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Finite-time road grade computation for a vehicle platoon

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***Abstract*— Given a platoon of vehicles traveling uphill, this paper considers the finite-time road grade computation problem. We propose a decentralized algorithm for an arbitrarily chosen vehicle to compute the road grade in a finite number of time-steps by using only its own successive velocity measurements. Simulations then illustrate the theoretical results. These new results can be applied to real-world vehicle platooning problems to reduce fuel consumption and carbon dioxide emissions.**

I. Introduction

Urban highways in major cities nowadays suffer from traffic congestion, which increases fuel consumption and air pollution. A recent study [1] by the International Transport Forum shows that the transport-sector carbon dioxide (CO_2) emission represents 23% globally and 30% within the OECD countries of the overall CO_2 emissions from fossil fuel combustion.

Vehicle platooning has been widely recognized as a promising solution to reduce fuel consumption and carbon dioxide emissions, to enhance the safety, and to improve highway utility (see for instance [2]–[5]). To study this problem, vehicle dynamics are quite often modeled as a second order differential equation [6]–[8], or modeled as a double integrator [5], [9], [10]. Regulation of the relative positions between neighboring vehicles, while maintaining a set velocity, is a key goal of vehicle platooning. Several control solutions can be traced back to the earlier work of [6], [7]. The control of vehicle platooning has attracted renewed interests [5], [8]–[15] due to the recent advances in manufactory vehicle industry, wireless communication, and distributed control.

Typically, heavy duty vehicles speed up down hill and loose speed climbing uphill. If the road grade ahead of the vehicle is known, the vehicle can adjust its speed in advance of uphill and downhill road segments. This road grade estimation problem was considered in [16], [17], where the authors proposed an algorithm to estimate the road grade. However, the considered model was a single vehicle rather than a platoon of vehicles.

This paper considers a platoon of multiple vehicles traveling uphill and proposes a decentralized algorithm for any vehicle in the platoon to compute the road grade in a finite number of time-steps using only its own velocity measurements. The analysis is built upon the work of [18], [19], which allows any agent to compute the consensus value for a connected graph in a finite number of steps using only its own state history. Comparing to the work of [16], the proposed algorithm exactly computes the road grade for a platoon of vehicles on-line. The computation of the road grade for a platoon of vehicles is important and can be used online in fuel-optimized collaborative cruise controllers [15].

The remainder of the paper is organized as follows: Section II introduces preliminary concepts and notations. Section III presents the motivation and formulates the finite-time road grade computation problem for a vehicle platoon. Section IV gives a necessary and sufficient condition on feedback gain parameters of a controller which utilizes the relative position errors and absolute velocity errors, for achieving the key goal of vehicle platooning. Section V proposes a decentralized algorithm for solving the finite-time road grade computation problem. Section VI illustrates the results with several examples. Finally, Section VII concludes the paper and points out future directions.

II. Preliminaries and Notations

Consider an undirected graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, with the set of nodes $\mathscr{V} = \{1, \ldots, N\}$ and the set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$. A matrix $W \in \mathbb{R}^{N \times N}$ is the corresponding adjacency matrix, with $W_{ij} = W_{ji} = 1$ if and only if the edge $(i, j) \in \mathscr{E}$ and $W_{ij} = 0$ otherwise. We assume that there is no self-loop, i.e., $W_{ii} = 0$ for $i \in \mathscr{V}$. The set of neighboring agents of agent i is defined as $\mathscr{N}_i = \{j \in \mathscr{V} | W_{ij} \neq 0\}$. A path from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathscr{E}$ for $j = 1, \ldots, k-1$. An undirected graph is said to be connected if there exists a path between any pair of distinct nodes.

For an undirected graph $\mathscr{G}$, a matrix $L = \{\ell\}_{ij} \in \mathbb{R}^{N \times N}$ with $\ell_{ii} = \sum_{j=1, j \neq i}^{N} W_{ij}$ and $\ell_{ij} = -W_{ij}$ for $j \neq i$, is called the Laplacian matrix associated with $\mathscr{G}$. It is well known that the Laplacian matrix has the property that all row sums are zero. If the undirected graph $\mathscr{G}$ is connected, then L has a simple eigenvalue at zero with the corresponding right eigenvector $\mathbf{1}$ and all other eigenvalues are strictly positive. According to Gershgorin disk theorem, all the eigenvalues can be ordered as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$.

