applications. Without having to specify a certain control implementation or carry out the actual control design, it is possible early in the development to answer structural questions, for instance, about number and location of sensors and actuators.

Many of the existing results on feedback performance limitations are frequency-domain results for linear systems, see [2,8,17,3,4,15,14] and references therein. However, in many cases time-domain bounds are more natural, for example, to answer questions about minimum rise time and settling time for a system. Such results were derived in [13] for SISO systems. For example, Middleton's results gave a bound on the undershoot of the set-point response in nonminimum-phase systems and a bound on the overshoot in unstable systems.

The main contribution of this paper is to generalize the time-domain results in [13] to multivariable systems. This gives new insight into the limitations multivariable zeros have on closed-loop responses. In contrast to scalar systems with right half-plane (RHP) zeros, a multivariable system must in general not have an inverse response. Instead there is a trade-off between the response time and the interaction. The trade-off depends both on the location of the zero and the zero direction. This paper presents time-domain results that support these facts. Counterparts in the frequency domain are presented in [5,14].

The outline of the paper is as follows. Some notation is introduced in Section 2. In Section 3 the main result of the paper on trade-off between settling time and interaction in nonminimum-phase systems is given. Section 4 presents a similar result for unstable systems. The results are illustrated on a new laboratory process in Section 5. The process is called the Quadruple-Tank Process and has a zero that can be placed in either the right or the left half-plane by simply adjusting a valve. The paper is concluded in Section 6.

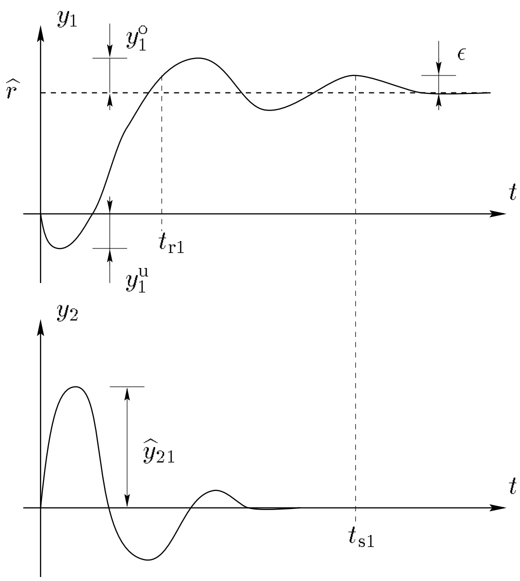

Fig. 1. Definition of settling time t_{s1}, settling level ϵ, rise time t_{r1}, overshoot y_1^{o}, undershoot y_1^{u}, and interaction $\widehat{y}_{21}$ in a 2×2 system with reference step $\widehat{r}$ in r_1.

2. PRELIMINARIES

Much of the notations and definitions in this paper are borrowed from the textbook [14]. Let

$$\begin{aligned} Y(s) &= G(s)U(s), \\ U(s) &= C(s)\big(R(s) - Y(s)\big) \end{aligned} \tag{1}$$

represent a stable closed-loop system with zero initial conditions. The process G and the controller C are $m \times m$ transfer function matrices. The variables Y, U, and R are Laplace transforms of the output y, the control signal u, and reference signal r, respectively, that is, $Y(s) = \int_0^\infty e^{-st} y(t)\, dt$ etc. Throughout the paper we make the assumptions that G is strictly proper and has full normal rank.

Definition 1. (Zeros and poles). $z \in \mathbf{C}$ is a zero of G with zero direction $\psi \in \mathbf{R}^m$, $|\psi| = 1$, if $\psi^* G(z) = 0$, where the asterisk denotes conjugate transpose.

$p \in \mathbf{C}$ is a pole of G with pole direction $\phi \in \mathbf{R}^m$, $|\phi| = 1$, if $G^{-1}(p)\phi = 0$. □

We assume that $G(s)$ looses only rank one at $s = z$ and that $G^{-1}(s)$ looses only rank one at $s = p$. Furthermore, it is assumed that the set of poles and the set of zeros of GC are disjoint and that the closed-loop system imposes no unstable cancellations.

We make the following definitions for a step response, see Figure 1.

