Automatica 59 (2015) 248–255

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Dynamic quantization of uncertain linear networked control systems ☆

Kun Liu [a,c], Emilia Fridman [b], Karl Henrik Johansson [c]

[a] School of Automation, Beijing Institute of Technology, 100081 Beijing, PR China

[b] School of Electrical Engineering, Tel Aviv University, 69978 Tel Aviv, Israel

[c] ACCESS Linnaeus Centre and School of Electrical Engineering, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden

ARTICLE INFO

Article history:
Received 26 September 2014
Received in revised form
23 April 2015
Accepted 15 June 2015

Keywords:
Networked control systems
Time-delay systems
Dynamic quantization
Lyapunov–Krasovskii method

ABSTRACT

This paper is concerned with the stability analysis of networked control systems with dynamic quantization, variable sampling intervals and communication delays. A time-triggered zooming algorithm for the dynamic quantization at the sensor side is proposed that leads to an exponentially stable closed-loop system. The algorithm includes proper initialization of the zoom parameter. More precisely, given a bound on the state initial conditions and the values of the quantizer range and error, we derive conditions for finding the initial value of the zoom parameter, starting from which the exponential stability is guaranteed by using "zooming-in" only. Polytopic type uncertainties in the system model can be easily included in the analysis. The efficiency of the method is illustrated on an example of an uncertain cart–pendulum system.

1. Introduction

Quantization always exists in computer-based control systems and quantization errors may have adverse effects on the system's stability and performance. In early 1990s, quantized state feedback to stabilize an unstable linear system was studied in Delchamps (1990). The values of the quantizer range and error were assumed to be fixed in advance and could not be changed by the control designer. Since then there has been research concerned with how the choice of quantization parameters affects the behavior of the system (Brockett & Liberzon, 2000; Elia & Mitter, 2001; Wong & Brockett, 1999). A more general type of quantizers with quantization regions having arbitrary shapes was introduced in Liberzon (2003). Recently, quantized feedback control of hybrid systems was studied in Liberzon, Nesic, and Teel (2014).

Networked control systems (NCSs) are systems with spatially distributed sensor, actuator and controller nodes which exchange data over a communication data channel. NCSs have attracted more and more attention in recent years. Three main approaches have been used to model NCSs as a discrete-time system (Fujioka, 2009), an impulsive/hybrid system (Naghshtabrizi, Hespanha, & Teel, 2010; Nesic & Liberzon, 2009) or a time-delay system (Fridman, Seuret, & Richard, 2004; Gao, Chen, & Lam, 2008). There have been a great number of results concerning networked and quantized control systems in the literature. To mention a few, static quantizer, such as logarithmic quantizer was addressed in Gao et al. (2008) and Yue, Peng, and Tang (2006), where the quantization error was treated as uncertainty or nonlinearity and bounded by using the sector bound approach (Fu & Xie, 2005). Dynamic quantizer was considered in Nesic and Liberzon (2009) for the stabilization of NCSs with scheduling protocols. Small communication delays (smaller than the sampling intervals) were further included in Heemels, Nesic, Teel, and van de Wouw (2009). The quantizer was assumed to take a finite set of values and incorporated an adjustable zoom variable (Liberzon, 2003, 2006). Recently, linear matrix inequality (LMI)-based conditions for stabilization with dynamic quantization and packet dropout were derived in Niu and Ho (2014) and Yan, Xia, and Li (2014). However, communication delays were not taken into account.

In the present paper, we develop a time-delay approach (see e.g., Chapter 7 of Fridman (2014)) for uncertain linear NCSs under dynamic quantization, variable sampling intervals and large communication delays (that may be larger than the sampling intervals). We follow Liberzon's framework (Liberzon, 2003) and

☆ This work was supported by the Knut and Alice Wallenberg Foundation (grant no. Dnr KAW 2009.088), the Swedish Research Council (grant no. VR 621-2014-6282) and the Israel Science Foundation (grant no. 754/10 and 1128/14). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Akira Kojima under the direction of Editor Ian R. Petersen. Kun Liu's work was done when he was with KTH Royal Institute of Technology.

E-mail addresses: kunliu@kth.se (K. Liu), emilia@eng.tau.ac.il (E. Fridman), kallej@kth.se (K.H. Johansson).

http://dx.doi.org/10.1016/j.automatica.2015.06.041
0005-1098/

model the closed-loop quantized system as a system with bounded disturbances. Sensor quantization is the focus of our study, cf., Gao et al. (2008). Communication delays lead to additional challenges: (1) The initial and level sets are defined in infinite-dimensional spaces, though saturation condition is given in terms of the delayed output vector. (2) The closed-loop system and the resulting solution bounds are formulated in terms of updating time instants at the actuators, while the zooming algorithm should be given in terms of sampling instants at the sensors. (3) The solution bounds include additional bounds on the first time interval of the delay length (Liu & Fridman, 2014). In this paper, the main contributions are as follows:

1. We suggest a time-triggered zooming algorithm for uncertain linear NCSs, which is implemented at the sensors although the solution bounds of the closed-loop system are given in terms of the updating time instants at the actuators. The zooming algorithm is formulated in terms of LMIs.
2. We present a direct Lyapunov approach for initialization of the zoom variable. More precisely, given a bound on the state initial conditions and the values of the quantizer range and error, we derive conditions for finding the initial value of the zoom variable to guarantee exponential stability of the closed-loop system.
3. The proposed framework can easily incorporate polytopic type uncertainties in the system model.

The rest of the paper is organized as follows. Section 2 presents the model of quantized NCSs. In Section 3 an LMI-based zooming algorithm for the dynamic quantization is proposed that leads to exponential stability of the resulting closed-loop system. In Section 4, the efficiency of the presented approach is illustrated by an uncertain cart–pendulum example. Finally, the conclusions and the future work are stated in Section 5.

Notations. Throughout the paper, the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. The space of functions $\phi : [a, b] \to \mathbb{R}^n$, which are absolutely continuous on $[a, b)$ (meaning that ϕ is continuous and its first-order derivative is Lebesgue integrable on $[a, b]$), have a finite limit $\lim_{\theta\to b^-} \phi(\theta)$ and have square integrable first-order derivatives, is denoted by $W[a, b]$ with norm $\|\phi\|_W = \max_{\theta\in[a,b]} |\phi(\theta)| + \left[\int_a^b |\dot\phi(s)|^2 ds\right]^{\frac{1}{2}}$. $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ denote the set of integers, non-negative integers and positive integers, respectively. $\lfloor x\rfloor$ denotes the largest integer k such that $k < x$, i.e., $\lfloor x\rfloor = \max\{k \in \mathbb{Z} : k < x\}$.

2. System model and preliminaries

2.1. Quantized NCSs

Consider the system architecture in Fig. 1 with plant

$$\dot x(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the control input, and A and B system matrices with appropriate dimensions. These matrices can be uncertain with polytopic type uncertainties. The NCS has N distributed sensors and quantizers, a controller node and an actuator node, which are all connected via two wireless networks. The measurements are given by $y_i(t) = C_i x(t) \in \mathbb{R}^{n_i}$, $i = 1,\dots,N$, $\sum_{i=1}^N n_i = n_y$. Denote $C = [C_1^T \cdots C_N^T]^T$ and $y(t) = [y_1^T(t) \cdots y_N^T(t)]^T \in \mathbb{R}^{n_y}$. Following Gao et al. (2008), in our consideration the quantization is performed at the sensor side.

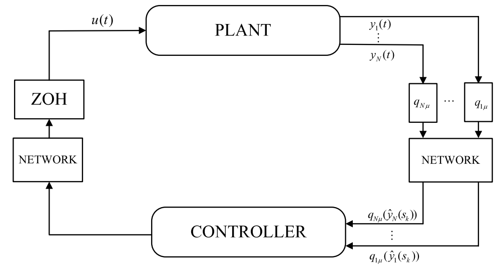


Fig. 1. Architecture of networked control systems with quantizers.

Let $z_i \in \mathbb{R}^{n_i}$, $i = 1,\dots,N$, be the vectors being quantized. The quantizers are piecewise constant functions $q_i : \mathbb{R}^{n_i} \to \mathbb{D}_i$, where $\mathbb{D}_i$ is a finite subset of $\mathbb{R}^{n_i}$, $i = 1,\dots,N$. Following Liberzon (2003), we assume that there exist real numbers $M_i > \Delta_i > 0$, $i = 1,\dots,N$, such that the following two conditions hold:

(a) if $|z_i| \le M_i$, then $|q_i(z_i) - z_i| \le \Delta_i$;

(b) if $|z_i| > M_i$, then $|q_i(z_i)| > M_i - \Delta_i$, where Δ_i and M_i are the quantization error bounds and ranges, respectively.

