A Distributed Algorithm for Economic Dispatch over Time-Varying Directed Networks with Delays

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***Abstract*—In power system operation, the economic dispatch problem (EDP) aims to minimize the total generation cost while meeting the demand and satisfying generator capacity limits. This paper proposes an algorithm based on the gradient push-sum method to solve the EDP in a distributed manner over communication networks potentially with time-varying topologies and communication delays. This paper shows that the proposed algorithm is guaranteed to solve the EDP if the time-varying directed communication network is uniformly jointly strongly connected. Moreover, the proposed algorithm is also able to handle arbitrarily large but bounded time-varying delays on communication links. Numerical simulations are used to illustrate and validate the proposed algorithm.**

***Index Terms*—Distributed algorithm, economic dispatch, gradient push-sum method, time-varying delays, time-varying networks.**

I. INTRODUCTION

THE economic dispatch problem (EDP) is one of important problems in power system operation. It is essentially an optimization problem where the objective is to minimize the total generation cost while meeting total demand and satisfying individual generator output limits. There exist many centralized approaches for solving the EDP, such as the lambda-iteration method and the gradient search method [1]. The centralized methods require a single control center that accesses the entire network's information, and therefore may be subject to performance limitations, such as high communication requirement and cost, substantial computational burden, and limited flexibility and scalability, and disrespect of privacy. It is thus desirable to develop distributed approaches to overcome these limitations and accommodate various resources in the future smart grid.

During the past few years, due to the rising of distributed control and multi-agent systems research [2]–[6], various distributed algorithms have been developed for power system applications [7]–[10]. As for the EDP, various consensus-based distributed algorithms have been proposed by choosing the generation incremental cost as the consensus variable. In these algorithms, each agent maintains a few variables and updates them through the information exchange with its neighboring agents. For instance, the authors of [11] propose a leader-follower consensus-based algorithm where the leader collects the mismatch between demand and generation, and then leads the updates of marginal cost in the system. To avoid the requirement of a leader, a two-level consensus-based algorithm is proposed in [12], where the upper level is the consensus and gradient algorithm, and the lower level executes the classical consensus by choosing the local mismatch as the consensus variable. In the algorithm proposed in [13], in addition to consensus part, an innovation term is introduced to ensure the balance between system generation and demand. All these three algorithms are only applicable to undirected communication networks, i.e., the information must be exchanged bidirectionally. Because directed communication networks cost less than their undirected counterparts [2], it is desirable to develop control and coordination algorithms that only require directed information flow. The capability of utilizing low-cost communication networks is favorable in the future smart grid. Realizing such a need, distributed algorithms have been proposed in [14], [15] to solve the EDP over both undirected and directed communication networks. In [14], the authors propose a ratio consensus based algorithm which relies on two linear iterations. The one in [15] estimates the mismatch with all the agents being participated. The authors of [16] propose minimum-time consensus-based algorithm to solve the EDP in a minimum number of time steps. As for generation cost functions, most of existing studies assume

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quadratic functions, whereas [17] considers general convex functions.

One common assumption on communication networks in the literature is that the communication links are time-invariant and are not subject to time delays. However, in practice, communication network topology may vary due to unexpected loss of communication links. In addition, time delays are ubiquitous in communication networks [18]. Therefore, it is desirable to investigate the potential impacts of imperfect communications on the existing distributed EDP algorithms, and develop algorithms that are robust to imperfect communications in the future smart grid. In [19], the authors study impacts of communication delays on the two-level consensus algorithm, and find that the algorithm may fail to converge due to time delays. In [20], the authors investigate the impacts of communication time delays on the algorithm proposed in [15]. Several potential negative impacts of time delays have been found, such as slower convergence rate, convergence to incorrect value, and divergence. The authors of [21] propose a nonnegative-surplus based distributed algorithm to solve the EDP over time-varying directed communication networks but without time delays.

The existing literature is inadequate to solve the EDP in imperfect communication networks that are subject to time-varying topologies and time delays. In order to better handle these practical restrictions, this paper proposes a distributed algorithm based on the gradient push-sum method. Compared with existing EDP studies, the main contributions of this paper are summarized as follows.

1) We propose a distributed algorithm to solve the EDP over time-varying directed communication networks that are uniformly jointly strongly connected. This is a mild condition on the connectivity of communication topologies, since the network can be disconnected at any time instance as long as the joint graph over a period of time is strongly connected. Therefore, the requirement on network topologies is more general compared to the fixed strongly connected topologies considered in the existing studies [11]–[17], [22].
2) While time-varying communication networks are also considered in [21], the proposed algorithm in this paper can handle the EDP with general convex cost functions, which are more general compared to the quadratic cost functions considered in [21].
3) The proposed algorithm can also handle arbitrarily large but bounded time-varying delays in addition to time-varying directed topologies, which is a distinguishing feature compared with many existing studies, such as [11], [12], [15], [16], [21]. To the best of our knowledge, this is the first algorithm that is capable to solve the EDP over time-varying directed communication networks with communication delays.

The remainder of the paper is organized as follows: In Section II, some preliminaries on graph theory and notations are introduced. Section III presents the EDP formulation and the centralized Lagrangian-based approach. In Section IV, a distributed algorithm based on the gradient push-sum method is proposed to solve the EDP over communication networks with imperfections, such as time-varying topologies and time delays. Case studies are presented in Section V to illustrate and validate the proposed algorithm. Finally, concluding remarks are offered in Section VI.

II. Preliminaries and Notations

This section first presents some background on graph theory [23], which is needed to describe the communication network. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a directed graph (digraph) with the set of nodes (agents) $\mathcal{V} = \{1, \ldots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A directed edge from node i to node j is denoted by $(i, j) \in \mathcal{E}$. For notational simplification, we assume that the digraph does not have any self loop, i.e., $(i, i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$ although each node i has an access to its own information. A directed *path* from node i_1 to node i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. If there exists a directed path from node i to node j, then node j is said to be reachable from node i. A digraph $\mathcal{G}$ is said to be *strongly connected* if every node is reachable from every other node.

In this paper, an agent is assigned to each bus in the power system. The topology of communication network could be different from the physical network, and is modeled as a time-varying directed graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where the edge set changes over time due to unexpected loss of communication links. All agents that can transmit information to node i directly at time t are said to be its in-neighbors and belong to the set $\mathcal{N}_i^{\text{in}}(t) = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}(t)\}$. The nodes which receive information from agent i at time t belong to the set of its out-neighbors, denoted by $\mathcal{N}_i^{\text{out}}(t) = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}(t)\}$. The cardinality of $\mathcal{N}_i^{\text{out}}(t)$ is called its out-degree at time t and is denoted by $d_i(t) = |\mathcal{N}_i^{\text{out}}(t)|$. The joint graph of $\mathcal{G}(t)$ in the time interval $[t_1, t_2)$ with $t_1 < t_2 \leq \infty$ is denoted as $\mathcal{G}([t_1, t_2)) = \cup_{t\in[t_1,t_2)}\mathcal{G}(t) = (\mathcal{V}, \cup_{t\in[t_1,t_2)}\mathcal{E}(t))$. A time-varying directed network $\mathcal{G}(t)$ is said to be *uniformly jointly strongly connected* if there exists a constant $T > 0$ such that $\mathcal{G}([t_0, t_0 + T))$ is strongly connected for any $t_0 \geq 0$.

