Distributed Freeway Ramp Metering: Optimization on Flow Speed

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***Abstract*— This paper studies the distributed freeway ramp metering problem, for which the cell transmission model (CTM) is utilized. Considering the jam density and the upper bounds on the queue lengths and the ramp metering, we first provide feasibility conditions with respect to the external demand to ensure the controllability of the freeway. Assuming that the freeway is controllable, we formulate an optimization problem which tradeoffs the maximum average flow speed and the minimum waiting queue for each cell. Although the cells of the CTM are dynamically coupled, we propose a distributed backward algorithm for the optimization problem and prove that the solution to the problem is a Nash equilibrium. Furthermore, if the optimization problem is simplified to only maximization of the average flow speed, we argue that the obtained explicit distributed controller is globally optimal. A numerical example is given to illustrate the effectiveness of the proposed control algorithm.**

I. INTRODUCTION

Increasing congestion on freeways is leading to intolerable travelling delays and economic loss. To mitigate traffic congestion, variable speed limits and on-ramp metering are two popular and effective control strategies [1], [2]. On-ramp metering is implemented by installing a traffic light to control the inflow of the freeway at each on-ramp.

Many optimization-based ramp metering strategies have been reported in the literatures, e.g., [6], [7]. One popular optimization objective is to minimize the total time spent by all the drivers on the freeway over a finite horizon. In [5] and [8], different optimization problems which are based on model predictive control (MPC) are proposed to control the traffic flow and density in a centralized form. Obviously, these methods encounter a large computational effort, a high communication cost, and the assumption of global model information. To overcome these adverse aspects, distributed or decentralized feedback controllers have been investigated for freeway ramp metering. In such framework, local controllers only rely on the local measurements and the limited exchange of information with the neighboring controllers. In [9] and [10], distributed MPC-based control strategies are formulated to coordinate the behaviours of different cells. Although distributed MPC can reduce the communication efforts and the computation time, the traffic demand cannot be predicted perfectly, thereby restricting the effectiveness of the controllers.

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The recent work [11] considers a distributed, non-predictive controller for ramp metering of CTM. The designed controller preserves monotonicity of the closed-loop system and achieves globally optimal performance, i.e., the minimum total time spent by all the drivers. Based on local communication and real-time measurements, this paper investigates distributed optimization-based ramp metering. We adopt the cell transmission model (CTM) of Daganzo [3], which is attractive due to its simple model equations, computational efficiency in the control problems, and guarantee of the nonnegative traffic speed [4]. Particularly, we consider a piece-wise affine fundamental diagram (FD) and a dynamical queue with limited length on each ramp [5]. Different from the optimization objective in [11], we tradeoff the average flow speed and the queue length (i.e., the number of waiting cars in each on-ramp) in our objective function. To sum up, the main contributions of this paper are:

(1) Considering the jam density and the upper bounds on the queue lengths and the ramp metering, we provide a feasibility condition with respect to the external demand to ensure the controllability of the freeway.
(2) We propose a backward algorithm for the coupled optimization problem and show that the solution by the backward optimization algorithm is a Nash equilibrium.
(3) We simplify the optimization problem to only maximization of the average flow speed, provide an explicit distributed controller, and prove that this controller is globally optimal.

This paper is organized as follows. Section II introduces the CTM and presents a feasibility condition for the external demand. Section III formulates an optimization problem and provides an algorithm. Section IV simplifies the optimization problem and derives a distributed controller. Section V simulates a numerical example. Section VI concludes.

Notation: The k-th element of a vector x is denoted by x_k. A cell of the CTM is indexed by k. The set of nonnegative integers is denoted by $\mathbb{N}$.

II. PRELIMINARIES

A. Cell Transmission Model

We use the asymmetric CTM [4], [5], [11] to describe a freeway in Fig 1 and Fig. 2. The freeway is partitioned into $n+1$ cells, each with one on- and one off-ramp. Denote by $\mathscr{C} = \{0,\ldots,n\}$ the index set of the cells. The model variables and parameters are listed in Table I. Assume that the split ratio of cell k is constant, which means that a fixed portion β_k of the total flow of cell k leaves from its off-ramp [5]. Define $\bar{\beta}_k = 1-\beta_k$. Following the mass conservation laws,

