Performance Analysis of a Network of Event-Based Systems

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***Abstract*—We consider a scenario where multiple event-based systems use a contention resolution mechanism (CRM) to communicate with their respective controllers over a wireless network. We present a Markov model that captures the joint interactions of the event-triggering policy and the CRM. This model is obtained by decoupling interactions between the different systems in the network, drawing inspiration from Bianchi's analysis of IEEE 802.11. We present Monte-Carlo simulations that validate our model under various network configurations, and verify the accuracy of the performance analysis.**

***Index Terms*—Contention resolution mechanism (CRM).**

I. Introduction

Event-based systems provide an alternative to periodic systems, wherein only measurements that qualify as "events" are sent to the controller. These systems could result in fewer transmissions [1], [2], which is an important consideration when multiple closed-loop systems use a shared network to communicate with their respective controllers, as shown in Fig. 1(a). The shared network may be able to support more event-based systems than time-triggered ones, with comparable system performances. To design such a network, the interaction of multiple event-based systems must be understood and analyzed.

We present a method to model the interaction between multiple event-based systems that use the Carrier Sense Multiple Access (CSMA) protocol to access the shared network. Each event-based system generates events in a distributed manner, and transmission requests from these event-based systems cannot be anticipated and prescheduled in such a network. Hence, we use a Contention Resolution Mechanism (CRM) like CSMA to arbitrate access in a distributed, non-coordinated manner, between the nodes in the network. An unavoidable consequence of distributed access decisions is packet collisions. To mitigate the impact of packet collisions, the event-triggering policy can be adapted to the CRM response, as shown in Fig. 1(b). However, such systems are inherently harder to analyze.

There are two main challenges to be addressed. First, the event-generation rate from an adaptive event-triggering policy [which is proportional to the data rate Λ_j in Fig. 1(b)] is a function of the probability of collision p_c^j. Thus, the performance of such a network can only be characterized through a joint analysis of the event-triggering policy and the CRM. Secondly, closed-loop systems with exogenous noise processes become correlated due to their network interactions [3], [4], making the joint analysis a difficult task.

Our main contribution is a joint analysis of the event-triggering policy and the CRM. We derive inspiration from Bianchi's much-acclaimed analysis of the Distributed Coordination Function in IEEE 802.11. Here, an approximated system with decoupled interaction between the nodes in steady state [5] is analyzed, and shown to result in remarkably accurate performance estimates compared to the real system. An initial version of our work was presented in [6], where we used a Markov model to analyze the performance of a slot-synchronous network with a simplified CRM. In the current technical note, we motivate the construction of the Markov model. We show that Bianchi's approximation restores a renewal property in our setup, enabling the use of a Markov model to represent the interactions in the event-based network. Furthermore, we extend our analysis to asynchronous slots with a fully functional CRM. We demonstrate the potential of Bianchi's approximation, and its suitability to this analysis, by comparing our analytical results to results obtained from Monte Carlo simulations in each of these network configurations.

Event-based control systems were proposed as a means to reduce congestion [1], [2], [7]. Early work showed that the same control performance can be achieved using fewer samples with event-based systems, for a single system [1], [8]. Various event-triggering policies have been proposed for different problem formulations, see [9] for a review of the field. However, the multiple access problem for event-based systems has not received as much attention. Many event-triggered formulations implicitly require dynamic real-time scheduling, which is not well-suited to wireless networks [10], [11]. In [3], different multiple access protocols were compared for event-based systems, and the use of CSMA/CA was adjudged to result in the best performance through Monte Carlo simulations. This work also highlighted the issue of network-induced correlations. Due to these correlations, analyzing the performance of an event-based system in conjunction with a CSMA-based multiple access protocol proves intractable. Since then, many authors have proposed simplified models or approximations to adequately describe the interaction of multiple event-based systems. In [4], event-based systems using Aloha were analyzed by assuming that packet losses due to collisions could be modelled by an independent Bernoulli process. In [12], a simple steady state model was presented and analyzed for an idealized CSMA-like protocol that resulted in no collisions. In [13], event-based systems using Aloha and Slotted Aloha have been analyzed for an event-triggering policy that is not adapted to the network. In this technical note, we use Bianchi's approximation as a technique to analyze the performance of a network of adaptive event-based systems to a highly accurate degree, while still modelling collisions and correlations.