Notations: In this paper, variables in boldface represent vectors or matrices. For a matrix $\mathbf{A}$, we use $\mathbf{A}_{ij}$ or $[\mathbf{A}]_{ij}$ to denote its (i, j)-th entry and $\mathbf{A}^{\mathsf{T}}$ to denote its transpose. A matrix is nonnegative if all its entries are equal to or greater than zero. A vector is a stochastic vector if all entries are nonnegative and sum up to 1. For a vector $\mathbf{x}$, we use $\mathbf{x}_i$ to denote its i-th entry. $\mathbf{0}$ and $\mathbf{1}$ denote the column vectors with all entries being 0 and 1, respectively. $\mathbf{0}$ and $\mathbf{I}$ denote the matrix with all entries being 0 and the identity matrix, respectively. The set of real (integer) numbers is denoted by $\mathbb{R}$ ($\mathbb{Z}$) and the set of nonnegative real (integer) numbers is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$).

III. Problem Formulation and Lagrangian-based Approach

This section first presents the mathematical formulation and then the centralized Lagrangian-based approach for the EDP.

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A. Formulation of EDP

The goal of EDP is to minimize the total generation cost while meeting total demand and satisfying individual generator output limits, as formulated in (1):

$$\min_{x_i} \quad \sum_{i=1}^{N} C_i(x_i) \tag{1a}$$

$$\text{subject to} \quad \sum_{i=1}^{N} x_i = D, \tag{1b}$$

$$x_i \in X_i := [x_i^{\min}, x_i^{\max}],\ i = 1, \ldots, N, \tag{1c}$$

where N is the number of generators, x_i is the power generation of generator i, $C_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is the cost function of generator i, $x_i^{\min}$ and $x_i^{\max}$ are respectively the lower and upper bounds of the power generation of generator i, and D is the total demand satisfying $\sum_{i=1}^{N} x_i^{\min} \le D \le \sum_{i=1}^{N} x_i^{\max}$ in order to ensure the feasibility of problem (1).

Compared to most studies [11]–[16], [21], [24] in the EDP literature where cost functions are assumed to be quadratic, this paper considers general convex cost functions that satisfy Assumption 1.

Assumption 1: For each $i \in \{1, \ldots, N\}$, the cost function $C_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly convex and continuously differentiable.

B. Centralized Lagrangian-based Approach

Since i) each cost function $C_i(\cdot)$ is convex, ii) the constraint (1b) is affine, and iii) the set $X_1 \times \cdots \times X_N$ is a polyhedral set, if we dualize problem (1) with respect to the constraint (1b), there is zero duality gap. Moreover, the dual optimal set is nonempty [25]. Consequently, solutions of the EDP can be obtained by solving its dual problem.

For convenience, let $\mathbf{x} = [x_1, \ldots, x_N]^{\mathrm{T}} \in \mathbb{R}_+^N$. Then, define the Lagrangian function

$$\mathcal{L}(\mathbf{x}, \lambda) = \sum_{i=1}^{N} C_i(x_i) - \lambda \left(\sum_{i=1}^{N} x_i - D \right).$$

The corresponding Lagrange dual problem is

$$\max_{\lambda \in \mathbb{R}_+} \sum_{i=1}^{N} \Psi_i(\lambda) + \lambda D, \tag{2}$$

where

$$\Psi_i(\lambda) = \min_{x_i \in X_i} C_i(x_i) - \lambda x_i. \tag{3}$$

Under Assumption 1, for any given $\lambda \in \mathbb{R}_+$, the right-hand side of (3) has a unique minimizer given by

$$x_i(\lambda) = \min\{\max\{\nabla C_i^{-1}(\lambda), x_i^{\min}\}, x_i^{\max}\}, \tag{4}$$

where ∇C_i^{-1} denotes the inverse function of ∇C_i, which exists over $[\nabla C_i(x_i^{\min}), \nabla C_i(x_i^{\max})]$ since ∇C_i is continuous and strictly increasing due to Assumption 1. Furthermore, there is at least one optimal solution to dual problem (2), and the unique optimal solution of the primal EDP is given by

$$x_i^* = x_i(\lambda^*), \quad \forall i = 1, 2, \ldots, N, \tag{5}$$

where λ^* is any dual optimal solution.

For any given $\lambda \in \mathbb{R}_+$, because of the uniqueness of $x_i(\lambda)$, the dual function $\sum_{i=1}^{N} \Psi_i(\lambda) + \lambda D$ is differentiable at λ and its gradient is given by $-(\sum_{i=1}^{N} x_i(\lambda) - D)$ [26]. We can then apply the gradient method to solve the dual problem in (2):

$$\lambda(t+1) = \lambda(t) - \gamma(t) \left(\sum_{i=1}^{N} x_i(\lambda(t)) - D \right), \tag{6}$$

where $\lambda(0) \in \mathbb{R}$ can be arbitrarily assigned and $\gamma(t)$ is the step-size at time step t. When designing a distributed algorithm based on (6), the main challenge is how to obtain the global quantity $\sum_{i=1}^{N} x_i(\lambda(t)) - D$ in a distributed manner. In this paper, we will propose a distributed algorithm to avoid the need of the global quantity.

IV. Main Results

This section proposes an algorithm that is capable to solve the EDP in a distributed fashion over time-varying communication networks potentially with arbitrarily large but bounded time delays. In Section IV-A, the dual problem in (2) is first converted to an agent-based distributed convex optimization problem. Then a distributed algorithm is proposed for the EDP based on the gradient push-sum method [27]. Section IV-B shows that the proposed algorithm is able to solve the EDP over time-varying directed communication networks. Finally, Section IV-C shows that the proposed algorithm is also robust to communication time delays.

A. Distributed Gradient Push-Sum Algorithm

The dual problem in (2) can be converted into

$$\max_{\lambda \in \mathbb{R}} \sum_{i=1}^{N} \Phi_i(\lambda), \tag{7}$$

where

$$\Phi_i(\lambda) = \min_{x_i \in X_i} C_i(x_i) - \lambda(x_i - D_i), \tag{8}$$

and D_i is a virtual local demand at each bus such that $\sum_{i=1}^{N} D_i = D$. The gradient of $\Phi_i(\lambda)$ is

$$\nabla \Phi_i(\lambda) = -\left(x_i(\lambda) - D_i\right). \tag{9}$$

Motivated by the gradient push-sum method [27], a distributed algorithm is proposed in Algorithm 1 to solve the equivalent dual problem (7). In the proposed algorithm, each agent i maintains scalar variables $v_i(t)$, $w_i(t)$, $y_i(t)$, $\lambda_i(t)$, $x_i(t)$, where $x_i(t)$ and $\lambda_i(t)$ are estimations of the primal and dual optimal solution, respectively. At each time step t, each agent $i \in \mathcal{V}$ updates its variables according to (10).

