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Stochastic Modeling and Optimal Control for Automated Overtaking

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***Abstract*— This paper proposes a solution to the overtaking problem where an automated vehicle tries to overtake a human-driven vehicle, which may not be moving at a constant velocity. Using reachability theory, we first provide a robust time-optimal control algorithm to guarantee that there is no collision throughout the overtaking process. Following the robust formulation, we provide a stochastic reachability formulation that allows a trade-off between the conservative overtaking time and the allowance of a small collision probability. To capture the stochasticity of a human driver's behavior, we propose a new martingale-based model where we classify the human driver as aggressive or nonaggressive. We show that if the human driver is nonaggressive, our stochastic time-optimal control algorithm can provide a shorter overtaking time than our robust algorithm, whereas if the human driver is aggressive, the stochastic algorithm will act on a collision probability of zero, which will match the robust algorithm. Finally, we detail a simulated example that illustrates the effectiveness of the proposed algorithms.**

I. INTRODUCTION

Due to the recent surge of interest in the introduction of automated vehicles into present-day traffic, many are tackling problems that arise when we introduce automated vehicles into mixed traffic settings where they must interact with human drivers [1]. The problem of overtaking a human-driven vehicle with an automated vehicle exemplifies one of these mixed traffic problems. Many researchers believe that by solving this problem safely, robustly, and efficiently, we will significantly progress the capabilities of automated vehicles [2].

The overtaking maneuver is a special maneuver that consists of several sub-maneuvers that observe both lateral and longitudinal movements an automated vehicle is expected to perform on a multi-lane road [3]. Several other works propose solutions for performing the overtaking maneuver safely and efficiently [4]–[7]. However, these solutions all assume the overtaken vehicle moves at a constant velocity, an assumption we avoid in this work.

Explicitly, the process we study in this work is the process of overtaking a human driven vehicle, which may not be moving at a constant velocity, with an automated vehicle. This overtaking maneuver requires the automated vehicle to change into an empty lane, then, longitudinally overtake the human-driven vehicle, and finally, merge back into the original lane in front of the human-driven vehicle. At each stage of the overtaking maneuver, many variables and factors introduce uncertainties that make overtaking complex and difficult to perform robustly and safely [8]. In this work, we focus on addressing the uncertainty around human driving behavior. We remark that our approach is different from [9], which considers the stochastic measurement uncertainties.

Our contributions towards solving the overtaking problem with a human-driven vehicle is two fold. First, we contribute two reachability analysis-based algorithms for safe overtaking: a robust algorithm for safe overtaking with strict guarantees of collision-avoidance and a stochastic algorithm for overtaking with a small collision probability, but possibly with shorter overtaking time. We discuss the trade-offs between the two algorithms in detail in Section V and Section VI. Second, we propose a new stochastic model based on martingales for modeling the expected behavior of the human-driven vehicle throughout the overtaking process. Moreover, to the extent of our knowledge, this is the first application of martingales for modeling human driving behavior.

We choose to use reachability analysis for the overtaking process due to its formal safety guarantees. Reachability analysis-based approaches are used in many applications in order to provide formal guarantees on the safety of vehicles [10]. Robust reachability analysis-based approaches maintain strict guarantees that there does not exist unsafe trajectories within a certain time horizon, assuming certain bounds on the system and its disturbances [11]. Stochastic reachability analysis-based approaches, instead, guarantee that there will not be an unsafe trajectory within a certain time horizon with a significantly large probability [12]. The stochastic reachability analysis-based approaches can be beneficial, since they permit a trade-off between collision probability and optimality (e.g., overtaking time). These approaches are also used for probabilistic collision detection in automated driving [13]. However, the introduction of stochasticity also introduces the problem of finding a good stochastic model for poorly understood phenomenon, such as human driving behavior. To address this challenge, we choose to utilize martingale theory for modeling a human's driving behavior during the overtaking process.

Historically, martingales are most often applied to gambling or pricing problems since they efficiently model the lack of arbitrage [14]. In [15], the author discusses the use of martingales in several classical stochastic control problems. However, in this paper, we show that they are a potentially useful model for modeling human behavior in certain driving scenarios. In particular, we find that martingales could be a strong choice for modeling the expected behavior of a

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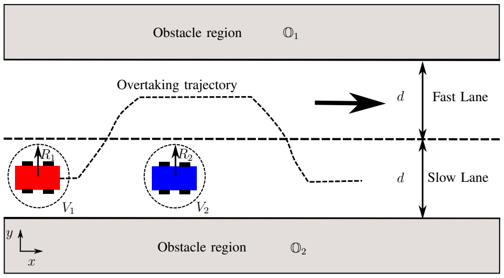


Fig. 1. Scenario where an automated vehicle V_1 is overtaking a human-driven vehicle V_2 on a road with two lanes.

human-driven vehicle. We remark that this martingale-based model is different from the Markov model used in [16], [17], which assumes a characteristic of the distribution of a human's decision-making process. In addition, unlike [18] which proposes an empirical method for ensuring safety in human-in-the-loop driving systems, our model is, by construction, agnostic to historical data.