The technical note is organized as follows. We formulate the problem in Section II. We derive the network interaction model using Bianchi's approximation in Section III. We present the performance analysis in Section IV and simulation results in Section V.

II. Problem Formulation

We consider a network of M event-based control systems, which communicate over a shared sensing link. A model for the jth

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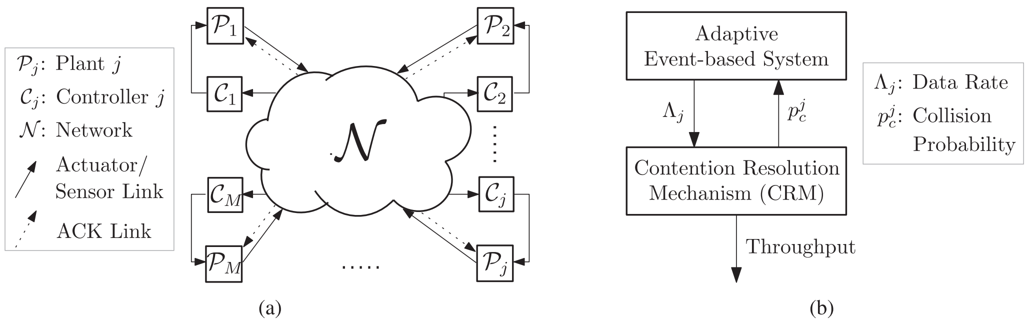

Fig. 1. On the left, a network of M closed-loop systems sharing a common medium on the sensing link. On the right, an adaptive event-based network stack showing that the input traffic (Λ_j) is adapted to the traffic in the network through p_c^j. (a) Control over a shared sensing link; (b) Event-based network stack.

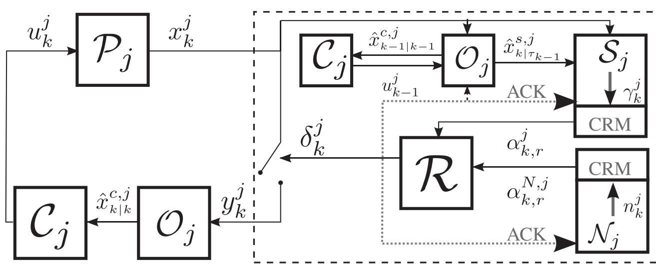

Fig. 2. Model of the jth event-based system in a network of other event-based systems. The event-triggering policy uses the prediction error to determine when to transmit. An ACK is needed to track the estimation error at the sensor node.

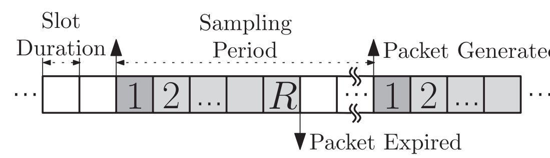

Fig. 3. Maximum of R retransmissions in R CRM slots, contained within a sampling period, is attempted for each packet.

event-based system in the network, for $j \in \{1, \ldots, M\}$, is depicted in Fig. 2 and explained below. A more detailed description can be found in [14].

Plant $\mathcal{P}_j$*:* The plant has state dynamics given by

$$x^j_{k+1} = A_j x^j_k + B_j u^j_k + w^j_k, \qquad \text{with } x^j_k \in \mathbb{R}^n \text{ and } u^j_k \in \mathbb{R}^m. \quad (1)$$

The initial state x^j_0 and the process noise w^j_k are i.i.d. zero-mean Gaussians with covariances R^j_0 and R^j_w, respectively. They are independent to all other initial states and process noises in the network. The plants are sampled T seconds apart, governed by a network clock.

