



# Distributed non-cooperative robust MPC based on reduced-order models

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## Abstract

In this paper, a non-cooperative distributed MPC algorithm based on reduced order model is proposed to stabilize large-scale systems. The large-scale system consists of a group of interconnected subsystems. Each subsystem can be partitioned into two parts: measurable part, whose states can be directly measured by sensors, and the unmeasurable part. In the online computation phase, only the measurable dynamics of the corresponding subsystem and neighbour-to-neighbour communication are necessary for the local controller design. Satisfaction of the state constraints and the practical stability are guaranteed while the complexity of the optimization problem is reduced. Numerical examples are given to show the effectiveness of this algorithm.

**Keywords:** Model predictive control, distributed control, building energy efficiency

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## 1 Introduction

Due to its capability to handle hard constraints on state, control input and output explicitly, and optimize the performance of the system with respect to the cost function [1–3], model predictive control (MPC) has received increasing attention in the last decades. At each time instant, the controller is required to solve a finite horizon optimal control problem and the first element

of the control sequence is applied to the plant. Then the optimization problem is reformulated when a new measurement comes and it is solved again. Traditionally, to compute the optimal control input, the controller needs the full model and state/output information of the system, and the optimization problem is formulated and solved in a centralized manner. However, when interconnected large scale systems such as power systems [4], traffic networks [5], biology systems [6]

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and building systems [7] are considered, it is usually not practical for the controller to know the full model and state/output information, and solve the optimization problem in reasonable time. To overcome these obstacles, distributed model predictive control (DMPC) is proposed in a lot of literatures [8–11].

Usually, in the DMPC setup, the model complexity mainly comes from the large number of interconnected subsystems. In some industrial control applications, each subsystem may also have complex dynamics. For example, the model of heating, ventilation and air-conditioning (HVAC) systems in buildings are usually obtained by discretization of partial differential equations [12, 13]. To guarantee high accuracy, the number of the grids should be large enough and the resulted ordinary differential equations usually have high dimension. Furthermore, because of the limited number of sensors, it is not practical to measure every state of a HVAC system. Therefore, even though the HVAC system in a building can be partitioned into connected subsystems by zones and rooms, the complexity of each subsystem should be reduced further. Model reduction techniques have been proposed in several literatures to simplify system dynamics at the expense of some modelling error [14, 15].

In this paper, we consider a distributed control problem for a large-scale interconnected system where its subsystem also has complex dynamics. Based on the similar idea in [16], the dynamics of each subsystem is assumed to be partitioned to the main dynamics and the minor one. Only the states of the main dynamics are assumed to be measurable. Then for each subsystem, an MPC controller is designed only based on the main dynamics, by which the complexity of the controller is reduced. The modelling errors are handled as disturbances. Then it is shown that this problem can be cast into robust distributed MPC by using an algorithm proposed in [11]. In [11] and [17], the authors studied the distributed algorithms and the implementation issues for large-scale linear systems. Compared with these two works, in this paper, we consider a more complex model and a model reduction technique is introduced to reduced the computational burden.

The remainder of this paper is organized as follows. In Section 2, the problem is formulated. In Section 3 and 4, the design and implementation procedures of the MPC controller are given. In Section 5, a numerical example is given to illustrate our algorithm. Finally, some conclusions are drawn in Section 6.

Some remarks on notations are given as follows.  $\mathbb{R}$  and  $\mathbb{N}$  are used to denote the real number set and the natural number set, respectively.  $\mathbf{0}$  denotes a zero matrix with proper dimension. The  $m \times n$  dimensional space is denoted as  $\mathbb{R}^{m \times n}$ . A matrix is Schur stable if its eigenvalues lie in the interior of the unit circle. The Minkowski sum of sets  $A$  and  $B$  is defined as  $C = A \oplus B = \{c = a+b, \forall a \in A \text{ and } b \in B\}$ . The Pontryagin difference of sets  $A$  and  $B$  is defined as  $C = A \ominus B = \{c | c+b \in A, \forall b \in B\}$ . For a discrete-time signal  $s_k$  and  $a, b \in \mathbb{N}$ ,  $a < b$ , we denote  $s[a : b]$  as  $(s_a, s_{a+1}, \dots, s_b)$ . Given a generic compact set  $\mathcal{X}$ ,  $\mathcal{Y} = \text{box}(\mathcal{X})$  is the smallest hyperrectangle containing  $\mathcal{X}$  with faces perpendicular to the cartesian axis.

## 2 Problem formulation

Consider a linear discrete-time system

$$x(k + 1) = Ax(k) + Bu(k) + Dw(k), \tag{1}$$

where  $x(k) \in \mathbb{R}^n$  is the state of the system,  $u(k) \in \mathbb{R}^m$  is the control input and  $w(k) \in \mathbb{R}^p$  is the external disturbance.

The system can be partitioned into  $N$  linear, discrete-time, interconnected non-overlapping subsystems. For each subsystem  $\mathcal{S}_i$ , the system dynamics is given by

$$x_i(k + 1) = A_{i,i}x_i(k) + B_iu_i(k) + \sum_{j \in \mathcal{N}_i^1} A_{i,j}x_j(k) + D_iw_i(k), \tag{2}$$

where  $x_i(k) \in \mathbb{R}^{n_i}$ ,  $u_i(k) \in \mathbb{R}^{m_i}$  and  $w_i(k) \in \mathbb{R}^{w_i}$  are the state, the control input and the bounded external disturbance for subsystem  $i$  and  $\mathcal{N}_i^1 = \{j | A_{i,j} \neq \mathbf{0}, j = 1, \dots, N, j \neq i\}$ . According to this decomposition, we have  $x(k) = (x_1(k), \dots, x_N(k))$ ,  $u(k) = (u_1(k), \dots, u_N(k))$ ,

$$w(k) = (w_1(k), \dots, w_N(k)), A = \begin{pmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{pmatrix}, B = \text{diag}\{B_1, \dots, B_N\} \text{ and } D = \text{diag}\{D_1, \dots, D_N\}.$$

