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Sensor-network-based robust distributed control and estimation

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ABSTRACT

This paper proposes a novel distributed estimation and control method for uncertain plants. It is of application in the case of large-scale systems, where each control unit is assumed to have access only to a subset of the plant outputs, and possibly controls a restricted subset of input channels. A constrained communication topology between nodes is considered so the units can benefit from estimates of neighboring nodes to build their own estimates. The paper proposes a methodology to design a distributed control structure so that the system is asymptotically driven to equilibrium with L_2-gain disturbance rejection capabilities. A difficulty that arises is that the separation principle does not hold, as every single unit ignores the control action that other units might be applying. To overcome this, a two-stage design is proposed: firstly, the distributed controllers are obtained to robustly stabilize the plant despite the observation errors in the controlled output. At the second stage, the distributed observers are designed aiming to minimize the effects of the communication noise in the observation error. Both stages are formulated in terms of linear matrix inequalities. The performance is shown on a level-control real plant.

1. Introduction

Distributed control is a relatively mature field of research, and nowadays constitutes a relevant and attractive field for its important applications and theoretical challenges. One of the main reasons is the applicability of these techniques to physical large-scale complex plants, where traditional centralized architectures are often hard or even impossible to implement.

Traditional procedures for analyzing systems and designing control strategies typically rely on the assumption of centrality: the information collected about the system, and the computations based upon this information, take place sufficiently close to each other, such that communication issues can be neglected.

In many today's complex systems applications it is preferable, if not unavoidable, to elude a centralized scheme for a number of reasons, for example, lower wiring costs, excessive computational burden required for centralized implementation, mitigation of failures by redundancy, increased flexibility, modularity, reconfigurability and reliability, etc. In other cases, as in geographically distributed systems, it is not realistic to assume that each control agent can use all the measurement signals of the system to generate its local control input. In other words, some constraints on information flow between agents must be considered.

Distributed estimation and control finds application in many fields, such as traffic systems, water delivery channels, oil/gas pipelines, electrical power grids water, manufacturing systems, large-scale structures, robotic systems, and multi-agent systems, among others. In all these cases the centrality assumption no longer holds, and a decentralized or distributed strategy is often more desirable.

Although decentralized control can be traced back to the late 70s (see Davison & Chang, 1990; Davison & Wang, 1973; Sandell, Varaiya, Athans, & Safonov, 1978 and references therein), in these first works most real-time control tasks were loosely distributed as they were carried out within individual modules without communication among them. Nowadays, recent advances in microelectronics and communications technologies provide us with a wealth of cheap, customizable, embedded sensors with wireless communication capacities. The advantage of Wireless Sensor Networks (WSNs) with respect to traditional technologies is enormous, as deploying and maintaining a geographically distributed wired network of thousands of nodes is impractical.

The state of the art concerning distributed control strategies comprises a vast number of techniques, taking different approaches depending on the problem nature and, in many cases, based on the area of expertise of the authors. It is possible, however, to group the works in a couple of main research lines: control of multi-agent systems and large-scale plants.

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The first line refers to the problems of controlling/monitoring a number of entities, called agents, that interact with the environment in the pursue of a control objective that must be collectively achieved. This line has revealed itself as a very productive topic of research with applications that have branched into a variety of fields as scheduling and planning (Colombo, Schoop, & Neubert, 2006; Zhang, Anosike, & Lim, 2007), diagnostics (Davidson, McArthur, McDonald, Cumming, & Watt, 2006), condition monitoring (Buse & Wu, 2004; McArthur, Strachan, & Jahn, 2004; Zhou, Chen, Zhang, Yan, & Chen, 2007), distributed control (Buse & Wu, 2004; Galdun, Takac, Ligus, Thiriet, & Sarnovsky, 2008; Zhou et al., 2007), hybrid control (Fregene, Kennedy, & Wang, 2005) and congestion control (Hwang, Tan, Hsiao, & Wu, 2005; Srinivasan & Choy, 2006) among others.

In the control field, consensus ideas have been especially prolific. The problem here consists in controlling a number of agents with identical dynamics. Many works in the field model the agents as integrators (Li, Xu, Chu, & Wang, 2008; Olfati-Saber, Fax, & Murray, 2007), though extensions to more complex systems can be found in Araki and Uchida (2008), Jin and Yang (2011), and Ni and Cheng (2010). Some other studies take into account the communication channel proposing event-triggered solutions (Dimarogonas & Johansson, 2009), or consensus with delays (Lu, Atay, & Jost, 2011).

The second main research line in distributed control refers to the problem of controlling large-scale systems. Typically, large-scale control systems have several local control stations, each one having access to some local outputs and controlling only some specific input channels. All the stations are involved, however, in controlling the overall system.

It is remarkable that many authors have followed ideas of Model Predictive Control, (Camponogara, Jia, Krogh, & Talukdar, 2002; Christofides, Liu, & Muñoz de la Peña, 2011; Dunbar, 2007; Liu, Chen, Muñoz de la Peña, & Christofides, 2010; Liu, Muñoz de la Peña, & Christofides, 2009; Maestre, 2011; Negemborn & Hellendoorn, 2008; Roshany-Yamchi et al., 2013; Scattolini, 2009; Venkat, Rawlings, & Wright, 2005). In Lynch, Law, and Blume (2002) a different solution is proposed based on semi-active control with applications to large-scale civil structures. A completely innovative idea is proposed in D'Andrea and Dullerud (2003), where the plant is modeled using small modular blocks that communicate with their neighbors and can be stacked building large-scale systems. The controllers are also modular and are associated to different plant modules. Most of these works, including Necoara, Nedelcu, and Dumitrache (2008), decompose the plant in smaller subsystems that are controlled by different nodes. This decoupling is referred to plant dynamics or control actions.

A closely related line of research is the so-called decentralized overlapping control, where different controllers are allowed to share control inputs of the plant. The decentralized overlapping control is fundamentally used in two cases. In the first case, the subsystems of a system (referred to as overlapping subsystems) share some states (Iftar, 1991, 1993; Siljak & Zecevic, 2005). In this case, it is usually desired that the structure of the controller matches the overlapping structure of the system (Siljak & Zecevic, 2005). The second situation considers some limitations on the availability of the states. In this case, only certain number of the system outputs are available for constructing each control signal.

In this work, a novel distributed control scheme for large-scale systems is proposed. The control scenario consists of an uncertain linear process which is to be controlled and monitored in a distributed fashion by a number of interconnected nodes with a given topology. Each node is assumed to have access to a limited subset of the plant outputs, and may possibly generate a control signal for a restricted subset of the control channels. The problem so formulated entails the design of an estimation and control structure for every node, such that the collective control action robustly asymptotically drives the system to equilibrium.

To this end, every node is assumed to run its own estimator of the plant states, resorting to a Luenberger-like observer structure improved with consensus strategies, that allows the nodes to benefit from the estimations of its neighbors. Local observability is not assumed, that is, no node is able to estimate the full plant states based only on its direct measurements of the plant. However, collective observability is a necessary assumption (Olfati-Saber et al., 2007). This means that the network of nodes is able, as a whole, to observe the complete state.

A difficulty that arises with this formulation of the problem is that the separation principle does not hold, as the nodes ignore the control signals that other actuator nodes are applying. To overcome this, a two-stage design is proposed. At the first stage, the distributed controllers are obtained to robustly stabilize the overall system despite uncertainties and observations errors. At a second stage, the observers are designed such that estimation errors are asymptotically stable with L_2-gain disturbance rejection capabilities. Both steps are formulated using the Lyapunov theory and solved in terms of Linear Matrix Inequalities (LMIs), for which efficient computational tools are widely available.

It is important to emphasize that the proposed method does not impose any specific constraint on the plant to be controlled (there is no need to be stable nor decomposable in any specific way). The design procedure, though requires a centralized off-line computation of controllers and observers, allows fully distributed implementation. Remarkably, the methodology accounts for overlapping control where different nodes can simultaneously provide control signals for the same control channel. This approach increases reliability and controllability of the overall plant. Delays and packet dropouts are not explicitly considered in the approach since there is a wealth of relevant practical applications where this limitation is not an issue, specially in the context of modern communications networks with increasing reliability and speed.