We consider quantized measurements of the form

$$q_{i\mu}(z_i) := \mu q_i\left(\frac{z_i}{\mu}\right), \quad i = 1,\dots,N, \tag{2}$$

where $\mu > 0$ is the zoom variable. The range of the quantizer $q_{i\mu}$, $i = 1,\dots,N$, is μM_i and the quantization error is $\mu\Delta_i$. Quantized measurements $q_{i\mu}(y_i)$ of the output y_i, $i = 1,\dots,N$, are available at the controller. The zoom variable μ will change dynamically at some discrete-time sampling instants in order to achieve exponential stability.

Let s_k denote the unbounded and monotonously increasing sequence of sampling instants, i.e.,

$$\begin{aligned} &0 = s_0 < s_1 < \cdots < s_k < \cdots, \quad k \in \mathbb{Z}_+,\\ &\lim_{k\to\infty} s_k = \infty, \quad s_{k+1} - s_k \le \text{MATI}, \end{aligned} \tag{3}$$

where MATI denotes the maximum allowable transmission interval. At each sampling instant s_k, all the outputs $y_i(t) \in \mathbb{R}^{n_i}$, $i = 1,\dots,N$, are sampled, quantized and transmitted over the networks. Assume that the data $q_\mu(y(s_k)) = [q_{1\mu}^T(y_1(s_k)) \cdots q_{N\mu}^T(y_N(s_k))]^T$, $k = 0, 1, \dots$ are transmitted in packets. We suppose that there is no data loss but the transmission over the two networks is subject to a variable delay η_k. Then $t_k = s_k + \eta_k$ is the updating time instant of the zero-order hold (ZOH) device. As in Liu and Fridman (2012) and Naghshtabrizi et al. (2010), we allow the delay to be large provided that the order of transmission of $q_\mu(y(s_k))$ is maintained at the reception. Assume that the network-induced delay η_k and the time span between the updating instant t_{k+1} and the current sampling instant s_k are bounded:

$$t_{k+1} - t_k + \eta_k \le \tau_M, \quad 0 \le \eta_m \le \eta_k \le \eta_M, \quad k \in \mathbb{Z}_+, \tag{4}$$

where η_m and η_M are known bounds and $\tau_M = \text{MATI} + \eta_M$.

We suppose that the controller and the actuator are event-driven. The first updating time t_0 corresponds to the time instant when the first data packet is received by the actuator, which means that $u(t) = 0$, $t \in [0, t_0)$. Then for $t \in [0, t_0)$, (1) is given by

$$\dot x(t) = Ax(t), \quad x(0) = x_0, \quad t \in [0, t_0). \tag{5}$$

We assume that x_0 may be unknown, but satisfies the bound $|x_0| < X_0$, where $X_0 > 0$ is known. Note that this assumption is common, e.g., for interval observer design (Polyakov, Efimov, Perruquetti, & Richard, 2013).

2.2. Closed-loop model and solution bounds

Assume that there exists a matrix $K = [K_1 \cdots K_N], K_i \in \mathbb{R}^{m\times n_i}$ such that $A + BKC$ is Hurwitz (with the real parts of eigenvalues strictly negative). Consider the static output feedback controller of the form

$$u(t) = \sum_{i=1}^{N} K_i q_{i\mu}(y_i(t_k - \eta_k)), \quad t \in [t_k, t_{k+1}),$$

where η_k is the communication delay. We thus obtain the closed-loop model as follows:

$$\dot{x}(t) = Ax(t) + A_1 x(t_k - \eta_k) + \sum_{i=1}^{N} B_i \omega_i(t), \quad t \in [t_k, t_{k+1}), \tag{6}$$

where $A_1 = BKC,\ B_i = BK_i,\ i = 1, \dots, N$, and $\omega_i(t) = q_{i\mu}(y_i(s_k)) - y_i(s_k),\ i = 1, \dots, N$, represent the quantization errors. If $|y_i(s_k)| \le \mu M_i$, then $|\omega_i(t)| \le \mu \Delta_i,\ i = 1, \dots, N$, for $t \in [t_k, t_{k+1})$.

Applying the time-delay approach to sampled-data control, denote $\tau(t) = t - t_k + \eta_k,\ t \in [t_k, t_{k+1})$. Then $\tau(t) \in [\eta_m, \tau_M]$ (cf., (4)) and $x(t_k - \eta_k) = x(t - \tau(t)),\ t \in [t_k, t_{k+1})$. The initial conditions for (6) are given by (5).

Consider first static quantizers with a constant zoom variable μ. We apply the following Lyapunov–Krasovskii functional (LKF) for delay-dependent analysis (Fridman & Dambrine, 2009; Park, Ko, & Jeong, 2011):

$$\begin{aligned} V(t, x_t, \dot{x}_t) &= x^T(t)Px(t) + \int_{t-\eta_m}^{t} e^{2\alpha(s-t)} x^T(s) S_0 x(s) ds \\ &\quad + \int_{t-\tau_M}^{t-\eta_m} e^{2\alpha(s-t)} x^T(s) S_1 x(s) ds, \\ &\quad + \eta_m \int_{-\eta_m}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) R_0 \dot{x}(s) ds d\theta \\ &\quad + (\tau_M - \eta_m) \int_{-\tau_M}^{-\eta_m} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta, \\ &P > 0,\ S_j > 0,\ R_j > 0,\ \alpha > 0, \\ &j = 0, 1,\ t \in [t_k, t_{k+1}),\ k \in \mathbb{Z}_+, \end{aligned} \tag{7}$$

where $x_t(\theta) \stackrel{\Delta}{=} x(t + \theta),\ \theta \in [-\tau_M, 0]$ and where (following Liu & Fridman, 2014) we define $x(t) = x_0,\ t < 0$. Following Fridman and Dambrine (2009) and using convex analysis of Park et al. (2011), we derive the following result (see Appendix A for the proof):

Lemma 1. *Given $0 \le \eta_m < \tau_M$, $\alpha > 0$, assume that there exist positive scalars $b_i, i = 1, \dots, N$, $n \times n$ matrices $P > 0$, $S_0 > 0$, $R_0 > 0$, $S_1 > 0$, $R_1 > 0$, S_{12}, such that the following LMIs are feasible:*

$$\Phi = \begin{bmatrix} R_1 & S_{12} \\ * & R_1 \end{bmatrix} \ge 0, \tag{8}$$

$$\Psi = \begin{bmatrix} \Sigma - F^T \Phi F e^{-2\alpha\tau_M} & \Xi^T H \\ * & -H \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{aligned} &\Sigma = F_1^T P \Xi + \Xi^T P F_1 + \Upsilon - F_2^T R_0 F_2 e^{-2\alpha\eta_m}, \\ &F_1 = [I_n\ 0_{n\times(3n+n_y)}], \qquad F_2 = [I_n\ -I_n\ 0_{n\times(2n+n_y)}], \\ &F = \begin{bmatrix} 0_{n\times n} & I_n & -I_n & 0_{n\times n} & 0_{n\times n_y} \\ 0_{n\times n} & 0_{n\times n} & I_n & -I_n & 0_{n\times n_y} \end{bmatrix}, \\ &H = \eta_m^2 R_0 + (\tau_M - \eta_m)^2 R_1, \\ &\Xi = [A\ 0_{n\times n}\ A_1\ 0_{n\times n}\ B_1 \cdots B_N], \end{aligned} \tag{10}$$

and $\Upsilon = \operatorname{diag}\{S_0 + 2\alpha P, -(S_0 - S_1)e^{-2\alpha\eta_m}, 0_{n\times n}, -S_1 e^{-2\alpha\tau_M}, -b_1 I_{n_1}, \dots, -b_N I_{n_N}\}$. Let $\mu > 0$ be constant and $|\omega_i(t)| \le \mu\Delta_i$, $i = 1, \dots, N$. Then the solutions of system (6) with the initial conditions $x_{t_0} \in W[-\tau_M, 0]$, satisfy the following inequality for $t \ge t_0$:

$$V(t, x_t, \dot{x}_t) \le e^{-2\alpha(t-t_0)} V(t_0, x_{t_0}, \dot{x}_{t_0}) + \frac{\mu^2}{2\alpha} \sum_{i=1}^{N} b_i \Delta_i^2. \tag{11}$$

Lemma 1 gives sufficient conditions for input-to-state stability. It will play a key role in developing the "zooming-in" algorithm for dynamic quantization. In what follows, based on Lemma 1 we will present the main results on dynamic quantization of NCSs. By defining the initial and level sets in Section 3.1, in Section 3.2 we will find an LMI-based time-triggered zooming algorithm (i.e., the choice of μ) for the stabilization of the closed-loop system (6). In Section 3.3, we will develop a novel Lyapunov-based method for initialization of the zoom parameter.

