# SOME PROPERTIES OF SWITCHED SYSTEMS

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Abstract: *Various aspects of dynamical systems with switches are studied. The two concepts "higher-order sliding" and "fast sliding" are defined and a result for stability of second order sliding is given. The linear case is described in particular detail. Several examples illustrate the results.* 

Keywords: Hybrid modes. Sliding mode control. Nonlinear control systems. Discontinuities.

## 1. INTRODUCTION

An important research issue is to give advice on how switched systems should be efficiently simulated, because these systems are often extremely hard to investigate by analytical methods. Therefore we need further understanding about the dynamics of switched systems. Several phenomena, which are not present in smooth control systems, may occur in systems with switches. They include sliding modes and arbitrarily fast switchings. Many of the proposed hybrid system models in the literature have restrictions to prevent infinitely fast switching between the discrete modes. If no such restrictions are imposed on the control system design, it is quite common to get infinitely fast mode changes. The resulting dynamics are then not always well defined, which may be an indication that the modeling must be refined. Fast switchings are always difficult to deal with for simulation tools.

In this paper some fundamental properties of switched control systems are pointed out. For a background see Filippov (1988); Utkin (1992) or Anosov (1959). Higher-order sliding modes have only to some extent been studied. Switching control laws for tracking is studied in

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Fridman and Levant (1996), see also Johansson (1997); Malmborg (1998).

Consider the non-smooth dynamical system

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a piecewise continuous function. The solution to this equation is interpreted as follows, see Filippov (1988):

*Definition 1.* (Filippov solution). An absolutely continuous function $x(t)$ is called a solution of (1) on $[t_0, t_1]$ if for almost all $t \in [t_0, t_1]$

$$\dot{x} \in \bigcap_{\delta>0} \bigcap_{\mu N=0} \overline{co}\, f\big(B(x,\delta) \setminus N\big),$$

where $\bigcap_{\mu N=0}$ denotes intersection over all sets $N$ of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(x,\delta)$ is a ball with center in $x \in \mathbf{R}^n$ and radius $\delta$.

Consider the set $S$ of discontinuity points of $f$. A solution $x(t)$ is called a sliding mode on the interval $[t_0, t_1]$, if $x(t) \in S$ for all $t \in [t_0, t_1]$. It was shown in Johansson et al. (1999) that a sliding mode can for instance be part of a stable limit cycle. Switched systems can have solutions close to a sliding mode and have very fast switching. As is shown in Section 3, the fast switchings can be stable or unstable in the sense that the solution approaches the sliding mode or not.

The outline of the paper is as follows: Section 2 illustrates the problem of non uniqueness of sliding solutions. In Section 3 we present necessary conditions for so called stable sliding of order two. Section 4 illustrates that such sliding can be part of the limit cycle of a linear system under relay feedback.

## 2. SIMULATION OF SWITCHED SYSTEMS

Switched systems are often extremely hard to analyze with analytical methods. Therefore efficient simulation methods are essential. However, several fundamental properties for solutions of ordinary differential equations are not valid if the vector field has jumps. This makes it more challenging to derive robust simulation tools for switched systems. The following is a simple example that illustrates that switched systems may have non-unique solutions.

*Example 1.* Consider the switched system

$$\dot{x}_1 = -\operatorname{sgn}(x_1 x_2),$$
$$\dot{x}_2 = -1.$$

The upper plot in Figure 1 shows its vector field. Filippov's definition of solution for the system does not give a unique solution for initial states with $|x_1(0)| \le x_2(0)$.

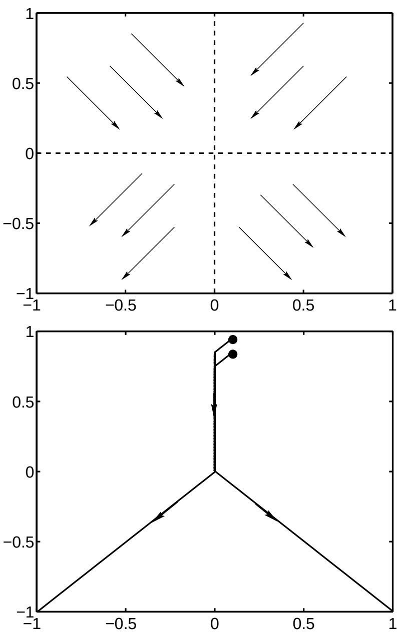

Fig. 1. The upper plot shows the vector field for system in Example 1. The lower plot shows two trajectories simulated in Simulink.