State-Based Scheduler: $\mathcal{S}_j$: A local scheduler situated in the sensor node generates a scheduler output $\gamma^j_k \in \{0, 1\}$, denoting the absence or presence of an event, respectively, as given by

$$\gamma^j_k = \begin{cases} 1 & \left| x^j_k - \hat{x}^{s,j}_{k|\tau_{k-1}} \right|^2 > \Delta_j(m^j_k), d^j_{k-1} < F \\ 1 & \left| x^j_k - \hat{x}^{s,j}_{k|k-F} \right|^2 > \Delta_j(m^j_k), d^j_{k-1} >= F \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The event threshold Δ_j depends on the memory index $m^j_k = \min(d^j_{k-1}, F)$, which tracks the delay since the last received packet d^j_{k-1} up to the maximum memory index F. Thus, we have a memory-limited event-triggering policy that adapts to the prediction error $x^j_k - \hat{x}^{s,j}_{k|\tau_{k-1}}$ for $d^j_{k-1} < F$, where $\hat{x}^{s,j}_{k|\tau_{k-1}}$ is the predicted estimate at the sensor node defined in (5) below and τ^j_{k-1} is the time index of the last successful transmission. For $d^j_{k-1} \geq F$, the state x_{k-F} is assumed to have been transmitted successfully, and the prediction error $x^j_k - \hat{x}^{s,j}_{k|k-F}$ is used to determine γ^k_j. To avoid notational overhead, we skip the index j for τ_k and d_k, when the context is clear.

CRM: The network uses p-persistent CSMA with R retransmissions [15]. The retransmissions occur in the CRM slots, as indicated in Fig. 3, and all retransmissions are completed before the next sampling period. The access indicator for the rth retransmission $\alpha^j_{k,r} \in \{0, 1\}$ is randomly set with persistence probability $p_{\alpha,r} = \mathbf{P}(\alpha^j_{k,r} = 1 | \gamma^j_k = 1)$.

Network Traffic $\mathcal{N}$*:* This block models traffic from all other event-based systems in the network. Events from other systems are indicated by $n^j_k \in \{0, 1\}$, with $\alpha^{N,j}_{k,r}$ denoting the access indicator for each retransmission attempt.

Sensing Channel: The channel access indicator $\delta^j_k \in \{0, 1\}$ denotes transmission failure or success, respectively, after R retransmission attempts. A transmission is successful if there is only one system accessing the channel per CRM slot, and thus $\sum_{j=1}^{M} \delta^j_k \leq R$. The channel access indicator is given by

$$\delta^j_k = \bigvee_{r=1}^{R} \left[\alpha^j_{k,r} \cdot \left(1 - \alpha^{N,j}_{k,r} \right) \right]. \quad (3)$$

Observer $\mathcal{O}_j$*:* The observer receives y^j_k, as given by

$$y^j_k = \begin{cases} x^j_k & \delta^j_k = 1 \\ \varepsilon & \text{otherwise} \end{cases}$$

where ε denotes a packet erasure when there is no transmission. The estimate $\hat{x}^{c,j}_{k|k}$ is given by

$$\hat{x}^{c,j}_{k|k} = \begin{cases} x^j_k & \delta^j_k = 1 \\ A_j \hat{x}^{c,j}_{k-1|k-1} + B_j u^j_{k-1} & \text{otherwise} \end{cases} \quad (4)$$

where $\hat{x}^{c,j}_{-1|-1} = 0$. In [14], $\hat{x}^{c,j}_{k|k}$ has been shown to be the minimum mean-square error estimate, with the estimation error defined as $\tilde{x}^j_k = x^j_k - \hat{x}^{c,j}_{k|k}$. The predicted estimate at the sensor node, for $k > \ell$, is given by

$$\hat{x}^{s,j}_{k|\ell} = A_j^{(k-\ell)} x^j_\ell + \sum_{l=\ell}^{k-1} A_j^{(k-l-1)} B_j u^j_l. \quad (5)$$

Controller $\mathcal{C}_j$*:* The controller is chosen to minimize the Linear Quadratic Gaussian (LQG) control cost, defined as $\lim_{N\to\infty}(1/N) \sum_{n=0}^{N-1} \mathbf{E}[x_n^T Q_1 x_n + u_n^T Q_2 u_n]$, where Q_1 and Q_2 are positive-definite state and control weighting matrices, respectively. Certainty