This paper is structured as follows: in Section II, we introduce our notation and background used throughout the paper; in Section III, we detail the problem we address in this work; in Section IV, we propose our robust reachability analysis-based algorithm for safe overtaking with strict guarantees; in Section V, we propose our stochastic reachability analysis-based algorithm for overtaking with small collision probabilities; in Section VI, we demonstrate the efficacy of our proposed algorithms and compare the different approaches; in Section VII, we discuss our results and future work.

II. Notations and Preliminaries

A. Notations

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{R}$ denote the set of real numbers. For some $q, s \in \mathbb{N}$ and $q < s$, let $\mathbb{N}_{[q,s]} = \{r \in \mathbb{N} \mid q \leq r \leq s\}$. For a set $\mathbb{X}$, cl($\mathbb{X}$) denotes its closure. For two sets $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{X} \oplus \mathbb{Y} = \{x + y \mid x \in \mathbb{X}, y \in \mathbb{Y}\}$. When $\leq$, $\geq$, $<$, and $>$ are applied to vectors, they are interpreted element-wise. For a set $\mathbb{X} \subseteq \mathbb{R}$, $\mathcal{B}(\mathbb{X})$ denotes the Boral space of $\mathbb{X}$. Pr denotes the probability and E the expectation.

B. Preliminaries

We introduce some definitions for submartingales and supermartingales, and one related inequality.

Definition 2.1: [19] A discrete-time integrable stochastic process $\{X_i, i \in \mathbb{N}\}$ with a filtration $\{\mathcal{F}_i, i \in \mathbb{N}\}$ and $\mathcal{F}_i \subseteq \mathcal{F}$ on the probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ is said to be (i) a submartingale if $\mathrm{E}[\mathrm{X_{i+1}}|\mathrm{X_i}] \geq \mathrm{X_i}$, $\forall i \in \mathbb{N}$; (ii) a supermartingale if $\mathrm{E}[\mathrm{X_{i+1}}|\mathrm{X_i}] \leq \mathrm{X_i}$, $\forall i \in \mathbb{N}$.

Lemma 2.1: [20] Consider a discrete-time supermartingale $\{X_i, i \in \mathbb{N}\}$ with a filtration $\{\mathcal{F}_i, i \in \mathbb{N}\}$ and $\mathcal{F}_i \subseteq \mathcal{F}$ on the probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$. If for all $i \in \mathbb{N}_{\geq 1}$, (i) $\mathrm{Var}[X_i|\mathcal{F}_{i-1}] \leq \sigma_i^2$, (ii) $X_i - \mathrm{E}[\mathrm{X_i}|\mathrm{X_{i-1}}] \leq \mathrm{M}$, then the following inequality holds

$$\mathrm{Pr}[\mathrm{X_i} \geq \mathrm{X_0} + \lambda] \leq \exp(-\frac{\lambda^2}{2\sum_{\mathrm{j=0}}^{\mathrm{i}}(\sigma_{\mathrm{i}}^2 + \mathrm{M}\lambda/3)}).$$

III. Problem Statement

As we introduced in Section I, we consider an overtaking scenario where an automated vehicle approaches a human-driven vehicle on a road with two lanes (illustrated in Fig. 1). We refer to the automated vehicle as V_1 and the human-driven vehicle as V_2. The road consists of two lanes: one slow lane and one fast lane. A vehicle typically keeps to the slow lane until it would like to overtake its preceding slower vehicle. We show a sample overtaking trajectory in Fig. 1.

A. Modeling of Road

The lateral width of each lane is d. The longitudinal velocity, v^x, of the vehicle moving along the slow lane (or the fast lane) is required to satisfy $v^S_{\min} \leq v^x \leq v^S_{\max}$ (or $v^F_{\min} \leq v^x \leq v^F_{\max}$). Without loss of generality, we assume that $0 < v^S_{\min} \leq v^F_{min}$ and $v^S_{\max} < v^F_{\max}$. Two regions beyond the slow and fast lanes are regarded as two obstacles $\mathbb{O}_1$ and $\mathbb{O}_2$.

B. Modeling of Automated Vehicle

We describe the dynamics of the automated vehicle, V_1, as

$$\mathbf{x}_1(k+1) = A_1\mathbf{x}_1(k) + B_1\mathbf{u}_1(k)$$

where $\mathbf{x}_1(k) = [p^x_1(k); p^y_1(k); v^x_1(k)]$, $\mathbf{u}_1(k) = [v^y_1(k); a^x_1(k)]$, $A_1 = [1, 0, \delta; 0, 1, 0; 0, 0, 1]$, and $B_1 = [0, 0; \delta, 0; 0, \delta]$. $p^x_1(k)$ and $p^y_1(k)$ are the longitudinal and lateral positions, respectively; $v^x_1(k)$ and $v^y_1(k)$ are the longitudinal and lateral velocities, respectively; $a^x_1(k)$ is the longitudinal acceleration.