The lower plot shows two trajectories of the system simulated with default routines in Simulink. The initial points are close and marked with dots. The trajectories of the simulation are close to the sliding mode on the segment with $x_2 > 0$. At the origin the simulation routine happens to choose the vector field in the third quadrant for one of the trajectories, while for the other trajectory it chooses the vector field in the fourth quadrant. The simulation result is, of course, due to that the signum function is approximated in Simulink because of limited accuracy in the numerical routines.

The example illustrates the importance for simulation packages to detect sliding modes and ambiguities of solutions. Most simulation programs today give a chattering solution instead of an exact sliding mode, due to limitations in numerical accuracy. A small step size gives an accurate solution close to the sliding mode. However, small step sizes also mean long simulation times. A solution to this problem is to introduce a state in the simulation routine that represents the sliding mode and a mechanism that detects when the conditions for an attractive sliding mode is fulfilled, see Malmborg (1998) and Mattsson (1996) for a discussion

For the simple system in Example 1 it is easy to predict the problem that will arise when the solution reaches the origin. However, in many applications it is far from trivial to detect such ambiguity points and they may very well arise from small errors in complicated models. It is possible in a simulation program to incorporate facilities to detect points with non-unique solutions in a similar way to the detection of sliding modes previously discussed.

## 3. FAST SWITCHING

In the following we only consider systems with one switch

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \\ u &= \operatorname{sgn} y, \end{aligned} \tag{2}$$

where $f : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ and $h : \mathbf{R}^n \mapsto \mathbf{R}$ are smooth functions and the differential equation is interpreted in the sense of Definition 1. We define sliding modes of order $r$ for these systems inspired by Fridman and Levant (1996).

*Definition 2.* (Higher-order sliding). Consider a point $\bar{x} \in \mathbf{R}^n$ with $h(\bar{x}) = 0$. Assume $L_f^k h(x)$ for $k = 1, \dots, r-1$ are smooth functions in a surrounding of $\bar{x}$ and introduce

$$H_r(x) = \begin{pmatrix} h(x) & L_f h(x) & \dots & L_f^{r-1} h(x) \end{pmatrix}^T .$$

Assume that $dH_r(\bar{x})/dx$ has full row rank. The $r$th-order sliding set is defined in a surrounding of $\bar{x}$ as the smooth set

$$S_r := \left\{ x : \; H_r(x) = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}^T \right\}.$$

A solution $x(t)$ of (2) is an $r$th-order sliding mode on $[t_0, t_1]$ if $x(t) \in S_r$ for all $t \in [t_0, t_1]$.

A first-order sliding mode is attractive if the vector fields on each side of the switching surface $h(x) = 0$ are pointing towards the surface. Instead of an exact higher-order sliding mode, it is common that a large number of fast switchings occur. For example, this may happen if the vector field is tangential to the switching surface.

*Definition 3.* (Fast switching). A system has fast switching if for every $\varepsilon > 0$ there exist two points $x_0, x_1 \in \mathbf{R}^n$ with $h(x_0) = h(x_1) = 0$, such that $x(t)$ is a solution of (2) for $t \in [t_0, t_1]$ with $x(t_0) = x_0$, $x(t_0 + \varepsilon) = x_1$, and $h\big(x(t)\big) \neq 0$ for $t \in (t_0, t_0 + \varepsilon)$.

A system has multiple fast switching if for every $\varepsilon > 0$ there exist three points $x_0, x_1, x_2 \in \mathbf{R}^n$ and $\varepsilon' \in (0, \varepsilon)$ with $h(x_0) = h(x_1) = h(x_2) = 0$, such that $x(t)$ is a solution of (2) for $t \in [t_0, t_1]$ with $x(t_0) = x_0$, $x(t_0 + \varepsilon') = x_1$, and $x(t_0 + \varepsilon) = x_2$, where $h\big(x(t)\big) \neq 0$ for $t \in (t_0, t_0 + \varepsilon') \cup (t_0 + \varepsilon', t_0 + \varepsilon)$.

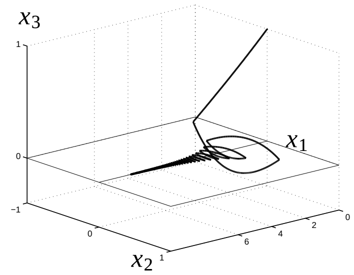

Fig. 2. Fast switching close to second-order sliding set.