Each  $x_i(k)$  can be further partitioned into two parts: measurable part  $x_i^1(k) \in \mathbb{R}^{n_{1,i}}$  and unmeasurable part  $x_i^2(k) \in \mathbb{R}^{n_{2,i}}$ , where  $n_{1,i} \ll n_{2,i}$ . The matrices  $A_{i,j}$ ,  $B_i$  and  $D_i$  have the corresponding structures:

$$A_{i,j} = \begin{pmatrix} A_{i,j}^{11} & A_{i,j}^{12} \\ A_{i,j}^{21} & A_{i,j}^{22} \end{pmatrix}, B_i = \begin{pmatrix} B_i^1 \\ B_i^2 \end{pmatrix}, D_i = \begin{pmatrix} D_i^1 \\ D_i^2 \end{pmatrix}.$$

The initial value of  $x_i^2(k)$  is bounded by a known poly-

hedron  $\Delta_i$ . The external disturbance  $w_i(k)$  is bounded by  $\mathcal{W}_i$ . The matrix  $A_{i,i}^{22}$  is Schur. The objective of this paper is to design a distributed control law to stabilize the system while enforcing the local constraints  $x_i^1(k) \in \mathcal{X}_i, u_i(k) \in \mathcal{U}_i$  and the joint constraints  $(x_1^1(k), \dots, x_N^1(k), u_1(k), \dots, u_N(k)) \in \mathcal{J}_l, l = 1, \dots, n_c$ . The sets  $\mathcal{W}_i, \mathcal{X}_i, \mathcal{U}_i$  and  $\mathcal{J}_l$  are polyhedron containing the origin in their interiors. If the projection of the polyhedron  $\mathcal{J}_l$  on  $(x_i^1, u_i)$  is not the whole hyperplane,  $(x_i^1, u_i)$  is called an argument of  $\mathcal{J}_l$ . If  $(x_i^1, u_i)$  and  $(x_j^1, u_j)$  are arguments of  $\mathcal{J}_l, \mathcal{J}_l$  is defined as a joint constraint on subsystem  $i$  and  $j$ . We call subsystem  $j$  a neighbor of subsystem  $i$  if  $A_{i,j} \neq 0$  and/or they have at least one common joint constraint  $h_m$ . We do not consider constraint on  $x_i^2(k)$ .

### 3 Controller design

For simplicity, in this section, we assume that  $D_i^2, A_{i,j}^{12}$  and  $A_{i,j}^{22}$  are 0,  $\forall i \neq j$ .

#### 3.1 Bounded unmeasurable state

First we rewrite the dynamics of  $x_i^2(k)$  into the following form:

$$\begin{aligned} x_i^2(k+1) &= A_{i,i}^{22}x_i^2(k) + d_i^2(k), \\ d_i^2(k) &= A_{i,i}^{21}x_i^1(k) + B_1^1u_i(k) + \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{21}x_j^1(k). \end{aligned}$$

Thus we obtain

$$x_i^2(k) = (A_{i,i}^{22})^k x_i^2(0) + \sum_{l=0}^{k-1} (A_{i,i}^{22})^{k-1-l} d_i^2(l). \quad (3)$$

Considering the bounds on  $x_i(k), x_j(k)$  and  $u_i(k)$ , we know that  $d_i^2(k) \in A_{i,i}^{21}\mathcal{X}_i \oplus B_1^1\mathcal{U}_i \oplus \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{21}\mathcal{X}_j \triangleq \mathcal{D}_i^2$ . Therefore by (3), we have  $x_i^2(k) \in (A_{i,i}^{22})^k \Delta_i \oplus \sum_{l=0}^{k-1} (A_{i,i}^{22})^l \mathcal{D}_i^2 \triangleq \mathcal{S}_i^2(k)$ . An over-approximation  $\mathcal{S}_i^2(\infty)$  for all  $\mathcal{S}_i^2(k), k \in \mathbb{N}$  can be easily given by  $\Delta_i \oplus (I - A_{i,i}^{22})^{-1} \mathcal{D}_i^2$ .

Rewrite the dynamics of  $x_i^1(k)$  in the following form:

$$x_i^1(k+1) = A_{i,i}^{11}x_i^1(k) + B_i^1u_i(k) + \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{11}x_j^1(k) + d_i^1(k), \quad (4)$$

$$d_i^1(k) = A_{i,i}^{12}x_i^2(k) + D_i^1w_i(k).$$

From the above discussion, we know that  $d_i^1(k) \in \mathcal{S}_i^2(\infty) \oplus D_i^1\mathcal{W}_i \triangleq \mathcal{D}_i^1$ . This fact implies that  $d_i^1(k)$  can

be treated as an additive bounded disturbance to  $x_i^1(k)$ . Therefore, distributed robust MPC can be designed to stabilize this system.

#### 3.2 Distributed robust MPC

Consider the nominal subsystem model

$$\hat{x}_i(k+1) = A_{i,i}^{11}\hat{x}_i(k) + B_i^1\hat{u}_i(k) + \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{11}\tilde{x}_j(k), \quad (5)$$

where  $\hat{x}_i(k)$  is the nominal state trajectory,  $\hat{u}_i(k)$  is the nominal control input and  $\tilde{x}_j(k)$  is the reference state trajectory which will all be specified later.

Before providing the distributed robust MPC algorithm, the following assumption on decentralized stabilizability is necessary.