$$w_i(t+1) = \frac{v_i(t)}{d_i(t)+1} + \sum_{j \in \mathcal{N}_i^{\text{in}}(t)} \frac{v_j(t)}{d_j(t)+1}, \tag{10a}$$

$$y_i(t+1) = \frac{y_i(t)}{d_i(t)+1} + \sum_{j \in \mathcal{N}_i^{\text{in}}(t)} \frac{y_j(t)}{d_j(t)+1}, \tag{10b}$$

$$\lambda_i(t+1) = \frac{w_i(t+1)}{y_i(t+1)}, \tag{10c}$$

$$x_i(t+1) = \min\{\max\{\nabla C_i^{-1}(\lambda_i(t+1)), x_i^{\min}\}, x_i^{\max}\}, \tag{10d}$$

$$v_i(t+1) = w_i(t+1) - \gamma(t+1)(x_i(t+1) - D_i). \tag{10e}$$

Algorithm 1 Distributed algorithm for the EDP

Input: The time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, the step-size $\gamma(t)$, an arbitrarily assigned $v_i(0)$, and $y_i(0) = 1$ for all $i \in \mathcal{V}$.
Output: The optimal incremental cost λ^* and the optimal generation x_i^*.
repeat
  for $i = 1$ **to** N **do**
    Run the update rule (10).
    Return $\lambda_i(t+1)$ and $x_i(t+1)$.
  end for
  Update t as $t := t + 1$.
until $|\lambda_i(t) - \lambda_i(t-1)| < \epsilon_1$ **and** $\max_{i,j \in \mathcal{V}} |\lambda_i(t) - \lambda_j(t)| < \epsilon_2$.

The step-size $\gamma(t+1)$ satisfies the following decay conditions:

$$\sum_{t=1}^{\infty} \gamma(t) = \infty, \quad \sum_{t=1}^{\infty} \gamma^2(t) < \infty,$$
$$\gamma(t) \leq \gamma(s) \text{ for all } t > s \geq 1. \tag{11}$$

One typical selection is $\gamma(t) = \frac{a}{t+b}$, where $a > 0$ and $b \geq 0$. In this algorithm, each agent i needs to know its out-degree $d_i(t)$ and sends the quantities $\frac{v_i(t)}{d_i(t)+1}$ and $\frac{y_i(t)}{d_i(t)+1}$ to all the agents j in its out-neighbors set. In initialization, $v_i(0)$ is assigned with an arbitrary value and $y_i(0) = 1$ for all $i \in \mathcal{V}$.

According to (9), $-(x_i(t+1) - D_i)$ in (10e) is the gradient of the function $\Phi_i(\lambda)$ at $\lambda = \lambda_i(t+1)$. Without (10d) and the gradient term in (10e), the algorithm would be reduced to a particular version of push-sum algorithm [28], or ratio consensus algorithm [29], [30] for computing the average of initial values in directed graphs. In this case, all $\lambda_i(t+1)$ converge to a common value. The inclusion of the gradient term in the update of $v_i(t+1)$ is to ensure that all $\lambda_i(t+1)$ converge to the optimal incremental cost λ^*.

Remark 1: Note that at each step t, agent i runs the update rule (10):

- $d_i^{\text{in}}(t) + 1$ multiplications are performed in each of (10a) and (10b), where $d_i^{\text{in}}(t)$ is the in-degree of agent i at time t. Note that $d_i^{\text{in}}(t) \leq N - 1$, where N is the size of the communication network.
- Only one division is performed in (10c).
- For quadratic cost functions $C_i(x_i) = a_i x_i^2 + b_i x_i + c_i$ where $a_i > 0$, b_i and c_i are cost parameters, the update equation (10d) has a closed form expression
$$x_i(t+1) = \min\{\max\{\frac{\lambda_i(t+1) - b_i}{2a_i}, x_i^{\min}\}, x_i^{\max}\},$$
therefore, only one multiplication is performed in (10d). For general convex cost functions, the update equation (10d) may not have a closed form expression. Nevertheless, its numerical solution can be obtained in a finite number of time steps by using the bisection method due to the continuity and strict monotonicity of $\nabla C_i^{-1}(\cdot)$.
- Only one multiplication is performed in (10e).

Moreover, it requires a finite number of time steps for the algorithm to converge based on the stopping criteria. Therefore, the computational complexity of the proposed algorithm is $\mathcal{O}(N)$.

B. Robustness to Time-Varying Communication Networks

In this subsection, we will show that the proposed distributed Algorithm 1 is capable to solve the EDP over time-varying directed communication networks which satisfy Assumption 2, as stated in Theorem 1.

Assumption 2: The time-varying directed communication network $\mathcal{G}(t)$ is uniformly jointly strongly connected, i.e., the jointly communication network $\mathcal{G}([t_0, t_0 + T))$ is strongly connected for any $t_0 \geq 0$ with some constant $T > 0$.

Theorem 1: Under Assumptions 1 and 2, distributed Algorithm 1 with the step-size $\gamma(t)$ satisfying conditions in (11) solves the EDP, i.e., $\lambda_i(t) \to \lambda^*$, and $x_i(t) \to x_i^*$ as $t \to \infty$ for all $i \in \mathcal{V}$.

Proof: Note that the equivalent dual problem (7) has the same form as the optimization problem considered in [27]. The only difference is that the dual problem is a maximization problem while the problem in [27] is a minimization problem. In order to apply [27, Theorem 1] to show Theorem 1, we need to verify that all the conditions are satisfied.

- The condition (a) is that the network is uniformly jointly strongly connected, which is satisfied in our case as assumed in Assumption 2.
- The condition (b) is that each function in the minimization problem is convex and the optimal set is nonempty. This is also satisfied in our case since each function $\Phi_i(\lambda)$ in the maximization problem (7) is concave and the optimal set is nonempty, which is guaranteed by Assumption 1.
- The condition (c) is that the (sub)gradient of each function in the problem is uniformly bounded. This is indeed satisfied in our case since it follows from (9) that the gradient of each function $\Phi_i(\lambda)$ is uniformly bounded, i.e.,

$$\Big|\nabla \Phi_i(\lambda_i(t+1))\Big| = \Big|-(x_i(\lambda_i(t+1)) - D_i)\Big| \leq \max_{i \in \mathcal{V}} x_i^{\max} + \max_{i \in \mathcal{V}} D_i. \tag{12}$$

Therefore, all the conditions are satisfied and the result follows. ■

Remark 2: The key result used in the proof of [27, Theorem 1] is [27, Lemma 3]. It shows that the matrix $\mathbf{A}(t)$ which captures the weights used in the update equations (10a) and (10b), defined as

$$\mathbf{A}_{ij}(t) = \begin{cases} \frac{1}{d_j(t)+1}, & \text{if } j \in \mathcal{N}_i^{\text{in}}(t) \cup \{i\}, \\ 0 & \text{otherwise}, \end{cases} \tag{13}$$

has the special property under Assumption 2, that is, for each $s \in \mathbb{Z}_+$, there is a stochastic vector $\boldsymbol{\phi}(s)$ such that for all i, j and $t \geq s$,

$$\Big|[\mathbf{A}^{\mathrm{T}}(t)\mathbf{A}^{\mathrm{T}}(t-1) \cdots \mathbf{A}^{\mathrm{T}}(s+1)\mathbf{A}^{\mathrm{T}}(s)]_{ij} - \boldsymbol{\phi}_j(s)\Big| \leq \alpha \beta^{t-s},$$

with

$$\alpha = 2, \quad \beta = \Big(1 - \frac{1}{N^{NT}}\Big)^{\frac{1}{NT}}, \tag{14}$$

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where T is the bound on the intercommunication interval given in Assumption 2.

