Quantised Control in Distributed Embedded Systems*

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Problem Presentation Traditional control design is based on ideal assumptions concerning the amount, type and accuracy of the information flow that can be circulated across the controller. Unfortunately, real implementation platforms exhibit non-idealities that may substantially invalidate such assumptions. As a result, the system's closed-loop performance can be severely affected and sometimes stability itself is jeopardised. These problems show up with particular strength when multiple feedback loops share a limited pool of computation and communication resources. In this case the designer is confronted with the challenging task of choosing at the same time the control law and the optimal allocation policy for the shared resources (control algortihm/system architecture co-design). An intriguing general discussion for this class of problem can be found in [2]. Investigations in this field have been developed ever since in several directions. A first prong of research activities has focused on the problem of resource sharing, i.e., finding optimal allocation policies for shared computation and communication resources [10, 7, 3]. However, these papers do not explicitly cope with quantisation and bit rate constraints that play an important role in complex distributed systems. A remarkable thread of papers has focused on the problem of stabilisation under bit rate constraints [4, 12, 5, 9, 6, 8, 1]. The main concern in these works is to find encoding-decoding schemes that make for an optimal use of the channel, when the latter is used in a control loop. In this perspective, the authors generally synthesize quantisation schemes instrumental to this goal. Albeit interesting from a theoretical point of view, this approach is to be verified from the standpoint of technological feasibility. In [11] a different view is taken. The authors analyse the attainable control performance when quantisation is a fixed element of the problem. An evident motivation for this work is the analysis of control systems where actuation and/or sensing are e.g., binary, thresholded or quantised sensors, actuators or converters.

In this work, we make the same assumption as in [11]: control loops are operated by quantised actuators, which are regarded as given hardware components to build on the top of. For the sake of simplicity, we restrict to the case of uniform quantisers. Moreover, a limited bandwidth channel is shared between several independent feedback loops. Each loop is used to control a first-order linear and time-invariant plant whose dynamics is described by

$$\dot{x}^{(i)} = a^{(i)}x^{(i)} + u^{(i)} + w^{(i)},\; i = 1, \ldots, N\,, \tag{1}$$

where $u^{(i)} \in \mathcal{U}^{(i)} \subseteq \epsilon^{(i)}\mathbb{Z}$ is the control variable and $w^{(i)} \in \mathbb{R}$ is an exogenous bounded noise term. As shown in [4] , the classical notion of stability has evident shortcomings when quantisation is in place. For this reason, we aim at practical stability for each loop: i.e., the ability to attract each system into a specified region $\Omega^{(i)}$ and to make it evolve therein. More precisely we consider two types of goals for each loop:

1. Controlled invariance of $\Omega^{(i)}$: $\forall\, x^{(i)}(0) \in \Omega^{(i)}$ there exists a control function $u^{(i)}(x^{(i)}, t) \in \mathcal{U}^{(i)}$ such that for any noise function $w^{(i)}(t)\,,\; x^{(i)}(t) \in \Omega^{(i)}\; \forall\, t > 0$.
2. $(\Omega^{(i)}, \omega^{(i)})$-stability: $\forall\, x^{(i)}(0) \in \Omega^{(i)}$ and for all possible noise functions $w^{(i)}(t)$, there exist a control function $u^{(i)}(x^{(i)}, t) \in \mathcal{U}^{(i)}$ and a real number $\bar{t} > 0$ such that $\forall\, t > 0,\; x^{(i)}(t) \in \Omega^{(i)}$ and $\forall\, t \geq \bar{t},\; x^{(i)}(t) \in \omega^{(i)}$.

Each controller is assumed to have exact knowledge of the state $x^{(i)}$, but values for the control commands are transmitted over a shared channel with maximum bit rate R. This set-up corresponds to distributed sensors and actuators with the control algorithms being implemented in the sensor nodes. Thereby, the bit rates $r^{(i)}$ devoted to each feedback loop have to comply with the following inequality:

$$\sum_i r^{(i)} \leq R.$$

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We restrict our investigation to the case of command functions generated by periodic sampling, i.e., every $T^{(i)}$ time units a sample of $x^{(i)}$ is collected and a new control value $u^{(i)}$ is computed and held constant up to the next period. Notably, the presence of the noise terms makes the problem non-trivial even in the case of open-loop stable systems ($a^{(i)} \leq 0$). To the purpose of limiting the bandwidth used for the transmission, some of the levels provided by the quantiser can be left unused. Therefore a design parameter is the subset $\mathcal{U}^{(i)} \subset \epsilon^{(i)}\mathbb{Z}$ that is necessary to accomplish the design goals.