TABLE I

MODEL VARIABLES AND PARAMETERS

Symbol	Name	Range	Unit
ρ_k	density	$[0, \bar{\rho}_k]$	cars/km
f_k	flow	$[0, F_k]$	cars/h
q_k	queue length	$[0, \bar{q}_k]$	cars
u_k	metering rate	$[0, \bar{u}_k]$	cars/h
r_k	external traffic demand	$[0, \infty)$	cars/h
ρ_k^c	critical density	$\frac{w_k}{v_k+w_k}\bar{\rho}_k$	cars/km
$\bar{\rho}_k$	jam density	$(0, \infty)$	cars/km
β_k	split ratio	$[0, 1)$	1
$\bar{\beta}_k$	$1-\beta_k$	$(0, 1]$	1
v_k	free flow speed	$(0, \infty)$	km/h
w_k	congestion wave speed	$(0, \infty)$	km/h
l_k	cell length	$(0, \infty)$	km
δ	sampling period	Assumption 2.1	h

the dynamics of cell $k \in \mathscr{C}$ in the CTM is governed by

$$\rho_k(t+1) = \rho_k(t) + \frac{\delta}{l_k}(f_{k-1}(t) + u_k(t) - \frac{1}{\bar{\beta}_k} f_k(t)), \quad (1a)$$

$$q_k(t+1) = q_k(t) + \delta(r_k(t) - u_k(t)). \quad (1b)$$

The flow $f(t)$ in the CTM is a function of the density $\rho(t)$. Particularly, the flow $f_k(t)$ is constrained by the demand $d_k(\rho_k(t))$, the capacity F_k, and the supply $s_{k+1}(\rho_{k+1}(t))$:

$$f_{-1}(t) = 0, \quad (2a)$$

$$f_k(t) = \min\{d_k(\rho_k(t)), F_k, s_{k+1}(\rho_{k+1}(t))\}, \ 0 \le k < n, \quad (2b)$$

$$f_n(t) = \min\{d_n(\rho_n(t)), F_n\} \quad (2c)$$

which corresponds to the so-called FD in Fig. 3. The demand $d_k(\rho_k(t)) = \bar{\beta}_k v_k \rho_k(t)$ is the number of the vehicles that want to move from cell k to $k+1$ at time t and the supply $s_k(\rho_k(t)) = w_k(\bar{\rho}_k - \rho_k(t))$ is the remaining free space of cell k at time t. The critical density is $\rho_k^c = \frac{w_k}{v_k+w_k}\bar{\rho}_k$. The capacity F_k of cell k satisfies $F_k = \min\{d_k(\rho_k^c), s_{k+1}(\rho_{k+1}^c)\}$. Cell k is said to be free at time t if $\rho_k(t) \in [0, \rho_k^c]$ and to be congested if $\rho_k(t) \in (\rho_k^c, \bar{\rho}_k]$.

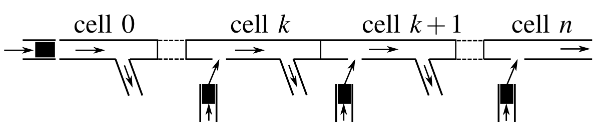

Fig. 1. A freeway with $n+1$ cells

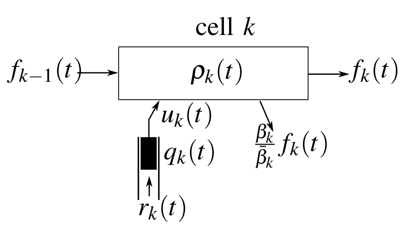

Fig. 2. Model of cell k

Remark 2.1: In this paper, the initial cell is equipped with one on-ramp, whose dynamics is the evolution of the queue length. Thus, the flow of the initial cell is constrained by FD. This is different from [11].

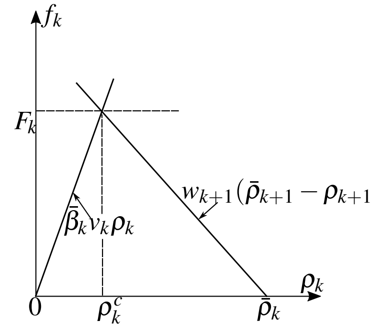

Fig. 3. Fundamental diagram

Assumption 2.1: The sampling period δ is a constant satisfying $0 < \delta < \frac{l_k}{v_k}$ for $\forall k \in \mathscr{C}$.

Typically, we have $v_k > w_k$ and thus $\delta < \frac{l_k}{w_k}$. Given the initial flow $\rho_k(0)$, the initial queue length $q_k(0)$, and the external traffic demand $r_k(t)$ for $k \in \mathscr{C}$ and $t \in \mathbb{N}$, the dynamical update of the states of the CTM is described by (1a)-(1b) and (2a)-(2c). The states and the control inputs for $\forall t \in \mathbb{N}$ are constrained as follows

$$\begin{cases} \rho_k(t) \in [0, \bar{\rho}_k], \ f_k(t) \in [0, F_k], \\ q_k(t) \in [0, \bar{q}_k], \ u_k(t) \in [0, \bar{u}_k]. \end{cases} \quad (3)$$

B. Dynamically Feasible External Demand

The following definitions state the feasibility of the external traffic demand and the controllability of the freeway.

Definition 2.1: The external traffic demand $r(t)$ is feasible at time t if there exists a control input $u_k(t) \in [0, \bar{u}_k]$ such that the constraints (3) are satisfied for $\forall k \in \mathscr{C}$.