Given a matrix A, A^{T} denotes its transpose, $\mathrm{rank}(A)$ denotes its rank and $\ker(A)$ denotes its nullspace. We denote by $A \otimes B$ the Kronecker product between matrices A and B. We denote e_r^{T} as the unit row vector whose r-th entry is one while all the other entries are zeros. I_N denotes the identity matrix of dimension $N \times N$. $\mathbf{1}_N$ denotes the column vector with each entry being 1. $\mathbb{N}$ denotes the set of all the positive

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integers. For column vectors $x_1,\dots,x_N$, the stacked column vector of $x_1,\dots,x_N$ is denoted by $[x_1;\dots;x_N]$.

Definition 1: (Minimal polynomial of a matrix) The minimal polynomial associated with the matrix $A\in\mathbb{R}^{n\times n}$ is the monic polynomial $q(t)\triangleq t^{D+1}+\sum_{i=0}^{D}\alpha_i t^i$ with smallest degree $D+1$ that satisfies $q(A)=0$.

Definition 2: (Minimal polynomial of a matrix pair) The minimal polynomial associated with the matrix pair (A,C_r) denoted by $q_r(t)\triangleq t^{D_r+1}+\sum_{i=0}^{D_r}\alpha_i^{(r)}t^i$ is the monic polynomial of smallest degree $D_r+1\le D+1$ that satisfies $C_r q_r(A)=0$.

III. Motivation and Problem Statement

The mathematical problem considered in this paper is motivated by vehicle platooning as introduced in Section I. Consider a platoon of N vehicles, as shown in Fig. 1, operating at close intermediate spacings and traveling up hill with a road grade α.

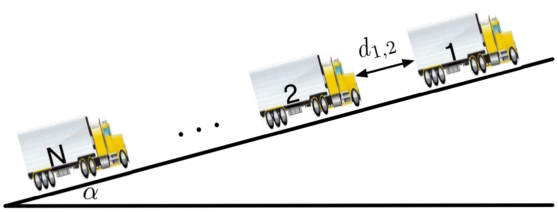

Fig. 1. A platoon of N vehicles traveling on a hilly road with grade α.

This unknown road grade (incline angle) α can be viewed as a constant disturbance to the vehicle's model. Each vehicle along the longitudinal direction is modeled by a discrete-time double integrator

$$x_i(k+1)=x_i(k)+v_i(k), \tag{1a}$$
$$v_i(k+1)=v_i(k)+u_i(k)-g\sin\alpha,\ i\in\{1,\dots,N\}, \tag{1b}$$

where $x_i(k),v_i(k),u_i(k)$ represent the (longitudinal) position, the velocity, and the control input of the vehicle i at time step k, respectively, and g is the gravity of the earth. This gravity does not affect our analysis, and to simplify the presentation we can rescale the variables and parameters to make $g=1$.

In addition to simplify the analysis, we assume that α is small so that $\alpha\approx\sin\alpha$, which can be easily relaxed. Thus from (1), we obtain the following dynamics.

$$x_i(k+1)=x_i(k)+v_i(k), \tag{2a}$$
$$v_i(k+1)=v_i(k)+u_i(k)-\alpha,\ i\in\{1,\dots,N\}. \tag{2b}$$

This model is the discrete-time counterpart of the continuous-time model considered in [8], [9] with drag coefficients per unit being zeros, and has also been considered in [20, Chapter 4]. We also assume that these vehicles are equipped with Radars, which allow them to measure relative positions to both preceding and succeeding vehicles. This scenario can be represented by a line graph given in Fig. 2.

In some cases, each vehicle is equipped with wireless sensor which can sense not only its own position relative to

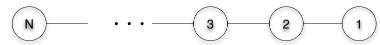

Fig. 2. One-hop neighbors vehicle communication topology.

that of one-hop neighbors, but also that of multi-hop neighbors via wireless communication, see for instance [5], [13]. For example, Fig. 3 shows the case where each vehicle can also sense its two-hop neighbors. In both one-hop neighbors

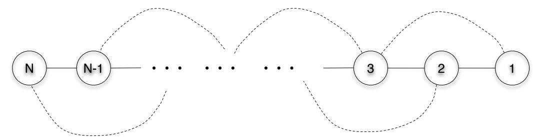

Fig. 3. Two-hop neighbors vehicle communication topology.

communications and multi-hop neighbors communications, communications among vehicles can be modelled as an undirected connected graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$.