As an application example, the proposed method has been successfully tested in a level-control real plant.

The rest of the paper is organized as follows. Section 2 describes the system set-up as well as the different devices involved. Section 3 formulates the problem under study. Section 4 deals with the controllers design problem and Section 5 with the observers design. Section 6 studies an application of the proposed distributed scheme to a coupled tank system. Finally, Section 7 summarizes the research in this paper.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, I is the identity matrix of appropriate dimensions, $\|\cdot\|$ stands for the Euclidean vector norm or the induced matrix 2-norm as appropriate. The notation $X>0$ (respectively, $X\geq 0$), for $X\in\mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric matrix positive definite (respectively, positive semi-definite). For an arbitrarily real matrix B and two real symmetric matrices A and C, $\begin{bmatrix} A & B \\ * & C\end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry. The symbol $\otimes$ stands for the Kronecker product. For any finite energy signal $a(t)$, $\|a(t)\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$-norm of $a(t)$, defined as $\|a(t)\|_{\mathcal{L}_2}=\int_0^\infty a^T(t)a(t)\,dt$.

2. Problem description

Consider the scheme depicted in Fig. 1, where Σ is an uncertain continuous-time plant being monitored/controlled through an interconnected sensor network. The notation related to the distributed scheme is summarized in Table 1.

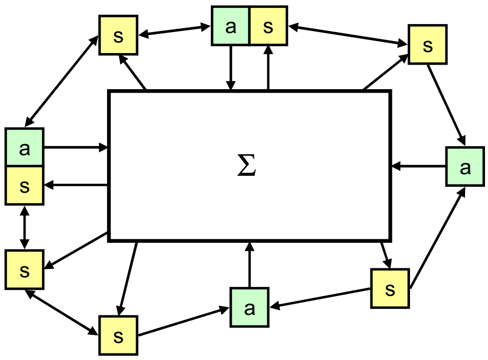

Fig. 1. Network of sensors (s) and actuators (a) for distributed control and observation.

Table 1
Notation

Variable	Description
x	State of the system
u_i	Control input by node i
y_i	Output measured by node i
$\hat{x}$	Estimated state
M_i	Luenberger-like gain
N_{ij}	Consensus matrix
K_i	Controller gain

In the following subsections, the different elements comprising the aforementioned system are described in detail.

2.1. Plant

The dynamics of the plant Σ to be controlled is given by the following equations:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f_n(t, x(t)) + B_\omega \omega(t), \\ z(t) = Dx(t), \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $u(t)\in\mathbb{R}^m$ is the control input, $z(t)\in\mathbb{R}^q$ is the controlled output, and $\omega(t)\in\mathbb{R}^n$ denotes an L_2 external perturbation. A, B, B_ω and D are some constant matrices of appropriate dimensions. The initial condition of the system is $x(t_0)=x_0$.

Function $f_n(t, x(t)) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ represents nonlinear uncertainties of the plant to be controlled. It is assumed that $f_n(t, x(t))$ is a piecewise-continuous nonlinear function in t and x that satisfies the following quadratic constraint condition:

$$f_n^T(t, x(t)) f_n(t, x(t)) \le \alpha^2 x^T(t) H^T H x(t), \quad \forall t \ge 0, \tag{2}$$

where $\alpha > 0$ is the bounding parameter of the uncertain function and H is a constant matrix. Some systems with mild nonlinearities operating in the proximity of a set-point can be adequately described with model (1). The four-coupled tank system used in this paper is an example, as it will be shown in Section 6.

Consider a partition of the control signal $u(t)$ as

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_p(t) \end{bmatrix}, \tag{3}$$

where $u_i\in\mathbb{R}^{d_i}$ $(i = 1, \ldots, p)$ is the control signal that actuator i applies to the system and p is the number of nodes in the network. It is assumed that $\sum_{i=1}^{p} d_i \ge m$, so that overlapping is considered. Control matrix B is consistently partitioned according to the dimensions of each individual control input u_i, that is, $B = [B_1\ B_2\ \cdots\ B_p]$.

The final objective of this work is to stabilize the plant (1) by applying suitable control inputs u_i $(i = 1, \ldots, p)$. In the developments to come, the following stability definition will be used.

Definition 1 (*Siljak & Stipanović, 2000*). System (1) is said to be robustly asymptotically stable with degree α if the equilibrium point $x(t)=0$ is globally asymptotically stable for all $f_n(t,x(t))$ verifying (2).

2.2. Network

The network in Fig. 1 is topologically defined by its graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with nodes $\mathcal{V} = \{1, 2, \ldots, p\}$ and links $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The set of nodes connected to node i is named the *neighborhood* of i and is denoted by $\mathcal{N}_i \equiv \{j : (i, j) \in \mathcal{E}\}$. Directed communications are considered so that link (i,j) implies that node i receives information from node j.

2.3. Nodes: sensors and actuators

Consider the distributed elements or nodes. As it has been already described, the nodes in the network can play the role of sensors, measuring local plant outputs, the role of controllers, providing a control signal to a subset of the plant control inputs, or both. Furthermore, the nodes require the information exchanged with their neighbors to observe the full plant state. Next, a common model valid for all nodes is presented.

A generic node i may receive information from the plant $y_i(t)\in\mathbb{R}^{r_i}$ and may apply some control input $u_i(t)\in\mathbb{R}^{d_i}$. The output and input vectors are defined as

$$y_i(t) = C_i x(t) + v_i(t), \tag{4}$$

$$u_i(t) = K_i \hat{x}_i(t), \tag{5}$$

where $\hat{x}_i\in\mathbb{R}^n$ denotes the estimation of node i and matrices C_i $(i\in\mathcal{V})$ are known. K_i $(i\in\mathcal{V})$ are the local controllers to be designed. The signal $v_i(t)\in L_2[t_0, \infty)$ represents an additive noise affecting the sensor measurements.

Local observability is not assumed, that is, the pairs (A,C_i) are neither observable nor detectable. However, a necessary assumption for the problem to be solvable is that *collective observability* holds, that is, the network as a whole is able to observe the state of the plant (see Olfati-Saber et al., 2007 for a formal definition of this concept). Mathematically, this assumption implies that the pair (A,C) is observable, where $C = [C_1^T\ C_2^T\ \cdots\ C_p^T]^T$.

Remark. In general, the nodes exhibit both sensing and actuation capabilities. However, in the present formulation this is not a necessary condition. By setting matrices $C_i\equiv 0$ or $B_i\equiv 0$, node i loses sensing or actuation ability, respectively.

In order to perform the estimation of the plant state, every node runs an observer described by

$$\begin{aligned} \dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + B\hat{u}_i(t) + M_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad + \sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(t) - \hat{x}_i(t)), \end{aligned} \tag{6}$$

where $\hat{u}_i(t)\in\mathbb{R}^m$ is an estimation of all the control actions applied to the plant at time t, defined by

$$\hat{u}_i(t) = K\hat{x}_i(t),$$

with controller $K^T = [K_1^T\ K_2^T\ \cdots\ K_p^T]$.

Looking at Eq. (6), each node has two different sources of information to correct its estimations. The output received from the plant is used in the same way as a classical Luenberger observer, $M_i(y_i(t)-C_i\hat{x}_i(t))$, being M_i, $i\in\mathcal{V}$, the observer gain to be designed.

On the other hand, the information received from neighboring nodes is also used to correct the estimations, $N_{ij}(\hat{x}_j(t)-\hat{x}_i(t))$, $\forall j\in\mathcal{N}_i$, where N_{ij}, $(i,j)\in\mathcal{E}$ are the consensus gains to be designed.

Using a compact notation, let $\mathcal{M},\mathcal{N},\mathcal{K}$ denote the sets of observers and controllers given by

$$\mathcal{M}=\{M_i, i\in\mathcal{V}\},$$
$$\mathcal{N}=\{N_{ij},(i,j)\in\mathcal{E}\},$$
$$\mathcal{K}=\{K_i, i\in\mathcal{V}\}.$$

The observation error is defined as

$$e_i(t)=x(t)-\hat{x}_i(t). \tag{7}$$

It is worth recalling here that no node knows exactly the actual control signal applied to the plant, as each actuator applies a different control signal based on its particular state estimation (5). However, each node needs a control signal to estimate the state of the plant according to (6).

