

Consensus Over Random Graph Processes: Network Borel–Cantelli Lemmas for Almost Sure Convergence

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Abstract—Distributed consensus computation over random graph processes is considered. The random graph process is defined as a sequence of random variables which take values from the set of all possible digraphs over the node set. At each time step, every node updates its state based on a Bernoulli trial, independent in time and among different nodes: either averaging among the neighbor set generated by the random graph, or sticking with its current state. The connectivity-independence and arc-independence are introduced to capture the fundamental influence of the random graphs on the consensus convergence. Necessary and/or sufficient conditions are presented on the success probabilities of the Bernoulli trials for the network to reach a global almost sure consensus, with some sharp threshold established revealing a consensus zero-one law. Convergence rates are established by the lower and upper bounds of the ϵ -computation time. We also generalize the concepts of connectivity/arc independence to their analogues from the \ast -mixing point of view, so that our results apply to a very wide class of graphical models, including the majority of random graph models in the literature, e.g., Erdős–Rényi, gossiping, and Markovian random graphs. We show that under \ast -mixing, our convergence analysis continues to hold and the corresponding almost sure consensus conditions are established. Finally, we further investigate almost sure finite-time convergence of random gossiping algorithms, and prove that the Bernoulli trials play a key role in ensuring finite-time convergence. These results add to the understanding of the interplay between random graphs, random computations, and convergence probability for distributed information processing.

Index Terms—Consensus algorithms, random graphs, zero-one law, gossiping.

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I. INTRODUCTION

IN RECENT years, there has been considerable research interest on distributed algorithms for information exchange, estimation, and computation over networks. Such algorithms have a variety of potential applications in sensor networks, peer-to-peer networks, wireless networks and networked control systems. Targeting the design of simple decentralized algorithms for computation or estimation, where nodes exchange information only with their neighbors, a distributed consensus algorithm serves as a primitive towards more sophisticated information processing algorithms, see [1]–[4].

The investigation of the consensus problem has a long history in several scientific fields including computer science [5], [6], engineering [7], [8], and social science [9]–[11]. Deterministic consensus algorithms have been extensively studied for both time-invariant and time-varying communication graphs in the literature, with efforts typically devoted to finding proper connectivity conditions that can ensure a desired collective convergence [12]–[15]. These efforts can indeed be traced back to the study of ergodicity of non-homogeneous Markov chains [16], [17] and distributed decision making [18]. In addition, researchers were also interested in the design of weighted averaging algorithms to reach a faster consensus, or to reach a consensus with asynchronous computations [19], [20].

On the other hand, the theory of random graphs [36] is fundamental for the study of in-network communication, information processing, and opinion aggregation, e.g., [10], [37], [39]. Consensus algorithms that are carried out over a random graph process have also drawn attention. In [21], the authors studied linear consensus dynamics with communication graphs defined as a sequence of independent, identically distributed (i.i.d.) Erdős–Rényi random graphs, and almost sure convergence was shown. Then in [22], the analysis was generalized to directed Erdős–Rényi graphs. Mean-square performance for consensus algorithms over i.i.d. random graphs was studied in [23], and the influence of random packet drop was investigated in [24]. In [25], a necessary and sufficient condition was presented on almost sure asymptotic consensus for i.i.d. models. In [26]–[28], the authors studied distributed average consensus in sensor networks with quantized data and random link failures. In [29], the communication graph was described as a finite-state

Markov chain where each graph corresponds to one state of the chain, and almost sure consensus was concluded by investigating the connectivity of the closed positive recurrent sets of the Markovian random graph. In [30], convergence to consensus was studied under more general linear consensus algorithms, where the random update and control matrices were determined by possibly non-stationary stochastic matrix processes coupled with disturbances. In [31], a general model was investigated for consensus computations over stationary random graph processes with a necessary and sufficient condition presented on the expected graphs regarding almost sure consensus. It is worth emphasizing that besides convergence to a (not necessarily average) consensus, the focus of [23] (and later [10]) was on the error with respect to the average of the initial values, which is far more challenging.

Classical random graph theory suggests that many important properties of static random graphs appear suddenly as some parameter is smoothly adjusted from a probabilistic point of view [36]. To be precise, there exists usually a function, called a threshold, of a parameter (e.g., the graph size), and a quantity (e.g., probability of edge appearance) such that a given behaviour appears asymptotically in the parameter if and only if such a quantity grows faster than the threshold. The phenomenon is called a zero-one law. Pioneered in [38], these sharp probabilistic phase transitions have been a central motif in the study of random graphs [39]–[42]. In fact, as early as in [25], a zero-one law has been shown to exist for consensus convergence over i.i.d. random graph processes. Naturally, one may wonder, with possible unwillingness or inability for a node to update, would there be any threshold condition which leads to a similar zero-one law with respect to collective convergence of consensus dynamics over random graphs? Although various results have been established to guarantee probabilistic consensus [21]–[24], [30], the literature still lacks a general model and analysis for consensus dynamics over random graphs which can accurately describe the fundamental influence of graph processes and whether such a zero-one law could arise.