The desired position of the vehicle i is given by $x_{d,i}=(i-1)\delta+v_d k$, where v_d is the desired velocity and δ is the desired distance between the neighboring vehicles. Every vehicle is assumed to have access to both v_d and δ. We consider the controller which utilizes relative position errors between neighboring vehicles and the absolute velocities errors

$$u_i=-f_1\sum_{j\in\mathscr{N}_i}W_{ij}(x_i-x_j-d_{j,i})-f_2(v_i-v_d), \tag{3}$$

where $d_{j,i}=(i-j)\delta$. This controller has also been considered in [10], [13] for a platoon of vehicle traveling with zero road grade.

We then define the position error $\bar{x}_i=x_i-x_{d,i}$ and the velocity error $\bar{v}_i=v_i-v_d$. In view of these deviation variables, the vehicle dynamics (2) become

$$\bar{x}_i(k+1)=\bar{x}_i(k)+\bar{v}_i(k), \tag{4a}$$
$$\bar{v}_i(k+1)=\bar{v}_i(k)+u_i(k)-\alpha,\quad i\in\{1,\dots,N\}, \tag{4b}$$

and the controller (3) becomes

$$u_i=-f_1\sum_{j\in\mathscr{N}_i}W_{ij}(\bar{x}_i-\bar{x}_j)-f_2\bar{v}_i. \tag{5}$$

In matrix form, the vehicle dynamics (4) can be represented as

$$\begin{bmatrix}\bar{x}(k+1)\\ \bar{v}(k+1)\end{bmatrix}=\begin{bmatrix}I_N & I_N\\ 0 & I_N\end{bmatrix}\begin{bmatrix}\bar{x}(k)\\ \bar{v}(k)\end{bmatrix}+\begin{bmatrix}0\\ I_N\end{bmatrix}(u(k)-\alpha\mathbf{1}), \tag{6}$$

and the controller (5) can be represented as

$$u(k)=-f_1L\bar{x}(k)-f_2\bar{v}(k), \tag{7}$$

where $\bar{x}(k)$, $\bar{v}(k)$, and $u(k)$ denote the position error, the velocity error, and the control input vectors at time instance k, respectively, e.g., $\bar{x}(k) = [\bar{x}_1(k); \ldots ; \bar{x}_N(k)]$. Thus, the closed-loop systems of (6) and (7) can be written as

$$\begin{bmatrix} \bar{x}(k+1) \\ \bar{v}(k+1) \end{bmatrix} = A \begin{bmatrix} \bar{x}(k) \\ \bar{v}(k) \end{bmatrix} - B\alpha \mathbf{1}, \tag{8}$$

where

$$A = \begin{bmatrix} I_N & I_N \\ -f_1 L & (1-f_2)I_N \end{bmatrix}, B = \begin{bmatrix} 0 \\ I_N \end{bmatrix}. \tag{9}$$

Our ultimate goal of this paper is to consider the finite-time road grade computation problem for a vehicle platoon. Thus the first goal is to guarantee that the key goal of vehicle platooning is achieved. This is formally stated as below:

Problem 1: Consider a platoon of N vehicles (2) and the controller (3). For a given set velocity v_d and desired inter-vehicle distance δ, the vehicle platooning problem is to appropriately choose feedback gain parameters f_1 and f_2 such that the key goal of vehicle platooning is achieved, that is,

$$\lim_{k\to\infty} v_i(k) = v_d + \varepsilon, i = 1, \ldots, N \tag{10a}$$

$$\lim_{k\to\infty} (x_{i+1}(k) - x_i(k+1)) = \delta, i = 1, \ldots, N-1, \tag{10b}$$

where ε is some constant which depends on α, f_1 and f_2.

Once Problem 1 is solved, we are able to obtain the value of the constant ε by using $\varepsilon = \lim_{k\to\infty} v_i(k) - v_d$. This implies that we can obtain the road grade asymptotically provided that the relationship between ε and α can be characterized explicitly. However, this steady-state needs a very large number of steps to achieve, moreover, it may lead to very complicated transient behavior which results in a high fuel consumption.