This fact constitutes a serious drawback in mixed control and estimation schemes. In order to make Eq. (6) realizable, the solution proposed in this work consists, roughly speaking, in allowing each node to run its observer as if all control inputs were decided based on its particular estimate. That is

$$B\hat{u}_i(t)=BK\hat{x}_i(t)=\sum_{j=1}^{p}B_jK_j\hat{x}_i(t).$$

The actual control signal applied to the plant is built based on the estimates of each actuator

$$Bu(t)=\sum_{j=1}^{p}B_jK_j\hat{x}_j(t).$$

In general, estimated and actual control signals differ. However, if the observers are designed in such a way that node estimations converge to the plant states, these differences progressively vanish.

Remark. Modern networked control strategies are nowadays implemented resorting to packet-based communications. Notice however that, as it is common practice in digital control and without loss of generality, the plant dynamics, observers running in the nodes, and the applied control actions are modeled as continuous-time processes. The remaining element, the communication links, can also be modeled as continuous-time processes as far as the communication characteristic times are negligible with respect to the plant's dynamics. This case is not uncommon in modern high-speed communications networks using error-free protocols. This assumption justifies the use of continuous-time formulation throughout the paper.

2.4. Preliminary results

So far, every element in the distributed scheme given in Fig. 1 has been introduced. The following propositions present the dynamics of the estimation error and the plant state according to the described setup. Let us define the augmented vectors $e^T(t)=[e_1^T(t)\ e_2^T(t)\ \cdots\ e_p^T(t)]^T$ and $v(t)=[v_1^T(t)\ v_2^T(t)\ \cdots\ v_p^T(t)]^T$.

Proposition 1. *The dynamics of the state of the plant $x(t)$ is given by*

$$\dot{x}(t)=(A+BK)x(t)+\Upsilon(\mathcal{K})e(t)+f_n(t,x(t))+B_\omega\omega(t), \tag{8}$$

where

$$\Upsilon(\mathcal{K})=[-B_1K_1\ -B_2K_2\ \cdots\ -B_pK_p].$$

The proof is immediate from Eq. (1).

Proposition 2. *The dynamics of the observation error vector $e(t)$ is given by*

$$\begin{aligned}\dot{e}(t)=&(\Phi(\mathcal{M})+\Psi(\mathcal{K})+\Lambda(\mathcal{N}))e(t)\\&+I\otimes(B_\omega\omega(t)+f_n(t,x(t)))-\Pi(\mathcal{M})v(t),\end{aligned} \tag{9}$$

where the matrix functions are defined by

$$\Phi(\mathcal{M})=diag\{(A-M_1C_1),\ldots,(A-M_pC_p)\},$$

$$\Psi(\mathcal{K})=diag\{BK,\ldots,BK\}+\begin{bmatrix}\Upsilon(\mathcal{K})\\ \vdots\\ \Upsilon(\mathcal{K})\end{bmatrix},$$

$$\Pi(\mathcal{M})=diag\{M_1,\ldots,M_P\},$$

$$\Lambda(\mathcal{N})=\sum_{(i,j)\in\mathcal{E}}\Theta(N_{ij}),$$

with

$$\Theta(N_{ij})=\begin{array}{c}\begin{array}{ccccccc}col. & & i & & j & & \end{array}\\ \begin{bmatrix}0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\ \vdots & & \vdots & & \vdots & & \vdots\\ 0 & \cdots & -N_{ij} & \cdots & N_{ij} & \cdots & 0\\ \vdots & & \vdots & & \vdots & & \vdots\\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0\end{bmatrix}\end{array}\begin{array}{c}\\ \\ row\ i\\ \\ \\ \end{array}$$

The proof is detailed in Appendix A.

3. Problem formulation

In this section, the problem to be solved is formally stated. Before proceeding, some preliminary issues are examined.

As it can be seen from Eq. (8), the dynamics of the plant is affected by external disturbances $\omega(t)$ and observation errors $e_i(t)$ $(i\in\mathcal{V})$. Since the control inputs are applied according to the node estimates, the observation errors can be viewed as external disturbances to the plant state, deviating their response from the ideal situation in which a centralized state feedback control is implemented. This way, the vector of disturbance signals for the plant is defined as $d_z(t)=[e^T(t)\ \omega^T(t)]^T$.

Similarly, the dynamics of the observation error (9) is affected by external disturbances $\omega(t)$ and measurement noises $v(t)$. In this case, the disturbance vector is defined as $d_e(t)=[\omega^T(t)\ v^T(t)]^T$.

Definition 2 (*Robust distributed control and observation problem*). Consider an uncertain plant with dynamics given by (1). The plant is being observed and controlled by a set of p nodes which are connected by means of a network represented by a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$. The dynamics of the nodes are given by (6). Each node may receive an output from the plant (4) and may apply a control signal according to (5).

The *robust distributed control and observation problem* consists of finding observers M_i, $i\in\mathcal{V}$, and N_{ij}, $(i,j)\in\mathcal{E}$, and controllers K_i, $i\in\mathcal{V}$, such that:

1. The dynamics of the system state $x(t)$ and the estimation errors are robustly asymptotically stable with degree α for $\omega(t)\equiv v(t)\equiv 0$.
2. Under the assumption of zero initial condition for the plant state, the effects of the external disturbances and the observation errors are attenuated in the controlled output by γ_x, such that $\|z(t)\|_{\mathcal{L}_2}\le\gamma_x\|d_z(t)\|_{\mathcal{L}_2}$.
3. Under the assumption of zero initial conditions for the estimation errors, the effects of the external disturbances and the measurement noises in the estimates are attenuated by γ_e, such that $\|e(t)\|_{\mathcal{L}_2}\le\gamma_e\|d_e(t)\|_{\mathcal{L}_2}$.

In the following sections a solution to this problem is presented. It consists of a two-stage design procedure. Firstly, stabilizing controllers are designed to satisfy the disturbance attenuation constraint γ_x. At the second step, the observers are designed to guarantee stable estimation errors and to minimize the attenuation index γ_e.

4. Controllers design

The first step of the procedure described above is presented in this section. In order to guarantee stability, a Lyapunov-based approach is employed. Concretely, the following classical Lyapunov function is chosen:

$$V_x(t) = x^T(t)P_x x(t), \tag{10}$$

where P_x is a positive definite matrix. The following theorem presents the design procedure to obtain the controllers K_i $(i\in\mathcal{V})$ according to the definition of the problem.

Theorem 1. *Given a positive scalar $\alpha > 0$, assume that a positive definite matrix X, any matrix Y, and a positive scalar ρ solve the following optimization problem*:

$$\max_{X,Y,\rho} \lambda_{\min}(X)$$

subject to

$$\begin{bmatrix} \phi_1 & \rho I & B_\omega X & \overline{\Upsilon}(Y) & XH^T & XD^T \\ * & -\rho I & 0 & 0 & 0 & 0 \\ * & * & -X & 0 & 0 & 0 \\ * & * & * & -I_{(p)}\otimes X & 0 & 0 \\ * & * & * & * & -\frac{\rho}{\alpha^2} I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0, \tag{11}$$

where

$$\overline{\Upsilon}(Y) = [-B_1S_1Y \quad -B_2S_2Y \quad \cdots \quad -B_pS_pY],$$

$$S_i = \begin{bmatrix} 0_{d_i\times d_1} & \cdots & I_{d_i\times d_i} & \cdots & 0_{d_i\times d_p}\end{bmatrix}, \quad i\in\mathcal{V}.$$

Then, by designing the distributed controllers as $K_i = S_iYX^{-1}$ $(i\in\mathcal{V})$, the system is robustly asymptotically stable with degree α for $d_z(t)\equiv 0$ and the $\mathcal{L}_2$ gain from $d_z(t)$ to $z(t)$ is given by $\gamma_x = 1/\lambda_{\min}(X)$.

Proof. The proof is based on the Lyapunov theory. As P_x is positive definite, the Lyapunov function (10) is positive for all $x(t)\neq 0$ and zero only for $x(t)\equiv 0$.