In this section we study fast switching close to a second-order sliding set. An example of such a system is given by

*Example 2.*

$$\begin{aligned} \dot{x}_1 &= \cos u \\ \dot{x}_2 &= -\sin u \\ \dot{x}_3 &= -x_3 + x_2 \\ y &= x_3 \\ u &= \operatorname{sgn} y \end{aligned}$$

see also Figure 2. The second order sliding set is given by $x_1 = x_2 = 0$.

To follow the notation in Filippov (1988) we transform the system so that the second order sliding set is given by $y = x = 0$ and write the dynamics as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{cases} \begin{pmatrix} P^+(x,y,z) \\ Q^+(x,y,z) \\ R^+(x,y,z) \end{pmatrix}, & y > 0, \\ \begin{pmatrix} P^-(x,y,z) \\ Q^-(x,y,z) \\ R^-(x,y,z) \end{pmatrix}, & y < 0, \end{cases}$$

where $\big(x(t), y(t), z(t)\big) : \mathbf{R} \mapsto \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-2}$ and $P^\pm, Q^\pm, R^\pm$ are smooth functions.

Assume that $Q^\pm(0,0;z) = 0, \forall z$, which is the case when there can be non-transversal sliding along the $x = y = 0$ subspace. With the sign conditions

$$xQ^+(x,0;z) < 0, \qquad xQ^-(x,0;z) < 0, \quad x \neq 0, \tag{3}$$

there is no sliding in the plane $y = 0$ unless possibly along the subspace $x = y = 0$. Further sign conditions assumed are

$$P^+(0,0;z) > 0, \qquad P^-(0,0;z) < 0. \tag{4}$$

From Equation 3 follows that $Q_x^\pm(0,0;z) \leq 0$. If this condition is sharpened to $Q_x^\pm(0,0;z) < 0$ the following theorem can be proved.

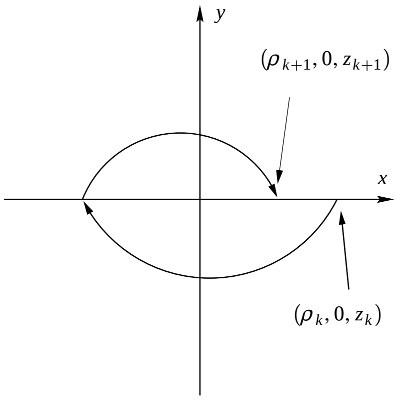


Fig. 3. The $y - x$-plane. The $z$-directions, $z \in \mathbf{R}^{n-2}$, are omitted for simplicity. The intersections with the $y = 0$ plane for $x > 0$ are denoted $\rho_k$.

*Theorem 1.* (Second order stable sliding). For the system described above, the subspace $x = y = 0$ is locally stable around the point $(0, 0; z)$ if $a_2(z) = A^+ - A^- < 0$, where

$$A^\pm = \left( \frac{P_x + Q_y}{P} - \frac{Q_{xx}}{2Q_x} + \frac{(Q_x P_z - P Q_{xz}) R}{Q_x P^2} \right)^\pm ,$$

where all functions are evaluated at $(0, \pm 0; z)$. The dynamics on the set $x = y = 0$ defined as in Definition 1 satisfies

$$\dot{z} = \alpha^+ R^+(0, 0; z) + \alpha^- R^-(0, 0; z), \qquad (5)$$

where $\alpha^+$ and $\alpha^-$ are uniquely defined by

$$\begin{aligned} \alpha^+ + \alpha^- &= 1 \\ \alpha^+ P^+ &= \alpha^- P^-. \end{aligned} \qquad (6)$$

Thus the convex definition is the natural solution if the dynamics on the subspace $x = y = 0$ should be the limit of dynamics just off it The series of intersections with the $y = 0$ plane for $x > 0$ is denoted $[0, \rho_k, z_k]$. The sequence $\rho_k$ is monotonously decreasing with

$$\rho_{k+1} = \rho_k + \frac{2}{3} a_2(z) \rho_k^2 + O(\rho_k^3). \qquad (7)$$

*Proof* The proof can be found in Malmborg (1998).

3.0.1. *Remark* In the two-dimensional case

$$A^\pm = \left( \frac{P_x + Q_y}{P} - \frac{Q_{xx}}{2Q_x} \right)^\pm .$$

This expression was derived on pages 234–238 in Filippov (1988).

*Linear systems*

We now consider switched systems with linear dynamics

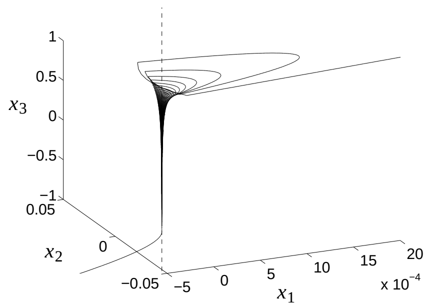


Fig. 4. Fast switching close to second-order sliding set.