To impose physical limits on the dynamics, we subject V_1 to the state and control input constraints:

$$\mathbf{x}_1(k) \in \mathbb{X}_1(k),\ \mathbf{u}_1(k) \in \mathbb{U}_1(k),$$

where $\mathbb{X}_1(k) = \{z \in \mathbb{R}^3 \mid \big[-\infty; 0; v^x_{1,\min}(k)\big] \leq z \leq \big[\infty; 2d; v^x_{1,\max}(k)\big]\}$, and $\mathbb{U}_1(k) = \{z \in \mathbb{R}_2 \mid \big[a^x_{1,\min}; v^y_{1,\min}\big] \leq z \leq \big[a^x_{1,\max}; v^y_{1,\max}\big]\}$; if $0 \leq p^y_1(k) \leq d$, $\big[v^x_{1,\min}(k); v^x_{1,\max}(k)\big] = \big[v^S_{\min}; v^S_{\max}\big]$, and if $d \leq p^y_1(k) \leq 2d$, $\big[v^x_{1,\min}(k); v^x_{1,\max}(k)\big] = \big[v^F_{\min}; v^F_{\max}\big]$. The occupancy of V_1 on the road is a circle with radius R_1: $\mathbb{S}_1(k) = \{z \in \mathbb{R}^2 \mid \|z - [p^x_1(k); p^y_1(k)]\|_2 \leq R_1\}$.

C. Modeling of Human-driven Vehicle

We describe the dynamics of the human-driven vehicle, V_2, as

$$\mathbf{x}_2(k+1) = A_2\mathbf{x}_2(k) + B_2\mathbf{u}_2(k) \tag{1}$$

where $\mathbf{x}_2(k) = [p^x_2(k); v^x_2(k)]$, $\mathbf{u}_2(k) = a^x_2(k)$, $A_2 = [1, \delta; 0, 1]$, $B_2 = [0; \delta]$. $p^x_2(k)$, $v^x_2(k)$, and $a^x_2(k)$ are the longitudinal position, velocity, and acceleration, respectively. During the overtaking process, we assume that V_2 keeps to the slow lane and maintains a lateral position of $p^y_2(k) = \frac{d}{2}$, $\forall k \in \mathbb{N}$.

For representing the physical limits and assumptions over V_2, we subject V_2 to the state and control input constraints:

$$\mathbf{x}_2(k) \in \mathbb{X}_2,\ \mathbf{u}_2(k) \in \mathbb{U}_2,$$

where $\mathbb{X}_2 = \{z \in \mathbb{R}^2 \mid [-\infty; v^S_{\min}] \leq z \leq [\infty; v^S_{\max}]\}$, and $\mathbb{U}_2 = \{z \in \mathbb{R} \mid a^x_{2,\min} \leq z \leq a^x_{2,\max}\}$. The occupancy of V_2 on the road is a circle with radius R_2: $\mathbb{S}_2(k) = \{z \in \mathbb{R}^2 \mid \|z - [p^x_2(k); p^y_2(k)]\|_2 \leq R_2\}$.

Assumption 3.1: The state $x_2(k)$ of the human-driven vehicle V_2 satisfies $x_2(k) \in \mathbb{X}_2$, $\forall k \in \mathbb{N}$.

D. Control Objective

Our control objective is to design a sequence of control inputs such that V_1 starts behind V_2 and ends in front of V_2 by taking a set of sequential maneuvers: lane-changing, lane-keeping, and merging. Throughout the entire process, we maintain the following safety constraints: collision avoidance between V_1 and V_2 and collision avoidance between V_1 and $\mathbb{O}_i$, $i = 1, 2$.

IV. Robust Time-Optimal Overtaking Algorithm

A. Reachable Set of Human-driven Vehicle

At time step k, $p^x_2(k)$ and $v^x_2(k)$ are the longitudinal position and velocity of the V_2, respectively. We define the reachable set as follows.

Definition 4.1: The reachable set of the V_2 predicted i steps ahead at time step k, denoted by $\mathbb{P}(k+i|k)$, evolves as

$$\begin{cases} \mathbb{P}(k+i+1|k) = (A_2\mathbb{P}(k+i|k) \oplus B_2\mathbb{U}_2) \cap \mathbb{X}_2, \\ \mathbb{P}(k|k) = \{\mathbf{x}_2(k)\}, \end{cases} \tag{2}$$

We define the projection of the reachable set on the longitudinal position as $\mathbb{P}_x(k+i|k) = \text{Proj}_1(\mathbb{P}(k+i|k))$ and the projection of the reachable set on the longitudinal velocity as $\mathbb{P}_v(k+i|k) = \text{Proj}_2(\mathbb{P}(k+i|k))$, where $\text{Proj}_j(\mathbb{Q})$ denotes the projection of the set $\mathbb{Q}$ on the jth dimension.