Definition 2.2: The freeway is controllable at time t if the external traffic demand $r(t)$ is feasible.

The next theorem establishes the conditions under which the freeway is recursively controllable.

Theorem 2.1: Under Assumption 2.1, provided that the freeway is controllable at time t, the sufficient condition to ensure the controllability of the freeway at time $t+1$ is

$$r_k(t) \le \frac{1}{\delta}(\bar{q}_k - q_k(t)) + h_k(t), \ \forall k \in \mathscr{C}, \quad (4)$$

where $h_k(t) = \min\{\bar{u}_k, (\frac{l_k}{\delta} - w_k)(\bar{\rho}_k - \rho_k(t))\}$.

Proof: If $\rho_k(t) \ge 0$, it follows from $f_k(t) \le d_k(\rho_k(t))$ and Assumption 2.1 that

$$\rho_k(t+1) \ge \rho_k(t) - \frac{\delta}{\bar{\beta}_k l_k} f_k(t) \ge (1 - \frac{\delta}{l_k} v_k)\rho_k(t) \ge 0. \quad (5)$$

Also the fact $f_{k-1}(t) \le s_k(\rho_k(t))$ yields that $\rho_k(t+1) \le \rho_k(t) + \frac{\delta}{l_k}(f_{k-1}(t) + u_k(t)) \le \rho_k(t) + \frac{\delta}{l_k}(w_k(\bar{\rho}_k - \rho_k(t)) + u_k(t))$. Given that $\rho_k(t) \le \bar{\rho}_k$, the sufficient condition to ensure that $\rho_k(t+1) \le \bar{\rho}_k$ is $u_k(t) \le (\frac{l_k}{\delta} - w_k)(\bar{\rho}_k - \rho_k(t))$. Similarly, given $0 \le q_k(t) \le \bar{q}_k$, the equivalent condition to ensure $0 \le q_k(t+1) \le \bar{q}_k$ is $\frac{1}{\delta}(q_k(t) - \bar{q}_k) + r_k(t) \le u_k \le \frac{1}{\delta} q_k(t) + r_k(t)$. Furthermore, considering $u_k(t) \le \bar{u}_k$, the sufficient condition to ensure the controllability of the freeway at time $t+1$ is $\frac{1}{\delta}(q_k(t) - \bar{q}_k) + r_k(t) \le (\frac{l_k}{\delta} - w_k)(\bar{\rho}_k - \rho_k(t))$ and $\frac{1}{\delta}(q_k(t) - \bar{q}_k) + r_k(t) \le \bar{u}_k$, which is equivalent to (4). ■

Remark 2.2: The sufficient condition (4) implies that the external demand of cell k cannot exceed the total remaining free space of this cell, which includes the free space available in the on-ramp and the minimum of the upper bound on u_k and the free space available in the mainline part. The underlying understanding is that if too many vehicles want to enter cell k, this cell will become uncontrollable. In this situation, the queue length of on-ramp is $\bar{q}_k$ and the density of the cell is the jam density $\bar{\rho}_k$.

Remark 2.3: In [11], the external demand is assumed to be bounded $0 \le r_k(t) \le \bar{u}_k$ for $\forall k \in \mathscr{C}$ and $t \in \mathbb{N}$. This assumption will simplify the sufficient condition (4) to be $r_k(t) \le \frac{1}{\delta}(\bar{q}_k - q_k(t)) + (\frac{l_k}{\delta} - w_k)(\bar{\rho}_k - \rho_k(t)),\ \forall k \in \mathscr{C}$.

Now let us unify the constraints (3) in a compact form. If the condition (4) is satisfied at time t, then the set containing the constraints (3) at time $t+1$ is nonempty. Given the variables $\rho_k(t), f_k(t), q_k(t), f_{k-1}(t)$ and $r_k(t)$, the constraints in (3) at time $t+1$ can be written as the following form by using the dynamics (1a)-(1b):

$$u_k(t) \le \frac{l_k}{\delta}(\bar{\rho}_k - \rho_k(t)) + \frac{1}{\bar{\beta}_k} f_k(t) - f_{k-1}(t),$$

$$\frac{1}{\delta}(q_k(t) - \bar{q}_k) + r_k(t) \le u_k(t) \le \frac{1}{\delta} q_k(t) + r_k(t).$$

We rewrite them in a compact form

$$u_k^1(t) \le u_k(t) \le u_k^2(t), \tag{6}$$

where

$$u_k^1(t) = \max\{0, \frac{1}{\delta}(q_k(t) - \bar{q}_k) + r_k(t)\}, \tag{7a}$$

$$u_k^2(t) = \min\{\bar{u}_k, \frac{l_k}{\delta}(\bar{\rho}_k - \rho_k(t)) + \frac{1}{\bar{\beta}_k} f_k(t) - f_{k-1}(t), \frac{1}{\delta} q_k(t) + r_k(t)\}. \tag{7b}$$

Observe that the constraints on the states (ρ_k, f_k, q_k) and the control input (u_k) are transformed into an equivalent constraint only in terms of the control input u_k.