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ u &= \operatorname{sgn} y. \end{aligned} \qquad (8)$$

It is well-known that this system has an attractive first-order sliding set if and only if $CB < 0$. It was shown in Johansson et al. (1999) that the system has multiple fast switching if and only if the first non-vanishing Markov parameter $CA^{r-1}B$ is negative. Systems with relative degree two may have an attractive second-order sliding set similar to the nonlinear case in previous section. This is illustrated in the following example.

*Example 3.* Consider the system

$$\dot{x} = \begin{pmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ -0.4 \\ 0.04 \end{pmatrix} u,$$

$$\begin{aligned} y &= -x_1, \\ u &= \operatorname{sgn} y. \end{aligned}$$

The linear dynamics have four poles in $-1$ and two zeros in 0.2.

It is possible to derive accurate estimates for fast switchings as the one shown in Example 3, see Johansson et al. (1997). Let the dynamics in (8) be given by

$$A = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ b_1 \\ \vdots \\ b_{n-2} \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Consider a solution of the closed-loop system with $x_1(0) = 0$, $x_2(t)$ small, and $|x_3(t)| < 1$ for $t \in [0, T]$ and $T > 0$. Then, $x_1$ satisfies

$$\frac{1}{|x_2(0)|} \max_{t \in [0,T]} |x_1(t)| \to 0,$$

as $|x_2(0)| \to 0$. The envelope of the peaks of $x_2$ is given by

$$x_2(t_k) = (-1)^k x_2(0) \exp\left[-(a_1 - b_1)t_k/3\right] \times \left(\frac{1 - x_3^2(t_k)}{1 - x_3^2(0)}\right)^{1/3} + \varepsilon_1(x_2(0); t_k), \quad (9)$$

where $\varepsilon_1(x_2(0); t_k)/x_2(0) \to 0$ as $x_2(0) \to 0$ for all $k$ with switch times $t_k \in [0, T]$. If $|x_3| \ll 1$, it follows from (9) that the amplitude of the oscillation in $x_2$ is decaying if $a_1 > b_1$. This local stability result for the second-order sliding set agrees with the condition given in Theorem 1, because in the linear case

$$A^+ - A^- = 2\frac{b_1 - a_1 + x_3^2(a_1 + b_1) - 2x_3x_4}{(x_3^2 - 1)^2}.$$

## 4. LIMIT CYCLES

The sliding modes and the fast switching discussed in previous section can be part of stable limit cycles. This was illustrated for linear systems with one switch in Johansson et al. (1999). Stability results for limit cycles with a first-order sliding mode as well as fast switchings close to a second-order sliding mode were derived in Johansson et al. (1997).

Next we show a nonlinear system with one switch that gives a limit cycle with six parts of fast switching every period.

*Example 4.* Consider the system

$$\dot{x} = \begin{pmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ -0.4 \\ 0.04 \end{pmatrix} u,$$
$$y = -\sin(11\pi x_1),$$
$$u = \operatorname{sgn} y.$$

The linear dynamics have four poles in $-1$ and two zeros in 0.2. Figure 5 shows the first two states of the system together with the relay output.

## 5. CONCLUSIONS

Fast switchings and sliding modes in switched control systems have been studied. The motivation for this was that these phenomena cause severe problems to simulation tools. Systems with one relay nonlinearity were studied. A stability result for second order fast switching has been given and a more detailed result for linear systems with relative degree two was also presented.

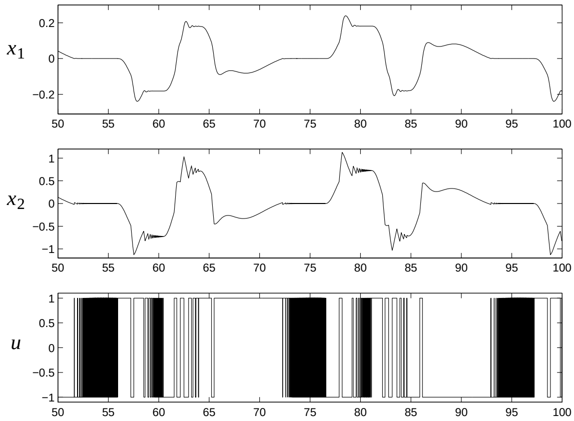


Fig. 5. Complicated limit cycle for system with one switch. Note the six parts of fast switchings that occur each limit cycle period.

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