3. Main results: dynamic quantization of NCSs

3.1. Initial and level sets

First, we define initial and level sets. Given positive scalar σ, define the region of initial conditions (initial set)

$$\begin{aligned} \mathcal{W}_\sigma = \{&x_{t_0} \in W[-\tau_M, 0] : V(t_0, x_{t_0}, \dot{x}_{t_0}) < \sigma, \\ &x^T(t)Px(t) < \sigma,\ t \in [t_0 - \eta_M, t_0]\}. \end{aligned} \tag{12}$$

Define the level set

$$\mathcal{X}_{t^*,\rho} = \{x_t \in W[-\tau_M, 0] : V(t, x_t, \dot{x}_t) < \rho,\ t \ge t^*\}.$$

Given positive numbers μ, M_0, $\beta < 1$ and $\nu < 1$, we derive conditions to guarantee the following: all solutions of (6) with $x_{t_0} \in \mathcal{W}_{\mu^2 M_0^2}$ will stay inside the region $\mathcal{X}_{t_0,(1+\beta\nu^2)\mu^2 M_0^2}$ for all $t \ge t_0$, and will enter a smaller region $\mathcal{X}_{t_0+\mathrm{T},\nu^2\mu^2 M_0^2}$ in a finite time T (see Appendix B for the proof).

Lemma 2. *Given $M_j > 0, j = 0, 1, \dots, N$, $\Delta_i > 0, i = 1, \dots, N$, $0 \le \eta_m < \tau_M$ and tuning parameters $\alpha > 0$, $0 < \nu < 1$, assume that there exist scalars $0 < \beta < 1$, b_i, $i = 1, \dots, N$, $n\times n$ matrices $P > 0$, $S_0 > 0$, $R_0 > 0$, $S_1 > 0$, $R_1 > 0$, S_{12}, such that the LMIs (8)–(9) and*

$$(1 + \beta\nu^2) M_0^2 C_i^T C_i < P M_i^2, \quad i = 1, \dots, N, \tag{13}$$

$$\frac{1}{2\alpha} \sum_{i=1}^{N} b_i \Delta_i^2 < \beta\nu^2 M_0^2 \tag{14}$$

hold. Let $\mu > 0$ be constant. Then the solutions of (6) that start in the region $\mathcal{W}_{\mu^2 M_0^2}$

(i) satisfy $|C_i x(t_k - \eta_k)| = |y_i(t_k - \eta_k)| < \mu M_i$, $k \in \mathbb{Z}_+$, (implying $|\omega_i(t)| \le \mu\Delta_i$ for all $t \ge t_0$, $i = 1, \dots, N$);

(ii) remain in the set $\mathcal{X}_{t_0,(1+\beta\nu^2)\mu^2 M_0^2}$;

(iii) enter a smaller set $\mathcal{X}_{t_0+\mathrm{T},\nu^2\mu^2 M_0^2}$ in a finite time T, where T is the solution of

$$e^{-2\alpha \mathrm{T}} = (1 - \beta)\nu^2. \tag{15}$$

Note that the second inequality in (12) with $\sigma = \mu^2 M_0^2$ allows us to guarantee the bounds on $y(s_k)$, $s_k < t_0$ by verifying (13).

The LMIs of Lemma 2 are feasible for small enough delay bound τ_M, large enough quantization ranges $M_1, \dots, M_N$ and small enough quantization errors $\Delta_1, \dots, \Delta_N$. Indeed, the LMIs (8) and (9) are feasible for $\tau_M = 0$ (i.e., in the absence of delay) since $A + BKC$ is Hurwitz. Hence, (8) and (9) are feasible for small enough τ_M. The LMIs (13) and (14) are feasible for large enough quantization ranges and small enough quantization errors.

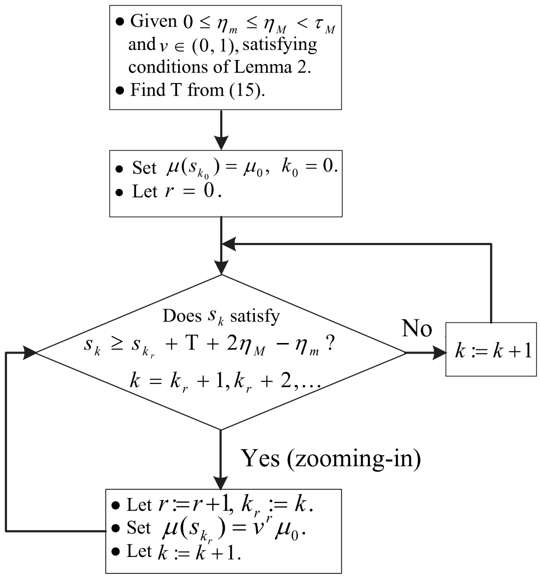


Fig. 2. The "zooming-in" algorithm for dynamic quantization.

3.2. Dynamic quantization and zooming algorithm

In this section, we consider dynamic quantizers with the zoom variable μ. Zooming is performed at the sensor side. Therefore, in the closed-loop system $\mu = \mu(s_k)$ is constant on $[t_k, t_{k+1})$.

Given $\mu_0 > 0$, let $\mu = \mu_0, x_{t_0} \in \mathcal{W}_{\mu^2 M_0^2} = \mathcal{W}_{\mu_0^2 M_0^2}$. We will show how to choose μ_0 in Theorem 1. Assume that the LMIs of Lemma 2 are feasible. We suggest a "zooming-in" algorithm shown in Fig. 2, where μ is decreased and, thus, the resulting quantization error is reduced in such a way as to drive the state of (6) to the origin exponentially.

Definition 1. The system (6) with $|\omega_i(t)| \le \mu \Delta_i$, $i = 1, \ldots, N$, is said to be exponentially stable for some choice of the zoom variable $\mu > 0$ if there exist constants $b > 0$, $\kappa > 0$ such that

$$|x(t)|^2 \le b e^{-2\kappa(t-t_0)} \mu_0^2 M_0^2, \quad \forall t \ge t_0$$

for the solutions of the system (6) initialized with $x_{t_0} \in \mathcal{W}_{\mu_0^2 M_0^2}$.

Proposition 1. *Assume that the LMIs of Lemma 2 are feasible. Given $\mu_0 > 0$, let $\mu = \mu_0$, $x_{t_0} \in \mathcal{W}_{\mu_0^2 M_0^2}$. Then under the algorithm in Fig. 2, the system (6) is exponentially stable with a decay rate $\kappa = -\frac{\ln\nu}{\mathrm{T}+\tau_M+2\eta_M-2\eta_m}$.*

Proof. Set $r = 0$. Since

$$t_{k_1} - \eta_M = s_{k_1} + \eta_{k_1} - \eta_M \ge t_0 + \mathrm{T} + \eta_{k_1} - \eta_m \ge t_0 + \mathrm{T},$$

application of Lemma 2 with $\mu = \mu_0$ leads to

$$x^T(t)Px(t) \le V(t, x_t, \dot{x}_t) < \nu^2 \mu_0^2 M_0^2, \quad \forall t \ge t_{k_1} - \eta_M.$$

Set $r = 1$. After zooming-in at s_{k_1}, the resulting closed-loop system has initial conditions

$$x_{t_{k_1}} \in W[-\tau_M, 0] : V(t_{k_1}, x_{t_{k_1}}, \dot{x}_{t_{k_1}}) < \nu^2 \mu_0^2 M_0^2. \tag{16}$$