Lemma 4.1: The set $\mathbb{P}(k+i|k)$ is a compact and convex set for all finite $i \in \mathbb{N}$. Furthermore, the sets $\mathbb{P}_x(k+i|k)$ and $\mathbb{P}_v(k+i|k)$ are two compact intervals for all finite $i \in \mathbb{N}$.

For notational simplicity, let $\mathbb{P}_x(k+i|k) = [p^x_{2,\min}(k+i|k), p^x_{2,\max}(k+i|k)]$ and $\mathbb{P}_v(k+i|k) = [v^x_{2,\min}(k+i|k), v^x_{2,\max}(k+i|k)]$. Denote all the possible occupancies of V_2 corresponding to $\mathbb{P}_x(k+i|k)$ as $\mathcal{S}_2(k+i|k)$, i.e., $\mathcal{S}_2(k+i|k) = \{z \in \mathbb{R}^2 \mid \|z - [p^x_2; \frac{d}{2}]\|_2 \leq R_2, p^x_2 \in \mathbb{P}_x(k+i|k)\}$.

B. Robust Time-Optimal Overtaking Controller

At time step k, $\mathbf{x}_1(k)$ is the state of V_1. Consider a finite horizon T. Given a sequence of control inputs $\{\mathbf{u}_1(k|k), \cdots, \mathbf{u}_1(k+i|k), \ldots, \mathbf{u}_1(k+T-1|k)\}$, we can have a state trajectory $\{\mathbf{x}_1(k|k), \cdots, \mathbf{x}_1(k+i|k), \ldots, \mathbf{x}_1(k+T|k)\}$ and thereby a sequence of occupancies $\{\mathbb{S}_1(k|k), \cdots, \mathbb{S}_1(k+i|k), \ldots, \mathbb{S}_1(k+T|k)\}$.

The robust time-optimal overtaking problem can be formulated as $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$:

$$\min_{\{\mathbf{u}_1(k+i|k)\}_{i=0}^{T-1}} \quad T \tag{3a}$$

subject to

$\forall i \in \mathbb{N}_{[0,T-1]}$:

$$\mathbf{x}_1(k+i+1|k) = A_1\mathbf{x}_1(k+i|k) + B_1\mathbf{u}_1(k+i|k), \tag{3b}$$

$$\mathbf{u}_1(k+i|k) \in \mathbb{U}_1(k+i|k), \tag{3c}$$

$\forall i \in \mathbb{N}_{[0,T]}$:

$$\mathbf{x}_1(k+i|k) \in \mathbb{X}_1(k+i|k), \tag{3d}$$

$$\mathbb{S}_1(k+i|k) \cap \mathbb{O}_j = \emptyset, j = 1, 2, \tag{3e}$$

$$\mathbb{S}_1(k+i|k) \cap \mathcal{S}_2(k+i|k) = \emptyset, \tag{3f}$$

$$\begin{cases} p^x_1(k+T|k) \geq p^x_{2,\max}(k+T|k) + R_1 + R_2, \\ p^y_1(k+T|k) = \frac{d}{2}, \end{cases} \tag{3g}$$

where $p^x_{2,\max}(k+T|k) = \max \mathbb{P}_x(k+T|k)$.

Denote the optimal objective function of $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ as T^*_k and the corresponding optimal solution as $\{\mathbf{u}^*_1(k+i|k)\}_{i=0}^{T^*_k-1}$. We call T^*_k the optimal overtaking time at time step k under our robust control formulation.

C. Robust Time-Optimal Overtaking Algorithm

We show the robust time-optimal overtaking algorithm in Algorithm 1. The initial feasibility of $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ determines the possibility of safe overtaking. The following theorem shows the recursive feasibility of $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$. Thus, if the problem $\mathcal{P}_1(\mathbf{x}_1(0), \mathbf{x}_2(0))$ is feasible, we solve $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ in a receding horizon way until the whole overtaking process is finished.

Algorithm 1 Robust Time-Optimal Overtaking Algorithm

```
Initialization: Set k = 0 and measure x1(k) and x2(k);
if P1(x1(k), x2(k)) is not feasible then
    Output: Infeasible overtaking;
else
    while p1^x(k) ≤ p2^x(k) + R1 + R2 or p1^y(k) ≠ d/2 do
        Solve P1(x1(k), x2(k));
        Implement u1*(k|k);
        Set k = k + 1 and measure x1(k) and x2(k);
    end while
    Output: Successful overtaking;
end if
```

Theorem 4.1: Suppose that Assumption 3.1 holds and the problem $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ is feasible at time step $k = 0$. Then, it is feasible for all $k \geq 1$ and the optimal overtaking time is decreasing, i.e., $T^*_{k+1} \leq T^*_k - 1$. Furthermore, the total overtaking time is no greater than T^*_0.