To this end, in this paper, we study the almost sure convergence of a randomized consensus algorithm over random graph processes. A general random graph process is adopted to model node interactions as a sequence of random variables which take values from the set of all possible graphs with the given node set. Another random node update process is independently built upon this random graph process, in that at each time step, every node independently updates its state as a weighted average of its neighbors' states or sticks with its current state. The choice is a Bernoulli trial with success probability P_k . We introduce connectivity-independence and arc-independence for the considered random graph processes. For connectivity-independent graphs, we show that $\sum_k P_k^{n-1} = \infty$ is a sufficient condition for almost sure consensus, where n is the number of nodes. For arc-independent graphs, we show that $\sum_k P_k = \infty$ is a sharp threshold, i.e., the consensus probability is zero for almost all initial conditions when the sum converges, while it is one for all initial conditions when the sum diverges. In other words,

a zero-one law is established for the presented randomized consensus processing over arc-independent graphs, and we see that the success probability of the node updates defines the parameter for the corresponding threshold function. Hence, also consensus computations over random graph processes show the sudden transition similar to Erdős–Rényi graphs, and other models in the literature [36].

A. Problem Definition

A directed graph (digraph) over V is defined as $\mathcal{G} = (V, \mathcal{E})$, where nodes are indexed in the set $V = \{1, 2, \dots, n\}$ and \mathcal{E} is the set of *arcs*, i.e., ordered pairs of distinct nodes [33]. An arc from i to j is denoted as (i, j) . There are $2^{n(n-1)}$ different digraphs with node set V . We label these graphs from 1 to $2^{n(n-1)}$ by an arbitrary order. In the following, we will identify an integer in $\mathcal{G} = \{1, \dots, 2^{n(n-1)}\}$ with the corresponding graph in this order. Time is slotted at $k = 0, 1, \dots$. We denote by $G = (V, E)$ a random graph, which by definition is a random variable (over some underlying probability space) that takes value in \mathcal{G} .

Let $G_k = (V, E_k), k = 0, 1, \dots$ be a sequence of random graphs over node set V . We call node j a *neighbor* of i at time k if $(j, i) \in G_k$. Each node is supposed to always be a neighbor of itself. Denote $N_i(k)$ as the set of neighbors of node i at time k . Each node i holds a state $x_i(k) \in \mathbb{R}$ at time k . Let $\langle P_k \rangle$ be a given deterministic sequence with $0 < P_k < 1$ for all k . Independent of the graph process, node states, and other nodes, the updating rule of node $x_i(k)$ is as follows:

$$x_i(k+1) = \begin{cases} \sum_{j \in N_i(k)} a_{ij}(k) x_j(k), & \text{with probability } P_k; \\ x_i(k), & \text{with probability } 1 - P_k. \end{cases} \quad (1)$$

Here $a_{ij}(k)$ represents the weight of arc (j, i) . We use the following weights rule as our standing assumption, see [1], [15], [43].

Assumption: (i) $\sum_{j \in N_i(k)} a_{ij}(k) = 1$ for all i and k ; (ii) there exists a constant $\eta > 0$ such that $a_{ij}(k) \geq \eta$ for all i, j and k .

Algorithm (1) can also be written as the following dynamics:

$$x_i(k+1) = \chi_i(k) \left(\sum_{j \in N_i(k)} a_{ij}(k) x_j(k) \right) + (1 - \chi_i(k)) x_i(k), \quad (2)$$

where $\chi_i(k), i = 1, \dots, n, k \in \mathbb{N}$ are independent Bernoulli random variables with $\mathbf{E}[\chi_i(k)] = P_k$ for all i , which are also independent with $\langle G_k \rangle$.