**Assumption 1** There exist matrices  $K_i, i = 1, \dots, N$  such that i)  $A_{K_i} = A_{i,i}^{11} + B_i^1K_i$  is Schur, ii)  $\hat{A} + \hat{B}K$  is

Schur, where  $\hat{A} = \begin{pmatrix} A_{1,1}^{11} & \dots & A_{1,N}^{11} \\ \vdots & & \vdots \\ A_{N,1}^{11} & \dots & A_{N,N}^{11} \end{pmatrix}, \hat{B} = \text{diag}\{B_1^1, \dots, B_N^1\}$  and  $K = \text{diag}\{K_1, \dots, K_N\}$ .

The control law for the  $i$ th subsystem (2) is given by

$$u_i(k) = \hat{u}_i(k) + K_i(x_i^1(k) - \hat{x}_i(k)). \quad (6)$$

By defining  $e_i(k) = x_i^1(k) - \hat{x}_i(k)$ , (4), from (5) and (6), we have

$$e_i(k+1) = A_{K_i}e_i(k) + \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{11}(x_j^1(k) - \tilde{x}_j(k)) + d_i^1(k). \quad (7)$$

Due to the stability of the matrix  $A_{K_i}$  and the boundedness of  $d_i(k)$ , if it can be further guaranteed that  $x_j^1(k) - \tilde{x}_j(k)$  are bounded,  $\forall j = 1, \dots, N$ , there exists a robust positively invariant set  $\mathcal{E}_i$  for (7), such that  $\forall e_i(k) \in \mathcal{E}_i, e_i(k+1) \in \mathcal{E}_i$  which implies that if  $\hat{x}_i(k) \rightarrow 0$  as  $k \rightarrow \infty, x_i^1(k) \rightarrow \mathcal{E}_i$  as  $k \rightarrow \infty$ .

At each time instant  $t$ , each subsystem  $\mathcal{S}_i$  transmits its future state and control reference trajectory over the prediction horizon  $N_c$  to its neighbors  $\mathcal{N}_i^1$ , and the MPC controller of each subsystem  $\mathcal{S}_i$  solves the following  $i$ th optimization problem:

$$\min_{\hat{x}_i(t), \hat{u}_i[t:t+N_c-1]} V_i^N$$

subject to (5),

$$x_i^1(t) - \hat{x}_i(t) \in \mathcal{E}_i, \quad (8)$$

and for  $k = t, \dots, t + N_c - 1$  and  $l = 1, \dots, n_c$ ,

$$\begin{aligned} \hat{x}_i(k) - \tilde{x}_i(k) &\in \mathcal{Z}_i, & (9) \\ \hat{u}_i(k) - \tilde{u}_i(k) &\in \mathcal{S}_i, & (10) \\ \hat{x}_i(k) &\in \hat{\mathcal{X}}_i \subseteq \mathcal{X}_i \ominus \mathcal{E}_i, \\ \hat{u}_i(k) &\in \hat{\mathcal{U}}_i \subseteq \mathcal{U}_i \ominus K_i \mathcal{E}_i, \\ (\hat{x}_i^1(k), \hat{u}_i(k), \tilde{x}_{-i}^1(k), \tilde{u}_{-i}(k)) &\in \hat{\mathcal{J}}_l, \\ \hat{x}_i(t + N_c) &\in \hat{\mathcal{X}}_i^F, \end{aligned}$$

where  $V_i^N = \sum_{k=t}^{t+N_c-1} l_i(\hat{x}_i(k), \hat{u}_i(k)) + V_i^F(\hat{x}_i(t + N_c))$ ,  $\mathcal{Z}_i$  is a convex set which contains the origin as its interior point,  $\hat{\mathcal{X}}_i^F$  is a nominal terminal set,  $l_i(\hat{x}_i(k), \hat{u}_i(k)) = \hat{x}_i(k)^T P_i \hat{x}_i(k) + \hat{u}_i(k)^T Q_i \hat{u}_i(k)$  is the stage cost,  $V_i^F(\hat{x}_i(t + N_c)) = \hat{x}_i(t + N_c)^T R_i \hat{x}_i(t + N_c)$  is the terminal cost,  $\hat{\mathcal{J}}_l$  is constructed such that if  $(\hat{x}_i^1(k), \hat{u}_i(k), \tilde{x}_{-i}^1(k), \tilde{u}_{-i}(k)) \in \hat{\mathcal{J}}_l$  then  $(x_1^1(k), \dots, x_N^1(k), u_1(k), \dots, u_N(k)) \in \mathcal{J}_l$  with  $\tilde{x}_{-i}^1(k) = (\tilde{x}_1^1(k), \dots, \tilde{x}_{i-1}^1(k), \tilde{x}_{i+1}^1(k), \dots, \tilde{x}_N^1(k))$  and  $\tilde{u}_{-i}^1(k) = (\tilde{u}_1^1(k), \dots, \tilde{u}_{i-1}^1(k), \tilde{u}_{i+1}^1(k), \dots, \tilde{u}_N^1(k))$ .

**Remark 1** (8) implies that  $e_i(t) \in \mathcal{E}_i$ . Based on the invariance of  $\mathcal{E}_i$  and by induction, it follows that  $e_i(k) \in \mathcal{E}_i, k = t, \dots, t + N_c - 1$ . Combining this fact with constraint (10), it can be obtained that  $\hat{x}_i^1(k) - \tilde{x}_i(k) \in \mathcal{E}_i \oplus \mathcal{Z}_i, k = t, \dots, t + N_c - 1$  which guarantees the boundedness of  $\sum_{j \in \mathcal{N}_i^1} A_{i,j}^{11}(x_j^1(k) - \tilde{x}_j(k)) + d_i^1(k)$  in (7). Considering (7), a sufficient condition of the sets  $\mathcal{E}_i$  and  $\mathcal{Z}_i, i = 1, \dots, N$  is that

$$A_{K_i} \mathcal{E}_i \oplus \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{11}(\mathcal{E}_j \oplus \mathcal{Z}_j) \oplus \mathcal{D}_i^1 \subseteq \mathcal{E}_i. \quad (11)$$

The following assumptions for the terminal constraint  $\hat{\mathcal{X}}_i^F$  and terminal cost function  $V_i^F(\cdot)$  are necessary to guarantee the recursive feasibility of the optimization problem and stability of the system.