Remark 3: In the proof of Theorem 1, we have built upon the recent developed result in [27]. However, our work is substantially different.

- We consider the EDP, which is a constrained optimization problem as shown in (1). We have converted the problem to an equivalent dual problem (7), which has the same form as the unconstrained problem considered in [27].
- In order to apply the result in [27], there are substantial details need to be verified as shown in Theorem 1.
- We will show in Theorem 2 that the proposed algorithm is also robust to arbitrarily large but bounded time-varying communication delays, which are not considered in [27].

Remark 4: Theorem 1 shows that the proposed distributed Algorithm 1 solves the EDP over time-varying directed communication networks that are uniformly jointly strongly connected. This is a mild condition on the connectivity of communication topologies, since the network can be disconnected at any time instance as long as the joint graph over a period of time is strongly connected. Therefore, the requirement on network topologies is more general than the existing studies [11]–[17], [22], where the fixed strongly connected topologies are required.

C. Robustness to Communication Time Delays

This subsection studies the impact of time delays on the proposed Algorithm 1. We first model time delays in the directed communication networks. In particular, the communication link (j, i) at time step t undergoes a priori unknown delay $\tau_{ji}(t) \in \mathbb{Z}_+$. We impose the following assumption on time-varying delays.

Assumption 3: The time-varying delays are uniformly bounded at all times, i.e., $0 \le \tau_{ji}(t) \le \bar{\tau}$ for all $t \in \mathbb{Z}_+$ with some finite $\bar{\tau} \in \mathbb{Z}_+$. Moreover, each agent has access to its own value without any time delay, i.e., $\tau_{ii}(t) = 0$ for all $i \in \mathcal{V}$ and for all time steps $t \in \mathbb{Z}_+$.

Note that in the proposed algorithm (10), only the updates of $w_i(t+1)$ and $y_j(t+1)$ rely on communications among the agents, while the updates of $\lambda_i(t+1)$, $x_i(t+1)$, and $v_i(t+1)$ are executed locally without the need of further communications. When communications are subject to time delays, each agent i updates the values of $w_i(t+1)$ and $y_i(t+1)$ by combining its own values and the delayed information received from its in-neighbors. More specifically, under time-delays, executing the update rule (10) results in:

$$w_i(t+1) = \frac{v_i(t)}{d_i(t)+1} + \sum_{j \in \mathcal{N}_i^{\text{in}}(t)} \frac{v_j\left(t-\tau_{ji}(t)\right)}{d_j(t)+1}, \tag{15a}$$

$$y_i(t+1) = \frac{y_i(t)}{d_i(t)+1} + \sum_{j \in \mathcal{N}_i^{\text{in}}(t)} \frac{y_j\left(t-\tau_{ji}(t)\right)}{d_j(t)+1}, \tag{15b}$$

$$\lambda_i(t+1) = \frac{w_i(t+1)}{y_i(t+1)}, \tag{15c}$$

$$x_i(t+1) = \min\{\max\{\nabla C_i^{-1}(\lambda_i(t+1)), x_i^{\min}\}, x_i^{\max}\}, \tag{15d}$$

$$v_i(t+1) = w_i(t+1) - \gamma(t+1)(x_i(t+1) - D_i). \tag{15e}$$

The following theorem shows that the proposed Algorithm 1 is able to solve the EDP over time-varying directed communication networks even when communication links are subject to arbitrarily large but bounded delays.

Theorem 2: Under Assumptions 1, 2, and 3, distributed Algorithm 1 with the step-size $\gamma(t)$ satisfying conditions in (11) solves the EDP even when communication links are subject to arbitrarily large but bounded delays, i.e., $\lambda_i(t) \to \lambda^*$, and $x_i(t) \to x_i^*$ as $t \to \infty$ for all $i \in \mathcal{V}$.

Proof: We first note that under our modeling on communication time delays, executing the update rule (10) results in (15). The proof is based on an augmented digraph representation which allows us to reduce the original system with bounded delays (15) to a system without delays. More specifically, for each agent i in the original graph, we introduce $\bar{\tau}$ virtual agents $i^{(1)}$, $i^{(2)}$, …, $i^{(\bar{\tau})}$, where at each time step t, virtual agent $i^{(r)}$ holds information that is destined to arrive to node i in r steps. Since time delays are bounded by $\bar{\tau}$, there are in total $N(\bar{\tau}+1)$ agents in the augmented digraph. In the augmented digraph, we enumerate the agents in the original digraph first and then the virtual agents. Moreover, the virtual agents are indexed so that the first N agents model the delay of 1 time step, the next N agents model the delay of 2 time steps, and so on.

We now describe how these agents communicate in the augmented digraph. In particular, at time step t, for each edge (j, i) in the original network, that edge also exists in the augmented digraph along with edges $(j, i^{(1)})$, $(j, i^{(2)})$, …, $(j, i^{(\bar{\tau})})$, and edges $(i^{(1)}, i)$, $(i^{(2)}, i^{(1)})$, …, $(i^{(\bar{\tau})}, i^{(\bar{\tau}-1)})$.

For each virtual agent $i^{(r)}$, where $r = 1, \ldots, \bar{\tau}$, we associate it with the states $v_i^{(r)}$, $y_i^{(r)}$ and $w_i^{(r)}$. We then define $\mathbf{v}^{(r)}(t)$, $\mathbf{y}^{(r)}(t)$ and $\mathbf{w}^{(r)}(t)$ as the column stack vectors of $v_i^{(r)}$, $y_i^{(r)}$ and $w_i^{(r)}$ where $i \in \mathcal{V}$, respectively. For example, $\mathbf{v}^{(r)}(t) = [v_1^{(r)}(t), \ldots, v_N^{(r)}(t)]^{\mathrm{T}}$. Finally, we define $\tilde{\mathbf{v}}(t) = [\mathbf{v}(t)^{\mathrm{T}}, \mathbf{v}^{(1)}(t)^{\mathrm{T}}, \ldots, \mathbf{v}^{(\bar{\tau})}(t)^{\mathrm{T}}]^{\mathrm{T}}$. Similarly, we define $\tilde{\mathbf{w}}(t)$ and $\tilde{\mathbf{y}}(t)$. For the agent in the original network $i \in \mathcal{V}$, the initial states are given by $v_i(0)$ and $y_i(0) = 1$, while for all the virtual agents, the initial states are given by $\mathbf{v}^{(r)}(0) = \mathbf{0}$ and $\mathbf{y}^{(r)}(0) = \mathbf{0}$ for all $r = 1, 2, \ldots \bar{\tau}$.