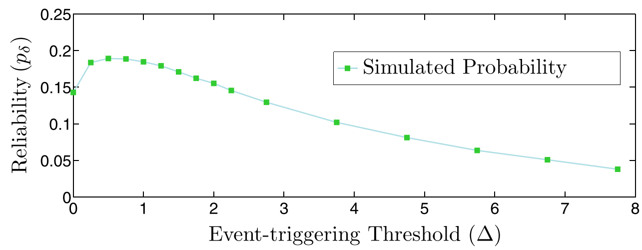

Fig. 4. Simulations show that the reliability-threshold curve cannot be approximated by simple modelling techniques.

equivalence has been shown to hold for this problem [14]. Thus, the controller uses the certainty equivalent control gain L_k^j to generate the control signal

$$u_k^j = -L_k^j \hat{x}_{k|k}^{c,j}. \tag{6}$$

We seek analytical expressions for the following two metrics that characterize the network performance and are a prerequisite for computing the LQG cost for a system in this network.

Definition 2.1 (Delay Distribution): The distribution of the delay between successive transmissions in steady-state is defined as $\mathbf{P}_d^j(\zeta) \triangleq \lim_{k\to\infty} \mathbf{P}(\tau_k^j - \tau_{k-1}^j = \zeta)$ for $\zeta \in \mathbb{Z}$.

Definition 2.2 (Reliability): The reliability is the steady state probability of a successful transmission due to the event-triggering policy and CRM, i.e., $p_\delta^j \triangleq \lim_{k\to\infty} \mathbf{P}(\delta_k^j = 1)$.

Next, we look at an example to motivate the methods used in this technical note.

Example 2.1: We consider a homogenous network of $M = 10$ nodes, with $R = 5$ retransmissions in the CRM. The network consists of identical first-order plants with $A = 1$ and $B = 1$. The event-triggering policy uses a constant threshold Δ. The persistence probability is $p_\alpha = 0.2$ for all retransmission attempts. We use Monte-Carlo simulations to numerically evaluate the reliability p_δ for $\Delta \in (0, 8)$, and plot the outcome in Fig. 4. The non-linear relationship in the figure shows that there is no simple loss process that captures the interaction of a single system with the rest of the network. We return to this example in Section V, and comment on the non-monotonic relationship and how our new model can capture it.

III. The Network Interference Model

We begin by examining the phenomenon of network-induced correlation. Then we introduce Bianchi's approximation and use this to construct a network interaction model.

A. Network-Induced Correlation and Bianchi's Approximation

Lemma 3.1: For the system (1)–(6), the estimation errors corresponding to different plants in the network are correlated, i.e., $\mathbf{P}(\tilde{x}_k^1, \ldots, \tilde{x}_k^M) \neq \prod_{j=1}^M \mathbf{P}(\tilde{x}_k^j)$.

Proof: A network that supports R retransmissions must satisfy the constraint $\sum_{j=1}^M \delta_k^j \leq R$. Due to this, and the definition of δ_k in (3), for each plant in the network, we have

$$\mathbf{P}\left(\delta_k^j | \gamma_k^1, \ldots, \gamma_k^j = 1, \ldots, \gamma_k^M\right) \neq \mathbf{P}\left(\delta_k^j | \gamma_k^j = 1\right). \tag{7}$$

Then, from the estimate in (4), the result follows. ■

Consequently, $e_k^j := \{\tilde{x}_{\tau_k}^j, \ldots, \tilde{x}^{j_k}\}$ is not a Markovian process and the inter-arrival times at the observer are not independent. In general, the stationary distributions of the system variables are not independent due to the network-induced correlations. To evaluate the control cost incurred by the control systems in the network, the joint distribution of all the states are required. Numerical methods such as gridding and evolving of the distributions are the only alternative to Monte Carlo simulations, as has been noted earlier [3], [4]. In contrast, we use an approximation motivated by Bianchi's seminal work [5] to simplify the analysis.

Approximation 3.1: Under Bianchi's approximation, the conditional probability of a collision in the rth retransmission in steady state is given by an independent probability p_r, i.e.

$$\lim_{k\to\infty} \mathbf{P}\left(\delta_k^j = 0 | \gamma_k^j = 1, \alpha_{k,r}^j = 1\right) = p_r^j \tag{8}$$

for all $j \in \{1, \ldots, M\}$ and all $r \in \{1, \ldots, R\}$.