Definition 2. (Set-point response). For the closed-loop system (1), consider a step in reference signal $i \in \{1, \ldots, m\}$, so that $r_i(t) = \widehat{r}$ and $r_j(t) = 0$ for all $j \neq i$ and $t > 0$. The settling time $t_{si} \in (0, \infty)$ is defined as

$$t_{si} = \max_{k \in \{1,\ldots,m\}} \inf_{\delta > 0} \{\delta : |y_k(t) - r_k(t)| \leq \epsilon, t > \delta\},$$

where $\epsilon \geq 0$ is a predefined settling level. The rise time is

$$t_{ri} = \sup_{\delta > 0} \{\delta : y_i(t) \leq \widehat{r}t/\delta, t \in (0, \delta)\}.$$

The overshoot in output i is denoted $y_i^{o} \geq 0$ and is defined as

$$y_i^{o} = \sup_{t > 0} \{y_i(t) - r_i(t), 0\}$$

and the undershoot $y_i^{u} \geq 0$ is defined as

$$y_i^{u} = \sup_{t > 0} \{-y_i(t), 0\}.$$

The interaction from r_i to output $k \neq i$ is denoted $\widehat{y}_{ki} \geq 0$ and is defined as

$$\widehat{y}_{ki} = \sup_{t > 0} \{|y_k(t)|\}.$$

□

By introducing coprime factorizations of G, it is straightforward to show that the sensitivity function $S = (I + GC)^{-1}$ and the complementary sensitivity function $T = GC(I + GC)^{-1}$ satisfy $S(p)\phi = 0$ and $\psi^* T(z) = 0$, respectively, where p is a pole of G and z is a zero, see [14].

3. RIGHT HALF-PLANE ZEROS

In this section a lower bound is derived on the undershoot and the interaction for a set-point step in one of the reference signals. A crucial observation is that if $z > 0$ is a real RHP zero of G, then

$$\psi^T T(z) = \psi^T G(z)C(z)\big(I + G(z)C(z)\big)^{-1} = 0$$

and therefore

$$\begin{aligned} \psi^T \int_0^\infty e^{-zt} y(t)\, dt &= \psi^T Y(z) \\ &= \psi^T T(z)R(z) = 0. \end{aligned} \tag{2}$$

There is thus a trade-off between the output responses $y_1, \ldots, y_m$ that is determined by the zero direction. The trade-off becomes more severe if the zero is located close to the origin. This is formalized in the following result.

Theorem 1. Consider the stable closed-loop system (1) with zero initial conditions at $t = 0$ and let $r(t) = (\widehat{r}, 0, \ldots, 0)^T$ for $t > 0$. Assume that G has a real RHP zero $z > 0$ with zero direction

$\psi \in \mathbf{R}^m$ and $\psi_1 > 0$. Then, the set-point response satisfies

$$\psi_1 y_1^{\mathrm{u}} + \sum_{k=2}^{m} |\psi_k| \widehat{y}_{k1} \geq \frac{1}{e^{z t_s} - 1} \left[\psi_1 (\widehat{r} - \epsilon) - \epsilon \sum_{k=2}^{m} |\psi_k| \right],$$

where y_1^{u} is the undershoot, $\widehat{y}_{k1}$ the interaction, ϵ the settling level, and t_{s1} the settling time, all as given in Definition 2.

Proof: Equation (2) gives

$$\sum_{k=1}^{m} \psi_k \int_0^{\infty} e^{-zt} y_k(t)\, dt = 0,$$

which is equivalent to

$$-\int_0^{t_{s1}} e^{-zt} \sum_{k=1}^{m} \psi_k y_k(t)\, dt = \int_{t_{s1}}^{\infty} e^{-zt} \sum_{k=1}^{m} \psi_k y_k(t)\, dt.$$