In the augmented digraph, the original system with delays (15) can be rewritten in a more compact form without delays as

$$\tilde{\mathbf{w}}(t+1) = \tilde{\mathbf{A}}(t)\tilde{\mathbf{v}}(t), \tag{16a}$$

$$\tilde{\mathbf{y}}(t+1) = \tilde{\mathbf{A}}(t)\tilde{\mathbf{y}}(t), \tag{16b}$$

$$\lambda_i(t+1) = \frac{w_i(t+1)}{y_i(t+1)}, \quad i \in \mathcal{V}, \tag{16c}$$

$$x_i(t+1) = \min\{\max\{\nabla C_i^{-1}(\lambda_i(t+1)), x_i^{\min}\}, x_i^{\max}\}, \tag{16d}$$

$$\tilde{\mathbf{v}}(t+1) = \tilde{\mathbf{w}}(t+1) - \gamma(t+1)[\mathbf{x}^{\mathrm{T}}(t) - \tilde{\mathbf{D}}^{\mathrm{T}}, \mathbf{0}_{N\bar{\tau}}^{\mathrm{T}}]^{\mathrm{T}}, \tag{16e}$$

where $\mathbf{x}(t) = [x_1(t), \ldots, x_N(t)]^{\mathrm{T}}$, $\tilde{\mathbf{D}} = [D_1, \ldots, D_N]^{\mathrm{T}}$, and

$$\tilde{\mathbf{A}}(t) = \begin{bmatrix} \mathbf{A}^{(0)}(t) & \mathbf{I}_{N\times N} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{A}^{(1)}(t) & \mathbf{0} & \mathbf{I}_{N\times N} & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{(\bar{\tau}-1)}(t) & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I}_{N\times N} \\ \mathbf{A}^{(\bar{\tau})}(t) & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}.$$

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Note that in (16), the update equations (16c) and (16d) are only for the agents in the original graph, which are identical to the update equations (10c) and (10d), respectively, for the case without communication delays. Also note that $\tilde{y}_i(t+1) > 0$ for all $t \in \mathbb{Z}_+$ and for all original agents $i \in \mathcal{V}$. This is not necessarily true for the virtual agents. Therefore, $\lambda_i(t+1)$ for $i \in \mathcal{V}$ in (16c) is a finite quality for all $t \in \mathbb{Z}_+$. $x_i(t+1)$ for $i \in \mathcal{V}$ is then updated according to (16d). These resulting values are finally used in the update equations (16e) for the agents in the original graph, while for the virtual agents, update equations (16e) reduces to $\tilde{v}_i(t+1) = \tilde{w}_i(t+1)$.

Herein, $\mathbf{A}^{(0)}(t)$, $\mathbf{A}^{(1)}(t)$, ..., $\mathbf{A}^{(\bar{\tau})}(t)$ are appropriately defined nonnegative matrices that depend on the communication link delays which are experienced by messages sent at time step t. Specifically, $\mathbf{A}^{(r)}(t)$ for $r = 0, \ldots, \bar{\tau}$ is a matrix associated only with the communication links for which the message was delayed by r steps at time step t, and satisfies

$$\mathbf{A}_{ij}^{(r)}(t) = \begin{cases} \mathbf{A}_{ij}(t), & \text{if } \tau_{ij}(t) = r, \ \ (i,j) \in \mathcal{E}(t), \\ 0, & \text{otherwise}, \end{cases}$$

with $\mathbf{A}_{ij}(t)$ given by (13).

Notice that at time step t, for each edge (j, i), only one of $\mathbf{A}_{ij}^{(0)}(t)$, ..., $\mathbf{A}_{ij}^{(\bar{\tau})}(t)$ is nonzero and is equal to $\mathbf{A}_{ij}(t)$. Thus, the special structure of the matrix $\tilde{\mathbf{A}}(t)$ allows us to analyze their products. More specially, it follows from [31, Lemma 5] that $\tilde{\mathbf{A}}(t)$ has the special property under Assumptions 2 and 3, that is, for each $s \in \mathbb{Z}_+$, there is a stochastic vector $\tilde{\boldsymbol{\phi}}(s)$ such that for all $i, j \in \{1, \ldots, N(\bar{\tau}+1)\}$ and $t \geq s$,

$$\left| [\tilde{\mathbf{A}}^{\mathrm{T}}(t)\tilde{\mathbf{A}}^{\mathrm{T}}(t-1)\cdots\tilde{\mathbf{A}}^{\mathrm{T}}(s+1)\tilde{\mathbf{A}}^{\mathrm{T}}(s)]_{ij} - \tilde{\boldsymbol{\phi}}_j(s) \right| \leq \alpha\tilde{\beta}^{t-s},$$

with

$$\alpha = 2, \quad \tilde{\beta} = \left(1 - \frac{1}{N^{NB}}\right)^{\frac{1}{NB}},$$

where $B = T + \bar{\tau}$, T is the bound on the intercommunication interval given in Assumption 2 and $\bar{\tau}$ is the upper bound of the communication time delays in Assumption 3. Therefore, the special property given in Remark 2 is satisfied with the matrix $\tilde{\mathbf{A}}$.

The rest of the proof follows from the proof of Theorem 1 by noticing that in the update equation (16e), the gradient of each function $\Phi_i(\lambda)$ for the original agent is uniformly bounded as shown in (12) and there is no perturbation for the virtual agents. ■

Remark 5: Theorem 2 shows that the proposed distributed Algorithm 1 also solves the EDP even when time-varying directed communication links are subject to arbitrarily large but bounded time-varying delays. In the EDP literature, various distributed algorithms have been proposed for fixed networks without time delays, e.g., [11], [13], [15], [17]. As shown in [19], [20], these algorithms fail to converge when the estimate of the mismatch is subject to time delays and when the time delays on the estimate of the optimal incremental cost are large. The authors of [21] propose a nonnegative-surplus based distributed algorithm to solve the EDP over time-varying communication networks but without time delays. To the best of our knowledge, our proposed algorithm is the first algorithm that is capable to solve the EDP over switching networks with arbitrarily large but bounded time-varying delays.

Remark 6: Note that the convergence rate of the existing algorithms for the EDP over fixed topologies in [11]–[13], [15], [17] depends on the real part of the dominant eigenvalue of the Laplacian matrix associated with network topology and the step-size. In our work, we consider switching topologies with time-varying communication delays, therefore the convergence rate of our proposed algorithm also depends on the nature of switching sequence and the nature of time delays. The explicit relationship is thus more complicated and is left as an interesting future direction.

Remark 7: Although the distributed Algorithm 1 is proposed to solve the EDP over switching communication networks with time-varying delays, it can also be applied to solve a particular type of the optimal resource allocation problem [32], [33], which can be formulated as (1).

V. Case Studies

In this section, various case studies are presented in order to illustrate and validate the proposed algorithm. Test systems have been developed and studied for distributed EDP algorithms in existing works, e.g., the IEEE 14-bus system and the IEEE-118 bus system for fixed communication networks [13], [15], [17] and a 4-bus system for time-varying communication networks [21]. These test systems are adopted to study the proposed algorithm with the corresponding type of communication networks, considering both without and with time delay scenarios.

A. Fixed Communication Networks

First, the IEEE 14-bus system is used to demonstrate the implementation of the proposed algorithm for a fixed directed communication network, which is modeled as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the edge set $\mathcal{E} = \{(i, i+1), (i, i+2) | 1 \leq i \leq 12\} \cup \{(13, 14), (13, 1), (14, 1), (1, 7), (2, 8), (3, 2), (3, 9), (4, 10), (5, 2), (5, 11), (6, 12)\}$. Note that generator buses are $\{1, 2, 3, 6, 8\}$, and load buses are $\{2, 3, 4, 5, 6, 9, 10, 11, 13, 14\}$. The generator parameters including the parameters of the quadratic cost functions are adopted from [13], [15], [17], which are given in Table I. When a bus does not contain generators, the power generation at that bus is set to zero. Thus, the update in (10d) simply becomes $x_i(t+1) = 0$ for $i \notin \{1, 2, 3, 6, 8\}$. The virtual local demands at each bus are given as $D_1 = 0$ MW, $D_2 = 9$ MW, $D_3 = 56$ MW, $D_4 = 55$ MW, $D_5 = 27$ MW, $D_6 = 27$ MW, $D_7 = 0$ MW, $D_8 = 0$ MW, $D_9 = 8$ MW, $D_{10} = 24$ MW, $D_{11} = 53$ MW, $D_{12} = 46$ MW, $D_{13} = 16$ MW, and $D_{14} = 40$ MW. The total demand is $D = \sum_{i=1}^{14} D_i = 380$ MW, which is unknown to the agent at each bus.