The randomized information processing (1) describes the possible unwillingness or inability of a node to update, even if it receives information from other nodes. For instance, in the opinion dynamics of social networks, forceful belief exchanges may happen randomly by Bernoulli decisions for individuals so that misinformation may be spread [10]. From an engineering viewpoint, in wireless communication nodes may be asleep or broken randomly due to the unpredictability of the environment and the unreliability of the networked communication [23], [27]. The above standing assumption

is adopted throughout the paper without specific mentioning. We note that in order to satisfy the standing assumption, the $a_{ij}(k)$ can be either deterministic or random, e.g., $a_{ij}(k) = 1/n$ for $j \neq i \in N_i(k)$ and $a_{ii}(k) = 1 - |N_i(k)|/n$; or $a_{ij}(k) = 1/|N_i(k)|$ for all $j \in N_i(k)$.

Our interest is in the convergence of the randomized consensus algorithm and the time it takes for the network to reach a consensus. Let $x(k) = (x_1(k) \dots x_n(k))^T \in \mathbb{R}^n$ be the random sequence driven by the randomized algorithm (1) for initial condition $x^0 = (x_1(0) \dots x_n(0))^T$. Denote

$$H(k) := \max_{i \in V} \{x_i(k)\}, \quad h(k) := \min_{i \in V} \{x_i(k)\}$$

as the maximal and minimal states among all nodes, respectively, and define $\mathcal{H}(k) := H(k) - h(k)$ as a consensus measure. Let \mathbf{P} be the probability measure capturing all the randomness in $\langle x(k) \rangle$, and whenever necessary we write \mathbf{P}_{x^0} indicating the probability measure for initial value x^0 .

Under our standing assumption there always holds that $\mathcal{H}(k+1) \leq \mathcal{H}(k)$. We introduce the following definition.

Definition 1: Global almost sure (a.s.) consensus is achieved for Algorithm (1) if $\mathbf{P}(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0) = 1$ for all $x^0 \in \mathbb{R}^n$. Moreover, for any $0 < \epsilon < 1$, the ϵ -computation time is defined as

$$\mathcal{T}_{\text{com}}(\epsilon) \doteq \sup_{x^0 \in \mathbb{R}^n} \inf \left\{ k : \mathbf{P}(\mathcal{H}(k) \geq \epsilon \mathcal{H}(0)) \leq \epsilon \right\}.$$

B. Main Results

Let x^0 be considered under the standard Lebesgue measure over \mathbb{R}^n . We first present the following result on the impossibility of consensus, which is built for arbitrary random graph process $\langle G_k \rangle$.

Theorem 1: (i). Global a.s. consensus can be achieved for Algorithm (1) only if $\sum_{k=0}^{\infty} P_k = \infty$, and a universal lower bound for $\mathcal{T}_{\text{com}}(\epsilon)$ is given by

$$\mathcal{T}_{\text{com}}(\epsilon) \geq \sup \left\{ k : \sum_{i=0}^{k-1} \log(1 - P_i)^{-1} \leq \frac{\log \epsilon^{-1}}{2} \right\}.$$

(ii). Suppose $\sum_{k=0}^{\infty} P_k < \infty$ and $\{a_{ij}(k) : i, j \in V, k \in \mathbb{N}\}$ contains at most countably many elements. Then one of the following must happen:

a) there exists a constant $p^b > 0$ such that $\mathbf{P}_{x^0}(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0) \geq p^b$ for all $x^0 \in \mathbb{R}^n$.

b) $\mathbf{P}_{x^0}(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0) = 0$ for almost all $x^0 \in \mathbb{R}^n$.

(iii). Suppose $\sum_{k=0}^{\infty} P_k < \infty$ and there exists a constant a_* such that $a_{ii}(k) \geq a_* > 1/2$ for all i and k . Then $\mathbf{P}(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0) = 0$ for almost all $x^0 \in \mathbb{R}^n$.

Recall that a digraph G is said to be *quasi-strongly connected* if G has a directed spanning tree [34]. We introduce the following definition.

Definition 2: The random graph process $\langle G_k \rangle$ is called connectivity-independent if the events

$$\{G_k \text{ is quasi-strongly connected}\}, \quad k \in \mathbb{N}$$

are mutually independent.

For connectivity-independent graphs, we present the following almost sure convergence result.

Theorem 2: Suppose $\langle G_k \rangle$ is connectivity-independent and there exists a constant $0 < q < 1$ such that $\mathbf{P}(G_k \text{ is quasi-strongly connected}) \geq q$ for all k . Assume in addition that $P_{k+1} \leq P_k$. Then Algorithm (1) achieves global a.s. consensus if $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$, and an upper bound of $\mathcal{T}_{\text{com}}(\epsilon)$ is given as

$$\mathcal{T}_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{i=0}^{M-1} \log \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot P_{(i+1)(n-1)^2}^{n-1} \right)^{-1} \geq \log \epsilon^{-2} \right\} \times (n-1)^2.$$

We believe that the convergence condition given in Theorem 2 is reasonably tight since the probability that all the arcs in Algorithm (1) are active at time k is P_k^n , while one single inactive arc may be enough to break the connectivity of the graph (note that even the condition $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$ allows P_k to decrease to zero). The condition $P_{k+1} \leq P_k$ is a technical assumption which enforces certain regularity of the node update frequencies.¹ While the sufficient convergence condition $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$ indicates it is more difficult to reach consensus over large networks.