This naturally leads to the key problem considered in this paper, that is, the finite-time road grade computation problem. To this end, we assume that each vehicle has memory to store its own velocity value over a range of time-steps. This is however not restricted since most heavy duty vehicles nowadays are equipped with computers. This problem is formally formulated as below:

Problem 2: Consider a platoon of N vehicles (2) and the controller (3). Assume that an arbitrarily chosen vehicle r observes its velocity. The problem is to compute the road grade α in a finite number of time-steps by using it own successive velocity measurements, observed over a range of time-steps.

In what follows, we shall present solutions to the two proposed problems in the subsequent sections, respectively.

IV. Key goal of vehicle platooning

In this section, we shall present our main result to the first proposed problem. The following proposition states a necessary and sufficient condition for solving Problem 1. The proof has been omitted due to the space limitation and can be found in [21].

Proposition 1: Consider a platoon of N vehicles (2) and the controller (3). The vehicle platooning is achieved, that is,

$$\lim_{k\to\infty} v_i(k) = v_d - \frac{\alpha}{f_2}, i \in \mathscr{V}, \tag{11a}$$

$$\lim_{k\to\infty} (x_{i+1}(k) - x_i(k)) = \delta, i = 1, \ldots, N-1, \tag{11b}$$

if and only if

$$0 < f_1 \lambda_N < f_2 < 2, \tag{12}$$

where λ_N is the largest eigenvalue of the Laplacian matrix associated with the communication graph $\mathscr{G}$.

V. Finite-time road grade computation

In this section we shall assume that the gain parameters f_1 and f_2 satisfy (12), so that Problem 1 is solved, that is, the goal of vehicle platooning is achieved. Our goal here is to consider Problem 2, where each vehicle r observes its own velocity, i.e.,

$$y_r(k) = C_r \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \tag{13}$$

with

$$C_r = \begin{bmatrix} 0 & e_r^{\mathrm{T}} \end{bmatrix}, \tag{14}$$

$x(k) = [x_1(k); \ldots ; x_N(k)]$ and $v(k) = [v_1(k); \ldots ; v_N(k)]$.

We shall show that for almost all initial conditions (except a set of initial conditions with Lebesgue measure zero), vehicle r can obtain the final value of the velocity $\phi_v = \lim_{k\to\infty} v_r(k)$ by using its own successive measurements $y_r(k)$ observed over a finite number of steps. The number of steps does not depend explicitly on the size of the network. We then use the value of ϕ_v to compute the road grade $\alpha = (v_d - \phi_v) f_2$ according to (11a).

Before presenting the main theorem, we present the following lemma whose proof is given in Appendix A.

Lemma 1: Consider a platoon of N vehicles (2) and the controller (3). Let $q_r(t) = \sum_{i=0}^{D_r+1} \alpha_i^{(r)} t^i$ be the minimal polynomial for the matrix pair (A, C_r) with A and C_r given by (9) and (14), respectively. Then the dynamics of observations $y_r(k)$ given by (13) satisfy the following regression equation

$$\sum_{i=0}^{d_r+1} \alpha_{r,i} y_r(k+i) = 0, \forall k \in \mathbb{N}, \tag{15}$$

where $d_r = D_r + 1$ and the coefficients

$$\alpha_{r,i} = \alpha_{i-1}^{(r)} - \alpha_i^{(r)}, i = 0, \ldots, d_r + 1, \tag{16}$$

with $\alpha_{-1}^{(r)} = \alpha_{D_r+2}^{(r)} = 0$ for the convention.

For simplicity of presentation, we make the following assumption for the rest of the paper. The assumption can be easily relaxed by using a similar analysis from [19].

Assumption 1: The matrix A given by (9) has full rank.

Now, we consider $2k+1$ successive observations of $y_r(k) = v_r(k)$ from agent r as

$$V_{0,1,\ldots,2k}^{(r)} = (v_r(0), v_r(1), \ldots, v_r(2k))$$

and define its associated Hankel matrix as

$$\Gamma\{V_{0,1,\ldots,2k}^{(r)}\} \triangleq \begin{bmatrix} v_r(0) & v_r(1) & \ldots & v_r(k) \\ v_r(1) & v_r(2) & \ldots & v_r(k+1) \\ \vdots & \vdots & \ddots & \vdots \\ v_r(k) & v_r(k+1) & \ldots & v_r(2k) \end{bmatrix}.$$

We are now ready to present our main theorem whose proof is given in Appendix B.