1) Without delay: This case represents ideal communication network, where the communication links are time-invariant and not subject to time delays. This most basic case has been used in many existing studies, and therefore is selected as a starting point for testing the proposed algorithm. The results with a step-size of $\gamma(t) = \frac{0.15}{t}$ are plotted in

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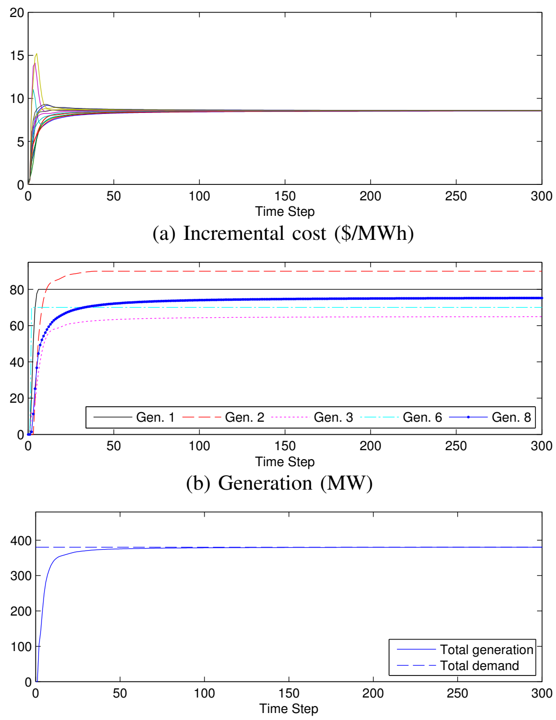
TABLE I
IEEE 14-BUS SYSTEM GENERATOR PARAMETERS

Bus	a_i(MW^{2}h)	b_i ($/MWh)	Range (MW)
1	0.04	2.0	[0,80]
2	0.03	3.0	[0,90]
3	0.035	4.0	[0,70]
6	0.03	4.0	[0,70]
8	0.04	2.5	[0,80]



(a) Incremental cost ($/MWh)

(b) Generation (MW)

(c) Total generation vs. demand (MW)

Fig. 1. Simulation results for the IEEE-14 bus over fixed communication networks without time delays.

Fig. 1, where Fig. 1(a) shows the evolution of incremental cost $\lambda_i(t)$, Fig. 1(b) shows the evolution of power generation $x_i(t)$, and Fig. 1(c) shows the evolution of the total generation in comparison of total demand. As can be seen, the incremental cost $\lambda_i(t)$ computed at each agent converges to the optimal value $\lambda^* = 8.52$ $/MWh. In particular, at time step $t = 300$, the maximum difference between all the λ_i is 0.0045 $/MWh, which is quite small. The generation at each generator bus also converges to their optimal values, which are $x_1^* = 80$ MW, $x_2^* = 90$ MW, $x_3^* = 64.65$ MW, $x_6^* = 70$ MW, and $x_8^* = 75.31$ MW. As $\lambda_i(t)$ and $x_i(t)$ converge to their optima, the total generation meets total demand $D = 380$ MW. The proposed algorithm is able to handle the power output constraints of individual generator and find the correct optimal solutions. For example, generators 1, 2 and 6 are at the upper bounds of their power output because they are cheaper than the other two generators and therefore provide generation as much as possible.

For this communication network, the proposed algorithm has also been studied for generators with non-quadratic cost functions. In particular, we adopt the following case from [17], where the generator at bus 6 is replaced with a fixed generation $x_6 = 100$ MW and the cost function of generators at bus 1 and 3 become:

$$C_1(x_1) = \frac{(x_1 + 25)^2}{25} + 50\exp(\frac{x_1 + 40}{100}),$$
$$C_3(x_3) = \frac{(x_3 + 57.14)^2}{28.58} + 7 \times 10^{-6} x_3^4.$$

Using the proposed algorithm, the incremental cost at each bus converges to 8.85 $/MWh, the generation at each generator bus respectively converges to $x_1 = 67.85$ MW, $x_2 = 90$ MW, $x_3 = 41.47$ MW, $x_6 = 100$ MW, and $x_8 = 79.96$ MW, which agrees with the centralized solution and the one found in [17].

2) With time-varying delays: In this case, the proposed algorithm is studied using the same communication network topology but with time delays. While there exist works that study the impacts of uniform fixed time delays on distributed EDP algorithm [19], [20], this work considers more general and challenging cases, where delays could be arbitrary time-varying with an upper bound. In this case study, we assume that the upper bound is $\bar{\tau} = 20$. In particular, at each time step, the time delay of each link has a probability of $1/21$ to be any integer in $\{0, 1, \ldots, 20\}$. Since time delays are stochastic, the iteration results at each agent vary from one simulation to another. Nevertheless, the proposed algorithm always converges to the optimal solutions. As an example, Fig. 2 plots the simulation results for a particular run. As can be seen, even in the presence of communication time delays, each variable eventually converges to the same value as the case without time delays. In particular, at time step $t = 5000$, the maximum difference between all the λ_i is 0.0412 $/MWh, which is only 0.48% of the optimal incremental cost λ^*. Compared with the results in Fig. 1, the optimal solutions are obtained with a slower rate.

B. Time-Varying Communication Networks

The test system and the communication topology are adopted from [21] for comparison purpose. Four generators are selected from three types, and the power output ranges and parameters of quadratic cost functions for each generator type are given in Table II. The communication network is modeled as a time-varying directed graph $\mathcal{G}(t)$ switching among three fixed topologies $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ shown in Fig. 3 at each time step. In particular,

$$\mathcal{G}(t) = \begin{cases} \mathcal{G}_1, & \text{if } t \in [0,1) \cup [3,4) \cdots \cup [3s, 3s+1) \cdots, \\ \mathcal{G}_2, & \text{if } t \in [1,2) \cup [4,5) \cdots \cup [3s+1, 3s+2) \cdots, \\ \mathcal{G}_3, & \text{if } t \in [2,3) \cup [5,6) \cdots \cup [3s+2, 3s+3) \cdots, \end{cases}$$

where $s \in \mathbb{Z}_+$. It is easy to check that each of the fixed topologies $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ is not strongly connected. For example, there is no directed path from agent 2 to agent 4 in $\mathcal{G}_1$. However, the time-varying directed graph $\mathcal{G}(t)$ is uniformly jointly strongly connected since the joint graph $\mathcal{G}([t_0, t_0 + T))$ is strongly connected for any $t_0 \geq 0$ with $T = 3$. Thus, Assumption 2 is satisfied with $T = 3$. According to Theorems 1 and 2, the proposed algorithm solves the EDP over the time-varying communication network without delays and with delays, respectively. To implement the proposed