The left-hand and the right-hand sides satisfies

$$-\int_0^{t_{s1}} e^{-zt} \sum_{k=1}^{m} \psi_k y_k(t)\, dt \leq \int_0^{t_{s1}} e^{-zt}\, dt \left[\psi_1 y_1^{\mathrm{u}} + |\psi_2| \widehat{y}_{21} + \cdots + |\psi_m| \widehat{y}_{m1} \right]$$

and

$$\int_{t_{s1}}^{\infty} e^{-zt} \sum_{k=1}^{m} \psi_k y_k(t)\, dt \geq \int_{t_{s1}}^{\infty} e^{-zt}\, dt \left[\psi_1 (\widehat{r} - \epsilon) - |\psi_2| \epsilon - \cdots - |\psi_m| \epsilon \right],$$

respectively. From

$$\int_0^{t_{s1}} e^{-zt}\, dt = \frac{1 - e^{-z t_{s1}}}{z}, \qquad \int_{t_{s1}}^{\infty} e^{-zt}\, dt = \frac{e^{-z t_{s1}}}{z},$$

it now follows that

$$e^{-z t_{s1}} \left[\psi_1 (\widehat{r} - \epsilon) - |\psi_2| \epsilon - \cdots - |\psi_m| \epsilon \right] \leq (1 - e^{-z t_{s1}}) \left[\psi_1 y_1^{\mathrm{u}} + |\psi_2| \widehat{y}_{21} + \cdots + |\psi_m| \widehat{y}_{m1} \right],$$

which gives the result. □

Remark 1. For a small settling level ϵ, it follows from Theorem 1 that approximately

$$\psi_1 y_1^{\mathrm{u}} + \sum_{k=2}^{m} |\psi_k| \widehat{y}_{k1} \geq \frac{\psi_1 \widehat{r}}{e^{z t_{s1}} - 1}.$$

So under the assumption that the right-hand side is larger than the sum in the left-hand side, we have a lower bound on the undershoot in y_1. The bound suggests that the undershoot will be large if the zero is close to the origin. Furthermore, it also suggests that if the interaction is small ($\widehat{y}_{k1} > 0$ is small), the undershoot has to be large.

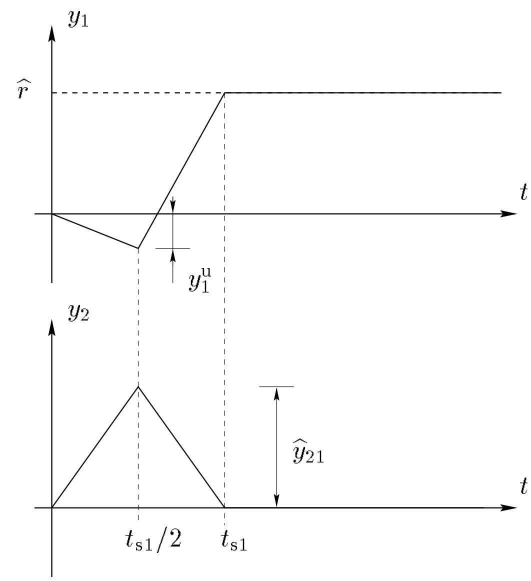

Fig. 2. By approximating the responses with straight lines, it is in many cases possible to derive better estimates for the relation between settling time, undershoot, and interaction.

There is hence an immediate trade-off between the undershoot in the considered set-point response loop and the interaction to the other loops.

Remark 2. Theorem 1 illustrates the importance of zero directions. A RHP zero in a SISO system is known to impose inverse set-point response. For MIMO systems, however, we see from Theorem 1 that it is only if all but one element of the zero direction ψ are zero that a RHP zero must give an inverse set-point response. Such zero is related to only one input–output pair and implies in that sense similar restrictions to the response for that loop as RHP zeros in scalar systems. This was illustrated in the frequency-domain in [5].

Remark 3. The bound given in Theorem 1 is in many cases conservative. This is, of course, due to the rough estimates used in deriving the formula. One possibility to get better estimates is to introduce some sort of approximate shape of the responses. Figure 2 shows an example of such shapes.

Remark 4. In the SISO case Theorem 1 reduces to Lemma 4 in [13] or Corollary 1.3.6 in [14]. Note that all these results are derived for control systems of one-degree of freedom. It is well-known that a two-degree of freedom controller can improve the set-point responses considerably. Theorem 1 gives suggestions when such an increased controller complexity is desirable for multivariable systems.