Previously, this approximation has been very successfully used to decouple interacting systems in steady state, particularly in adaptive networks. A primary example of this is the analysis of the IEEE 802.11 protocol [5]. A theoretical validation of Bianchi's approximation has been presented in [16]. In this technical note, we use Bianchi's approximation in analyzing networked control systems, and demonstrate the accuracy of our analytical results in Section V. Now, we use this approximation to establish a few desirable properties.

Theorem 3.2: For a system (1)–(6), under Bianchi's approximation, $e_k^j = \{\tilde{x}_{\tau_k}^j, \ldots, \tilde{x}_k^j\}$ is a Markov process, i.e., $\lim_{k\to\infty} \mathbf{P}(e_k^j | e_0^j, \ldots, e_{k-1}^j) = \lim_{k\to\infty} \mathbf{P}(e_k^j | e_{k-1}^j)$, for all $j \in \{1, \ldots, M\}$.

Proof: Re-examining (7) with (8), we get

$$\mathbf{P}\left(\delta_k^j | \gamma_k^1, \ldots, \gamma_k^j = 1, \ldots, \gamma_k^M\right) = \sum_{\alpha_k^j \in \{0,1\}} \mathbf{P}\left(\delta_k^j | \gamma_k^j = 1, \alpha_k^j\right) \cdot \mathbf{P}\left(\alpha_k^j | \gamma_k^j = 1\right)$$

which is equal to $p^j \cdot p_\alpha + 1 \cdot q_\alpha$ for $\delta_k^j = 0$, and $q^j \cdot p_\alpha$ for $\delta_k^j = 1$. Thus, the dependence on the other scheduler outputs vanishes, and the estimation error is independent of its past following a successful transmission, i.e., $\lim_{k\to\infty} \mathbf{P}(\tilde{x}_{\tau_k+1|\tau_k+1}^j | \tilde{x}_{\tau_k|\tau_k}^j) = \lim_{k\to\infty} \mathbf{P}(\tilde{x}_{\tau_k+1|\tau_k+1}^j)$. Thus, we establish the Markovian property of e_k^j. ■

Consequently, the inter-arrival times at the observer for each plant are independent. Bianchi's approximation hence restores a renewal property of the estimation error, which was lost due to network-induced correlations. We now proceed to analyze the performance of this network.

B. Markov Chain Representation

We use Bianchi's approximation to construct the Markov model in Fig. 5 for the event-triggering policy and p-persistent CSMA with R retransmissions. We skip the index j as we present a model for a single system in the network. We denote each state in the Markov model by (S, m) or (S, m, r), and its stationary probability by $\pi_{(S,m)}$ or $\pi_{(S,m,r)}$, respectively. The index m denotes the memory of the scheduler, r denotes the retransmission attempt in the CRM and S denotes the following states: *a)* $S = I$ is the *idle* state reached before the next sampling instant, *b)* $S = N$ is the *non-event* state reached when the scheduler output $\gamma_k = 0$, *c)* $S = E$ is the *event* state reached when $\gamma_k = 1$, and *d)* $S = T$ is the *transmission* state reached when $\alpha_{k,r} = 1$.

In Fig. 5, $p_{\alpha,r}$ denotes the persistence probability of the CRM and p_r denotes Bianchi's conditional collision probability. The event probability $p_{\gamma,m+1}$ is given by

$$\mathbf{P}\left(\left|\sum_{l=k-(m+1)}^{k-1} A^{(k-l-1)} w_l\right|^2 > \Delta(m_k) \middle| d_{k-1} = m_k\right) \tag{9}$$