**Assumption 2** Denote  $\hat{\mathcal{X}} = \prod_{i=1}^N \hat{\mathcal{X}}_i, \hat{\mathcal{U}} = \prod_{i=1}^N \hat{\mathcal{U}}_i, \hat{\mathcal{X}}^F = \prod_{i=1}^N \hat{\mathcal{X}}_i^F$  and  $\hat{x} = (\hat{x}_1, \dots, \hat{x}_N)$ .

- i)  $\hat{\mathcal{X}}^F \subseteq \hat{\mathcal{X}}$  is an invariant set for  $\hat{x}(k+1) = (\hat{A} + \hat{B}K)\hat{x}(k)$ ;
- ii)  $K\hat{x}(k) \in \hat{\mathcal{U}}$  for all  $\hat{x}(k) \in \hat{\mathcal{X}}^F$ ; and
- iii) for all  $\hat{x}(k) \in \hat{\mathcal{X}}^F$  and for a given constant  $\alpha > 0$ ,

$$\begin{aligned} &\sum_{i=1}^N [V_i^F(\hat{x}_i(k+1)) - V_i^F(\hat{x}_i(k))] \\ &\leq -(1 + \alpha) \sum_{i=1}^N l_i(\hat{x}_i(k), \hat{u}_i(k)). \end{aligned}$$

At each time instant  $t$ , a pair  $(\hat{x}_i(t), \hat{u}_i[t : t + N_c - 1])$

can be obtained by  $\mathcal{S}_i$  by solving the  $i$ th problem. Based on (5) and (6), the future state trajectory  $\hat{x}_i[t + 1 : t + N_c]$  can also be computed. Then set  $\tilde{x}_i(t + N_c) = \hat{x}_i(t + N_c)$  and  $\tilde{u}_i(t + N_c) = \hat{u}_i(t + N_c)$  as the last elements of the reference state and control trajectory and transmitted to the neighbours of the  $i$ th subsystem for updating and reformulating the local optimization problems at time instant  $t + 1$ .

Based on the above setup, we have the following convergence result of the distributed MPC algorithm.

**Theorem 1** If Assumptions 1 and 2 hold, then  $(x_1^1(k), \dots, x_N^1(k))$  exponentially converges to the invariant set of system  $x(k + 1) = (\hat{A} + \hat{B}K)x(k) + d(k)$ , where  $d(k) = (d_1^1(k), \dots, d_N^1(k))$ .

The proof goes along the line of Theorem 1 in [11] so it is omitted here.

## 4 Implementation

In this section, some algorithms for the computation of the weighted matrices, the feedback gains, the invariant sets and the initial feasible reference trajectory will be briefly introduced [17].

### 4.1 Feedback gains and weighted matrices

Denote  $R = \text{diag}\{R_1, \dots, R_N\}$ . Define matrices  $S = R^{-1}$  and  $Y = KS$ . Considering  $S$  and  $Y$  as feasible solutions to the following LMI, the feedback gain  $K_i$  and terminal weighted matrices  $R_i$  which satisfy Assumption 1 with  $R - (\hat{A} + \hat{B}K)^T R (\hat{A} + \hat{B}K) > 0$  can be obtained by the definition of  $S$  and  $Y$ .

$$\begin{pmatrix} S & (\hat{A}S + \hat{B}Y)^T \\ \hat{A}S + \hat{B}Y & S \end{pmatrix} > 0, \quad (12)$$

$$\begin{pmatrix} S_{i,i} & (\hat{A}_{i,i}S_{i,i} + \hat{B}_{i,i}Y_{i,i})^T \\ \hat{A}_{i,i}S_{i,i} + \hat{B}_{i,i}Y_{i,i} & S_{i,i} \end{pmatrix} > 0, \quad (13)$$

$$S_{i,j} = 0, \quad \forall i, j = 1, \dots, N (i \neq j), \quad (14)$$

$$Y_{i,j} = 0, \quad \forall i, j = 1, \dots, N (i \neq j), \quad (15)$$

where  $S_{i,j}$  and  $Y_{i,j}, \forall i, j = 1, \dots, N$  are the  $(i, j)$ th block of  $S$  and  $Y$ .

Once  $K$  and  $R$  are computed,  $P = \text{diag}\{P_1, \dots, P_N\}$  and  $Q = \text{diag}\{Q_1, \dots, Q_N\}$  can be constructed to satisfy Assumption 2 iii) by considering the following inequality:

$$R - (\hat{A} + \hat{B}K)^T R (\hat{A} + \hat{B}K) - (Q + K^T P K)(1 + \alpha) > 0.$$

Due to the positiveness of  $R - (\hat{A} + \hat{B}K)^T R (\hat{A} + \hat{B}K)$ , the above inequality can be satisfied by choosing sufficiently small  $\alpha$ ,  $P$  and  $Q$ .

### 4.2 Computation of sets

Sets  $\mathcal{D}_i^1, i = 1, \dots, N$  can be easily computed following the discussion in Section 3. Then it needs to find  $\mathcal{Z}_i \neq \emptyset, \mathcal{E}_i \subset \mathcal{X}_i$  and  $K_i \mathcal{E}_i \subset \mathcal{U}_i$  which satisfy (11) for all  $i = 1, \dots, N$ . By using the  $\text{box}(\cdot)$  operator, sets  $\mathcal{E}_i$  and  $\mathcal{Z}_i, i = 1, \dots, N$  can be constructed by the following algorithm.