The derivative of the Lyapunov function is given by

$$\dot{V}_x(t) = 2x^T(t)P_x\dot{x}(t).$$

Using the evolution of $\dot{x}(t)$ in Proposition 1, the derivative can be written as follows:[2]

$$\dot{V}_x = 2x^TP_x(A+BK)x + 2x^TP_x\Upsilon(\mathcal{K})e + 2x^TP_x(f_n+\omega).$$

Now, some null terms are added to the derivative

$$\dot{V}_x = 2x^TP_x(A+BK)x + 2x^TP_x\Upsilon(\mathcal{K})e + 2x^TP_x(f_n+\omega) \pm \epsilon f_n^Tf_n \pm \omega^TP_x\omega \pm e^T\overline{P}_xe \pm z^Tz,$$

where ϵ is a positive scalar and $\overline{P}_x \triangleq I_{(p)}\otimes P_x$.

Defining an augmented state vector as $\xi^T = [x^T\ f_n^T\ \omega^T\ e^T]$, previous equation can be rewritten as

$$\dot{V}_x = \xi^TF_x\xi + \epsilon f_n^Tf_n + \omega^TP_x\omega + e^T\overline{P}_xe - z^Tz,$$

where

$$F_x = \begin{bmatrix} P_xA_K + A_K^TP_x + D^TD & P_x & P_xB_\omega & P_x\Upsilon(\mathcal{K}) \\ * & -\epsilon I & 0 & 0 \\ * & * & -P_x & 0 \\ * & * & * & -\overline{P}_x \end{bmatrix},$$

with $A_K = A + BK$.

The term $\epsilon f_n^Tf_n$ can be bounded by $\epsilon\alpha 2x^TH^THx$. Then, it turns out that the derivative of the Lyapunov function can also be bounded as follows:

$$\dot{V}_x \le \xi^T\Xi_x\xi + \omega^TP_x\omega + e^T\overline{P}_xe - z^Tz, \tag{12}$$

where

$$\Xi_x = F_x + \begin{bmatrix} \epsilon\alpha^2H^TH & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}. \tag{13}$$

Assume now that Ξ_x is negative definite.

- For $\omega(t), e(t)\equiv 0, \forall t$, the following holds:

  $$\dot{V}_x \le \xi^T\Xi_x\xi - z^Tz. \tag{14}$$

  As Ξ_x is negative definite, one can obtain that $V_x(t)$ decreases for all t. Then $\dot{V}_x(t) \le -\delta\|x(t)\|^2$ for a sufficient small $\delta > 0$, which ensure asymptotic stability of system with degree α.
- Taking into account $\Xi_x < 0$, the term $\xi^T\Xi_x\xi$ is negative definite. Thus, for $\omega, e\neq 0$ and under zero initial conditions,

  $$\dot{V}_x \le -z^Tz + \omega^TP_x\omega + e^T\overline{P_x}e. \tag{15}$$

  Integrating both sides of (15) from t_0 to t, one can see that

  $$V_x(t) - V_x(t_0) \le -\int_{t_0}^t z^T(s)z(s)\,ds + \int_{t_0}^t (\omega^T(s)P_x\omega(s) + e^T(s)\overline{P}_xe(s))\,ds.$$

  Then, letting $t\to\infty$ and taking into account that under zero initial condition $V_x(t_0) = 0$ and the positive definitiveness of the functional, it can be shown that

  $$\int_{t_0}^\infty z^T(s)z(s)\,ds \le \int_{t_0}^\infty (\omega^T(s)P_x\omega(s) + e^T(s)\overline{P}_xe(s))\,ds.$$

  The quadratic terms on the right-hand side of the equation can be bounded using the property $x^TPx \le \lambda_{\max}(P)x^Tx$, for $P > 0$. Therefore

  $$\|z(t)\|_{\mathcal{L}_2} \le \lambda_{\max}(P_x)(\|\omega(t)\|_{\mathcal{L}_2} + \|e(t)\|_{\mathcal{L}_2}) \le \lambda_{\max}(P_x)\|d_z(t)\|_{\mathcal{L}_2},$$

  where it has been used that $\lambda_{\max}(P_x) = \lambda_{\max}(\overline{P}_x)$.

Hence, if $\Xi_x < 0$ the asymptotic stability of the system is guaranteed and the $\mathcal{L}_2$ gain from $d_z(t)$ to $z(t)$ is $\gamma_x = \lambda_{\max}(P_x) = 1/\lambda_{\min}(X)$.

It remains to prove that matrix Ξ_x is indeed negative definite. To do so, Schur complements are applied to the inequality $\Xi_x < 0$ to eliminate the quadratic terms D^TD and $\epsilon\alpha^2H^TH$ from the element (1,1) of Ξ_x. Then, pre- and post-multiplying the resulting inequality by $diag\{P_x^{-1}, \epsilon^{-1}, P_x^{-1}, \overline{P}_x^{-1}, I, I\}$ and its transpose, an inequality with the same structure of (11) is obtained by defining $\rho\equiv\epsilon^{-1}$, $X\equiv P_x^{-1}$ and $Y\equiv KP_x^{-1}$. Therefore, if LMI (11) is satisfied, then $\Xi_x < 0$ holds. □

Theorem 1 solves the first two points of the robust distributed control and observation problem given in Definition 2. The controllers

[2] Time references have been removed to alleviate the notation.

are synthesized to attenuate disturbances due to external perturbations and observation errors.

The optimization problem with linear constraints proposed in Theorem 1 can be solved using efficient interior point algorithms, as for instance `mincx` in Matlab. The interested reader may find some examples in Boyd, El Ghaoui, Feron, & Balakrishnan (1994).

5. Observers design

This section is devoted to the second stage of the design procedure. The objective is the synthesis of the observers such that the estimation errors are asymptotically stable. Additionally, the effects of the measurement noises are attenuated.

As before, a Lyapunov-based approach is followed. In this case, the Lyapunov function includes terms related to the observation error and the state of the system, as both dynamics are coupled:

$$\dot{V}_e(t) = x^T(t)P_x x(t) + e^T(t)P_e e(t),$$

where P_x was designed in the previous section and P_e is a block diagonal matrix

$$P_e = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_p \end{bmatrix},$$

where matrices $P_i \in \mathbb{R}^{n\times n}$ $(i \in \mathcal{V})$ are positive definite.

Recalling the dynamics of the observation error given in Proposition 2, it is worth mentioning that the separation principle does not hold here. The main implication of this fact is that the design of the observers depends on the controller gains previously designed through Theorem 1. The following theorem presents the synthesis procedure for M_i and N_{ij} $(i \in \mathcal{V}, j \in \mathcal{N}_i)$.

Theorem 2. *Given scalars $\alpha, \gamma_e > 0$, a positive definite matrix P_x, and controllers K_i, $i \in \mathcal{V}$, assume that the LMI (16) has a feasible solution for a positive definite matrix P_e, any matrices W_i, X_{ij} $(i \in \mathcal{V}, j \in \mathcal{N}_i)$ and a positive scalar ϵ. Then, if the observers are designed as $M_i = P_i^{-1}W_i$ and $N_{ij} = P_i^{-1}X_{ij}$, $i \in \mathcal{V}$, $j \in \mathcal{N}_i$, the estimation errors of all the nodes are robustly asymptotically stable with degree α for $d_e(t) \equiv 0$ and the $\mathcal{L}_2$ gain from $d_e(t)$ to $e(t)$ is lower than γ_e:*

$$\left[\begin{array}{cccc|ccc|cc} \theta_{11} & P_x & P_x B_\omega & 0 & -P_x B_1 K_1 & \dots & -P_x B_p K_p & 0 & \varepsilon H^T \\ * & -\varepsilon I & 0 & 0 & P_1 & \dots & P_p & 0 & 0 \\ * & * & -\gamma_e^2 I & 0 & B_\omega^T P_1 & \dots & B_\omega^T P_p & 0 & 0 \\ * & * & * & -\gamma_e^2 I & -W_1^T & \dots & -W_p^T & 0 & 0 \\ \hline * & * & * & * & \multicolumn{3}{c|}{\theta_{55a} + \theta_{55b} + \theta_{55c} + (\theta_{55a} + \theta_{55b} + \theta_{55c})^T} & I & 0 \\ \hline * & * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & * & -\frac{\varepsilon}{\alpha^2} I \end{array}\right] < 0, \tag{16}$$

where

$$\theta_{11} = P_x(A+BK) + (A+BK)^T P_x,$$

$$\theta_{55a} = \begin{bmatrix} P_1 A + A^T P_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_p A + A^T P_p \end{bmatrix},$$

$$\theta_{55b} = \begin{bmatrix} P_1 BK & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_p BK \end{bmatrix} - \begin{bmatrix} P_1 B_1 K_1 & \cdots & P_1 B_p K_p \\ \vdots & \ddots & \vdots \\ P_p B_1 K_1 & \cdots & P_p B_p K_p \end{bmatrix},$$

$$\theta_{55c} = \sum_{(i,j)\in\mathcal{E}} \begin{array}{c} \begin{array}{ccccccc} \mathit{col.} & & i & & j & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & -X_{ij} & \cdots & X_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \mathit{row}\ i. \\ \\ \\ \end{array}$$