Remark 2.4: The lower bound $u_k^1(t)$ is to guarantee that $u_k(t) \ge 0$ and the queue length $q_k(t+1) \le \bar{q}_k$. The upper bound $u_k^2(t)$ is to ensure that the input $u_k(t) \le \bar{u}_k$, the density $\rho_k(t+1) \le \bar{\rho}_k$, and the queue length $q_k(t+1) \ge 0$.

III. DISTRIBUTED OPTIMIZATION PROBLEM

A. Problem Formulation

In this section, we will formulate a distributed optimization problem which aims to tradeoff the average flow speed and the queue length $q_k(t)$ for each cell. As in [12], the average flow speed ξ_k of the traffic flow of cell k at time t can be derived by

$$\xi_k(t) = \frac{f_k(t)}{\rho_k(t)}, \text{ for } \rho_k(t) > 0. \tag{8}$$

Remark 3.1: If the flow of cell k is $f_k(t) = d_k(\rho_k(t))$, the average flow speed $\xi_k(t)$ is $\bar{\beta}_k v_k$. Otherwise, it is less than $\bar{\beta}_k v_k$. By incorporating the average flow speed into the optimization objective, we aim at controlling the cell k to be free as much as possible.

Without loss of generality, we assume that the traffic density $\rho_k(t) > 0$ for $\forall k \in \mathscr{C}$ and for $\forall t \in \mathbb{N}$. Assuming that the freeway is controllable at time t and the external demand $r_k(t)$ satisfies the condition (4), the optimization problem $\mathscr{P}_k(t)$ of cell k at time t is formulated by

$$\max_{u_k(t)} \ J_k(t+1) = \xi_k(t+1) - \lambda_k q_k(t+1) \tag{9a}$$

$$\text{s.t. } \rho_k(t+1) = \rho_k(t) + \frac{\delta}{l_k}(f_{k-1}(t) + u_k(t) - \frac{1}{\bar{\beta}_k} f_k(t)), \tag{9b}$$

$$q_k(t+1) = q_k(t) + \delta(r_k(t) - u_k(t)), \tag{9c}$$

$$f_k(t) = \begin{cases} \min\{d_k(\rho_k(t)), F_k, s_{k+1}(\rho_{k+1}(t))\}, & k \ne n, \\ \min\{d_n(\rho_n(t)), F_n\}, & k = n, \end{cases} \tag{9d}$$

$$u_k^1(t) \le u_k(t) \le u_k^2(t), \tag{9e}$$

$$\rho_k(t), q_k(t), r_k(t) \text{ given}, \tag{9f}$$

where $u_k^1(t)$ and $u_k^2(t)$ are respectively defined in (7a)-(7b); $\lambda_k \ge 0$ is a given tuning scalar to balance the average flow speed and the queue length for each cell.

Remark 3.2: In the optimization problem above, we assume that each cell is equipped with one on- and one off-ramp. Although the practical application is not always this case, this formulation contains all the possibilities. In practice, we can set $\beta_k = 0$ for the cell k which has no off-ramp. Similarly, we can select $\bar{q}_k = \bar{u}_k = r_k(t) = 0$, $\forall t \in \mathbb{N}$, for cell k which has no on-ramp. Notice that there is no ramp metering for cell without on-ramp, resulting in no optimization for this cell.

Remark 3.3: The typical optimization objective for the freeway in the works [5], [11] is to minimize the total travel time (TTT) and the total waiting time (TWT) over a finite horizon T: $TTT = \delta \sum_{t=0}^{T} \sum_{k=0}^{n} l_k \rho_k(t)$ and $TWT = \delta \sum_{t=0}^{T} \sum_{k=0}^{n} q_k(t)$. Actually, the essence of our objective function coincides with the above items. Particularly, the average flow speed corresponds to the travel time.

B. Backward Algorithm

It is well-known that the cells of the CTM are dynamically coupled in the cascaded way due to the constraint (9d). The solution to $\mathscr{P}_k(t)$ depends on the ramp metering $u_{k+1}(t)$ of the next cell $k+1$. Thus, all the cells cannot implement the optimization at the same time or synchronously. Notice that the flow of the last cell only relies on its own demand and capacity. In order to solve the optimization problem $\mathscr{P}_k(t)$ for $\forall k \in \mathscr{C}$ in the distributed fashion, we propose a backward algorithm, as stated in Algorithm 1. The algorithm is implemented in a backward sequence, i.e. from the last cell to the first cell, within one sampling period δ. In the next subsection, we will analyze the complexity of the local optimization problem and argue that the backward algorithm is applicable in some situations.