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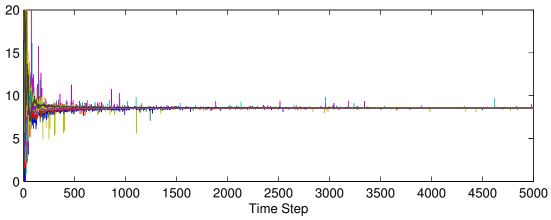
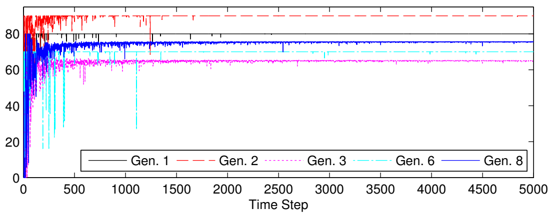
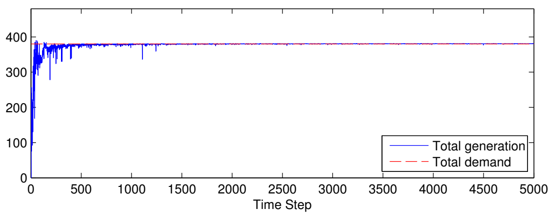


(a) Incremental cost (\$/MWh)



(b) Generation (MW)



(c) Total generation vs. demand (MW)

Fig. 2. Simulation results for the IEEE-14 bus over fixed communication networks with time-varying delays.

TABLE II
GENERATOR PARAMETERS

Type	A (Gen. 1&2)	B (Gen. 3)	C (Gen. 4)
Range (MW)	[150,600]	[100,400]	[50,200]
a_i(MW^{2}h)	0.00142	0.00194	0.00482
b_i (\$/MWh)	7.2	7.85	7.97
c_i (\$/h)	510	310	78

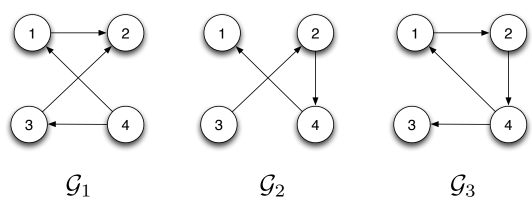
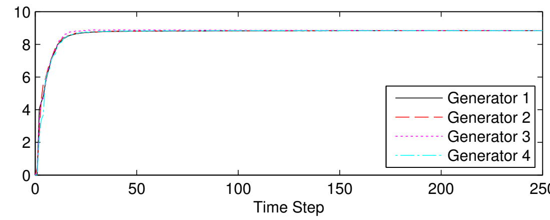


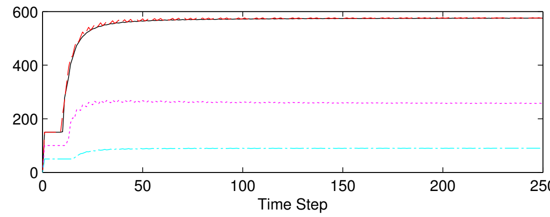
Fig. 3. Time-varying directed communication network.

algorithm, we first choose the virtual local demands at each bus as $D_1 = 500$ MW, $D_2 = 500$ MW, $D_3 = 350$ MW, and $D_4 = 150$ MW. The total demand is $D = \sum_{i=1}^{4} D_i = 1500$ MW, which is unknown to the agent at each bus.

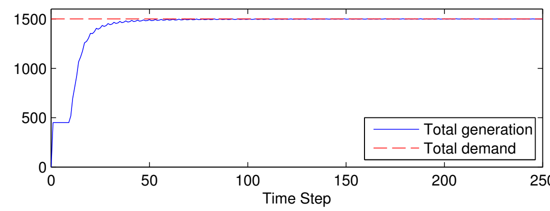
1) Without delay: First, we consider the case where communication links are not subject to delays. With a step-size of $\gamma(t) = \frac{0.01}{t}$, the simulation results are shown in Fig. 4. As can be seen, $\lambda_i(t)$ converge to the optimal incremental cost $\lambda^* = 8.84$ \$/MWh, and x_i converges to the optimal generation $x_1^* = 577.46$ MW, $x_2^* = 577.46$ MW, $x_3^* = 255.16$ MW, and $x_4^* = 90.25$ MW as shown in Fig. 4(b), which agree with the centralized solution and the one obtained in [21]. In particular, at time step $t = 250$, the maximum difference between all the λ_i is 0.0126 \$/MWh, which is quite small. The total generation



(a) Incremental cost (\$/MWh)



(b) Generation (MW)



(c) Total generation vs. demand (MW)

Fig. 4. Simulation results for the system of four generators over time-varying communication networks without time delays.

gradually meets the total demand 1500 MW.

2) With time-varying delays: Note that the nonnegative-surplus based algorithm proposed in [21] cannot handle time delays since it is build upon the algorithm in [15] which is not robust to even uniformly constant delays as shown in [20]. On the other hand, the proposed algorithm in this paper is robust to time-varying delays in addition to time-varying topologies as shown in Theorem 2. In order to demonstrate this distinguishing feature, we herein consider the same time-varying communication topology but with time delays. In this case study, we consider the case where time-varying delays are upper bounded by $\bar{\tau} = 3$ and the probability mass function of time delay on any communication link is given by $P_\tau(\tau) = 0.5$ for $\tau = 0$, $P_\tau(\tau) = 0.35$ for $\tau = 1$, $P_\tau(\tau) = 0.1$ for $\tau = 2$, and $P_\tau(\tau) = 0.05$ for $\tau = 3$.

Since time delays are stochastic, the dynamics at each agent vary from one simulation to another. Nevertheless, the proposed algorithm always converges to the optimal solutions. Simulation results for a particular run are plotted in Fig. 5, As can been seen, even in the presence of communication time delays, each variable still converges to the same value as the case without time delays. In particular, at time step $t = 600$, the maximum difference between all the λ_i is 0.0126 \$/MWh, which is quite small. Compared with the results in Fig. 4, the optimal solutions are obtained with a slower rate.

C. IEEE-118 Bus System

In this case we test our algorithm to the IEEE 118-bus system to further show the effectiveness of the proposed algorithm.