#### Algorithm 1

**Step 1** Initialize  $\mathcal{Z}_i, i = 1, \dots, N$  as arbitrarily hyper-rectangles.

**Step 2** Initialize  $\mathcal{E}_i = \sum_{j \in \mathcal{N}_i^1} \text{box}(A_{i,j}^{11} \mathcal{Z}_j) \oplus \text{box}(\mathcal{D}_i^1),$   
 $i = 1, \dots, N.$

**Step 3** Compute

$$\mathcal{E}_i^+ = \text{box}(A_{K_i} \mathcal{E}_i) \oplus \sum_{j \in \mathcal{N}_i^1} \text{box}(A_{i,j}^{11} (\mathcal{E}_j \oplus \mathcal{Z}_j)) \oplus \text{box}(\mathcal{D}_i^1),$$

$i = 1, \dots, N.$

**Step 4** For all  $i = 1, \dots, N$ , if  $\mathcal{E}_i^+ \subset \mathcal{E}_i$ , which means that  $\mathcal{E}_i$  satisfies (11), then go to Step 5. Otherwise set  $\mathcal{E}_i = \mathcal{E}_i^+$  and repeat Step 3.

**Step 5** If  $\mathcal{E}_i \subset \mathcal{X}_i$  and  $K_i \mathcal{E}_i \subset \mathcal{U}_i$  then stop. Otherwise set  $\mathcal{Z}_i = \beta \mathcal{Z}_i$  with  $\beta \in (0, 1)$  and go to Step 2.

Finally,  $\hat{\mathcal{X}}_i^F$  can be simply chosen as  $\beta \mathcal{E}_i$  with  $\beta \in (0, 1)$ .

### 4.3 Initial feasible reference trajectories

Denote the current measurement of system state is  $\tilde{x}(0) = (\tilde{x}_1(0), \dots, \tilde{x}_N(0))$ . The algorithm to find the initial feasible reference trajectories of each subsystem  $\mathcal{S}_i$  is based on the one-step prediction.

#### Algorithm 2

**Step 1** For all  $i = 1, \dots, N$ , initialize  $\tilde{x}_i(0)$  and  $v = 0$ .

**Step 2** For all  $i = 1, \dots, N$ , each subsystem  $\mathcal{S}_i$  transmits its state  $\hat{x}_i(v)$  to its neighbours  $\mathcal{S}_j, j \in \mathcal{N}_i^1$  and solves the following optimization problem.

$$\min_{\hat{u}_i(v)} \|\tilde{x}_i(v+1)\|^2$$

subject to

$$\tilde{x}_i(v+1) = A_{i,i}^{11} \tilde{x}_i(v) + B_i^1 \hat{u}_i(v) + \sum_{j \in \mathcal{N}_i^1} A_{i,j}^{11} \tilde{x}_j(v),$$

$$\tilde{x}_i(v+1) \in \hat{\mathcal{X}}_i,$$

$$(\hat{x}_i^1(k), \hat{u}_i(k), \tilde{x}_{-i}^1(k), \tilde{u}_{-i}(k)) \in \hat{\mathcal{J}}_i,$$

$$\hat{u}_i(v) \in \hat{\mathcal{U}}_i.$$

**Step 3** For  $i = 1, \dots, N$ , if  $\tilde{x}_i(v+1) \in \hat{\mathcal{X}}_i^F$ , set prediction horizon as  $v+1$  and stop. Otherwise, set  $v = v+1$  and go to Step 2.

## 5 Numerical example

In this section, the proposed control algorithm is tested on a building temperature regulation problem proposed in [17] with slightly modification.

Consider four connected rooms A, B, C and D as in Fig. 1. Rooms A and B are combined together as one apartment while rooms C and D as the other apartment. The air temperature  $T_A, T_B, T_C$  and  $T_D$  are considered as the measurable states and the control objective is to regulate them to the set point. Each room is equipped with an air-conditioner  $q_A, q_B, q_C$  and  $q_D$ . The heat transfer coefficient between rooms A-C and B-D is  $k_1^t = 5.8 \text{ W/m}^2\text{K}$ , the one between rooms A-B and C-D is  $k_2^t = 5.8 \text{ W/m}^2\text{K}$ , and the one between each room and the external environment is  $k_e^t = 3 \text{ W/m}^2\text{K}$ . The nominal external temperature  $\bar{T}_E$  is  $35^\circ\text{C}$  and solar radiation is not considered for simplicity. The volume of each room is  $V = 48 \text{ m}^3$ , and the wall surfaces between the rooms are all equal to  $s_r = 12 \text{ m}^2$ , while those between the rooms and the environment are equal to  $s_e = 24 \text{ m}^2$ . Air density and heat capacity are  $\rho = 1.225 \text{ kg/m}^3$  and  $c = 1005 \text{ J/kgK}$ , respectively. Letting  $\phi = \rho c V$ , the dynamic model is as follows:

$$\phi \frac{dT_A}{dt} = s_r k_2^t (T_B - T_A) + s_r k_1^t (T_C - T_A) + s_e k_e^t (T_E - T_A) + \sum_{i=1}^3 s_A^i k_A^i (T_A^i - T_A) - q_A,$$

$$\phi \frac{dT_B}{dt} = s_r k_2^t (T_A - T_B) + s_r k_1^t (T_D - T_B) + s_e k_e^t (T_E - T_B) + \sum_{i=1}^3 s_B^i k_B^i (T_B^i - T_B) - q_B,$$

$$\phi \frac{dT_C}{dt} = s_r k_2^t (T_A - T_C) + s_r k_1^t (T_D - T_C) + s_e k_e^t (T_E - T_C) + \sum_{i=1}^3 s_C^i k_C^i (T_C^i - T_C) - q_C,$$

$$\phi \frac{dT_D}{dt} = s_r k_2^t (T_B - T_D) + s_r k_1^t (T_C - T_D) + s_e k_e^t (T_E - T_D) + \sum_{i=1}^3 s_D^i k_D^i (T_D^i - T_D) - q_D,$$