Proof: Assume that the problem $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ is feasible at time step k, the optimal overtaking time is T^*_k, and the corresponding optimal solution is $\{\mathbf{u}^*_1(k+i|k)\}_{i=0}^{T^*_k-1}$. At time step $k+1$, it follows from (2) that $\mathbb{P}(k+1+i|k+1) \subseteq \mathbb{P}(k+1+i|k)$ and $\mathcal{S}_2(k+1+i|k+1) \subseteq \mathcal{S}_2(k+1+i|k)$. It yields that $p^x_{2,\max}(k+T^*_k|k+1) \leq p^x_{2,\max}(k+T^*_k|k)$. We have that the solution sequence $\{\mathbf{u}^*_1(k+1+i|k)\}_{i=0}^{T^*_k-1}$ still guarantees that the constraints (3b)–(3g) are satisfied at time step $k+1$. That is, the problem $\mathcal{P}_1(\mathbf{x}_1(k+1), \mathbf{x}_2(k+1))$ is feasible and the optimal overtaking time T^*_{k+1} is no greater than $T^*_k - 1$. Furthermore, it follows that the total overtaking time is no greater than T^*_0. ■

V. Stochastic Time-Optimal Overtaking Algorithm

A. Stochastic Model of Human-driven Vehicle

Given any state $\mathbf{x}_2(k) \in \mathbb{X}_2$ at time step k, assume that the predicted velocity $\{v^x_2(k+i), i \in \mathbb{N}\}$ of V_2 is a stochastic process with a filtration $\{\mathcal{B}(\mathbb{P}_v(k+i|k), i \in \mathbb{N}\}$ on the probability space $([v^S_{\min}, v^S_{\max}], \mathcal{B}([v^S_{\min}, v^S_{\max}]), \Pr)$. In this sense, the future state $\mathbf{x}_2(k+i|k)$, $i \geq 0$, is also a stochastic process.

Definition 5.1: Given any state $\mathbf{x}_2(k) \in \mathbb{X}_2$ at time step k, a human driver is said to be

- *aggressive* if

$$\forall i \in \mathbb{N}, \begin{cases} \mathbf{x}_2(k+i|k) \in \mathbb{P}(k+i|k), \\ \mathrm{E}[\mathrm{v^x_2(k+i+1|k)|v^x_2(k+i|k)}] \geq \mathrm{v^x_2(k+i|k)}, \end{cases}$$

- *nonaggressive* if

$$\forall i \in \mathbb{N}, \begin{cases} \mathbf{x}_2(k+i|k) \in \mathbb{P}(k+i|k), \\ \mathrm{E}[\mathrm{v^x_2(k+i+1|k)|v^x_2(k+i|k)}] \leq \mathrm{v^x_2(k+i|k)}. \end{cases}$$

We assume that the type of the human driver (aggressive or nonaggressive) does not change throughout the overtaking process. If the driver is nonaggressive, the set $\mathbb{P}^\alpha_v(k+i|k)$ is defined as

$$\mathbb{P}^\alpha_v(k+i|k) = \{z \in \mathbb{R} \mid z \in \mathbb{P}_v(k+i|k), \Pr[\tilde{\mathrm{z}} \geq \mathrm{z}] \leq \alpha\},$$

where $\tilde{z} \in \mathbb{P}_v(k+i|k)$ is a random variable, $\alpha \in [0, \bar{\alpha}]$, and $\bar{\alpha}$ is a positive constant smaller than 1.

Next let us consider how to compute the set $\mathbb{P}^\alpha_v(k+i|k)$.

Proposition 5.1: If the driver is nonaggressive, the set $\mathbb{P}^\alpha_v(k+i|k) = [\min\{v^x_2(k)+\lambda, v^x_{2,\max}(k+i|k)\}, v^x_{2,\max}(k+i|k)]$ where

$$\lambda = \frac{1}{3} M \ln(\frac{1}{\alpha}) + \sqrt{\frac{1}{9} M^2 (\ln(\frac{1}{\alpha}))^2 + 2iM^2 \ln(\frac{1}{\alpha})},$$
$$M = \max\{\delta |a^x_{2,\min}|, \delta a^x_{2,\max}\}.$$

Proof: From Lemma 4.1, the set $\mathbb{P}_v(k+i|k)$ is a compact interval for all finite i. Construct a filtration $\mathcal{F}_i = \sigma(\{\mathbb{P}_v(k+j|k)\}_{j \in \mathbb{N}_{\leq i}})$. From Definition 5.1, we have that if the driver is nonaggressive, the stochastic process $v^x_2(k+i|k)$, $i \geq 0$, is a supermartingale. It follows from Popoviciu's inequality that the standard variance of $v_2(k+i|k)$, which is conditional on $v_2(k+i-1|k)$, is upper bounded by M^2, which implies the condition (i) in Lemma 2.1. By the constraint on the longitudinal acceleration of Vehicle 2, it gives $v^x_2(k+i|k) - \mathrm{E}[\mathrm{v^x_2(k+i-1|k)}] \leq \mathrm{M}$, which implies the condition (ii) in Lemma 2.1. By Lemma 2.1 and setting $\alpha = \exp(-\frac{\lambda^2}{2\sum_{j=0}^{i}(\sigma_i^2 + M\lambda/3)})$, we have $\lambda = \frac{1}{3} M \ln(\frac{1}{\alpha}) + \sqrt{\frac{1}{9} M^2 (\ln(\frac{1}{\alpha}))^2 + 2iM^2 \ln(\frac{1}{\alpha})}$, which implies that the set $\mathbb{P}^\alpha_v(k+i|k) = [\min\{v^x_2(k)+\lambda, v^x_{2,\max}(k+i|k)\}, v^x_{2,\max}(k+i|k)]$. ■