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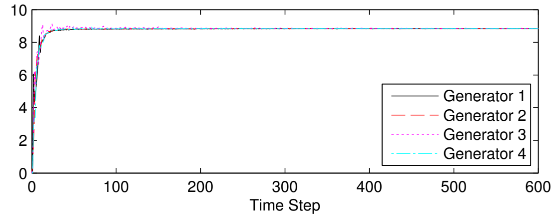
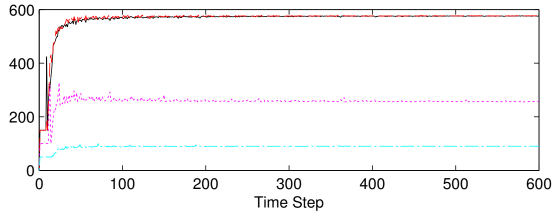
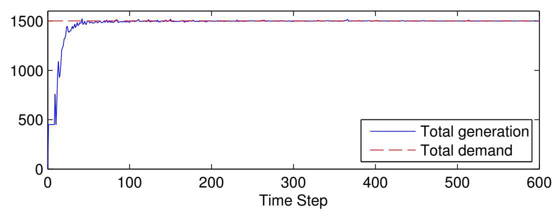


(a) Incremental cost (\$/MWh)

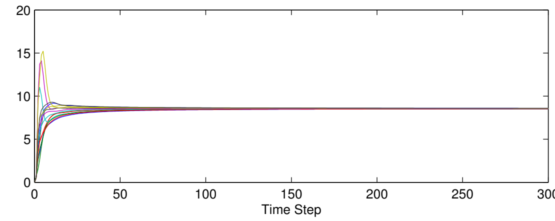


(b) Generation (MW)

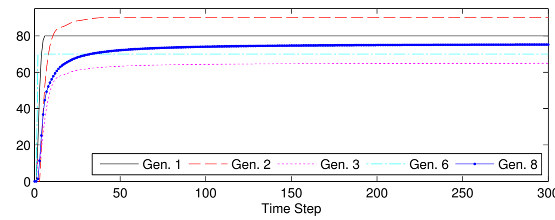


(c) Total generation vs. demand (MW)

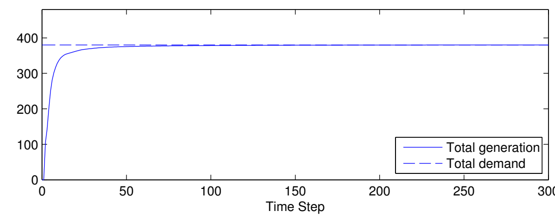
Fig. 5. Simulation results for the system of four generators over time-varying communication networks with time delays.



(a) Incremental cost (\$/MWh)



(b) Generation (MW)



(c) Total generation vs. demand (MW)

Fig. 6. Simulation results for the IEEE 118-bus over fixed networks.

1) Fixed topologies without delays: We first assume that the communication topology is the same as the physical network. The results with a step size of $\gamma(t) = \frac{0.6}{t}$ are plotted in Fig. 6, where Fig. 6(a) shows the evolution of incremental cost $\lambda_i(t)$, Fig. 6(b) shows the evolution of power generation $x_i(t)$, and Fig. 6(c) shows the evolution of the total generation in comparison of total demand. As can be seen, the incremental cost $\lambda_i(t)$ computed at each agent converges to the optimal value $\lambda^* = 39.38$ \$/MWh, which agrees with the result obtained by running the MATPOWER [34]. The generation at each generator bus also converges to the same optimal values as obtained by the MATPOWER. As $\lambda_i(t)$ and $x_i(t)$ converge to their optima, the total generation meets total demand $D = 4242$ MW.

2) Time-varying topologies with time-varying delays: In this case, the proposed algorithm is evaluated using a time-varying communication network with time delays. The communication network is modeled as a time-varying graph switching among two fixed topologies. In particular,

$$\mathcal{G}(t) = \begin{cases} \mathcal{G}_1, & \text{if } t \in [0,1) \cup [2,3) \cdots \cup [2s, 2s+1) \cdots, \\ \mathcal{G}_2, & \text{if } t \in [1,2) \cup [3,4) \cdots \cup [2s+1, 2s+2) \cdots, \end{cases}$$

where $s \in \mathbb{Z}_+$, $\mathcal{G}_1$ is the graph obtained by disconnecting Zone 1 and Zone 2 in [35, Figure 6], where IEEE 118-bus system have been partitioned into three different zones, and $\mathcal{G}_2$ is the graph obtained by disconnecting Zone 2 and Zone 3 in [35, Figure 6]. It is easy to check that each of the fixed topologies $\mathcal{G}_1$ and $\mathcal{G}_2$ is not connected. However, the time-varying graph $\mathcal{G}(t)$ is uniformly jointly connected since the joint graph $\mathcal{G}([t_0, t_0 + T))$ is connected for any $t_0 \geq 0$ with $T = 2$. Thus, Assumption 2 is satisfied with $T = 2$. According to Theorem 2, the proposed algorithm solves the EDP over the time-varying communication network with arbitrary time-varying communication delays with an upper bound. In this case study, we assume that the upper bound is $\bar{\tau} = 3$. In particular, at each time step, the time delay of each link has a probability of $1/4$ to be any integer in $\{0, 1, 2, 3\}$. Since time delays are stochastic, the dynamics at each agent vary from one simulation to another. Nevertheless, the proposed algorithm always converges to the optimal solutions. Simulation results for a particular run are plotted in Fig. 7. As can been seen, even in the presence of both time-varying communication links and communication time delays, each variable still converges to the same value as the case for the fixed communication network without time delays.

VI. Conclusions

This paper proposes a distributed algorithm based on the gradient push-sum method to solve the EDP over time-varying directed communication networks with time-varying delays. The cost functions are assumed to be convex rather than quadratic as in most existing studies. The proposed algorithm is fully distributed, without requiring global information of the system. Both theoretical proofs and simulation results showed that the proposed algorithm can solve the EDP over time-varying directed communication networks provided that the network is uniformly jointly strongly connected. Moreover, the proposed algorithm is also robust to arbitrarily large but bounded time-varying communication delays. One future work will focus on the robustness of the proposed distributed algorithm against unreliable communication links that may drop packets. Another interesting direction is to extend the

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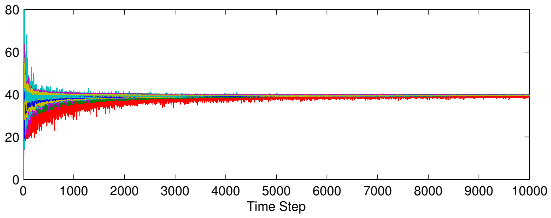
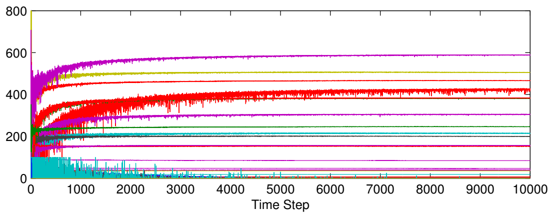
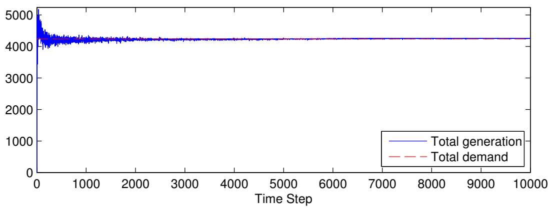


(a) Incremental cost ($/MWh)



(b) Generation (MW)



(c) Total generation vs. demand (MW)

Fig. 7. Simulation results for the IEEE 118-bus over time-varying networks with delays.

proposed distributed algorithm to accommodate additional physical constraints, such as transmission line loss and power flow and transmission line flow constraints.

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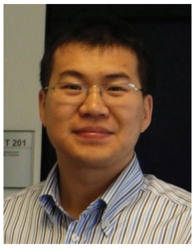

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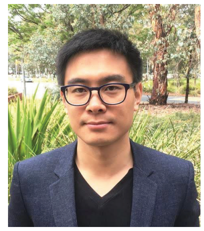

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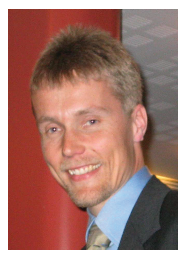

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