Proof. The proof is very similar to that of Theorem 1. The derivative of the Lyapunov function is

$$\dot{V}_e(t) = 2x^T(t)P_x\dot{x}(t) + 2e^T(t)P_e\dot{e}(t).$$

Using the evolution of $x(t)$ in Proposition 1 and of $e(t)$ in Proposition 2, the derivative can be written as follows[3]:

$$\begin{aligned} \dot{V}_e = {} & 2x^T P_x(A+BK)x + 2x^T P_x \Upsilon(\mathcal{K})e + 2x^T P_x f_n \\ & + 2x^T P_x B_\omega \omega + 2e^T P_e(\Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N}))e \\ & + 2e^T P_e(I \otimes f_n) + 2e^T P_e(I \otimes B_\omega \omega) - 2e^T P_e \Pi(\mathcal{M})v. \end{aligned}$$

Adding some null terms to the derivative, it yields

$$\begin{aligned} \dot{V}_e = {} & 2x^T P_x(A+BK)x + 2x^T P_x \Upsilon(\mathcal{K})e + 2x^T P_x f_n \\ & + 2x^T P_x B_\omega \omega + 2e^T P_e(\Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N}))e \\ & + 2e^T P_e(I \otimes f_n) + 2e^T P_e(I \otimes B_\omega \omega) - 2e^T P_e \Pi(\mathcal{M})v \\ & \pm e^T e \pm \gamma_e^2(\omega^T \omega + v^T v) \pm \epsilon f_n^T f_n. \end{aligned}$$

Now, defining a different augmented vector ζ as $\zeta^T = (x^T\ f_n^T\ \omega^T\ v^T\ e^T)$, the last equation can be rewritten in the following form:

$$\dot{V}_e = \zeta^T \Xi_{ex} \zeta - e^T e,$$

where

$$\Xi_{ex} = \left[\begin{array}{cccc|ccc} \theta_{11} & P_x & P_x B_\omega & 0 & -P_x B_1 K_1 & \dots & -P_x B_p K_p \\ * & -\varepsilon I & 0 & 0 & P_1 & \dots & P_p \\ * & * & -\gamma_e^2 I & 0 & B_\omega^T P_1 & \dots & B_\omega^T P_p \\ * & * & * & -\gamma_e^2 I & -M_1^T P_1 & \dots & -M_p^T P_p \\ \hline * & * & * & * & \multicolumn{3}{c}{\Xi_{ex}^{55}} \end{array}\right]$$

with $\Xi_{ex}^{55} = P_e(\Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N})) + (\Phi(\mathcal{M}) + \Psi(\mathcal{K}) + \Lambda(\mathcal{N}))^T P_e$.

Using the bound on $\epsilon f_n^T f_n$, $\dot{V}_e(t)$ can be bounded as it was done for $\dot{V}_x(t)$ in the proof of Theorem 1. Then, Schur complements are applied following the same procedure. The application of Schur complements together with the changes of variables $M_i P_i = W_i$ and $N_{ij} P_i = X_{ij}$, allow to obtain that Ξ_{ex} is negative definite if the LMI (16) holds. This way, it is straightforward to follow the rest of the steps in the proof of Theorem 1 to deduce the robust stability of the estimation error as well as the bound $\|e(t)\|_{\mathcal{L}_2} \le \gamma_e \|d_e(t)\|_{\mathcal{L}_2}$. □

The results given in Theorems 1 and 2 deserve some comments concerning the practical implementation of the proposed scheme.

[3] Time references have been removed to alleviate the notation.

In particular, the computation of the controllers and observers makes implicit use of the network connectivity ($\mathcal{G}$) and the input and outputs channels of all agents (B_i,C_i). Therefore, this implies that both problems must be solved in a centralized way, which can be computationally complex when the system dimension, the agents, and the number of connections are large. However, the design problem is solved only once and this is made offline.

Once the controllers and observers are synthesized, every node requires only local information to carry out its tasks: plant output $y_i(t)$ and state estimations from neighboring nodes $\hat{x}_j(t)$, $j \in \mathcal{N}_i$. Hence, the implementation is completely distributed.

6. Application example

This section presents an application of the proposed distributed scheme to test its performance in a real system. The plant and the experimental setup are described, providing all the considerations related to the distributed scheme. Later, simulation and experimental results are presented.

6.1. Plant description

The quadruple-tank process introduced by Johansson (2000) has received a great attention because it exhibits interesting properties representative of relevant problems in both, research and industry. The system exhibits complex dynamics, including interactions and a tunable transmission zero location.

The experiments have been performed in the 33-041 Coupled Tanks System of Feedback Instruments, see Instruments (2012). A picture of the platform is given in Fig. 2. It comprises four tanks, each one with a pressure sensor to measure the water level. The couplings between the tanks can be modified using seven manual valves. Water is delivered to the tanks by two independently controlled, submerged pumps. Drain flow rates can be modified using easy-to-change orifice caps. Notation related with the plant is given in Table 2.

The coupled tanks are controlled using Simulink and an Advanced PCI1711 Interface Card. The system is highly configurable, due to the numerous available valves. For the experiments, the following configuration is chosen (see Fig. 3):

- Input water is delivered to the upper tanks. Pump 1 feeds tank 1 and pump 2 feeds tank 3.
- Tanks 1 and 3 are coupled by opening the corresponding valve.

The distributed scheme proposed in this paper can find a possible application in large-scale chemical plants, where coupled processes (represented by the coupled tanks) can be located hundred of meters away from each other. In these situations, communication between local sensors and controllers can be expensive using classical point-to-point wired networks, so only neighboring devices should be able to communicate.

In this experiment, a reduced network with four nodes is proposed, two of them being sensors and the other two sensor +actuators. Fig. 4 shows a block diagram of the whole system. Each node has been tagged from 1 to 4 according to the number of the tank whose level it is measuring. Node 1 (respectively 3) applies the control signal to pump 1 (2). The nodes communicate by means of a network with topology 2⇔1⇔3⇔4. Note that no node can estimate the whole plant state based only on the available local measurements of the plant.

The objective of the experiments is twofold. First, the state of the plant must be monitored from every node. Second, the water level of the two lower tanks is to be controlled.

Fig. 2. Plant of four-coupled tanks.

Table 2
Notation related to the plant.