Let $\mathbb{P}^{1-\alpha}_v(k+i|k) = \mathrm{cl}(\mathbb{P}_v(k+i|k) \setminus \mathbb{P}^\alpha_v(k+i|k))$, i.e., $\mathbb{P}^{1-\alpha}_v(k+i|k) = [v^x_{2,\min}(k+i|k), \min\{v^x_2(k)+\epsilon, v^x_{2,\max}(k+i|k)\}]$. The corresponding set can be obtained from $\mathbb{P}^{1-\alpha}(k+i|k) = \{z \in \mathbb{R}^2 \mid z \in \mathbb{P}(k+i|k), \mathrm{Proj}_2(z) \in \mathbb{P}^{1-\alpha}_v(k+i|k)\}$, and the projection on the longitudinal position is denoted as $\mathbb{P}^{1-\alpha}_x(k+i|k) = \mathrm{Proj}_1(\mathbb{P}^{1-\alpha}(k+i|k))$. Denoted by $\mathcal{S}^{1-\alpha}_2(k+i|k)$, all the possible occupancies of the vehicle V_2 corresponding to $\mathbb{P}^{1-\alpha}_x(k+i|k)$, i.e., $\mathcal{S}^{1-\alpha}_2(k+i|k) = \{z \in \mathbb{R}^2 \mid \|z - \left[p^x_2; \frac{d}{2}\right]\|_2 \leq R_2, p^x_2 \in \mathbb{P}^{1-\alpha}_x(k+i|k)\}$.

Remark 5.1: The parameter α represents the tolerated collision probability in the prediction. The introduction of α releases a larger overtaking region than that in the robust case for V_1 and thereby generates a smaller overtaking time if the human driver is non-aggressive.

B. Stochastic Time-Optimal Overtaking Controller

If the driver is nonaggressive, at time step k, the stochastic time-optimal overtaking problem can be formulated as $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$:

$$\min_{\{\mathbf{u}_1(k+i|k)\}_{i=0}^{T-1}} \quad T \tag{4a}$$

subject to

$\forall i \in \mathbb{N}_{[0,T-1]}$:

$$\mathbf{x}_1(k+i+1|k) = A_1 \mathbf{x}_1(k+i|k) + B_1 \mathbf{u}_1(k+i|k), \tag{4b}$$
$$\mathbf{u}_1(k+i|k) \in \mathbb{U}_1(k+i|k), \tag{4c}$$

$\forall i \in \mathbb{N}_{[0,T]}$:

$$\mathbf{x}_1(k+i|k) \in \mathbb{X}_1(k+i|k), \tag{4d}$$
$$\mathbb{S}_1(k+i|k) \cap \mathbb{O}_j = \emptyset, j = 1, 2, \tag{4e}$$
$$\mathbb{S}_1(k+i|k) \cap \mathcal{S}^{1-\alpha}_2(k+i|k) = \emptyset, \tag{4f}$$
$$\begin{cases} p^x_1(k+T|k) \geq p^{x,1-\alpha}_{2,\max}(k+T|k) + R_1 + R_2, \\ p^y_1(k+T|k) = \frac{d}{2}, \end{cases} \tag{4g}$$

where $p^{x,1-\alpha}_{2,\max}(k+T|k) = \max \mathbb{P}^{1-\alpha}_x(k+T|k)$.

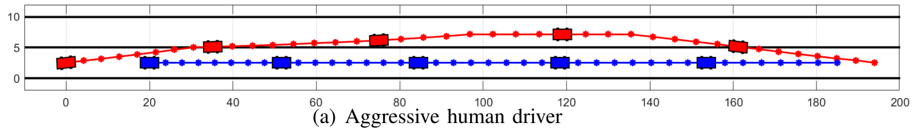
Proposition 5.2: For a nonaggressive driver, if $\alpha = 0$, then the problem $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$ is equivalent to $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$.