Then Lemma 2 is applied with $\mu = \mu_0 \nu$, where t_0 and η_0 are changed by t_{k_1} and η_{k_1}, respectively. Thus, the solutions of (6) initiated by (16) remain in a region $\mathcal{X}_{t_{k_1}, (1+\beta\nu^2)\nu^2\mu_0^2 M_0^2}$ for all $t \ge t_{k_1}$. Since $s_{k_1} = t_{k_1} - \eta_{k_1} \ge t_{k_1} - \eta_M$, from (13) it follows that

$$x^T(s_k) C_i^T C_i x(s_k) < \frac{x^T(s_k)Px(s_k) \cdot \nu^2 \mu_0^2 M_i^2}{(1+\beta\nu^2)\nu^2\mu_0^2 M_0^2} < \nu^2 \mu_0^2 M_i^2,$$
$$i = 1, \ldots, N, \ \forall k \ge k_1,$$

and thus,

$$|\omega_i(t)| \le \nu \mu_0 \Delta_i, \quad i = 1, \ldots, N, \ t \ge t_{k_1} = s_{k_1} + \eta_{k_1}.$$

Therefore, for $t \ge t_{k_2} - \eta_M \ge t_{k_1} + \mathrm{T}$,

$$\begin{aligned} V(t, x_t, \dot{x}_t) \le\ & e^{-2\alpha(t - s_{k_1} - \eta_{k_1})} V(t, x_t, \dot{x}_t)_{|t = s_{k_1} + \eta_{k_1}} \\ & + \frac{\nu^2\mu_0^2}{2\alpha} \sum_{i=1}^{N} b_i \Delta_i^2 \\ & \le e^{-2\alpha\mathrm{T}} V(t, x_t, \dot{x}_t)_{|t = s_{k_1} + \eta_{k_1}} + \frac{\nu^2\mu_0^2}{2\alpha} \sum_{i=1}^{N} b_i \Delta_i^2 \\ & < (1-\beta)\nu^2 \cdot \nu^2 \mu_0^2 M_0^2 + \beta\nu^2\mu_0^2 M_0^2 \cdot \nu^2 \\ & = \nu^4 \mu_0^2 M_0^2. \end{aligned}$$

Similarly, for $r = 2, 3, \ldots$ we have $V(t, x_t, \dot{x}_t) < \nu^{2r}\mu_0^2 M_0^2$ for all $t \in [t_{k_r} - \eta_M, t_{k_{r+1}} - \eta_M)$. Noting that

$$\begin{aligned} & r\mathrm{T} + (r-1)(2\eta_M - \eta_m) + t_0 \le t_{k_r} - \eta_M \le t \\ & \le t_{k_{r+1}} - \eta_M < (r+1)(\mathrm{T} + \tau_M + 2\eta_M - 2\eta_m) + t_0, \end{aligned}$$

we obtain

$$\begin{aligned} V(t, x_t, \dot{x}_t) &< \nu^{2r}\mu_0^2 M_0^2 < \nu^{2[\frac{t-t_0}{\mathrm{T}+\tau_M+2\eta_M-2\eta_m} - 1]} \mu_0^2 M_0^2 \\ &= \nu^{-2} e^{\frac{2\ln\nu}{\mathrm{T}+\tau_M+2\eta_M-2\eta_m}\cdot(t-t_0)} \mu_0^2 M_0^2, \\ & t \in [t_{k_r} - \eta_M, t_{k_{r+1}} - \eta_M), \ r \in \mathbb{N}. \end{aligned}$$

Then the following holds for $t \ge t_0$

$$|x(t)|^2 \le \nu^{-2} [\lambda_{\min}(P)]^{-1} e^{\frac{2\ln\nu}{\mathrm{T}+\tau_M+2\eta_M-2\eta_m}\cdot(t-t_0)} \mu_0^2 M_0^2. \quad \square$$

For the implementation of the "zooming-in" algorithm, we set a counter $\theta(t)$ at the sensor in terms of sampling instants s_k. The counter triggers the zooming in whenever $\theta(s_k) \ge \mathrm{T} + 2\eta_M - \eta_m$. At the triggering times ($t = s_{k_r}$), it is reset to zero.

Remark 1. In the above reasoning, we assumed that packet loss does not occur. However, if the number of successive packet dropouts is upper bounded by $\bar{d}$, we could accommodate for packet dropouts by modeling them as prolongations of the transmission interval and replace T by $\mathrm{T} + 2\bar{d} \cdot \mathrm{MATI}$ in the algorithm.

3.3. Initialization of the zoom variable

The algorithm of the previous section is given in terms of the initial set $\mathcal{W}_{\mu_0^2 M_0^2}$ that involves the bound on $V(t_0, x_{t_0}, \dot{x}_{t_0})$. In this section, we find the ball of initial conditions $x(0) = x_0$, starting from which the solutions of (5)–(6) remain in the initial set $\mathcal{W}_{\mu_0^2 M_0^2}$. Following Liu and Fridman (2014), we derive a bound on $V(t_0, x_{t_0}, \dot{x}_{t_0})$ in terms of x_0 in the next lemma:

Lemma 3 (*Liu & Fridman, 2014*). *Consider LKF $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ given by (7) and denote $V_0(t) = x^T(t)Px(t)$. Under the constant initial condition $x(t) = x_0$, $t < 0$, let there exist $\alpha > 0$ and $\delta > 0$ such that the following inequalities*

$$\dot{V}_0(t) - 2\delta V_0(t) \le 0, \tag{17a}$$

$$\dot{\bar{V}}(t) + 2\alpha\bar{V}(t) - 2\delta V_0(t) \le 0, \tag{17b}$$

hold for $0 \le t < t_0$ along (5). Then we have

$$\begin{aligned} & V_0(t) \le \lambda_{\max}(e^{2\delta\eta_M} P)|x_0|^2, \quad 0 \le t < t_0, \\ & \bar{V}(t_0) \le \lambda_{\max}(e^{2\delta\eta_M} P + \Omega)|x_0|^2, \end{aligned} \tag{18}$$

where

$$\Omega = \eta_m S_0 + e^{-2\alpha\eta_m}(\tau_M - \eta_m) S_1. \tag{19}$$

We are in a position to formulate our main result:

Theorem 1. *Given $M_j > 0, j = 0, 1, \ldots, N, \Delta_i > 0, i = 1, \ldots, N$, $0 \le \eta_m \le \eta_M < \tau_M$ and tuning parameters $\alpha > 0, 0 < \nu < 1$, $\delta > 0$, assume that there exist scalars $0 < \beta < 1, b_i, i = 1, \ldots, N$, $n \times n$ matrices $P > 0, S_0 > 0, R_0 > 0, S_1 > 0, R_1 > 0, S_{12}$, such that LMIs (8)–(9), (13)–(14) and the following LMIs are feasible:*

$$\tilde{\Psi}_1 = P(A - \delta I_n) + (A - \delta I_n)^T P < 0, \tag{20}$$

$$\tilde{\Psi}_2 = \begin{bmatrix} \tilde{\Sigma} - \tilde{F}^T \Phi \tilde{F} e^{-2\alpha\tau_M} & \tilde{\Xi}^T H \\ * & -H \end{bmatrix} < 0, \tag{21}$$

where

$$\begin{aligned} &\tilde{\Sigma} = \tilde{F}_1^T P \tilde{\Xi} + \tilde{\Xi}^T P \tilde{F}_1 + \tilde{\Upsilon} - \tilde{F}_2^T R_0 \tilde{F}_2 e^{-2\alpha\eta_m}, \\ &\tilde{F}_1 = [I_n \; 0_{n\times 3n}], \quad \tilde{F}_2 = [I_n \; -I_n \; 0_{n\times 2n}], \quad \tilde{\Xi} = [A \; 0_{n\times 3n}], \\ &\tilde{F} = \begin{bmatrix} 0_{n\times n} & I_n & -I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & -I_n \end{bmatrix}, \end{aligned} \tag{22}$$

$\tilde{\Upsilon} = \mathrm{diag}\{S_0 + 2\alpha P - 2\delta P, -(S_0 - S_1)e^{-2\alpha\eta_m}, 0_{n\times n}, -S_1 e^{-2\alpha\tau_M}\}$, and the notations Φ and H are given by (8) and (10), respectively. Assume that the initial condition satisfies the inequality $|x_0| < X_0$, where $X_0 > 0$ is known. Then the "zooming-in" algorithm of Section 3.2 starting with $\mu(s_0) = \mu_0$ with μ_0 given by

$$\mu_0^2 = \frac{\lambda_{\max}(e^{2\delta\eta_M} P + \Omega)}{M_0^2} X_0^2 \tag{23}$$

exponentially stabilizes system (5)–(6), where Ω is given by (19).