Variable	Description
h_i	Water level of tank i
v_i	Voltage of pump i
h_i^0	Reference level of tank i
v_i^0	Reference voltage of pump i
Δh_i	Increment of h_i with respect to h_i^0
Δv_i	Increment of v_i with respect to v_i^0
s	Output to be tracked
r	Output reference for s
Δh_r	Reference level with respect to h^0
Δv_r	Reference voltage with respect to v^0

6.2. Plant modeling

The coupled tanks can be easily modeled by means of the following nonlinear model:

$$\frac{dh_1(t)}{dt} = -\frac{a_1}{A}\sqrt{2gh_1(t)} + \eta v_1(t) - \frac{a_{13}}{A}\sqrt{2g(h_1(t)-h_3(t))},$$
$$\frac{dh_2(t)}{dt} = \frac{a_1}{A}\sqrt{2gh_1(t)} - \frac{a_2}{A}\sqrt{2gh_2(t)},$$
$$\frac{dh_3(t)}{dt} = -\frac{a_3}{A}\sqrt{2gh_3(t)} + \eta v_2(t) + \frac{a_{13}}{A}\sqrt{2g(h_1(t)-h_3(t))},$$
$$\frac{dh_4(t)}{dt} = \frac{a_3}{A}\sqrt{2gh_3(t)} - \frac{a_4}{A}\sqrt{2gh_4(t)},$$

where $h_i(t)$ $(i=1,\dots,4)$ denote the water level in the tanks; v_i $(i=1,2)$ are voltages applied to the pumps; a_i $(i=1,\dots,4)$ are the outlet area of the tanks; a_{13} is the outlet area between tanks 1 and 3; η is a constant relating the control voltage with the water flow

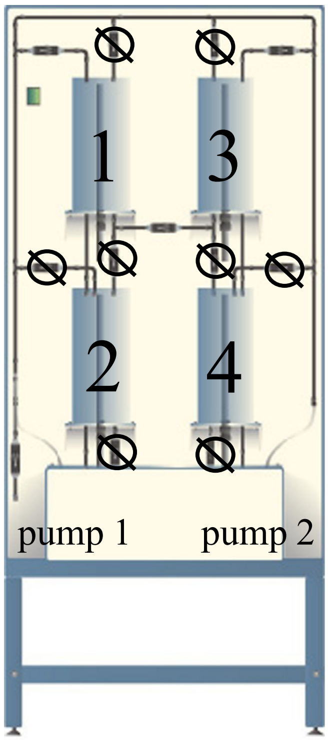

Fig. 3. Schematic configuration of the coupled tanks.

from the pump; A is the cross-sectional area of the tanks; and g is the gravitational constant.

This system is linearized around the equilibrium point given by $h_i{}^0$ and $u_i{}^0$, yielding

$$\dot{\Delta h}(t) = A\Delta h(t) + B\Delta v(t) + f_n(t, \Delta h(t)), \tag{17}$$

where $\Delta h(t) = [h_1(t) - h_1^0 \cdots h_4(t) - h_4^0]^T$ and $\Delta v(t) = [v_1(t) - v_1^0 \; v_2(t) - v_2^0]^T$. Matrices A and B are obtained by using a Taylor expansion of the nonlinear equations of the model (18).

$$A = \begin{bmatrix} -\frac{a_1 g}{A\sqrt{2gh_1^0}} - \frac{a_{13} g}{A\sqrt{2g(h_1^0 - h_3^0)}} & 0 & \frac{a_{13} g}{A\sqrt{2g(h_1^0 - h_3^0)}} & 0 \\ \frac{a_1 g}{A\sqrt{2gh_1^0}} & -\frac{a_2 g}{A\sqrt{2gh_2^0}} & 0 & 0 \\ \frac{a_{13} g}{A\sqrt{2g(h_1^0 - h_3^0)}} & 0 & -\frac{a_3 g}{A\sqrt{2gh_3^0}} - \frac{a_{13} g}{A\sqrt{2g(h_1^0 - h_3^0)}} & 0 \\ 0 & 0 & \frac{a_3 g}{A\sqrt{2gh_3^0}} & -\frac{a_4 g}{A\sqrt{2gh_4^0}} \end{bmatrix},$$

$$B = \begin{bmatrix} \eta & 0 \\ 0 & 0 \\ 0 & \eta \\ 0 & 0 \end{bmatrix}$$

The nonlinear term $f_n(t, \Delta h(t))$ in (17) includes the linearization errors. For each tank i, let R_i denote the linearization error of this tank. This error is given by (see Phillips & Taylor, 1996)

$$R_i = \sum_j \frac{g_j^{(2)}(\varsigma_j)}{2} (\Delta h_j(t))^2,$$

where functions g_j represent the influence of the tank level j on the dynamics of level i. Variable ς_j is an unknown number belonging to the interval of interest relative to tank level j. For instance, the linearization error of tank 2 is given by

$$R_2 = -\frac{a_1}{A}\sqrt{2g}\frac{1}{4\varsigma_1^{3/2}}(\Delta h_1(t))^2 + \frac{a_2}{A}\sqrt{2g}\frac{1}{4\varsigma_2^{3/2}}(\Delta h_2(t))^2.$$

Given the interval of interest, the maximum value of $|R_2|$ can be found, which is an upper bound of the linearization error. For the other tanks, an equivalent procedure can be used to obtain the maximum of $|R_1|, |R_3|, |R_4|$.

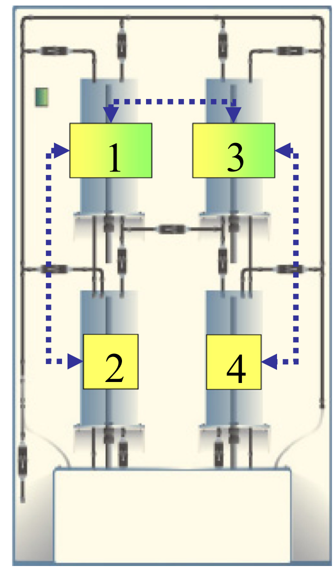

Fig. 4. Distributed control scheme with four nodes. Nodes 1 and 3 are sensor +actuators; nodes 2 and 4 are sensors. Dotted lines represent the communication links.

Note that the maximum of $|R_i|$ depends quadratically on $\Delta h(t)$. Recalling the model of the nonlinear uncertainties in (2), the maximum of $|f_n(t, \Delta h(t))|$ depends linearly on $\Delta h(t)$. As it is always possible to upper bound a quadratic function using a linear one around the equilibrium point, suitable values for H and α can be found in order to take into account the linearization errors. Needless to mention, the larger interval of interest, the larger value for α (given a fixed H). For the rest of the section,[4] $H = I_{(4)}$ and $\alpha = 0.01$.

The objective is not only to stabilize the plant around the linearization point, but also to track references. To do so, the system output is set as $s \triangleq C_r \Delta h$, where C_r is a matrix that selects the water level of tanks 2 and 4. The references are given by the vector r. At the equilibrium points, it should be verified $s \simeq r$ and $\dot{\Delta h}_r \simeq 0$. To perform the tracking task, the incremental equilibrium points $(\Delta h_r, \Delta v_r)$ associated with references r are found as follows:

$$0 = A\Delta h_r + B\Delta v_r,$$

$$r = C_z \Delta h_r.$$

Rewriting the equation above in blocks, it yields

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} A & B \\ C_z & 0 \end{bmatrix} \begin{bmatrix} \Delta h_r \\ \Delta v_r \end{bmatrix},$$

so that the incremental equilibrium point associated with r can be obtained as

$$\begin{bmatrix} \Delta h_r \\ \Delta v_r \end{bmatrix} = \begin{bmatrix} A & B \\ C_z & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

It is assumed that the references are reachable by the system, that is, the inverse above does exist. Finally, to track references, we must stabilize the following system:

$$\dot{x}(t) = Ax(t) + Bu(t) + f_n(t, x(t)),$$

where $x(t) \triangleq \Delta h(t) - \Delta h_r$ and $u(t) \triangleq \Delta v(t) - \Delta v_r$. Note that this system has the same structure that the one described in (1).

[4] Observe that the multiplicative constants on R_2 are of order 10^{-4}.

Table 3
Parameters of the plant. The terms in parentheses are related to the simulation experiments.

Variable/parameter	Value	Unit	Description
h_i	0–25	cm	Water level of tank i
v_i	0–5	V	Voltage level of pump i
A	0.01389	m^2	Cross-sectional area
a_i	50.265e−6	m^2	Outlet area of tank i
a_{13}	50.265e−6	m^2	Outlet area between tanks 1 and 3
η	0.22	cm/V s	Constant relating voltage and flow
h_1^0	9.55 (12.6)	cm	Reference level of tank 1
h_2^0	16.9 (12.6)	cm	Reference level of tank 2
h_3^0	7.6 (11)	cm	Reference level of tank 3
h_4^0	14.1 (11)	cm	Reference level of tank 4
v_1^0	3.3 (3.5)	V	Voltage level of pump 1
v_2^0	2.6 (1.5)	V	Voltage level of pump 2

6.3. Simulation results

In the simulation example, the objective consists in tracking the following reference:

- From $t = 100$ s to $t = 500$ s, the water level in tanks 2 and 4 should rise 4 and 2 cm, respectively.
- From $t = 500$ s to $t = 900$, the water level in both tanks should go to the equilibrium point.
- From $t = 900$ s to $t = 1300$ s, the water level in tanks 2 and 4 should rise 1 and 1.5 cm, respectively.
- From $t = 1300$ s, both tanks should reach the equilibrium point.