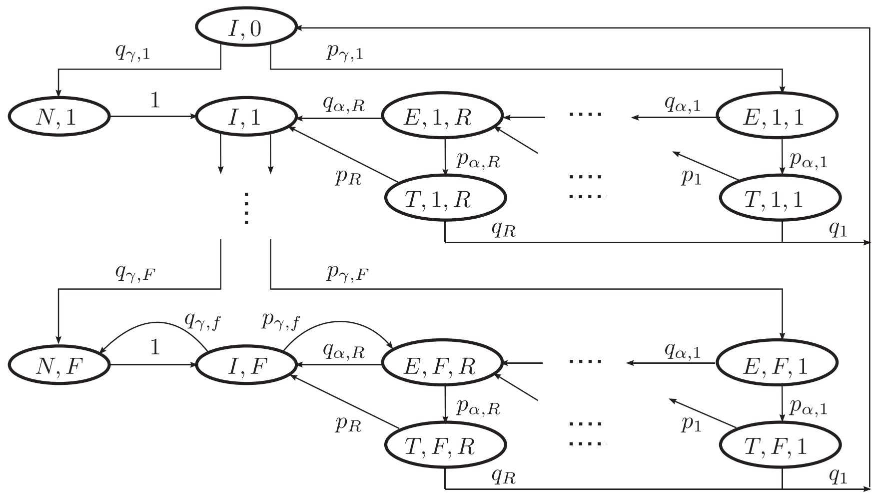

Fig. 5. Markov chain representation for the event-triggering policy in (2) and the p-persistent CSMA with R retransmissions.

for $m < F$, with $p_{\gamma,f} = p_{\gamma,F+1}$ and complimentary probabilities $q_{\gamma,m}$. Note the one-to-one correspondence between the sets of event thresholds and event probabilities. The event probabilities can be computed numerically by noting that $\sum_{m=1}^{F} A^{(k-m)} w_m$ corresponds to a Gaussian process recursively truncated at the event thresholds. Note that each system in the network has its own Markov chain, as in Fig. 5, and the interactions between these systems result in the collision process in (8).

IV. Steady State Performance Analysis

We now use the stationarity aspect of Bianchi's approximation to derive a steady state analysis for the Markov model in Fig. 5. We also extend our analysis to asynchronous networks.

A. Performance Analysis

Theorem 4.1: The reliability p_δ^j of a system (1)–(6), under Bianchi's approximation, is given by

$$\frac{1}{1 + p_{\mathrm{crm},R} p_{\gamma,1} + \cdots + p_{\mathrm{crm},R}^{F-1}\, p_{\gamma,1:m} + p_{\mathrm{crm},R}^{F}\, p_{\gamma,1:m+1}} \tag{10}$$

where $p_{\mathrm{crm},N} = \prod_{r=1}^{N}(1 - p_{\alpha,r} q_r)$ for $N \leq R$, $p_{\gamma,1:m} = \prod_{m=1}^{F-1} p_{\gamma,m}$ and $p_{\gamma,1:m+1} = \prod_{m=1}^{F} p_{\gamma,m}/(1 - p_{\mathrm{crm},R} p_{\gamma,f})$.

Proof: We begin by evaluating the stationary probabilities in the Markov chain, and obtain $\pi_{(I,m)} = p_{\mathrm{crm},R} p_{\gamma,m} \pi_{(I,m-1)}$ for $0 < m < F$ and $\pi_{(I,F)} = (p_{\mathrm{crm},R} p_{\gamma,F})/(1 - p_{\mathrm{crm},R} p_{\gamma,f}) \cdot \pi_{(I,F-1)}$. Then, $\pi_{(T,m,r)} = p_{\mathrm{crm},r-1} p_{\alpha,r} p_{\gamma,m} \pi_{(I,m-1)}$. The conditional collision probability in the rth retransmission attempt is given by

$$p_r^j = 1 - \prod_{i \neq j, i=1}^{M} \left(1 - \left(\sum_{m=1}^{F} \pi_{(T,m,r)}\right)^i\right), \text{ for } r \in \{1, \ldots, R\}. \tag{11}$$

At any sampling instant, a node must be in one of the (I, m) states, giving us $\sum_{m=0}^{F} \pi_{(I,m)} = 1$. Solving the above equations, and using the fact that a successful node transitions to the state $(I, 0)$, we get the expression for p_δ^j in (10). ■

The delay distribution can be derived from the above Markov model, as shown below.