Proof. As follows from Liu and Fridman (2014), the LMIs (8), (20) and (21) guarantee (17) along (5) for $0 \le t < t_0$. Therefore, if the initial condition satisfies the inequality $|x_0| < X_0$, then

$$\begin{aligned} \max\{V_0(t), \bar{V}(t_0)\} &\le \lambda_{\max}(e^{2\delta\eta_M} P + \Omega) X_0^2 \\ &= \mu_0^2 M_0^2, \quad t \in [0, t_0], \end{aligned}$$

meaning that $x_{t_0} \in \mathcal{X}_{\mu_0^2 M_0^2}$. The result then follows from Proposition 1. □

Remark 2. Given $\alpha > 0$, there always exists $\delta > 0$ such that $A - (\delta - \alpha)I_n$ is Hurwitz, i.e., $\dot{V}_0 + 2(\alpha + \varepsilon - \delta)V_0 \le 0$ holds for some $P > 0$ with small enough $\varepsilon > 0$. Then LMI (20) is feasible, and for small enough $\tau_M > 0$, by standard arguments for delay-dependent methods (see e.g., Fridman, 2014) the LMIs (8) and (21) are satisfied.

Remark 3. Note that given a bound $X_0 > 0$ on the state initial conditions and the values of the quantizer range $M_i > 0$ and error $\Delta_i > 0, i = 1, \ldots, N$, Eq. (23) defines the initial value of the zoom variable, starting from which the exponential stability is guaranteed by using "zooming-in" only.

Remark 4. If the initial value of the zoom variable is given by μ_0, then the "zooming-in" algorithm starting with $\mu(s_0) = \mu_0$ exponentially stabilizes all the solutions of (5)–(6) starting from the initial ball

$$|x_0| < X_0, \quad X_0 = \frac{\mu_0 M_0}{\sqrt{\lambda_{\max}(e^{2\delta\eta_M} P + \Omega)}}. \tag{24}$$

In order to maximize the initial ball (24), i.e., to minimize $\lambda_{\max}(e^{2\delta\eta_M} P + \Omega)$, the condition $e^{2\delta\eta_M} P + \Omega - \gamma I_n < 0$ can be added to the conditions of Theorem 1, where $\gamma > 0$ is to be minimized.

Remark 5. Consider the case where all the conditions of Theorem 1 are satisfied, but the initial ball $|x_0| < \bar{X}_0$ is larger: $\bar{X}_0 > X_0$, where X_0 is given by (24). Then we change μ_0 in the algorithm by $\bar{\mu}_0 = M_0^{-1} \bar{X}_0 \sqrt{\lambda_{\max}(e^{2\delta\eta_M} P + \Omega)}$ and zoom-out by resetting $\bar{M}_i = \bar{\mu}_0 M_i$, $\bar{\Delta}_i = \bar{\mu}_0 \Delta_i$, $i = 1, \ldots, N$. Therefore, we can start with the quantizer $q_{i\bar{\mu}_0}(z_i)$ (corresponding to $\mu(s_0) = \bar{\mu}_0$) whose range and quantization error are given by $\bar{M}_i$ and $\bar{\Delta}_i$, $i = 1, \ldots, N$, respectively. After this initial "zooming-out", "zooming-in" is used as suggested in the algorithm of Section 3.2. This "zooming-in"–"zooming-out" algorithm was originally proposed by Liberzon (2003).

Since the LMIs (9), (20) and (21) are affine in the system matrices, the conditions of Theorem 1 can be applied to the case where these matrices are uncertain. Consider next system (1) with the polytopic type uncertainties. Denote $\Theta = \begin{bmatrix} A & B \end{bmatrix}$ and assume that

$$\Theta = \sum_{j=1}^{M} g_j(t)\Theta_j, \quad 0 \le g_j(t) \le 1, \quad \sum_{j=1}^{M} g_j(t) = 1, \tag{25}$$

where g_j are uncertain time-varying parameters and where the M vertices of the polytope are described by $\Theta_j = \begin{bmatrix} A^{(j)} & B^{(j)} \end{bmatrix}$, $j = 1, \ldots, M$.

Suppose that the following LMIs for $j = 1, \ldots, M$, are feasible with the same decision matrices:

$$\Psi^{(j)} = \begin{bmatrix} \Sigma^{(j)} - F^T \Phi F e^{-2\alpha\tau_M} & (\Xi^{(j)})^T H \\ * & -H \end{bmatrix} < 0, \tag{26}$$

$$\tilde{\Psi}_1^{(j)} = P(A^{(j)} - \delta I_n) + (A^{(j)} - \delta I_n)^T P < 0, \tag{27}$$

$$\tilde{\Psi}_2^{(j)} = \begin{bmatrix} \tilde{\Sigma}^{(j)} - \tilde{F}^T \Phi \tilde{F} e^{-2\alpha\tau_M} & (\tilde{\Xi}^{(j)})^T H \\ * & -H \end{bmatrix} < 0, \tag{28}$$

where

$$\begin{aligned} &A_1^{(j)} = B^{(j)} K C, \quad B_i^{(j)} = B^{(j)} K_i, \quad i = 1, \ldots, N, \\ &\Sigma^{(j)} = F_1^T P \Xi^{(j)} + (\Xi^{(j)})^T P F_1 + \Upsilon - F_2^T R_0 F_2 e^{-2\alpha\eta_m}, \\ &\Xi^{(j)} = [A^{(j)} \; 0_{n\times n} \; A_1^{(j)} \; 0_{n\times n} \; B_1^{(j)} \cdots B_N^{(j)}], \\ &\tilde{\Sigma}^{(j)} = \tilde{F}_1^T P \tilde{\Xi}^{(j)} + (\tilde{\Xi}^{(j)})^T P \tilde{F}_1 + \tilde{\Upsilon} - \tilde{F}_2^T R_0 \tilde{F}_2 e^{-2\alpha\eta_m}, \\ &\tilde{\Xi}^{(j)} = [A^{(j)} \; 0_{n\times 3n}], \end{aligned} \tag{29}$$

and where notations are given by (8), (10), (22). Then we obtain

$$\sum_{j=1}^{M} g_j(t)\Psi^{(j)} = \Psi < 0 \quad \text{and}$$

$$\sum_{j=1}^{M} g_j(t)\tilde{\Psi}_i^{(j)} = \tilde{\Psi}_i < 0, \quad i = 1, 2,$$

which mean that (9), (20) and (21) are feasible. The following statement holds:

Theorem 2. *Given $M_j > 0, j = 0, 1, \ldots, N, \Delta_i > 0, i = 1, \ldots, N$, $0 \le \eta_m \le \eta_M < \tau_M$ and tuning parameters $\alpha > 0, 0 < \nu < 1$, $\delta > 0$, assume that there exist scalars $0 < \beta < 1, b_i, i = 1, \ldots, N$, $n \times n$ matrices $P > 0, S_0 > 0, R_0 > 0, S_1 > 0, R_1 > 0, S_{12}$, such that LMIs (8), (13)–(14), (26)–(28) are feasible for $j = 1, \ldots, M$, where notations are given by (8), (10), (22) and (29). Assume that the initial condition satisfies the inequality $|x_0| < X_0$, where $X_0 > 0$ is known. Then the "zooming-in" algorithm of Section 3.2 starting with $\mu(s_0) = \mu_0$ with μ_0 given by (23) exponentially stabilizes the uncertain system (5)–(6) with (25).*

Remark 6. Theorems 1 and 2 focus on the stability analysis of NCSs with dynamic quantization, variable communication delays and variable sampling intervals. For the static output-feedback stabilization problem, one possible solution is to apply the approach introduced in Liu and Fridman (2012) together with the descriptor method (Fridman & Shaked, 2003).