The equilibrium point is defined in Table 3 in parentheses. The result of the simulation is shown in Fig. 5.

The references are tracked and the control performance is satisfactory. The plant has a characteristic rise time of 300 s. With the distributed control strategy, it is reduced to approximately 100 s. Note that the control objective is to track references in tanks 2 and 4, so that overshooting in tanks 1 and 3 is allowed to improve the tracking performance. By properly tuning the controller, it is possible to obtain slower response with less overshooting.

Furthermore, it can be observed that the observer performance is also adequate. Fig. 6 shows the estimation in node 1 of the water level of tanks 2 and 4. It is worthwhile to recall that node 1 measures only the level in tank 1. In order to estimate the height of the water column in tanks 2 and 4, node 1 needs to communicate with its neighbors. The stabilization of the estimation error is faster than the tracking. Node 1 achieves a tolerable estimation of the water levels in tanks 2 and 4 in 30–40 s.

6.4. Experimental results

This section shows the experimental results obtained in the FeedBack Coupled Tank System. The references in the first experiment are identical to those of the previous simulation. The linearization point is different, see Table 3. Fig. 7 depicts the evolution of the water level for the four tanks. The estimates in node 1 of the levels in tanks 2 and 4 are shown in Fig. 8.

The control performance is similar to that obtained in simulation. Again, with overshooting in tanks 1 and 3, a rise time of approximately 100 s is achieved. The estimator in node 1 also shows a good performance.

In the experiments in Figs. 9 and 10, some disturbances are introduced. Concretely:

- An additional valve between tanks 3 and 4 is opened.
- 50 cl of extra water is added in tank 4.

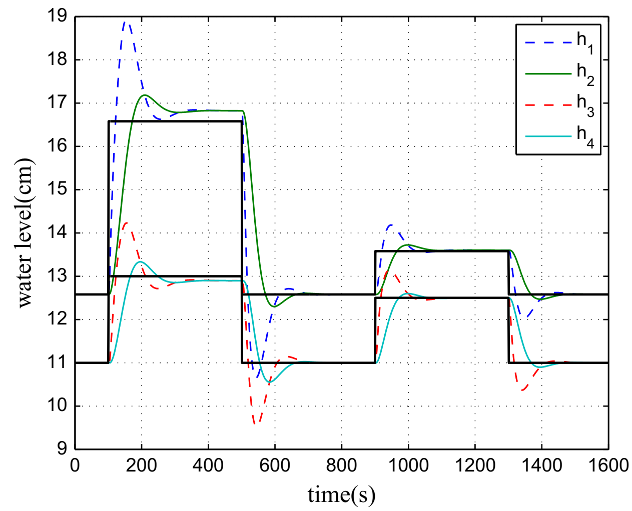

Fig. 5. Water level of the four tanks in simulation.

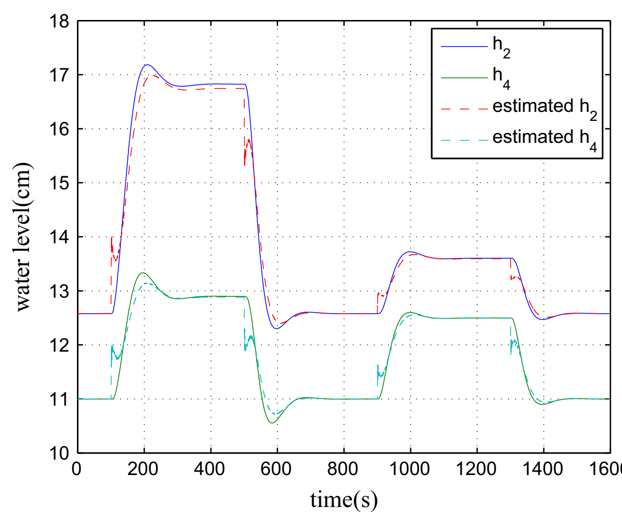

Fig. 6. Water level of tanks 2 and 4 and the estimates in node 1 in simulation.

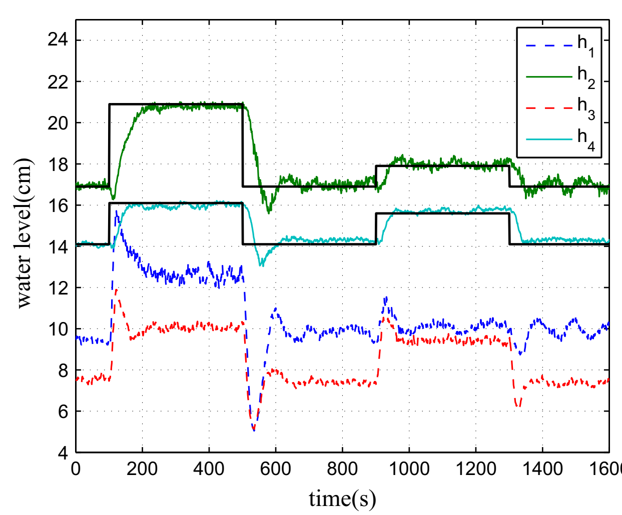

Fig. 7. Water level of the four tanks.

It can be seen that the distributed controller exhibits a good disturbance rejection in both cases.

In the last experiment, the importance of the coupling effect is showed. Tank 2 is asked to track references whereas tank 4 is asked to maintain the water level at the equilibrium point. In order to vary the level of tank 2, tank 1 must be filled or emptied. Due to the coupling valve, tank 3 varies its level, affecting to tank 4. Fig. 11 depicts the system response. The controller achieves notable decoupling of the closed-loop dynamics. The control signal applied to the pumps is shown in Fig. 12.

7. Conclusions

In this paper a novel method for distributed estimation and control is proposed. The method is intended to be of application in the case of large-scale uncertain plants where the control is geographically distributed among a number of units. Each individual unit is assumed to have access to a subset of the plant states, and possibly controls only a restricted subset of plant control channels. A communication network between nodes is also considered so that the units use the estimates of neighboring nodes to build their own estimates of the plant states.

The objective is designing a control structure for every unit (distributed control), so that collective control actions robustly asymptotically drive the system to equilibrium with L_2-gain disturbance rejection capabilities. A difficulty that readily arises when the problem is so formulated is that the separation principle does not hold, as every single unit ignores the control action that other units might be applying. To overcome this, a two-stage design is proposed: in a first stage, the control gains are obtained to robustly stabilize the plant despite the observation errors. At the second stage, the observer gains for every unit are designed to minimize an H_∞ index to reduce the effects of the communication noise in the observation error. Both steps are formulated and solved in terms of LMIs. The performance is shown both by simulations and experimentally on a four-tank level-control system.

It is worth mentioning that the proposed methodology exhibits the following novel characteristics:

- Both, control and estimation, are tackled in a unified way providing a robust design that takes into account nonlinear

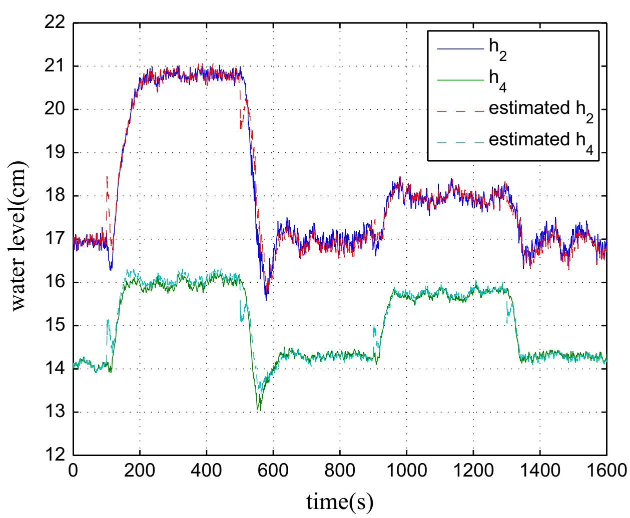

Fig. 8. Water level of tanks 2 and 4 and the estimates in node 1.