Notice that Algorithm 1 is not a traditional iterative algorithm, which requires the multiple iterations among the cells within δ. Comparatively, Algorithm 1 only requires one communication for each cell within δ and the optimization problem $\mathscr{P}_k(t)$ of cell k only involves the local information.

Algorithm 1 Backward Algorithm

1: Initialize $k = n$.
2: Solve the optimization problem $\mathscr{P}_k(t)$. And obtain the the solution $u_k^*(t)$ and the corresponding $\rho_k^*(t+1)$. If $k = 0$, stop.
3: Transmit $s(\rho_k^*(t+1))$to cell $k-1$.
4: Set $k = k-1$.
5: Go to step 2.

Now we will discuss the property of the solution by the backward algorithm from the perspective of the game theory. Let $U_k(t) = [u_k^1(t), u_k^2(t)]$. For the fixed time t, each cell is a player, $J_k(t+1)$ in (9a) is the utility function and $u_k(t)$ is the action variable in the corresponding action set $U_k(t)$. Considering the dynamical couplings between the adjacent cells, we denote by $B_k(u_{k+1}(t))$ the best-response correspondence of cell k at time t:

$$B_k(u_{k+1}(t)) = \{u_k(t) \mid u_k(t) = \arg\max_{u_k(t)\in U_k(t)} J_k(t+1)\}.$$

Theorem 3.1: The solution by Algorithm 1 is a Nash equilibrium for the freeway at time t.

Proof: The solution $u_n^*(t)$ is obviously optimal since it is independent of the other cells. The solutions of other cells by Algorithm 1 satisfy $u_k^*(t) \in B_k(u_{k+1}^*(t))$ for $0 \le k < n$. Thus, $u^*(t)$ is a Nash equilibrium at time t (see Definition 1.10 in [13]). ∎

C. Analysis of $\mathscr{P}_k(t)$ Given $u_{k+1}(t)$

Given the ramp metering $u_{k+1}(t)$, the supply $s_{k+1}(\rho_{k+1}(t+1))$ can be determined. Then the optimization problem $\mathscr{P}_k(t)$ only involves the decision variable $u_k(t)$. For the ease of the notation, we define $\rho_k(t+1) = \rho_k(t) + \frac{\delta}{l_k}(f_{k-1}(t) + u_k(t) - \frac{1}{\beta_k} f_k(t)) \triangleq G_k(u_k(t))$. Denote by $u_k^3(t)$ the solution of the following equation

$$d_k(G_k(u_k^3(t))) = \min\{F_k, s_{k+1}(\rho_{k+1}(t+1))\}. \tag{10}$$

Notice that when $u_k(t) = u_k^3(t)$, the flow is $f_k(t) = d_k(G_k(u_k^3(t)))$. The introduction of $u_k^3(t)$ aims to justify the possible values of the flow under the control of $u_k(t) \in U_k(t)$. Recall (6) and (7a)-(7b). Then, the optimization problem $\mathscr{P}_k(t)$ of cell k can be discussed in the following cases.

- Case 1: If $u_k^2(t) \le u_k^3(t)$, the optimization problem $\mathscr{P}_k(t)$ becomes

$$\max_{u_k(t)} \; J_k^1(t+1) = \lambda_k \delta u_k(t) + H_k^1(t)$$
$$\text{s.t. } u_k^1(t) \le u_k(t) \le u_k^2(t),$$

  where $H_k^1(t) = \bar{\beta}_k v_k - \lambda_k(q_k(t) + \delta r_k(t))$ is independent of the decision variable $u_k(t)$. Obviously, the optimal solution is $u_k(t) = u_k^2(t)$.

- Case 2: If $u_k^3(t) \le u_k^1(t)$, the optimization problem $\mathscr{P}_k(t)$ becomes

$$\max_{u_k(t)} \; J_k^2(t+1) = \frac{A_k(t)}{B_k(t) + \alpha_k u_k(t)} + \lambda_k \delta u_k(t) + H_k^2(t)$$
$$\text{s.t. } u_k^1(t) \le u_k(t) \le u_k^2(t),$$

  where $A_k(t) = \min\{F_k, s_{k+1}(\rho_{k+1}(t+1))\}$, $B_k(t) = \rho_k(t) + \frac{\delta}{l_k}(f_{k-1}(t) - \frac{1}{\beta_k} f_k(t))$, $\alpha_k = \frac{\delta}{l_k}$, and $H_k^2(t) = -\lambda_k(q_k(t) + \delta r_k(t))$. All of these variables are independent of the decision variable $u_k(t)$. This optimization can be easily solved by checking the monotonicity of the $J_k^2(t+1)$ with respect to $u_k(t)$.