4. Example: uncertain inverted pendulum

Consider an inverted pendulum mounted on a car. Following Geromel, Korogui, and Bernussou (2007), we assume that the friction coefficient between the air and the car, f_c, and the air and the bar, f_b, are not exactly known and time-varying: $f_c(t) \in [0.15, 0.25]$ and $f_b(t) \in [0.15, 0.25]$. The linearized model can be written as (1), where matrix $A = E^{-1}A_f$ is determined from

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3/2 & -1/4 \\ 0 & 0 & -1/4 & 1/6 \end{bmatrix},$$

$$A_f = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(f_c+f_b) & f_b/2 \\ 0 & 5/2 & f_b/2 & -f_b/3 \end{bmatrix}.$$

Matrix B is given by $B = E^{-1}B_0$ with $B_0 = [0\ 0\ 1\ 0]^T$. Note that A can be described by a polytope with four vertices. The pendulum can be stabilized by the state feedback $u(t) = Kx(t) = YQ^{-1}x(t)$, where $Y \in \mathbb{R}^{1\times 4}$ and $0 < Q \in \mathbb{R}^{4\times 4}$ satisfy

$$AQ + QA^T + 2\alpha Q + BY + Y^TB^T < 0 \tag{30}$$

in the vertices of polytope for a tuning parameter $\alpha > 0$.

Consider $N = 2$ and

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The quantizer is chosen as

$$q_\mu(y^i) = \begin{cases} 100\mu\ \mathrm{sgn}(y^i), & \text{if } |y^i| > 100\mu, \\ \mu\left\lfloor \dfrac{y^i}{\mu} + 0.1 \right\rfloor, & \text{if } |y^i| \le 100\mu, \end{cases}$$

where y^i is the ith component of y, $i = 1, \ldots, n_y$. Therefore, we can take $M_1 = M_2 = 100$, $\Delta_1 = \Delta_2 = 0.1$. Choose $\mu_0 = 1$, $M_0 = 100$, $\nu = 0.8$, $\delta = 10$.

First, choosing $\alpha = 0.3$, from (30) we obtain the controller gain

$$K = [K_1\ \ K_2], \tag{31}$$

where

$$K_1 = \begin{bmatrix} 25.1319 & -222.9722 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 28.7826 & -44.2075 \end{bmatrix}.$$

Application of Theorem 2 with $\tau_M = 0.02$, $\eta_m = 0.011$, $\eta_M = 0.015$ leads to $\mathrm{T} = -\frac{\ln(1-\beta)+2\ln\nu}{2\alpha} = 1.0002$ from (15). Then the "zooming-in" algorithm of Section 3.2 with $\mathrm{T} = 1.0002$ and $\nu = 0.8$ exponentially stabilizes all the solutions of (5)–(6) with (25) starting from the initial ball $|x_0| < 5.2693$. The evolution of the zoom variable μ is shown in Fig. 3.

Moreover, we find that the system is exponentially stable with a decay rate $\kappa = -\frac{\ln\nu}{\mathrm{T}+\tau_M+2\eta_M-2\eta_m} = 0.2170$. Let the initial state $x_0 = [1\ 3\ 2\ -1]^T$. The evolution of the control input and the state is shown in Fig. 4.

If all the conditions of Theorem 2 are satisfied, but the initial ball is $|x_0| < 15$, which is out of $|x_0| < 5.2693$, we substitute $\mu_0 = 15/5.2693 = 2.85$ for 1 and zoom-out by resetting $\bar{M}_i = \mu_0 M_i = 285$, $\bar{\Delta}_i = \mu_0\Delta_i = 0.285$, $i = 1, 2$. After this initial "zooming-out", "zooming-in" is used by Theorem 2 and the algorithm of Section 3.2.

Next, taking a larger $\alpha = 0.7$, from (30) we have another controller gain

$$\bar{K} = [\bar{K}_1\ \ \bar{K}_2], \tag{32}$$

where

$$\bar{K}_1 = \begin{bmatrix} 15.6527 & -105.6658 \end{bmatrix}, \qquad \bar{K}_2 = \begin{bmatrix} 16.0894 & -22.0086 \end{bmatrix}.$$

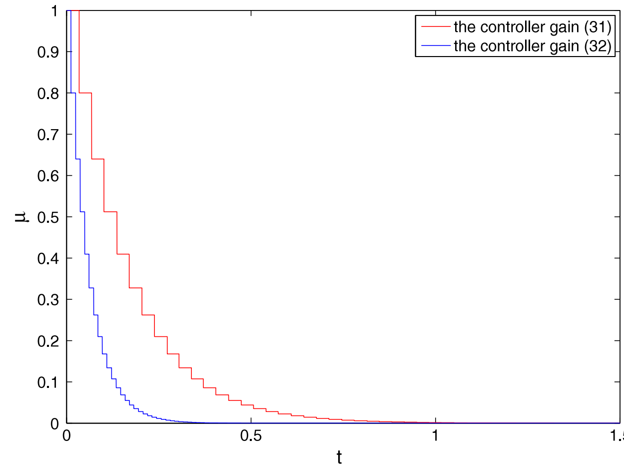


Fig. 3. Evolution of the zoom variable μ in the "zooming-in" algorithm.

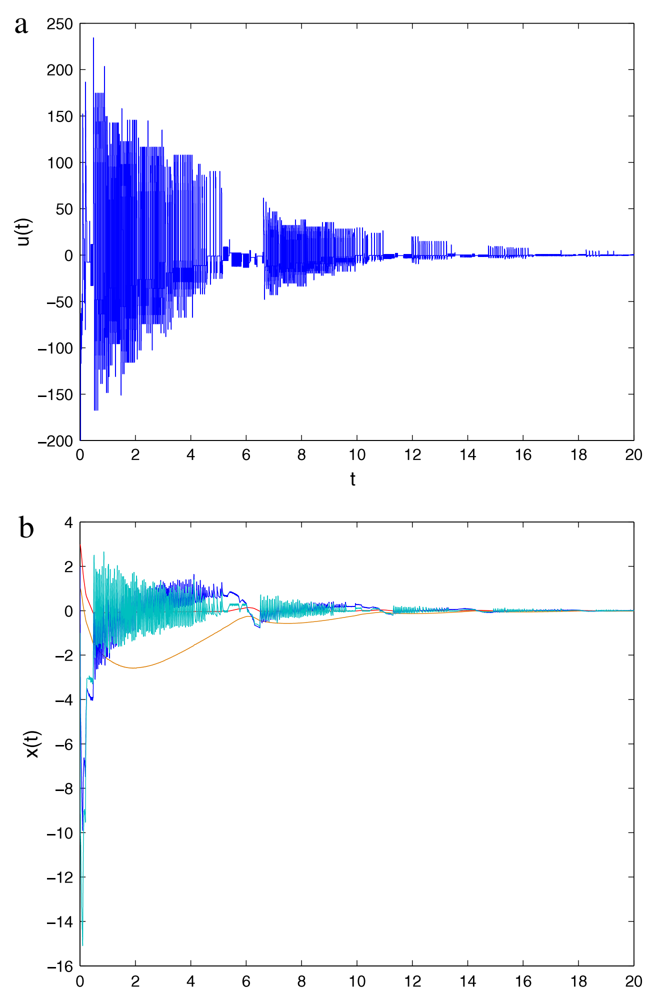


Fig. 4. (a) Evolution of the control input with the gain (31); (b) Trajectory of the closed-loop system with the controller gain (31).

We find that given $\tau_M = 0.02$, $\eta_m = 0.011$, $\eta_M = 0.015$, the "zooming-in" algorithm of Section 3.2 with a smaller $\mathrm{T} = 0.3457$ and the same $\nu = 0.8$ exponentially stabilizes all the solutions of (5), (6) and (25) starting from a larger initial ball $|x_0| < 6.2782$ with a larger decay rate $\kappa = 0.5972$. The evolution of the zoom variable μ and the evolution of the control input, the state with the

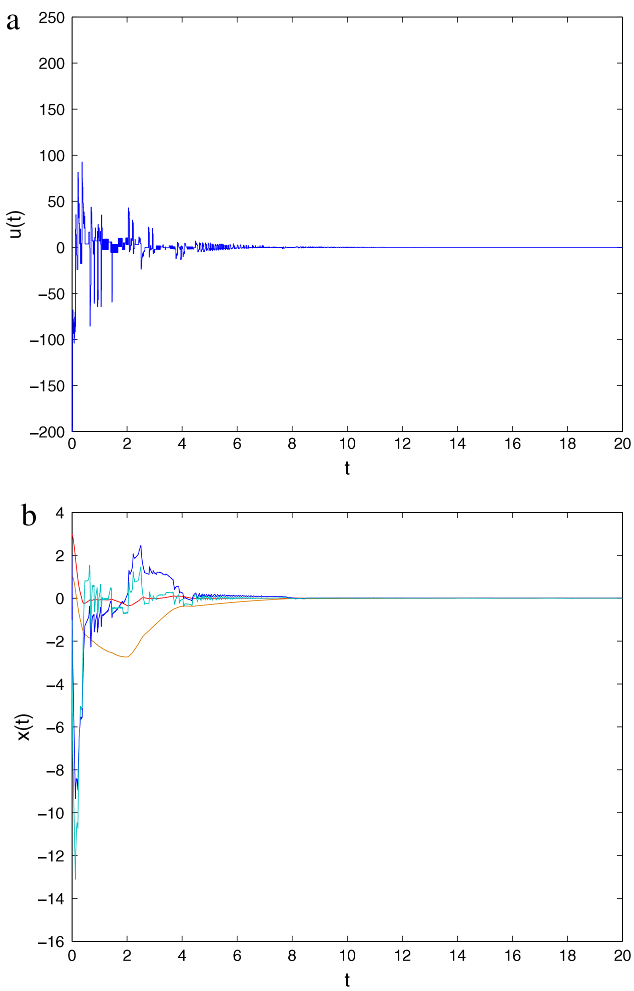


Fig. 5. (a) Evolution of the control input with the gain (32); (b) Trajectory of the closed-loop system with the controller gain (32).

initial state x_0 are shown in Figs. 3 and 5, respectively (confirming the theoretical results).