Presented Results As a preliminary issue, we show that for the considered class of systems and controllers the analysis of practical stability can be carried out on the discrete-time counterparts of the systems in Equation (1):

$$x_{k+1}^{(i)} = \Phi^{(i)} x_k^{(i)} + \Gamma^{(i)} u_k^{(i)} + w_k^{(i)}, \tag{2}$$

where $\Phi^{(i)} = e^{a^{(i)}T^{(i)}}$ and $\Gamma^{(i)} = (e^{a^{(i)}T^{(i)}} - 1)/a^{(i)}$ if $a^{(i)} \neq 0$ (the case $a^{(i)} = 0$ can be dealt with in a similar fashion).

The main problem we consider is concerned with $\big(I(\Delta^{(i)}), I(\delta^{(i)}\big)$-stability on each loop, where $I(\Delta^{(i)}) = [-\Delta^{(i)}/2, \Delta^{(i)}/2]$, $I(\delta^{(i)}) = [-\delta^{(i)}/2, \delta^{(i)}/2]$, $\Delta^{(i)} \geq \delta^{(i)} > 0$. Moreover, we aim at optimising performance in terms of attainable $(\delta^{(i)}, \Delta^{(i)})$ pairs over the possible choices of design parameters for the different controllers. The problem can be formalised as follows:

$$\begin{array}{ll} \min_{\sigma^{(1)},\sigma^{(2)},\ldots,\sigma^{(N)}} & f(\delta^{(1)}, .., \delta^{(N)}, \Delta^{(1)}, \ldots, \Delta^{(N)}) \\ \text{subj. to} & \big(\delta^{(i)} \leq \overline{\delta^{(i)}}\big) \wedge \big(\Delta^{(i)} \geq \overline{\Delta^{(i)}}\big),\ i = 1, .., N \\ & \sum_i r^{(i)} \leq R, \end{array} \tag{3}$$

where f is the cost function, $\overline{\delta^{(i)}}$ and $\overline{\Delta^{(i)}}$ are desired boundary values for the stability sets, $\sigma^{(i)} = [T^{(i)}, r^{(i)}, \delta^{(i)}, \Delta^{(i)}]$ are the design parameters available for each loop. As a special case, it is possible to investigate the existence of feasible solution by choosing $f \equiv 0$. To cope with this problem, we first investigate conditions under which an assigned pair $[T^{(i)}, r^{(i)}]$ attains $\big(I(\Delta^{(i)}), I(\delta^{(i)})\big)$-stability for specified values of $\Delta^{(i)}$ and $\delta^{(i)}$. Assuming that the system is noise-free, we provide necessary and sufficient conditions for the case of $a^{(i)}T^{(i)} \geq \log 2$: to this aim we take advantage of the chaotic controller proposed by Fagnani-Zampieri [6]. In case $a^{(i)}T^{(i)} \leq \log 3$, we find sufficient conditions but the attained bound is close to necessity. Another important contribution consists in the calculation, for a given pair $(\delta^{(i)}, \Delta^{(i)})$, of a close estimate of the minimum bit rate $r^{(i)}$ allowing $\big(I(\Delta^{(i)}), I(\delta^{(i)})\big)$-stability. This result determines a very compact and effective formulation for the design problem in Equation (3) and shows that the bit rate $r^{(i)}$ is sufficient to characterise the system's performance. Moreover, such an issue allows us to provide a solving algorithm for the problem in Equation (3) consisting of an automated procedure for the optimal allocation of the channel capacity among the different loops.

Although the investigation of the controlled invariance problem can be seen as a special case of the problem just reported, in this framework the presence of noise is deeply analysed. The performance metric that we consider is related to the regions $I(\delta^{(i)})$ that can be made invariant. The problem is formulated as follows:

$$\begin{array}{ll} \min_{\sigma^{(1)},\sigma^{(2)},\ldots,\sigma^{(N)}} & f(\delta^{(1)}, \ldots, \delta^{(N)}) \\ \text{subj. to} & \delta^{(i)} \leq \overline{\delta^{(i)}},\ i = 1, .., N \\ & \sum_i r^{(i)} \leq R, \end{array} \tag{4}$$

where the design parameters are now $\sigma^{(i)} = [T^{(i)}, r^{(i)}, \delta^{(i)}]$. As for the previous problem, our first concern is finding conditions for which a pair $[T^{(i)}, r^{(i)}]$ makes the region $I(\delta^{(i)})$ controlled invariant for a specified value of $\delta^{(i)}$. In this case a good starting point is the work of Baillieul [1], that we revisit and extend to achieve a necessary and sufficient test in the context $\mathcal{U}^{(i)} \subseteq \epsilon^{(i)}\mathbb{Z}$ and in the presence of bounded noise. Moreover, also in this case, for each $\delta^{(i)}$ it is possible to find the minimum bit rate attaining controlled invariance of $I(\delta^{(i)})$ thus allowing for a compact formulation of the problem in Equation (4).

Extensions The results described above are based on the simplifying assumption that the command values are applied immediately after computation. In fact, bit rate limitations entail transmission delays that have to be accounted for. We show how our control approaches can be made robust against transmission delays by properly including memory elements in the controller.

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