Theorem 1: Consider a platoon of N vehicles (2) and the controller (3). Let $q_r(t) = \sum_{i=0}^{D_r+1} \alpha_i^{(r)} t^i$ be the minimal polynomial for the matrix pair (A, C_r) with A and C_r given by (9) and (14), respectively. Suppose that an arbitrarily chosen vehicle r has successive observations of its own velocity given by (13). Then starting from an arbitrary step, vehicle r can compute the final value for the velocity ϕ_v by

$$\phi_v = \frac{y_{d_r}^{\mathrm{T}} \beta^{(r)}}{\mathbf{1}^{\mathrm{T}} \beta^{(r)}}, \tag{17}$$

and thus the road grade by

$$\alpha = (v_d - \phi_v) f_2, \tag{18}$$

where $y_{d_r}^{\mathrm{T}} = \begin{bmatrix} y_r(0) & y_r(1) & \ldots & y_r(D_r+1) \end{bmatrix}$ and

$$\beta^{(r)} = \begin{bmatrix} \beta_0^{(r)} & \ldots & \beta_{D_r}^{(r)} & 1 \end{bmatrix}^{\mathrm{T}}$$

for $\beta_i^{(r)} = \alpha_i^{(r)}, i = 0, \ldots, D_r$. Moreover, the coefficients $\beta_i^{(r)}$ can be obtained as the normalized kernel of the Hankel matrix $\Gamma\{\bar{V}_{0,1,\ldots,2(D_r+1)}^{(r)}\}$ with $\{\bar{V}_{0,1,\ldots,2k}^{(r)}\}$ is given in (20), i.e.,

$$\begin{bmatrix} \beta_0^{(r)} & \ldots & \beta_{D_r}^{(r)} & 1 \end{bmatrix}^{\mathrm{T}} \in \ker(\Gamma\{\bar{V}_{0,1,\ldots,2(D_r+1)}^{(r)}\}).$$

The proof of Theorem 1 naturally leads a decentralized algorithm for vehicle r to compute the road grade α by using its own successive velocity observations in $2(D_r+2)$ number of time-steps, which is minimum as shown in [19]. We summarize the procedure in Algorithm 1.

Algorithm 1 Finite-time road grade computation

Data: Successive observations of $y_r(k) = v_r(k),\ k \in \mathbb{N}$.
Result: The road grade α.
Step 1: Compute the vector of differences $\bar{V}_{0,1,\ldots,2k}^{r}$.
Step 2: Increase the dimension k of the Hankel matrix $\Gamma\{\bar{V}_{0,1,\ldots,2k}^{(r)}\}$ until it loses rank and store the first defective Hankel matrix.
Step 3: Compute the kernel $\beta^{(r)} = \begin{bmatrix} \beta_0^{(r)} & \ldots & \beta_{D_r}^{(r)} & 1 \end{bmatrix}^{\mathrm{T}}$ of the first defective Hankel matrix.
Step 4: Compute the velocity consensus value ϕ_v using (17).
Step 5: Compute the road grade α using (18).

Next we shall relate the required number of steps $2(D_r+2)$ to the required number of steps [19] for computing final consensus value. We have the following result:

Proposition 2: Consider the observability matrices

$$\Omega_{r,v} = \begin{bmatrix} C_r \\ C_r A \\ \vdots \\ C_r A^{2N-1} \end{bmatrix}, \quad \Theta_r = \begin{bmatrix} e_r^{\mathrm{T}} \\ e_r^{\mathrm{T}} L \\ \vdots \\ e_r^{\mathrm{T}} L^{N-1} \end{bmatrix},$$

then $D_r + 2 = \operatorname{rank}(\Omega_{r,v}) + 1 = 2\operatorname{rank}(\Theta_r)$.