The simulation results listed above show that the larger value of α and the resulting controller gain give rise to a larger decay rate. The choice of α to be 1.58 and the corresponding controller gain $\tilde{K} = [\tilde{K}_1 \ \ \tilde{K}_2]$, where $\tilde{K}_1 = [39.8549 \ \ -141.5345]$, $\tilde{K}_2 = [30.1820 \ \ -30.7271]$, lead to the maximum value of the decay rate $\kappa = 1.0689$.

5. Conclusions

In this paper, a time-delay approach was developed for the stability analysis of uncertain linear NCSs with dynamic quantization, variable communication delays and variable sampling intervals. An LMI-based time-triggered zooming algorithm was presented for the dynamic quantization that leads to the exponential stability of the closed-loop system. A novel Lyapunov-based method was proposed for initialization of the zoom parameter. Future work will involve analysis of more general NCS models, including quantized input and stochastic communication delays.

Appendix A

Proof of Lemma 1. Consider $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$ and define $\xi(t) = \text{col}\{x(t), x(t-\eta_m), x(t-\tau(t)), x(t-\tau_M), \omega_1(t), \cdots, \omega_N(t)\}$. Differentiating V along (6) and applying Jensen's inequality, we have

$$\begin{aligned}
\eta_m \int_{t-\eta_m}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds &\geq \int_{t-\eta_m}^{t} \dot{x}^T(s)dsR_0 \int_{t-\eta_m}^{t} \dot{x}(s)ds \\
&= \xi^T(t)F_2^T R_0 F_2 \xi(t),
\end{aligned}$$

$$\begin{aligned}
&-(\tau_M - \eta_m)\int_{t-\tau_M}^{t-\eta_m} \dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad = -(\tau_M - \eta_m)\int_{t-\tau(t)}^{t-\eta_m} \dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad\quad -(\tau_M - \eta_m)\int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad \leq -\frac{\tau_M - \eta_m}{\tau(t) - \eta_m}\xi^T(t)\Big[[I_n \ 0_{n\times n}]F\Big]^T R_1 [I_n \ 0_{n\times n}]F\xi(t) \\
&\quad\quad - \frac{\tau_M - \eta_m}{\tau_M - \tau(t)}\xi^T(t)\Big[[0_{n\times n} \ I_n]F\Big]^T R_1 [0_{n\times n} \ I_n]F\xi(t) \\
&\quad \leq -\xi^T(t)F^T\Phi F\xi(t).
\end{aligned}$$

The latter inequality holds if (8) is feasible (Park et al., 2011). Then

$$\begin{aligned}
&\frac{d}{dt}V + 2\alpha V - \sum_{i=1}^{N} b_i|\omega_i(t)|^2 \\
&\quad \leq \xi^T(t)[\Sigma + \Xi^T H \Xi - F^T\Phi F e^{-2\alpha\tau_M}]\xi(t) \leq 0,
\end{aligned} \tag{33}$$

if $\Sigma + \Xi^T H \Xi - F^T \Phi F e^{-2\alpha\tau_M} < 0$, i.e., by Schur complement, if (9) is feasible.

Since $|\omega_i(t)| \leq \mu\Delta_i$, $i = 1, \ldots, N$, by the comparison principle (Khalil & Grizzle, 2002), (33) implies for $t \in [t_k, t_{k+1})$

$$\begin{aligned}
V(t, x_t, \dot{x}_t) &\leq e^{-2\alpha(t-t_k)}V(t_k, x_{t_k}, \dot{x}_{t_k}) \\
&\quad + \mu^2 \sum_{i=1}^{N} b_i\Delta_i^2 \int_{t_k}^{t} e^{-2\alpha(t-s)}ds \\
&\leq e^{-2\alpha(t-t_{k-1})}V(t_{k-1}, x_{t_{k-1}}, \dot{x}_{t_{k-1}}) \\
&\quad + \mu^2 \sum_{i=1}^{N} b_i\Delta_i^2 \int_{t_{k-1}}^{t} e^{-2\alpha(t-s)}ds \\
&\ \ \vdots \\
&\leq e^{-2\alpha(t-t_0)}V(t_0, x_{t_0}, \dot{x}_{t_0}) \\
&\quad + \mu^2 \sum_{i=1}^{N} b_i\Delta_i^2 \int_{t_0}^{t} e^{-2\alpha(t-s)}ds \\
&\leq e^{-2\alpha(t-t_0)}V(t_0, x_{t_0}, \dot{x}_{t_0}) + \frac{\mu^2}{2\alpha}\sum_{i=1}^{N} b_i\Delta_i^2,
\end{aligned}$$

that completes the proof. □

Appendix B

Proof of Lemma 2. For all $x_t \in \mathcal{X}_{t_0,(1+\beta\nu^2)\mu^2M_0^2}$ starting from $\mathcal{W}_{\mu^2M_0^2}$, we have

$$x^T(t)Px(t) \leq V(t, x_t, \dot{x}_t) < (1 + \beta\nu^2)\mu^2M_0^2,$$

and, thus, (13) guarantees that

$$x^T(s_k)C_i^T C_i x(s_k) < \frac{x^T(s_k)Px(s_k)\cdot\mu^2M_i^2}{(1+\beta\nu^2)\mu^2M_0^2} < \mu^2M_i^2, \quad k \in \mathbb{Z}_+.$$

Hence, there is no saturation for the sensor node $y_i(s_k) = C_i x(s_k)$, $k \in \mathbb{Z}_+$ ($|C_i x(s_k)| < \mu M_i$), which implies $|\omega_i(t)| \leq \mu\Delta_i$,

$t \geq t_0, i = 1, \dots, N$ whenever $x_t \in \mathcal{X}_{t_0,(1+\beta\nu^2)\mu^2 M_0^2}$ and $x_{t_0} \in \mathcal{X}_{\mu^2 M_0^2}$.

Let $x_{t_0} \in \mathcal{X}_{\mu^2 M_0^2}$, then solutions of (6) satisfy $V(t, x_t, \dot{x}_t) < (1+\beta\nu^2)\mu^2 M_0^2$ for $t \in [t_0, t')$ for some $t' > t_0$. We will show next that solutions of (6) with $x_{t_0} \in \mathcal{X}_{\mu^2 M_0^2}$ stay in $\mathcal{X}_{t_0,(1+\beta\nu^2)\mu^2 M_0^2}$ for all $t \geq t_0$ if LMIs (8)–(9), (13)–(14) are feasible.

Assume, on the contrary, that there exists a finite time $t' > t_0$ such that $V(t, x_t, \dot{x}_t) < (1+\beta\nu^2)\mu^2 M_0^2$ for $t \in [t_0, t')$ and $V(t', x_{t'}, \dot{x}_{t'}) = (1+\beta\nu^2)\mu^2 M_0^2$. Then under (13) we have $|\omega_i(t)| \leq \mu\Delta_i, i = 1, \dots, N$ for $t \in [t_0, t']$. From (11)–(14), it follows that

$$\begin{aligned} V(t, x_t, \dot{x}_t) &\leq e^{-2\alpha(t-t_0)} V(t_0, x_{t_0}, \dot{x}_{t_0}) + \frac{\mu^2}{2\alpha} \sum_{i=1}^{N} b_i \Delta_i^2 \\ &< (1+\beta\nu^2)\mu^2 M_0^2, \quad t \in [t_0, t'], \end{aligned}$$

which contradicts to $V(t', x_{t'}, \dot{x}_{t'}) = (1+\beta\nu^2)\mu^2 M_0^2$.