Proof: If $\alpha = 0$, then the set $\mathbb{P}^\alpha_v(k+i|k) = \{v^x_{2,\max}(k+i|k)\}$, which implies that $\mathbb{P}^{1-\alpha}_x(k+i|k) = \mathbb{P}_x(k+i|k)$. Thus, the problems $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ and $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$ enjoy the same constraints, which proves the equivalence. ■

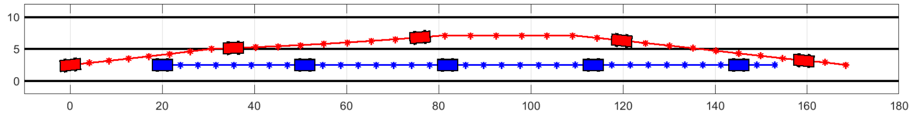
Denote the optimal objective function of $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$ as $\bar{T}^*_k$ and the corresponding optimal solution as $\{\bar{\mathbf{u}}^*_1(k+i|k)\}_{i=0}^{\bar{T}^*_k-1}$. We call $\bar{T}^*_k$ the optimal overtaking time at time step k under our stochastic control formulation.

Theorem 5.1: If the driver is nonaggressive, then the feasibility of $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ implies the feasibility of $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$ for any $\alpha \in [0, \bar{\alpha}]$. Furthermore, if $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ is feasible, then the optimal overtaking time satisfies $\bar{T}^*_k \leq T^*_k$.

Proof: Since $\mathbb{P}^{1-\alpha}_x(k+i|k) \subseteq \mathbb{P}_x(k+i|k)$, the constraints satisfaction of $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ always ensures the the constraints satisfaction of $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$. Thus, the feasibility of $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ implies the feasibility of $\mathcal{P}^\alpha_2(\mathbf{x}_1(k), \mathbf{x}_2(k))$, which leads to the second result. ■



(a) Aggressive human driver



(b) Non-aggressive human driver

Fig. 2. (a) Position trajectory under Algorithm 1 for aggressive human driver; (b) Position trajectory under Algorithm 2 for non-aggressive human driver.

C. Stochastic Time-Optimal Overtaking Algorithm

The stochastic time-optimal overtaking algorithm is shown in Algorithm 2. We offline identify that the human driver is nonaggressive or aggressive. If the driver is aggressive, then we solve the robust time-optimal overtaking problem $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$. Otherwise, we solve $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(k), \mathbf{x}_2(k))$. The initial feasibility of $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(k), \mathbf{x}_2(k))$ determines the possibility of overtaking. Thus, if the problem $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(0), \mathbf{x}_2(0))$ is feasible, then we solve $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(k), \mathbf{x}_2(k))$ in a receding horizon fashion until the whole overtaking process is finished.

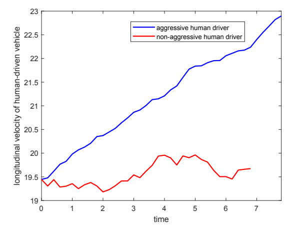
Algorithm 2 Stochastic Time-Optimal Overtaking Algorithm

Offline: Identify that the human driver is nonaggressive or aggressive.
Online:

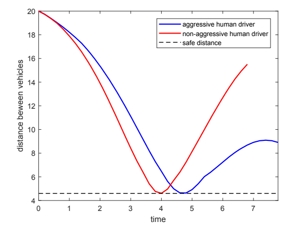
if The human driver is aggressive **then**
  Implement Algorithm 1.
else
  Initialization: Choose α. Set $k = 0$ and measure $\mathbf{x}_1(k)$ and $\mathbf{x}_2(k)$;
  if $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(k), \mathbf{x}_2(k))$ is not feasible **then**
    Output: Infeasible overtaking;
  else
    while $p_1^x(k) < p_2^x(k)$ or $p_1^y(k) \neq \frac{d}{2}$ **do**
      Solve $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(k), \mathbf{x}_2(k))$;
      Implement $\mathbf{u}_1^*(k|k)$;
      Set $k = k+1$ and measure $\mathbf{x}_1(k)$ and $\mathbf{x}_2(k)$;
    end while
    Output: Successful overtaking;
  end if
end if

TABLE I
CASE STUDY PARAMETERS

Lane	Vehicle 1	Vehicle 2
$v_{\min}^S = 60\text{km/h}$	$v_{1,\min}^y = 2\text{m/s}$	$a_{1,\min}^x = 1\text{m/s}^2$
$v_{\max}^S = 90\text{km/h}$	$v_{1,\max}^y = -2\text{m/s}$	$a_{1,\max}^x = -1\text{m/s}^2$
$v_{\min}^F = 60\text{km/h}$	$a_{1,\min}^x = 2\text{m/s}^2$	$R_2 = 2.3m$
$v_{\max}^F = 100\text{km/h}$	$a_{1,\max}^x = -2\text{m/s}^2$	
$d = 5\text{m}$	$R_1 = 2.3\text{m}$	
$\delta = 0.2\text{s}$		



(a)



(b)

Fig. 3. (a) Longitudinal velocities of human-driven vehicle; (b) Inter-vehicle distance under Algorithm 1 for aggressive human driver and Algorithm 2 for non-aggressive human driver.