Proof : The first equality follows from [19, Proposition 2] and the fact that $d_r = D_r + 1$ from Lemma 1. To show the second equality, we note that it follows from [19, Theorem 2] that $\operatorname{rank}(\Theta_r) = N - \mu_r$, where μ_r is the number of eigenvalues shared between L and L_r, and L_r is the submatrix of L obtained by deleting the r-th row and r-th column. Thus, it is equivalent to show that $\dim(\ker(\Omega_{r,v})) = 2\mu_r + 1$. This can be seen as follows. We first note that the vector $[\mathbf{1}^{\mathrm{T}}, 0^{\mathrm{T}}]^{\mathrm{T}}$ is in the null space of the observability matrix $\Omega_{r,v}$. Let p_i for $i = \{1, \ldots, \mu_r\}$ be the μ_r vectors which span the null space of the observability matrix Θ_r, i.e., $\Theta_r p_i = 0$. It is then easy to check that the vector $[p_i^{\mathrm{T}}, 0^{\mathrm{T}}]$ and $[0^{\mathrm{T}}, p_i^{\mathrm{T}}]$ are in the null space of the observability matrix $\Omega_{r,v}$. Hence, the result follows. ■

Remark 1: It follows from Proposition 2 that the number of steps for vehicle r to compute the consensus velocity value and thus the road grade is $2(D_r+2) = 4\operatorname{rank}(\Theta_r)$, while the required number of steps to compute the consensus value for the single-integrator case is $2\operatorname{rank}(\Theta_r)$ as given in [19, Proposition 1]. Hence, in our case it takes exactly twice the time-steps compared to the single-integrator case.

VI. Motivating example revisit

In this section, we illustrate our results by revisiting the motivating examples in Section III.

A. One-hop neighbors communications

Consider the case given in Fig. 1 for $N = 6$ vehicles, and the vehicle communication topology is given by Fig. 2 with $N = 6$. The desired inter-vehicle distance is $\delta = 20$, the set velocity is $v_d = 40$, and the road grade is $\alpha = 0.2$.

We choose $f_1 = 0.35$ and $f_2 = 1.9$ such that the condition (12) is satisfied. We choose the initial states as $x(0) = [6; 3; 2; 4; 1; 5]$, $v(0) = [7; 8; 9; 10; 11; 12]$.

The evolution of the velocity v_i and inter-vehicle distance $d_{i,i+1}$ are plotted in Fig. 4 and Fig. 5, respectively. From Fig. 4, we see that the steady-state value for the velocity is 39.8947, which agrees with the value computed by (11a). Fig. 5 clearly shows that in steady state, the inter-vehicle distances $d_{1,2} = d_{2,3} = d_{3,4} = d_{4,5} = \delta = 20$.

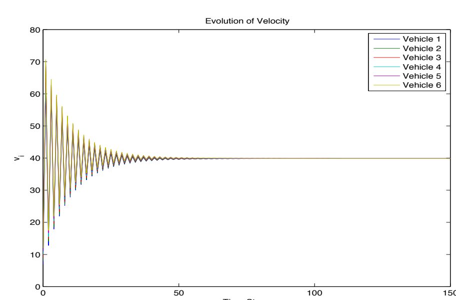

Fig. 4. Evolution of the velocity of all vehicles in one-hop topology.

Suppose that vehicle r has successive observations of its own velocity given by (13). By applying Algorithm 1 to

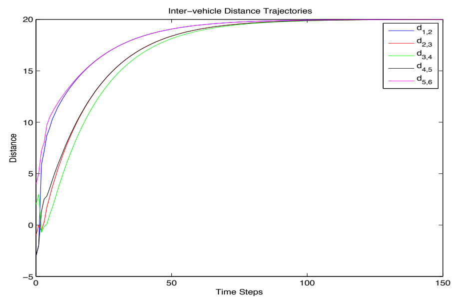

Fig. 5. The inter-vehicle distance trajectories versus the number of time-steps for the one-hop communication topology.

each vehicle, we see that vehicles 1, 3, 4, and 6 compute the consensus value $\phi_v = 39.8947$ in $2\times 12 = 24$ steps, while vehicles 2 and 5 compute the consensus value in $2\times 10 = 20$ steps, compared to roughly 70 steps as shown in Fig. 4. Each vehicle can then compute the road grade by (18) and obtain 0.2, which agrees with the fact that $\alpha = 0.2$.