Then (11)–(15) yield

$$\begin{aligned} V(t, x_t, \dot{x}_t) &\leq e^{-2\alpha \mathrm{T}} V(t_0, x_{t_0}, \dot{x}_{t_0}) + \frac{\mu^2}{2\alpha} \sum_{i=1}^{N} b_i \Delta_i^2 \\ &< (1-\beta)\nu^2 \cdot \mu^2 M_0^2 + \beta\nu^2\mu^2 M_0^2 \\ &= \nu^2 \mu^2 M_0^2, \quad t \geq t_0 + \mathrm{T}, \end{aligned}$$

that completes the proof. □

References

Brockett, R. W., & Liberzon, D. (2000). Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic control*, *45*(7), 1279–1289.

Delchamps, D. F. (1990). Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic control*, *35*(8), 916–924.

Elia, N., & Mitter, S. K. (2001). Stabilization of linear systems with limited information. *IEEE Transactions on Automatic control*, *46*(9), 1384–1400.

Fridman, E. (2014). *Introduction to time-delay systems*. Birkhäuser.

Fridman, E., & Dambrine, M. (2009). Control under quantization, saturation and delay: an LMI approach. *Automatica*, *45*(10), 2258–2264.

Fridman, E., Seuret, A., & Richard, J. P. (2004). Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, *40*(8), 1441–1446.

Fridman, E., & Shaked, U. (2003). Delay-dependent stability and H_∞ control: constant and time-varying delays. *International Journal of Control*, *76*(1), 48–60.

Fu, M., & Xie, L. (2005). The sector bound approach to quantized feedback control. *IEEE Transactions on Automatic control*, *50*(11), 1698–1711.

Fujioka, H. (2009). A discrete-time approach to stability analysis of systems with aperiodic sample-and-hold devices. *IEEE Transactions on Automatic control*, *54*(10), 2440–2445.

Gao, H., Chen, T., & Lam, J. (2008). A new delay system approach to network-based control. *Automatica*, *44*(1), 39–52.

Geromel, J. C., Korogui, R. H., & Bernussou, J. (2007). H_2 and H_∞ robust output feedback control for continuous time polytopic systems. *IET Control Theory & Applications*, *1*(5), 1541–1549.

Heemels, W.P.M.H., Nesic, D., Teel, A.R., & van de Wouw, N. (2009). Networked and quantized control systems with communication delays. In *Proceedings of the 48th IEEE conference on decision and control*. Shanghai, China.

Khalil, H., & Grizzle, J. (2002). *Nonlinear systems*. Prentice hall.

Liberzon, D. (2003). Hybrid feedback stabilization of systems with quantized signals. *Automatica*, *39*(9), 1543–1554.

Liberzon, D. (2006). Quantization, time delays, and nonlinear stabilization. *IEEE Transactions on Automatic control*, *51*(7), 1190–1195.

Liberzon, D., Nesic, D., & Teel, A. R. (2014). Lyapunov-based small-gain theorems for hybrid systems. *IEEE Transactions on Automatic control*, *59*(6), 1395–1410.

Liu, K., & Fridman, E. (2012). Networked-based stabilization via discontinuous Lyapunov functionals. *International Journal of Robust and Nonlinear Control*, *22*(4), 420–436.

Liu, K., & Fridman, E. (2014). Delay-dependent methods and the first delay interval. *Systems & Control Letters*, *64*(1), 57–63.

Naghshtabrizi, P., Hespanha, J. P., & Teel, A. R. (2010). Stability of delay impulsive systems with application to networked control systems. *Transactions of the Institute of Measurement and Control*, *32*(5), 511–528.

Nesic, D., & Liberzon, D. (2009). A unified framework for design and analysis of networked and quantized control systems. *IEEE Transactions on Automatic control*, *54*(4), 732–747.

Niu, Y., & Ho, D. (2014). Control strategy with adaptive quantizer's parameters under digital communication channels. *Automatica*, *50*(10), 2665–2671.

Park, P. G., Ko, J. W., & Jeong, C. (2011). Reciprocally convex approach to stability of systems with time-varying delays. *Automatica*, *47*(1), 235–238.

Polyakov, A., Efimov, D., Perruquetti, W., & Richard, J. P. (2013). Output stabilization of time-varying input delay systems using interval observation technique. *Automatica*, *49*(11), 3402–3410.

Wong, W. S., & Brockett, R. W. (1999). Systems with finite communication bandwidth constraints. II. stabilization with limited information feedback. *IEEE Transactions on Automatic control*, *44*(5), 1049–1053.

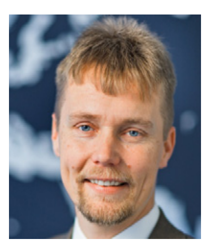
Yan, J., Xia, Y., & Li, L. (2014). Stabilization of fuzzy systems with quantization and packet dropout. *International Journal of Robust and Nonlinear Control*, *24*, 1563–1583.

Yue, D., Peng, C., & Tang, G. Y. (2006). Guaranteed cost control of linear systems over networks with state and input quantisations. *IEE Proceedings: Control Theory and Applications*, *153*(6), 658–664.

Kun Liu received the Ph.D. degree in the Department of Electrical Engineering and Systems from Tel Aviv University, Tel Aviv, Israel, in December 2012. From February 2013 to February 2015 he was a postdoctoral researcher at the ACCESS Linnaeus Centre, KTH Royal Institute of Technology, Stockholm, Sweden. From March 2015 to August 2015 he held Researcher, Visiting, and Research Associate positions at, respectively, KTH Royal Institute of Technology, Stockholm, Sweden, CNRS, Laboratory for Analysis and Architecture of Systems, Toulouse, France, and The University of Hong Kong, Hong Kong. Since September 2015 he is an Associate Professor at the School of Automation, Beijing Institute of Technology, China. His research interests include time-delay systems, networked control, quantized systems, and robust control.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in Mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published about 100 articles in international scientific journals. She is the author of the monograph "Introduction to Time-Delay Systems: Analysis and Control" (Birkhauser, 2014). In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Currently she serves as Associate Editor in Automatica and SIAM Journal on Control and Optimization.



Karl Henrik Johansson is Director of the ACCESS Linnaeus Centre and Professor at the School of Electrical Engineering, KTH Royal Institute of Technology, Sweden. He is a Wallenberg Scholar and has held a six-year Senior Researcher Position with the Swedish Research Council. He is also heading the Stockholm Strategic Research Area ICT The Next Generation. He received M.Sc. and Ph.D. degrees in Electrical Engineering from Lund University. He has held visiting positions at UC Berkeley (1998–2000) and California Institute of Technology (2006–2007). His research interests are in networked control systems, cyber–physical systems, and applications in transportation, energy, and automation systems. He has been a member of the IEEE Control Systems Society Board of Governors and the Chair of the IFAC Technical Committee on Networked Systems. He has been on the Editorial Boards of several journals, including Automatica, IEEE Transactions on Automatic Control, and IET Control Theory and Applications. He is currently on the Editorial Board of IEEE Transactions on Control of Network Systems and the European Journal of Control. He has been Guest Editor for special issues, including one issue of IEEE Transactions on Automatic Control on cyber–physical systems and one of IEEE Control Systems Magazine on cyber–physical security. He was the General Chair of the ACM/IEEE Cyber–Physical Systems Week 2010 in Stockholm and IPC Chair of many conferences. He has served on the Executive Committees of several European research projects in the area of networked embedded systems. He received the Best Paper Award of the IEEE International Conference on Mobile Ad-hoc and Sensor Systems in 2009 and the Best Theory Paper Award of the World Congress on Intelligent Control and Automation in 2014. In 2009 he was awarded Wallenberg Scholar, as one of the first ten scholars from all sciences, by the Knut and Alice Wallenberg Foundation. He was awarded Future Research Leader from the Swedish Foundation for Strategic Research in 2005. He received the triennial Young Author Prize from IFAC in 1996 and the Peccei Award from the International Institute of System Analysis, Austria, in 1993. He received Young Researcher Awards from Scania in 1996 and from Ericsson in 1998 and 1999. He is a Fellow of the IEEE.