Corollary 4.2: The delay distribution for (1)–(6), under Bianchi's approximation, is given by

$$\mathbf{P}_d^j(\zeta) = \begin{cases} \pi_{(I,\zeta)}^j p_{\gamma,\zeta+1}^j (1 - p_{\mathrm{crm},R}), & \zeta < F \\ \pi_{(I,F-1)}^j \bar{p}_{\gamma,F}^j \left(\bar{p}_{\gamma,f}^j\right)^{\zeta - F} p_{\gamma,f}^j (1 - p_{\mathrm{crm},R}), & \zeta \geq F. \end{cases} \tag{12}$$

where $\bar{p}_{\gamma,F}^j = (q_{\gamma,F}^j + p_{\gamma,F}^j p_{\mathrm{crm},R})$ and $\bar{p}_{\gamma,f}^j = (q_{\gamma,f}^j + p_{\gamma,f}^j p_{\mathrm{crm},R})$.

The above result can be shown by applying Definition 2.1, and using the expressions from the proof of Theorem 4.1.

B. Asynchronous Networks

In an asynchronous network, each system may initiate sampling at a different CRM slot. When the sampling slots are spread apart, performance is typically improved as the interference in each slot is limited to the nodes that sampled within the previous $R - 1$ slots. The sampling slots chosen by all the nodes must be known to analyze the performance in a combinatorial manner. By assuming a uniform selection of the initial sampling instants, we can compute the conditional collision probability by averaging across all possible combinations of interactions in each retransmission state. This gives us a uniform conditional collision probability across all retransmission attempts, i.e., $p_r^j = p^j$ for $r = 1, \ldots, R$, as derived below.

Theorem 4.3: In an asynchronous network of systems (1)–(6), under Bianchi's approximation

$$p^j = 1 - \prod_{i \neq j, i=1}^{M} \left(1 - \frac{1}{R} \sum_{r=1}^{R} \sum_{m=1}^{F} \pi_{(T,m,r)}^i\right). \tag{13}$$

Proof: To evaluate p^j, we average across all possible transmissions seen by the state (T, m, r) of the jth node, for some m and r. There are R^{M-1} different combinations of interactions between the R retransmission stages of the other $M - 1$ nodes in the network, due to different initial sampling slots. Suitably averaging across these, we obtain the above result. ■

The reliability and delay distribution can be computed using (13) in place of (11).

V. Examples

We present a number of examples to validate our analysis and to demonstrate the potential of Bianchi's approximation. In each case, we present the reliability obtained through Monte-Carlo simulations, and compare it to the analytical value obtained using the analysis presented above. We show that the differences are negligible, thus indicating that Bianchi's approximation may prove to be a valid model for the interactions in a network of event-based systems. We use the LQG cost as the performance metric for a given event-triggering policy.

Parameter	Simulation	Analysis
p_δ	0.1840	0.1872
p_1	0.5937	0.5944
p_2	0.5655	0.5620
p_3	0.5367	0.5277
p_4	0.5076	0.4917
p_5	0.4778	0.4542

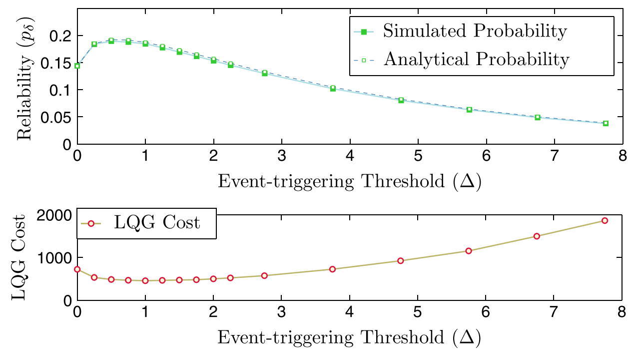

Fig. 6. Table on the left compares analytical and simulated values of the conditional collision probability. The figure on the right compares simulated and analytical values of reliability versus the event threshold. This example validates the use of Bianchi's approximation as a modelling technique for this networked control problem with CRM.