B. Two-hop neighbors communications

We now consider the case where the vehicle communications are given by Fig. 3 with $N = 6$. The inter-vehicle distance, the set velocity, and the road grade are the same.

For this case, the same feedback gain parameters $f_1 = 0.35$ and $f_2 = 1.9$ also satisfy the condition (12). We also choose the same initial states as before. The evolution of the velocity v_i and inter-vehicle distance $d_{i,i+1}$ are plotted in Fig. 6 and Fig. 7, respectively.

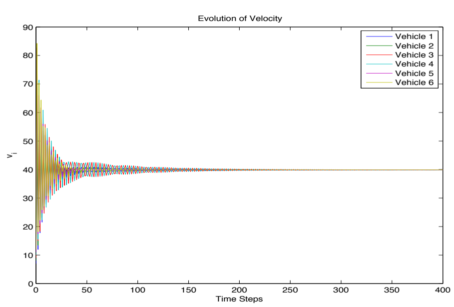

Fig. 6. Evolution of the velocity in two-hop communication topology.

By applying Algorithm 1 to each vehicle, we see that vehicles 1 and 6 compute the consensus value $\phi_v = 39.8947$ in $2\times 10 = 20$ steps, while vehicles 2, 3, 4 and 5 compute the consensus value in $2\times 12 = 24$ steps, compared to roughly 200 steps as shown in Fig. 6. Each vehicle can then compute the road grade by (18) and obtain 0.2, which agrees with the fact that $\alpha = 0.2$.

C. Comparison

As seen from both cases, the number of steps for each vehicle to compute the final consensus value and thus

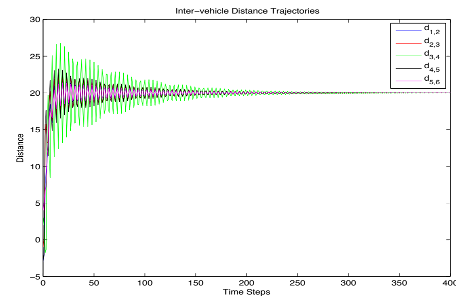

Fig. 7. The inter-vehicle distance trajectories versus the number of time-steps for the two-hop communication topology.

road grade has been dramatically reduced. With this newly introduced algorithm, it can avoid undesirable oscillatory behavior for vehicle platooning.

By comparing Fig. 5 and Fig. 7, we see that the inter-vehicle distances in case of two-hop neighbor communications have oscillations before they settle down to the steady-state. By comparing Fig. 4 and Fig. 6, we see that the velocities converge to the same steady-state value faster in case of one-hop neighbor communications. For the computation of the road grade, in case of two-hop neighbor communications, the number of time-steps for vehicles 1 and 6 have been reduced from 24 to 20, while the number of time-steps for vehicles 2 and 5 increase from 20 to 24, for vehicles 3 and 5 stay the same. These time-steps are related to the communication topology. We are currently carrying out the detailed graph-theoretical analysis regarding this issue.

VII. CONCLUSION

This paper proposed a decentralized algorithm for any vehicle traveling uphill in a platoon to compute the road grade in a finite number of steps by using its own successive velocity measurements. We assumed that these measurements are perfect. We are currently investigating the case where these measurements are imperfect, in particular, the case where they are subject to packet loss.

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APPENDIX

A. Proof of Lemma 1

Let us define $q(k) = [\bar{x}(k); \bar{v}(k)]$. Since $q_r(t) = \sum_{i=0}^{D_r+1} \alpha_i^{(r)} t^i$ is the minimal polynomial for the matrix pair (A, C_r), it then follows from Definition 2 that $C_r q_r(A) = C_r \sum_{i=0}^{D_r+1} \alpha_i^{(r)} A^i = 0$. By using this and the coefficients $\alpha_{r,i}$ given in (16), we obtain that