VI. Example

This section will provide a case study to demonstrate the effectiveness of our theoretical results and compare our results with the existing overtaking where the vehicle V_2 drives at a constant velocity. The parameters used in the case study are listed in Table I. We choose the initial state: $\mathbf{x}_1(0) = [0\text{m}; 2.5\text{m}; 75\text{km/h}]$ and $\mathbf{x}_2(0) = [20\text{m}; 70\text{km/h}]$.

We consider two overtaking scenarios in which the human driver is aggressive or non-aggressive. We implement the robust time-optimal overtaking algorithm (Algorithm 1), for an aggressive driver and the stochastic time-optimal overtaking algorithm (Algorithm 2) for a non-aggressive human driver. For Algorithm 2, we choose $\alpha = 0.2$. First, we verify that two problems $\mathcal{P}_1(\mathbf{x}_1(k), \mathbf{x}_2(k))$ and $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(k), \mathbf{x}_2(k))$ are feasible at $k = 0$. In Fig. 2, we show the position trajectories of two vehicles under Algorithm 1 and Algorithm 2. In subfigure (a) of Fig. 3, we show the longitudinal velocities of the vehicles driven by the aggressive and non-aggressive human drivers. The total overtaking time for the first scenario is $7.8s$, while it is $6.8s$ for the second scenario.

We use the inter-vehicle distance as a safety certificate: $D_{veh} = \sqrt{(p_1^x(k) - p_2^x(k))^2 + (p_1^y(k) - \frac{d}{2})^2}$. As shown in subfigure (b) of Fig. 3, the distance between the vehicles in both scenarios is always greater than the safe distance $R_1 + R_2 = 4.6\text{m}$ throughout the overtaking process. In particular, the safety is also ensured even though some predictive collision is permitted with small probability in the stochastic overtaking setup. One explanation for this is that the receding horizon implementation provides feedback by taking into account the velocity of the human-driven vehicle, while the

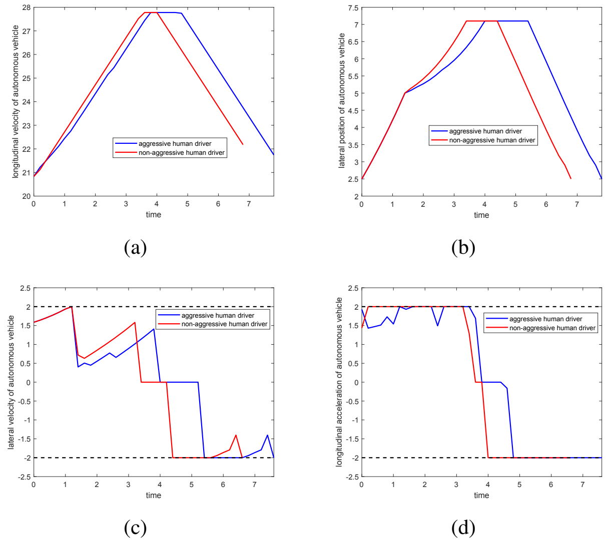


Fig. 4. State trajectories and control inputs of the automated vehicle under Algorithm 1 for the aggressive human driver and Algorithm 2 for the non-aggressive human driver: (a) longitudinal velocities; (b) lateral positions; (c) lateral velocities; (d) longitudinal acceleration.

short-term prediction does not increase the collision risk. In subfigures (a)–(b) of Fig. 4, we show the corresponding longitudinal velocities and lateral positions of the automated vehicle for these two scenarios. In subfigures (c)–(d) of Fig. 4, we show that the control inputs of the automated vehicle (lateral velocities and longitudinal velocities) satisfy the control input constraints.

We also apply Algorithm 1 to a second overtaking scenario, where the human driver is non-aggressive. The initial optimal overtaking time by solving the problem $\mathcal{P}_1(\mathbf{x}_1(0), \mathbf{x}_2(0))$ is 9.8s, as expected, which is 8.89% greater than that by solving the problem $\mathcal{P}_2^{\alpha}(\mathbf{x}_1(0), \mathbf{x}_2(0))$.

VII. Conclusion and Future Directions

In this work, we study the overtaking problem where an automated vehicle attempts to overtake a human-driven vehicle. Here, we do not require the conventional assumption that the human-driven vehicle moves at a constant velocity. Using a reachability analysis-based approach, we first provide a robust time-optimal control algorithm with the guarantee of collision-free overtaking. To capture the stochasticity of a human driver's behavior, we propose a new model for classifying the human driver as aggressive or nonaggressive. Based on this model, we provide a stochastic time-optimal control algorithm which allows a trade off between the conservative overtaking time of the robust formulation and the allowance of a small collision probability. Finally, we illustrate the effectiveness of the proposed algorithms in a simulated case study.

There are several future directions of great interest. First, we plan to investigate an approach with lower computational complexity to solve the robust or stochastic time-optimal overtaking problems. Second, we would like to model the time-varying stochasticity of human drivers and online identify the states of human drivers. Furthermore, we are interested in studying the interplay between the automated vehicle and the human-driven vehicle in more detail and incorporating the findings into our formulation.

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