$$\phi \frac{dT_A^i}{dt} = 60 s_A^i k_A^i (T_A - T_A^i), \quad i = 1, 2, 3,$$

$$\begin{aligned} \phi \frac{dT_B^i}{dt} &= 60s_B^i k_B^i (T_B - T_B^i), \quad i = 1, 2, 3, \\ \phi \frac{dT_C^i}{dt} &= 60s_C^i k_C^i (T_C - T_C^i), \quad i = 1, 2, 3, \\ \phi \frac{dT_D^i}{dt} &= 60s_D^i k_D^i (T_D - T_D^i), \quad i = 1, 2, 3, \end{aligned}$$

$$\text{Matrices } K_i = \begin{pmatrix} -0.0193 & -0.0002 \\ -0.0002 & -0.0193 \end{pmatrix} \text{ and}$$

$$R_i = \begin{pmatrix} 2.9249 \cdot 10^7 & 118.2915 \\ 118.2915 & 2.9249 \cdot 10^7 \end{pmatrix}$$

where  $T_A^i, T_B^i, T_C^i$  and  $T_D^i, i = 1, 2, 3$  are used to represent the thermal dynamics of furniture and walls which are assumed unmeasurable,  $s_A^i, s_B^i, s_C^i$  and  $s_D^i$  are the equivalent surfaces chosen randomly, and  $k_A^i, k_B^i, k_C^i$  and  $k_D^i$  are the equivalent heat transfer coefficients chosen randomly.

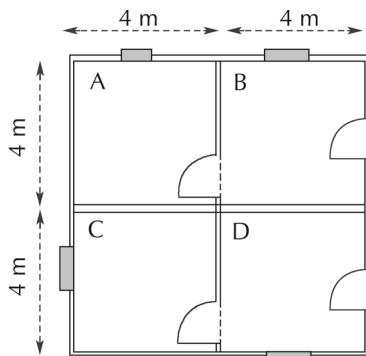


Fig. 1 Schematic representation of a building with two apartments.

The considered equilibrium point is:  $q_A = q_B = q_C = q_D = \bar{q} = s_e k_e^t (\bar{T} - \bar{T}_E) = 1081.4 \text{ W}$ , with  $T_A = T_B = T_C = T_D = \bar{T} = 20^\circ\text{C}$ . Let  $\delta T_A = T_A - \bar{T}, \delta T_B = T_B - \bar{T}, \delta T_C = T_C - \bar{T}, \delta T_D = T_D - \bar{T}, \delta q_A = (q_A - \bar{q})/c\rho V, \delta q_B = (q_B - \bar{q})/c\rho V, \delta q_C = (q_C - \bar{q})/c\rho V$  and  $\delta q_D = (q_D - \bar{q})/c\rho V$ .

The corresponding discrete-time model of the form (1) is obtained by mE-ZOH discretization [18] with sampling time  $h = 10 \text{ s}$ . The partition of inputs and measurable states is:

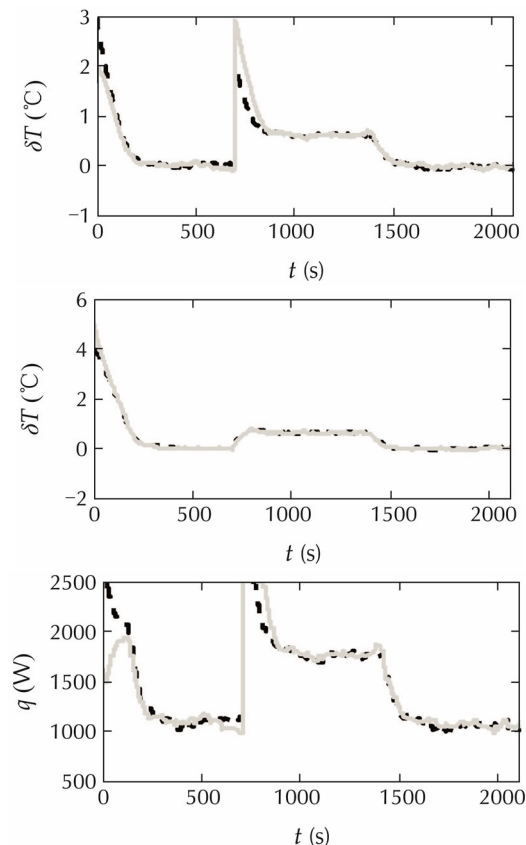
$$\begin{aligned} x^1 &= [\delta T_A \quad \delta T_B]^T, \quad u^1 = [\delta q_A \quad \delta q_B]^T, \\ x^2 &= [\delta T_C \quad \delta T_D]^T, \quad u^2 = [\delta q_C \quad \delta q_D]^T. \end{aligned}$$

The constraints on the inputs and the states of the discretized system is

$$\begin{aligned} x_{\min}^1 &= [-10 \quad -10]^T, \quad x_{\max}^1 = [10 \quad 10]^T, \\ x_{\min}^2 &= [-10 \quad -10]^T, \quad x_{\max}^2 = [10 \quad 10]^T, \\ u_{\min}^1 &= [-2500 \quad -2500], \quad u_{\max}^1 = [2500 \quad 2500], \\ u_{\min}^2 &= [-2500 \quad -2500], \quad u_{\max}^2 = [2500 \quad 2500], \\ \|u^1\|_1 + \|u^2\|_1 &\leq 8000. \end{aligned}$$

are constructed by finding feasible solutions to LMI (12)–(15). The selected weighting matrices are  $P_i = I_2$  and  $Q_i = 100I_2$ . Algorithm 1 is used to find the invariant sets and the initial feasible reference trajectory is constructed by using Algorithm 2 with  $N_c = 6$ .