4. RIGHT HALF-PLANE POLES

In this section systems with RHP poles are considered. It is shown that such poles imply constraints on interaction similar to RHP zeros. If $p > 0$ is a real RHP pole of G, then

$$S(p)\phi = \big(I + G(p)C(p)\big)^{-1}\phi = 0.$$

Consider m responses to set-point steps $\widehat{r}$ in reference signals r_1 to r_m, respectively. They give the control error matrix $E = R - Y = SR$, where

$$R(s) = \begin{bmatrix} \widehat{r}/s & 0 & \dots & 0 \\ 0 & \widehat{r}/s & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \widehat{r}/s \end{bmatrix}.$$

The control error satisfies

$$E(p) = \int_0^\infty e^{-pt} e(t)\, dt \cdot \phi,$$

so that

$$E(p)\phi = S(p)R(p)\phi = S(p)\phi/p = 0. \qquad (3)$$

There is thus a trade-off between the errors for a certain output for input steps in various reference signals. The trade-off is determined by the pole direction.

Theorem 2. Consider the stable closed-loop system (1) with zero initial conditions at $t = 0$. Assume that G has a real RHP pole $p > 0$ with pole direction $\phi \in \mathbf{R}^m$ and $\phi_1 > 0$. Consider m independent set-point responses with $r_i(t) = \widehat{r}$ for $t > 0$. Then, these responses satisfy

$$\phi_1(\widehat{r} + y_1^{\circ}) + \sum_{k=2}^{m} |\phi_k| \widehat{y}_{1k} \geq \frac{\widehat{r} p t_{r1}}{2}\phi_1 - \big(e^{p t_{r1}} - 1\big) \sum_{k=2}^{m} |\phi_k| \widehat{y}_{1k},$$

where y_1° and t_{r1} are the overshoot and the rise time for set-point response in r_1, respectively, and $\widehat{y}_{1k}$ is the interaction to y_1 with set-point response in r_k, all as given in Definition 2.

Proof: Let e_{1k} be the response in the first error signal for a set-point step in $r_k(t) = \widehat{r} > 0$. Equation (3) gives

$$\sum_{k=1}^{m} \phi_k \int_0^\infty e^{-pt} e_{1k}(t)\, dt = 0,$$

which is equivalent to

$$-\int_{t_{r1}}^\infty e^{-pt} \sum_{k=1}^{m} \phi_k e_{1k}(t)\, dt = \int_0^{t_{r1}} e^{-pt} \sum_{k=1}^{m} \phi_k e_{1k}(t)\, dt.$$

The left-hand and the right-hand sides satisfies

$$-\int_{t_{r1}}^\infty e^{-pt} \sum_{k=1}^{m} \phi_k e_{1k}(t)\, dt \leq \int_{t_{r1}}^\infty e^{-pt}\, dt \Big[\phi_1 y_1^{\circ} + |\phi_2|\widehat{y}_{12} + \dots + |\phi_m|\widehat{y}_{1m}\Big]$$

and

$$\begin{aligned}\int_0^{t_{r1}} e^{-pt} \sum_{k=1}^{m} \phi_k e_{1k}(t)\, dt &\geq \int_0^{t_{r1}} e^{-pt} \Big[\phi_1 \widehat{r}\Big(1 - \frac{t}{t_{r1}}\Big) - |\phi_2|\widehat{y}_{12} - \dots - |\phi_m|\widehat{y}_{1m}\Big]\, dt \\ &= \frac{p t_{r1} - 1 + e^{-p t_{r1}}}{p^2 t_{r1}}\phi_1 \widehat{r} - \frac{1 - e^{-p t_{r1}}}{p}\Big[|\phi_2|\widehat{y}_{12} + \dots + |\phi_m|\widehat{y}_{1m}\Big],\end{aligned}$$

respectively. From this together with

$$\frac{(p t_{r1} - 1)e^{p t_{r1}} + 1}{p t_{r1}} \geq \frac{p t_{r1}}{2}$$

the result now follows. □

Remark 5. Note that in Theorem 2 we consider the set-point response in y_1 for r_1 together with the responses in y_1 for set-point steps in $r_2, \dots, r_m$.