$$
\begin{aligned}
&\sum_{i=0}^{d_r+1} \alpha_{r,i} y_r(k+i) \\
&= \sum_{i=0}^{d_r+1} \alpha_{r,i} \bar{v}_r(k+i) \\
&= C_r \sum_{i=0}^{d_r+1} \alpha_{r,i} A^i q(k) - C_r \sum_{i=0}^{d_r+1} \alpha_{r,i} \sum_{j=0}^{i-1} A^j B \alpha \mathbf{1} \\
&= C_r \sum_{i=0}^{d_r+1} (\alpha_{i-1}^{(r)} - \alpha_i^{(r)}) A^i q(k) \\
&\quad - C_r \sum_{i=0}^{d_r+1} (\alpha_{i-1}^{(r)} - \alpha_i^{(r)}) \sum_{j=0}^{i-1} A^j B \alpha \mathbf{1} \\
&= -C_r \sum_{i=0}^{D_r+1} \alpha_i^{(r)} A^i (I - A) q(k) - C_r \sum_{i=0}^{D_r+1} \alpha_i^{(r)} A^i B \alpha \mathbf{1} = 0.
\end{aligned}
$$

Hence, the dynamics of observation $y_r(k) = v_r(k)$ satisfy the regression equation (15).

B. Proof of Theorem 1

Let us denote the z-transform of $y_r(k)$ as $Y_r(z) \triangleq \mathbb{Z}(y_r(k))$. From (15) and the time-shift property of the $z-$transform, it is easy to show that

$$
Y_r(z) = \frac{\sum_{i=1}^{d_r+1} \alpha_{r,i} \sum_{\ell=0}^{i-1} y_r(\ell) z^{i-\ell}}{q_r(z)} \triangleq \frac{H(z)}{q_r(z)}. \tag{19}
$$

Note that $q_r(t) = 0$ does not possess any unstable root apart from single one at 1 since the matrix A given by (9) has only a single unstable eigenvalue at 1, which has been shown in the proof of Theorem 1. We then define the polynomial $p_r(z) \triangleq \frac{q_r(z)}{z-1}$. Then we can show that $p_r(z) = \sum_{i=0}^{d_r} \beta_i^{(r)} z^i$, for $\beta_i^{(r)} = \begin{bmatrix} \beta_0^{(r)} & \ldots & \beta_{d_r-1}^{(r)} & \beta_{d_r}^{(r)} \end{bmatrix}^{\mathrm{T}}$ with $\beta_i^{(r)} = -\sum_{j=0}^{i} \alpha_{r,j}$, $i = 0, \ldots, d_r$. This together with $d_r = D_r + 1$ and (16) implies that $\beta_i^{(r)} = \alpha_i^{(r)}$ for $i = 0, \ldots, D_r + 1$. We can then obtain the final value ϕ_v for the velocity by applying the final-value theorem and some simple algebra

$$
\phi_v = \lim_{k \to \infty} y_r(k) = \lim_{z \to 1} (z-1) Y_r(z) = \frac{y_{d_r}^{\mathrm{T}} \beta^{(r)}}{\mathbf{1}^{\mathrm{T}} \beta^{(r)}}.
$$

We see that if we can obtain the unknown coefficients $\beta_i^{(r)}$, then an arbitrarily chosen vehicle $r \in \mathscr{V}$ can compute the consensus value for the velocities by using only a finite number of successive observations of its own velocity.

Next, we shall show how vehicle r can compute the unknown coefficients $\beta_i^{(r)}$ in (17) and hence compute the final value for the velocity. It then follows from (11a) that the road grade can be compute by (18).

We assemble the difference between successive values of $y_r(k) = v_r(k)$ as

$$
\bar{V}_{0,1,\ldots,2k}^{(r)} = (v_r(1) - v_r(0), \ldots, v_r(2k+1) - v_r(2k)). \tag{20}
$$

By using (15) and the fact that $\sum_{i=0}^{d_r} \alpha_{r,i} = -1$, we can show that for any $k \in \mathbb{N}$,

$$
\begin{aligned}
&v_r(k + d_r + 1) - v_r(k + d_r) \\
&= -\beta_{d_r}^{(r)} (v_r(k + d_r) - v_r(k + d_r - 1)) \\
&\quad - \ldots - \beta_0^{(r)} (v_r(k+1) - v_r(k)).
\end{aligned}
$$

When $\Gamma\{\bar{V}_{0,1,\ldots,2k}^{(r)}\}$ loses rank, we can then compute the kernel which gives $\begin{bmatrix} \beta_0^{(r)} & \ldots & \beta_{d_r-1}^{(r)} & 1 \end{bmatrix}^{\mathrm{T}}$.