The initial condition of this numerical example is  $\delta T_A = 3^\circ\text{C}, \delta T_B = 2^\circ\text{C}, \delta T_C = 4^\circ\text{C}$  and  $\delta T_D = 5^\circ\text{C}$ . The initial temperature of the unmeasurable states are randomly chosen between  $[22^\circ\text{C}, 25^\circ\text{C}]$ . The real external temperature is assumed to randomly vary between  $[25^\circ\text{C}, 45^\circ\text{C}]$ . At time instant  $t = 700 \text{ s}$ , there is a sudden increase of temperature  $T_A$  and  $T_B$  representing the opening of doors and windows. Over time interval  $[700 \text{ s}, 1400 \text{ s}]$  additional heat sources with  $30^\circ\text{C}$  are added in both rooms. The control input is calculated by using the discretized model and applied to the continuous-time model. The simulation results are shown in Fig. 2.



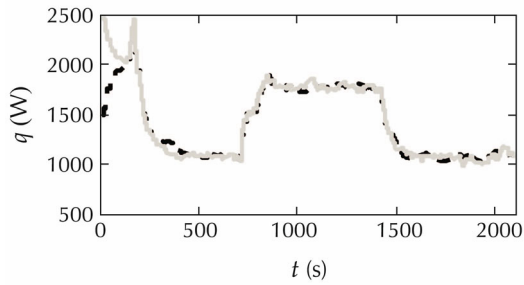


Fig. 2 State and input trajectory of distributed algorithm.

To compare the performance of the proposed distributed algorithm, a decentralized MPC and a centralized MPC with complete state information are also used to regulate the temperature. The simulation results are shown in Figs. 3 and 4.

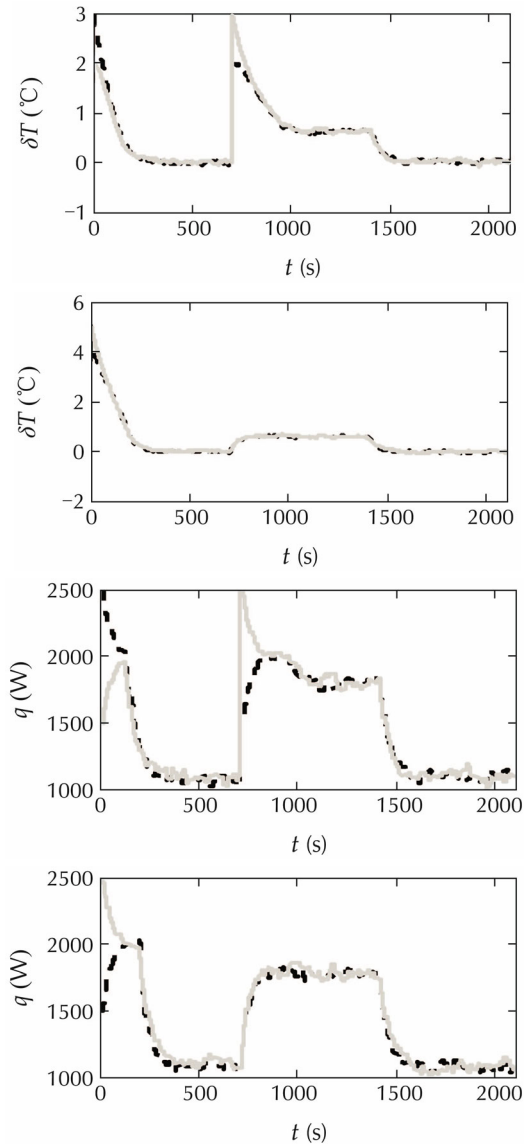


Fig. 3 State and input trajectory of decentralized algorithm.

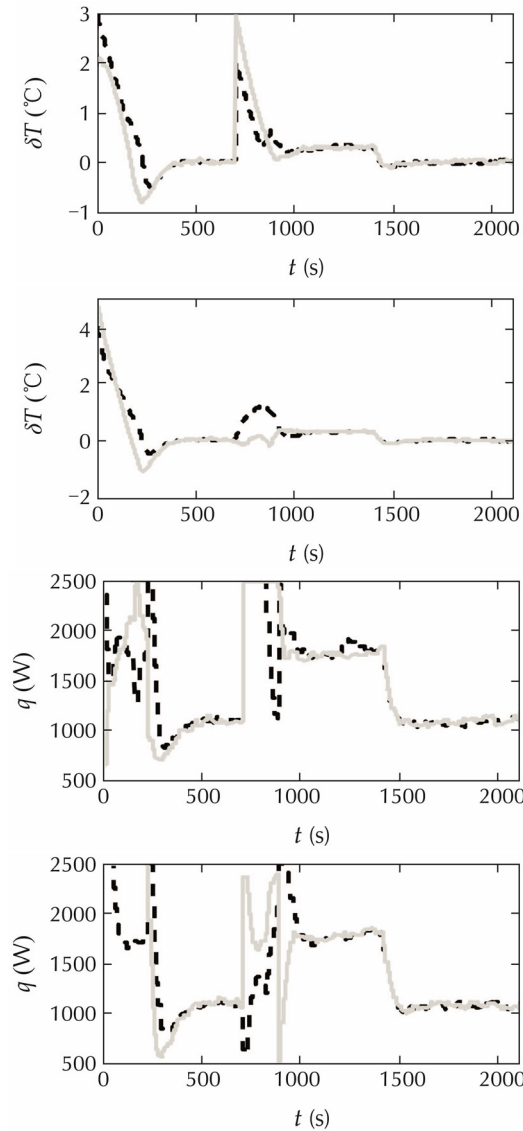


Fig. 4 State and input trajectory of centralized full order algorithm.

To show the difference among these algorithms more clearly, the temperature deviation between distributed algorithm and the decentralized algorithm are shown in Figs. 5 and 6. The temperature deviation between distributed algorithm and the centralized algorithm are shown in Figs. 7 and 8.