- Case 3: If $u_k^1(t) \le u_k^3(t) \le u_k^2(t)$, the optimization problem $\mathscr{P}_k(t)$ is divided into two subproblems:

$$\begin{cases} \max\limits_{u_k(t)} \; J_k^1(t+1) \\ \text{s.t. } u_k^1(t) \le u_k(t) \le u_k^3(t), \end{cases} \qquad \begin{cases} \max\limits_{u_k(t)} \; J_k^2(t+1) \\ \text{s.t. } u_k^3(t) \le u_k(t) \le u_k^2(t). \end{cases}$$

  Then compare the two solutions obtained and determine the optimal solution.

Remark 3.4: The above analysis shows that given $u_{k+1}(t)$, the optimization problem $\mathscr{P}_k(t)$ can be solved efficiently. Thus, the backward algorithm is applicable for the case in which the sample period satisfies the communication time and the computational time.

IV. Maximizing the Average Flow Speed

If $\lambda_k = 0$, the objective of the optimization problem $\mathscr{P}_k(t)$ is only to maximize the average flow speed $\xi_k(t)$. The simplified optimization problem, denoted by $\mathscr{P}_k^s(t)$, is written as

$$\max_{u_k(t)} \; J_k(t+1) = \xi_k(t+1)$$
$$\text{s.t. } (9b), (9d), \text{ and } (9e).$$

As mentioned before, the optimization problem $\mathscr{P}_k(t)$ for $\forall k \in \mathscr{C}$ can not be solved synchronously. If we simplify it to $\mathscr{P}_k^s(t)$, it is shown that an explicit optimal solution to $\mathscr{P}_k^s(t)$ can be obtained synchronously for $\forall k \in \mathscr{C}$. Further, this solution is also globally optimal.

Before Theorem 4.1, we introduce some notations. Let

$$\rho_k^1(t+1) = G_k(u_k^1(t)) \text{ and } \rho_k^2(t+1) = G_k(u_k^2(t)).$$

And denote by $u_k^s(t)$ the solution of the following equation

$$d_k(G_k(u_k^s(t))) = \min\{F_k, s_{k+1}(\rho_{k+1}^1(t+1))\}, \tag{11}$$

which is similar to (10).

Theorem 4.1: The set of the optimal solution to the simplified optimization problem $\mathscr{P}_k^s(t)$, denoted by $U_k^*(t)$, is given

- Case 1: If $u_k^2(t) \le u_k^s(t)$, then $U_k^*(t) = [u_k^1(t), u_k^2(t)]$;
- Case 2: If $u_k^1(t) \le u_k^s(t) < u_k^2(t)$, then $U_k^*(t) = [u^1(k), u^s(k)]$;
- Case 3: If $u_k^s(t) < u_k^1(t)$, then $U_k^*(t) = \{u_k^1(t)\}$.

Proof: It follows that

$$\begin{aligned}\xi_k(t+1) &= \frac{\min\{d_k(\rho_k(t+1)),F_k,s_{k+1}(\rho_{k+1}(t+1))\}}{\rho_k(t+1)}\\ &\le \frac{\min\{d_k(\rho_k(t+1)),F_k,\max\{s_{k+1}(\rho_{k+1}(t+1))\}\}}{\rho_k(t+1)}\\ &= \frac{\min\{d_k(\rho_k(t+1)),F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}}{\rho_k(t+1)}. \quad (12)\end{aligned}$$

Since the average flow speed $\xi_k(t)$ for $\forall k\in\mathscr{C}$ is nonincreasing with respect to $\rho_k(t)$, $\rho_{k+1}(t)$, $u_k(t)$, and $u_{k+1}(t)$ at each time t, the optimal solution to the optimization problem $\mathscr{P}^s_k(t)$ can be analyzed:

- Case 1: If $d_k(G_k(u_k(t)))\le\min\{F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}$ for $u^1_k(t)\le u_k(t)\le u^2_k(t)$, i.e. $u^2_k(t)\le u^s_k(t)$, the optimal solution set to the optimization problem $\mathscr{P}^s_k(t)$ is $U^*_k(t)=[u^1_k(t),u^2_k(t)]$.
- Case 2: If $d_k(G_k(u_k(t)))\le\min\{F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}$ for $u^1_k(t)\le u_k(t)\le u^s_k(t)$ and $d_k(G_k(u_k(t)))>\min\{F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}$ for for $u^s_k(t)<u_k(t)\le u^2_k(t)$, i.e. $u^1_k(t)\le u^s_k(t)<u^2_k(t)$, the optimal solution set to the optimization problem $\mathscr{P}^s_k(t)$ is $U^*_k(t)=[u^1_k(t),u^s_k(t)]$.
- Case 3: If $d_k(G_k(u_k(t)))>\min\{F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}$ for $u^1_k(t)\le u_k(t)\le u^2_k(t)$, i.e. $u^s_k(t)<u^1_k(t)$, the optimal solution set to the optimization problem $\mathscr{P}^s_k(t)$ is $U^*_k(t)=\{u^1_k(t)\}$.