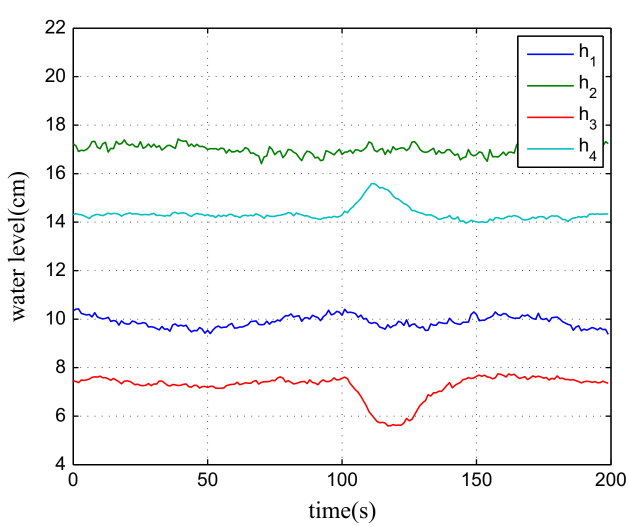

Fig. 9. Disturbance rejection when an additional valve between tanks 3 and 4 is opened from $t=100$ to $t=110$ s.

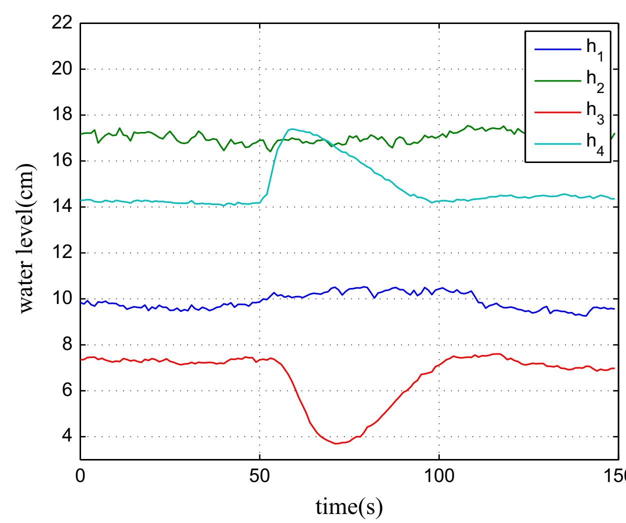

Fig. 10. Disturbance rejection when 50 cl of water is added to the tank 4 at $t=50$ s.

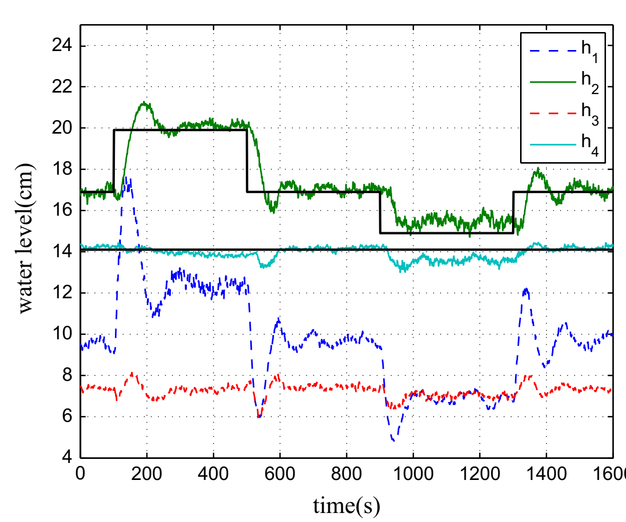

Fig. 11. Coupling effect in tank 4 when the level of tank 2 is modified.

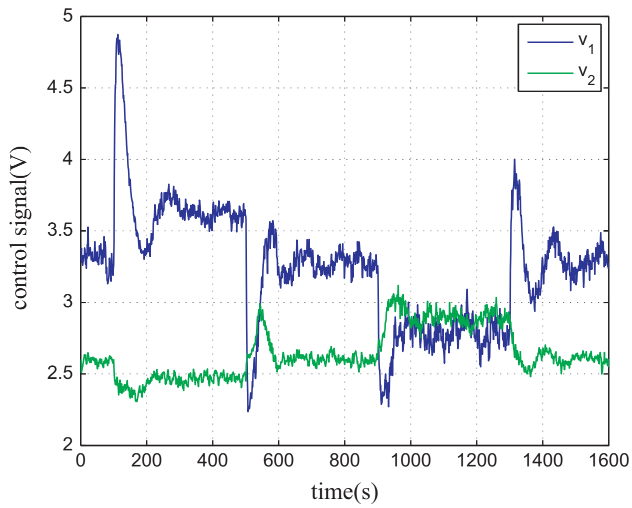

Fig. 12. Control signal applied to the pumps.

time-varying model uncertainties and L_2-gain disturbance rejection capabilities.

- The design procedure, though centralized in conception, allows fully decentralized implementation. The solution is obtained in terms of LMIs for which efficient computational algorithms are widely available. The distributed design, which would contribute to reduce the computational complexity, is an interesting open problem and it will be matter of future research.
- The methodology allows the consideration of two types of units: sensor units which only build their estimate of the plant states, and sensor+actuator nodes which both estimate plant states and generate control actions.
- Remarkably, the methodology accounts for overlapping control where different units can simultaneously provide control signals for the same control channel. This approach increases reliability and controllability of the overall plant.

Appendix A. Proof of Proposition 2

The observation error of node i can be obtained using Eq. (7) and Proposition 1:

$$\begin{aligned}\dot{e}_i(t) &= \dot{x}(t) - \dot{\hat{x}}_i(t)\\ &= (A+BK)x(t) + \Upsilon(\mathcal{K})e(t) + f_n(t,x(t)) + B_\omega \omega(t)\\ &\quad -(A+BK)\hat{x}_i(t) - M_i(y_i(t) - C_i\hat{x}_i(t))\\ &\quad -\sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(t) - \hat{x}_i(t)).\end{aligned} \tag{A.1}$$

We can write $\dot{e}_i(t) = (tr1)_i + (tr2)_i + (tr3)_i$, where $(tr1)_i$ includes the terms of (A.1) which do not depend on the neighbors, $(tr2)_i$ are related to other nodes, and $(tr3)_i$ depends on external signals. Consider first the terms $(tr1)_i$:

$$\begin{aligned}(tr1)_i &\triangleq (A+BK)e_i(t) - M_iC_ie_i(t) + \Upsilon(\mathcal{K})e(t)\\ &= (A - M_iC_i + BK)e_i(t) + \Upsilon(\mathcal{K})e(t).\end{aligned} \tag{A.2}$$

Consider now $(tr2)_i$:

$$\begin{aligned}(tr2)_i &\triangleq \sum_{j\in\mathcal{N}_i} N_{ij}(\hat{x}_j(t) - \hat{x}_i(t))\\ &= \sum_{j\in\mathcal{N}_i} N_{ij}(e_i(t) - e_j(t)).\end{aligned} \tag{A.3}$$

Lastly, the external inputs are given by

$$(tr3)_i \triangleq f_n(t,x(t)) + B_\omega\omega(t) - M_i v_i(t). \tag{A.4}$$

Recall the definition of the augmented observation error $e^T(t) = [e_1^T(t) \cdots e_p^T(t)]$ and the augmented noise vector $v(t) = [v_1^T(t) \cdots v_p^T(t)]^T$. Making some mathematical manipulations, it can be checked that the following equalities hold:

$$\begin{bmatrix}(tr1)_1\\(tr1)_2\\ \vdots \\ (tr1)_p\end{bmatrix} = (\Phi(\mathcal{M}) + \Psi(\mathcal{K}))e(t),$$

$$\begin{bmatrix}(tr2)_1\\(tr2)_2\\ \vdots \\ (tr2)_p\end{bmatrix} = \Lambda(\mathcal{N})e(t),$$

$$\begin{bmatrix}(tr3)_1\\(tr3)_2\\ \vdots \\ (tr3)_p\end{bmatrix} = I\otimes(f_n(t,x(t)) + B_\omega\omega(t)) - \Pi(\mathcal{M})v(t).$$

By adding the three vectors above it is immediate to obtain that the derivative of $e(t)$ can be written as in (9).

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