Remark 6. Theorem 2 suggests that if the pole direction is such that $\phi_1 \gg |\phi_k|$ for $k = 2, \dots, m$, then a real RHP far from the origin must necessarily give a large overshoot if the rise time is long. In general, however, the pole direction gives freedom in the design to improve the performance. In the SISO case Theorem 2 reduces to Lemma 3 in [13] or Corollary 1.3.5 in [14].

5. EXAMPLE

Consider the Quadruple-Tank Process [9,11]. This laboratory process, which is shown in Figure 3, has two inputs and two outputs. The inputs are voltages to the pumps and the outputs are the levels in the lower two tanks. The Quadruple-Tank Process has two valves that are set prior to an experiment. They are used to make the process more or less difficult to control. The parameters $\gamma_1, \gamma_2 \in [0, 1]$ define how the valves are set, such that the flows to the lower two tanks are proportional to them. For example, if $\gamma_1 = 1$ all flow from Pump 1 goes to Tank 1 and if $\gamma_1 = 0$ all flow goes to Tank 4.

It is possible to show that the linearized dynamics of the quadruple-tank process have no RHP zeros if $\gamma_1 + \gamma_2 \in (1, 2)$ and one RHP zero if $\gamma_1 + \gamma_2 \in (0, 1)$, see [11]. In the following we

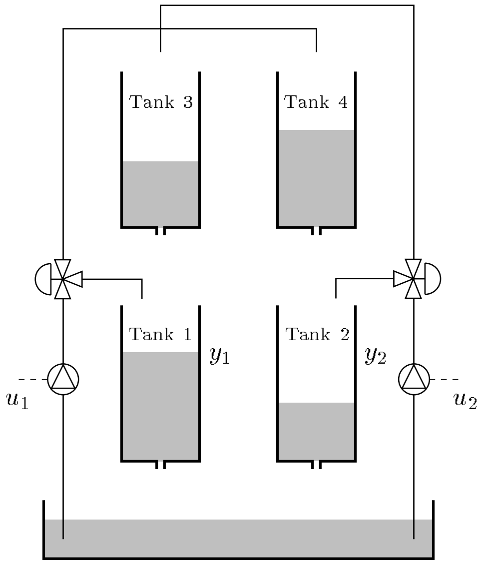

Fig. 3. The quadruple-tank laboratory process. The water levels in Tank 1 and Tank 2 are controlled by two pumps. When changing the position of the valves, the location of a multivariable zero for the linearized model is moved.

study two particular settings of the valves: the minimum-phase setting $(\gamma_1, \gamma_2) = (0.70, 0.60)$ and the nonminimum-phase setting $(0.43, 0.34)$. System identification experiments give the following two models:

$$G_-(s) = \begin{bmatrix} \frac{3.11}{1+95.57s} & \frac{2.04}{(1+32.05s)(1+95.57s)} \\ \frac{1.71}{(1+38.90s)(1+98.67s)} & \frac{3.24}{1+98.67s} \end{bmatrix}$$

and

$$G_+(s) = \begin{bmatrix} \frac{1.69}{1+76.75s} & \frac{3.33}{(1+52.30s)(1+76.75s)} \\ \frac{3.11}{(1+56.36s)(1+111.55s)} & \frac{1.97}{1+111.55s} \end{bmatrix}.$$

The transfer function matrix G_- has zeros in -0.012 and -0.045, while G_+ has zeros in 0.014 and -0.051. Hence, G_- has no RHP zeros, but G_+ has one in $z = 0.014$.

Because G_- is stable and minimum phase, theoretically it can be arbitrarily tight controlled. This is not the case for G_+. Theorem 1 gives a trade-off between settling time, undershoot, and interaction for a set-point response. The zero $z = 0.014$ of G_+ has zero direction $\psi = (\psi_1, \psi_2)^T = (0.64, -0.77)^T$. With settling level $\epsilon = 0$, Theorem 1 gives that

$$\psi_1 y_1^{\mathrm{u}} + |\psi_2| \widehat{y}_{21} \geq \frac{\psi_1}{e^{z t_{s1}} - 1}$$

for a unit step in r_1. So the trade-off can be written as

$$y_1^{\mathrm{u}} + 1.20 \widehat{y}_{21} \geq \frac{1}{e^{0.014 t_{s1}} - 1}.$$

For a settling time of $t_{s1} = 100$, we get

$$y_1^{\mathrm{u}} \geq -1.20 \widehat{y}_{21} + 0.32.$$

Therefore, a sufficiently small interaction imposes an undershoot of at least 0.32.