The proof is completed. ∎

For the ease of the notation, we introduce $\hat{\rho}_{k+1}$ such that $s_{k+1}(\hat{\rho}_{k+1})=F_k$, i.e. $\hat{\rho}_{k+1}=\bar{\rho}_{k+1}-\frac{F_k}{w_{k+1}}$. We can verify that $\hat{\rho}_{k+1}\ge\rho^c_{k+1}$ by the monotonicity of s_{k+1} and the definition of F_k. Hence, it follows that $F_{k+1}\le d_{k+1}(\hat{\rho}_{k+1})$.

Theorem 4.2: The solution set $U^*(t)=U^*_0(t)\times\ldots\times U^*_n(t)$ is globally optimal, i.e. any element in the solution set $U^*(t)$ is the optimal solution to the centralized problem:

$$\max_{u_k(t),\forall k\in\mathscr{C}} J(t+1)=\sum_{k=0}^{n}\xi_k(t+1)$$
$$\text{s.t. } (9b),(9d), \text{ and } (9e),\ \forall k\in\mathscr{C}.$$

Proof: To verify that the solution set $U^*(t)=U^*_0(t)\times\ldots\times U^*_n(t)$ is globally optimal, it is sufficient to show that the upper bound (12) on the average flow speed of each cell can be achieved. It is equivalent to verify that the solution set to $\mathscr{P}^s_{k+1}(t)$ can achieve $\min\{F_k,s_{k+1}(G_{k+1}(u_{k+1}(t)))\}=\min\{F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}$ for $u_{k+1}(t)\in U^*_{k+1}(t)$. In order to show this, we need to discuss the following cases:

- Case 1: If $u^2_{k+1}(t)\le u^s_{k+1}(t)$, we have $\rho^1_{k+1}(t)\le\rho^2_{k+1}(t)\le\rho^c_{k+1}\le\hat{\rho}_{k+1}$. Thus, $\min\{F_k, s_{k+1}(G_{k+1}(u_{k+1}(t)))\}=F_k$ for $u_{k+1}(t)\in U^*_{k+1}(t)$.
- Case 2: If $u^1_{k+1}(t)\le u^s_{k+1}(t)<u^2_{k+1}(t)$, we have that $\rho^1_{k+1}(t)\le\rho^c_{k+1}\le\hat{\rho}_{k+1}$. Thus, $\min\{F_k, s_{k+1}(G_{k+1}(u_{k+1}(t)))\}=F_k$ for $u_{k+1}(t)\in U^*_{k+1}(t)$.
- Case 3: If $u^s_{k+1}(t)<u^1_{k+1}(t)$, the optimal solution set of $U^*_{k+1}(t)$ only has one element $u^1_{k+1}(t)$, which implies that $\min\{F_k,s_{k+1}(G_{k+1}(u_{k+1}(t)))\}=\min\{F_k,s_{k+1}(\rho^1_{k+1}(t+1))\}$ for $u_{k+1}(t)\in U^*_{k+1}(t)$.

The proof is completed. ∎

TABLE II
PARAMETER VALUES

cell k	$\bar{\rho}_k$	β_k	v_k	w_k	l_k	F_k	$\bar{u}_k$	$\bar{q}_k$
0	250	0.15	90	21	0.6	4119.2	2200	50
1	250	0.10	90	28	0.8	4682.8	1800	50
2	250	0.17	90	25	0.8	4256.8	1800	50
3	250	0	90	21	0.8	4100.0	1800	50

Remark 4.1: For the last cell, the definitions of u^3_n and u^s_n only depends on F_n. The corresponding proof in Theorem 4.1 for the last cell can be further simplified.

Remark 4.2: The distributed, non-predictive controller in [11] belongs to the solution set $U^*(t)$ of this paper. This can be interpreted as below: the controller in [11] aims to move the local density to the critical density as fast as possible while the controller in our paper aims to move the state of each cell to be free as far as possible (including the critical density). Notice that the controller in [11] is the maximum point of the solution set $U^*_k(t)$ for $\forall k\in\mathscr{C}$.

V. NUMERICAL EXAMPLE

We consider a simple freeway with 4 cells. The parameter values are listed in Table II. The sampling period δ is 15 seconds and the simulation time is 60 minutes. The external demands are $r_0(t)=1500+500\times\text{rand}(0,1)$, $r_1(t)=1000+500\times\text{rand}(0,1)$, $r_2(t)=1000+500\times\text{rand}(0,1)$, $r_3(t)=800+800\times\text{rand}(0,1)$, where $\text{rand}(0,1)$ denotes a random value in $(0,1)$. The initial density and the queue length of each cell are $\rho_0(0)=\rho_2(0)=100$, $\rho_1(0)=\rho_3(0)=50$, $q_0(0)=q_1(0)=q_2(0)=q_3(0)=5$, respectively. The tuning parameter λ_k in (9a) is set to be $\lambda_k=0.48$ for all k. We compare the balanced controller which aims to balance the average flow speed and the queue length proposed in Section III and the maximum speed controller which only aims to maximize the average flow speed in Section IV.