In Figs. 2–4, the black dashed lines represent the corresponding main states and control inputs of rooms A and C while the grey solid lines represent those of rooms C and D. In Fig. 5 the black dashed line represents  $\delta T_{A,dis} - \delta T_{A,dec}$  and the grey solid line represents  $\delta T_{B,dis} - \delta T_{B,dec}$ , and in Fig. 6 they represent  $\delta T_{C,dis} - \delta T_{C,dec}$  and  $\delta T_{D,dis} - \delta T_{D,dec}$ , respectively. In Fig. 7 the black dashed line represents  $\delta T_{A,dis} - \delta T_{A,c}$  and

the grey solid line represents  $\delta T_{B,dis} - \delta T_{B,c}$ , and in Fig. 8 they represent  $\delta T_{C,dis} - \delta T_{C,c}$  and  $\delta T_{D,dis} - \delta T_{D,c}$ , respectively. The subscripts dis, dec, c represent distributed algorithm, decentralized algorithm and centralized algorithm, respectively. For clarity reason, the minor dynamics are not shown in these figures.

In the decentralized algorithm, there is no information exchange between controllers. Therefore, the joint constraint  $\|u^1\|_1 + \|u^2\|_1 \leq 8000$  is tightened as  $\|u^1\|_1 \leq 4000$  and  $\|u^2\|_1 \leq 4000$ . The dynamics coupling terms are treated as unknown bounded disturbance and the method in [19] are used to design local controller. For the centralized algorithm, we also follow the way in [19] to design the centralized controller.

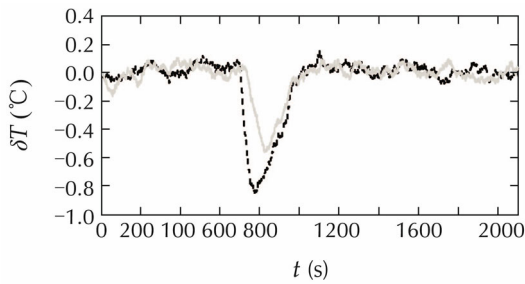


Fig. 5 Temperature deviation between distributed and decentralized algorithm (rooms A and B). In this example, the air-conditioning system is working on cooling mode. Therefore, the distributed algorithm has faster response than the decentralized one since the former one has lower temperature.

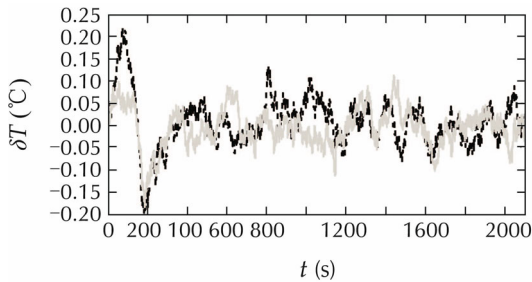


Fig. 6 Temperature deviation between distributed and decentralized algorithm (rooms C and D).

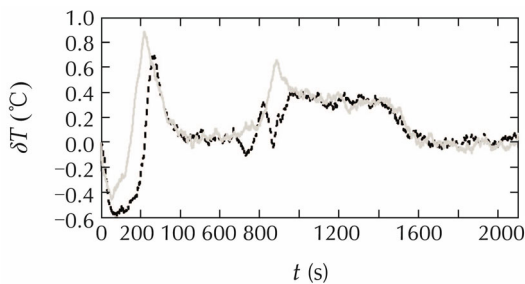


Fig. 7 Temperature deviation between distributed and centralized algorithm (rooms A and B).

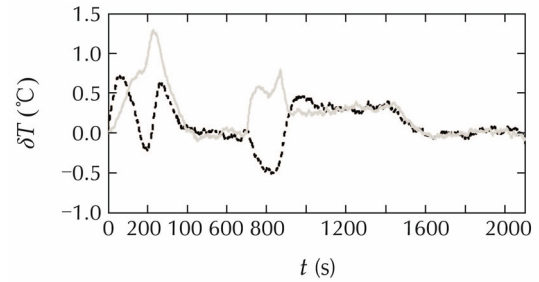


Fig. 8 Temperature deviation between distributed and centralized algorithm (rooms C and D).

From Figs. 2–4 it can be observed that with the additional heat source, all the three algorithms have some off-set. From Figs. 5 and 6 one can see that distributed algorithm has faster response than the decentralized one. This is due to the fact that in the distributed algorithm, controllers can communicate with each other. Therefore, controllers for rooms A and B know how much power will be used by the other two controllers and the overall power, which is restricted to be less or equal to 8000 W, can be used more efficiently. This fact can also be observed in Figs. 2 and 3. In Fig. 2, controllers for rooms A and B attain the maximal power 2500 W while in Fig. 3, their power has to satisfy the constraint  $q_A + q_B \leq 4000$  W. The off-set of both algorithms are similar. Figs. 7 and 8 show that centralized algorithm has faster response than the distributed one and its off-set is also smaller than that of the distributed one. This is not surprising because the centralized algorithm utilizes the complete state information.

Finally, we compare the energy, which is defined as  $\int_0^{2110} \sum_{A,B,C,D} \|q_i\| dt$ , and the average computational time consumed by the three algorithms in Table 1.

Table 1 Cost and average computational time of the three algorithms.

	Time (s)	Cost (kJ)
Centralized (full model)	0.47	732
Distributed (reduced order)	0.23	758
Decentralized (reduced order)	0.35	893

### 6 Conclusions

In this paper, a distributed robust MPC algorithm has been proposed. Each subsystem can be partitioned into measurable main dynamics and unmeasurable minor dynamics. Only the main dynamics is used to predict the future trajectory while the minor dynamics is not



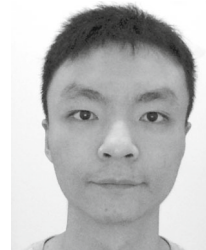
handled explicitly in the online computation, by which the complexity of the optimization problem is significantly reduced. Numerical examples has been given to illustrate the effectiveness of our algorithm.

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