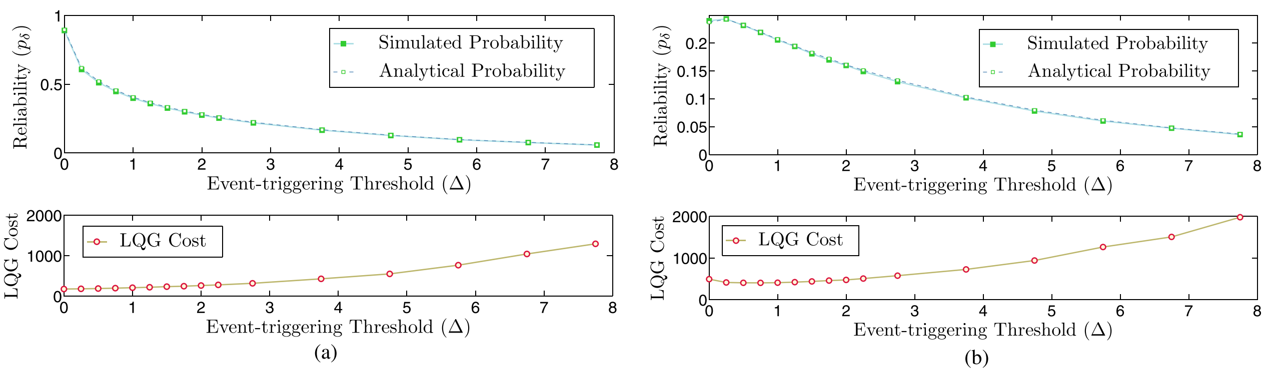

Fig. 7. Comparison of the analytical and simulated values of the reliability versus the scheduler threshold, with retransmissions in the CRM for two different networks. The example on the left validates Approximation 3.1 for an unsaturated synchronous network. The example on the right validates Approximation 3.1 for an asynchronous network. (a) Synchronous unsaturated network; (b) Asynchronous network.

Example 5.1: We now return to Example 2.1, and apply our analysis to this setup. The event probabilities are given to be $p_{\gamma,m} = [0.3171\ 0.5138]$ for $m = 1, \ldots, M$. To implement the event-triggering policy, we need to identify event thresholds corresponding to $p_{\gamma,m}$. Computing these thresholds explicitly is not easy, as the estimation error distributions are truncated Gaussians. We numerically select thresholds that result in these event probabilities by simulating the evolution of the estimation error in this network. In fact, $\Delta = 1$ achieves $p_{\gamma,m}$.

The simulated values of the conditional collision probability agree closely with the analytical values computed using Theorem 4.1, as shown in the table in Fig. 6. A comparison of analytical and simulated values of the reliability versus the threshold for this synchronized network is shown by the plot in Fig. 6, again illustrating the close match. The LQG control performance obtained from the network is, as expected, poor due to synchronization and congestion. Low thresholds cause many packets to generate events, resulting in congestion and low reliability. High thresholds result in low reliability due to insufficient transmissions.

Example 5.2: We consider a network with $M = 2$ nodes and $R = 5$ retransmissions in the CRM. The plant model and event-triggering policy are the same as before. A comparison of the reliability obtained for different thresholds is shown in Fig. 7(a), validating the use of Approximation 3.1 even in sparse traffic.

Example 5.3: We consider an asynchronous network, with $M = 5$ nodes, a sampling period of $T = 3$ slots and $R = 2$ retransmissions in the CRM. The plant model and event-triggering policy are again the same. The persistence probabilities of the CRM are $p_{\alpha,1} = p_{\alpha,2} = 0.4$. A comparison of the reliabilities obtained for various thresholds is shown in Fig. 7(b), thus validating the accuracy of the formula in Theorem 4.3. Note that the highest reliability obtainable from this system may far exceed the average reliability shown in Fig. 7(b).

Similar examples validate Bianchi's approximation for unsaturated asynchronous networks and heterogeneous networks [17]. We have not included these here due to space constraints.

VI. Conclusion

We have presented a method to analyze the performance of a network of event-based systems that use the CSMA protocol to access the shared network. We have shown that a Markov model can be constructed using Bianchi's approximation to represent the interaction of multiple event-based systems in this network. Our simulation results demonstrated the accuracy of this analysis, validating for the first time the use of Bianchi's approximation as a modelling technique for networked control systems. For future work, we wish to use the insights obtained from this work to design adaptive event-triggering policies.

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