The evolutions of density, flow, queue length, and average flow speed under two different controllers are respectively shown in Fig. 4 - Fig. 7. The difference between the balanced controller and the maximum speed controller is clearly seen. The average flow speed is maximized by the maximum speed controller in a greedy way, i.e. by sacrificing the queue length, while the balanced controller can tradeoff the average flow speed and the queue length. To quantify the control performance, we use two evaluation indexes $TWT=\delta\sum_{t=0}^{T}\sum_{k=0}^{n}q_k(t)$ and $DIS=\delta\sum_{t=0}^{T}\sum_{k=0}^{n}\xi_k(t)$ where TWT is the total waiting time and DIS is the total travelling distance. The saving of the total waiting time by the balanced controller is

$$\frac{TWT_{Maxspeed}-TWT_{balanced}}{TWT_{Maxspeed}}=\frac{106.40-37.92}{106.40}=64.36\%,$$

while the reduction of the total travelling distance is

$$\frac{DIS_{balanced}-DIS_{Maxspeed}}{DIS_{balanced}}=\frac{229.93-183.19}{229.93}=20.33\%.$$

Furthermore, if we increase the tuning parameter λ_k from 0.48 to 2.4, the saving of the total waiting time by the balanced controller will be 85.64%, which implies almost no waiting time by the balanced controller. The above numerical results are consistent with the essences of the two controllers and the control algorithms of this paper can achieve a distributed ramp metering with guaranteed performance.

VI. CONCLUSION

The distributed ramp metering problem of the CTM has been studied in this paper by optimizing the weighted average flow speed of each cell and the queue length of each on-ramp. We have proposed the feasible external demand to ensure the controllability of the freeway. Under the assumption of controllability, we have formulated an optimization problem which achieves a trade-off between the maximum average flow speed and the minimum queue length. A backward algorithm has been proposed for this problem and the corresponding solution is a Nash equilibrium. We have further simplified the optimization problem to only maximize the average flow speed and shown that the resulting explicit distributed controller is globally optimal.

On-going work is the distributed control of the CTM subject to the uncertain FD. The challenge of this problem lies in the synthesis of the dynamics of the traffic flow and the uncertain model of the FD.

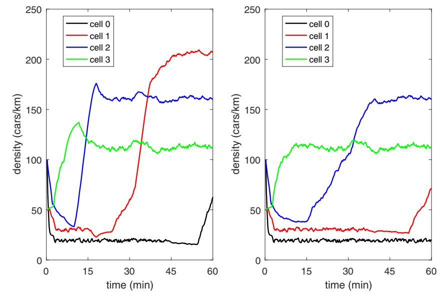

Fig. 4. Evolutions of density under the balanced controller (left) and the maximum speed controller (right)

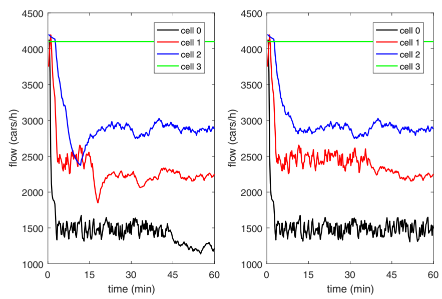

Fig. 5. Evolutions of flow under the balanced controller (left) and the maximum speed controller (right)

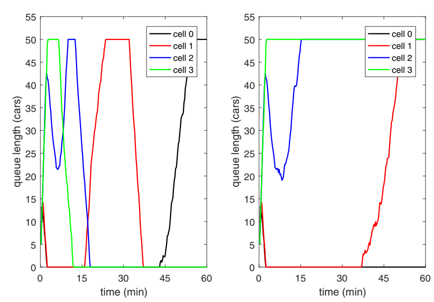

Fig. 6. Evolutions of queue under the balanced controller (left) and the maximum speed controller (right)

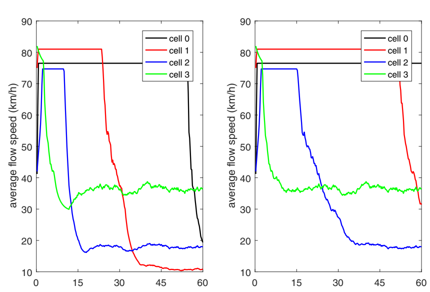

Fig. 7. Evolutions of average flow speed under the balanced controller (left) and the maximum speed controller (right)

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