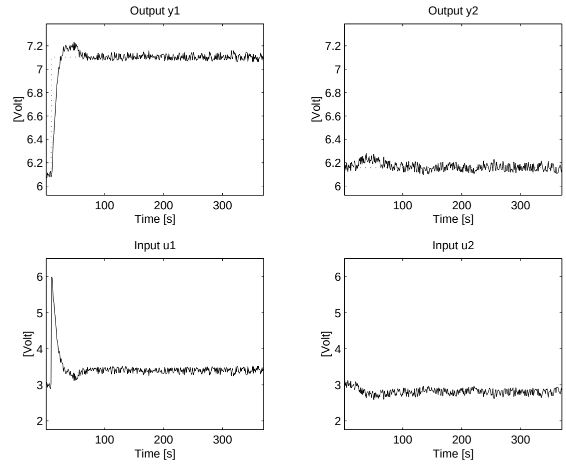

Fig. 4. Responses for decentralized PI control of the quadruple-tank process in minimum-phase setting. The input is a unit reference step in r_1.

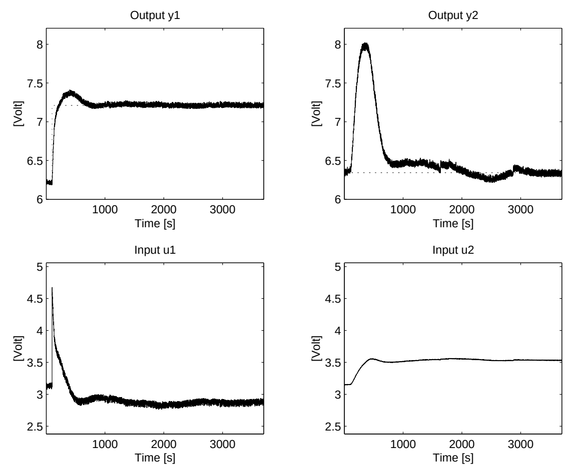

Fig. 5. Responses for decentralized PI control of the quadruple-tank process in nonminimum-phase setting. The input is a unit reference step in r_1. Note that the settling time is about ten times longer than for the minimum-phase setting shown in Figure 4.

Two decentralized PI controllers were manually tuned for the two process settings. Figure 4 shows the responses for the minimum-phase setting of the true process for a unit reference step in r_1. The settling time with settling level $\epsilon \approx 0$ is approximately 60 seconds.

The responses for the nonminimum-phase setting are shown in Figure 5. The settling time is about 600 seconds, which is about ten times longer than for the minimum-phase case. The interaction in Figure 5 is much worse than predicted from the linear model G_+ and Theorem 1. This may indicate that a much better performance can be achieved with a centralized controller or it may also indicate that the bound in the theorem is rough.

6. CONCLUSIONS

Performance limitations in linear multivariable systems with controllers of one degree of freedom were discussed. It was shown that there is trade-off for nonminimum-phase systems between the closed-loop output responses and the zero direction of the open-loop system. The trade-off becomes more severe if the RHP zero is close to the origin. Similar results for unstable open-loop systems were also derived.

The results were illustrated on the Quadruple-Tank Process. The process has an adjustable zero, which can be located in either the left or the right half-plane. It was shown that the control performance of the nonminimum-phase setting with a decentralized controller was much worse than predicted by Theorem 1. Ongoing work include improved control of the quadruple-tank process with a centralized multivariable controller [7,10]. These results show, however, that a centralized controller only gives slightly faster responses. It seems to be possible to derive much better estimates of the settling time and other variables by using approximate shapes of the responses as described in Remark 3. This work will be presented in future reports.

Choosing control structure is a difficult problem, but of great interest to process industry [15]. There exist, however, few results even for the simplified problem on when for linear systems a decentralized controller is outperformed by a centralized one. Results on when decentralized control is sufficient is given in [18,12]. The bounds derived in this paper can be used to judge how much can be gained by applying centralized control.

Acknowledgment

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