Nevertheless, connectivity is a global property of a graph, and indeed it does not rely on any specific arc. The next definition is on the independence of the existence of the arcs in the graph process.

Definition 3: Let $\mathcal{G}^\dagger = (V, \mathcal{E}^\dagger)$ be a (deterministic) digraph with $\mathcal{E}^\dagger = \{(i_\tau, j_\tau) : \tau = 1, \dots, |\mathcal{E}^\dagger|\}$. Denote $\mathcal{E}_k(\tau) := \{(i_\tau, j_\tau) \in G_k\}$ for $\tau = 1, \dots, |\mathcal{E}^\dagger|$ and $k \in \mathbb{N}$. Then $\langle G_k \rangle$ is called arc-independent with respect to \mathcal{G}^\dagger if the events

$$\mathcal{E}_k(\tau_k), \quad k \in \mathbb{N}$$

are mutually independent for any deterministic sequence $\langle \tau_k \rangle$ with each τ_k taking value from $\{1, \dots, |\mathcal{E}^\dagger|\}$. We call \mathcal{G}^\dagger a basic graph of the random graph process $\langle G_k \rangle$.

For arc-independent graphs, the following result holds.

Theorem 3: Suppose $\langle G_k \rangle$ is arc-independent with respect to the basic graph $\mathcal{G}^\dagger = (V, \mathcal{E}^\dagger)$. Let \mathcal{G}^\dagger be quasi-strongly connected and assume that there exists a constant $0 < q_* < 1$ such that $\mathbf{P}((i, j) \in E_k) \geq q_*$ for all k and $(i, j) \in \mathcal{E}^\dagger$. Then Algorithm (1) achieves global a.s. consensus if and only if $\sum_{k=0}^{\infty} P_k = \infty$, and there holds

$$\mathcal{T}_{\text{com}}(\epsilon) \leq \inf \left\{ k : \sum_{i=0}^{k-1} (1 - (1 - P_i)^n) \geq \frac{(n-1)|\mathcal{E}^\dagger|}{\log A} \log(A\epsilon^2/n) \right\}$$

where $A = 1 - (\eta q_*/n)^{(n-1)|\mathcal{E}^\dagger|}$ and $|\mathcal{E}^\dagger|$ represents the number of elements in \mathcal{E}^\dagger .

C. Remarks

Theorems 1 and 3 combined show that $\sum_{k=0}^{\infty} P_k = \infty$ is a sharp threshold for Algorithm (1) to reach consensus

¹The condition $P_{k+1} \leq P_k$ is also to make the problem more interesting since the main challenge is to reach consensus with as small P_k as possible. Certainly the counter condition $P_{k+1} \geq P_k$ will ensure convergence straightforwardly.

convergence under quite general conditions leading to a *Zero-One Law*: consensus convergence holds true with probability one for all initial conditions when the infinite sum diverges, and otherwise consensus is achieved with probability zero for almost all initial conditions. These results are essentially due to the Borel-Cantelli Lemma, where the condition $\sum_{k=0}^{\infty} P_k = \infty$ ensures that a.s. every node updates its states infinitely often. Consensus convergence however requires much more than that: it is crucial that every node updates infinitely often (cf., Theorem 1), but more importantly, these node updates should essentially comply with the underlying random graph process (cf., Theorems 2 and 3).

The convergence time estimates given in Theorems 2 and 3 are established with the help of Markov's inequality, where the random graph process and the Bernoulli trials together cumulatively contribute to the upper bound. Moreover, from the proofs it is straightforward to see that all the results can be easily generalized to the case when nodes have distinct successful update probabilities, i.e., the P_k become P_k^i depending on $i \in V$. For the ease of presentation we let $\langle P_k \rangle$ apply to all nodes.

The proposed concepts of connectivity and arc independence directly apply to many random graph models in the literature, for which a detailed discussion is provided in Subsection IV-A illustrating the usefulness of the derived results for several fundamental random graph processes. In fact, we manage to generalize the connectivity/arc independence to connectivity/arc *mixing* of random graph processes, and show that the same analysis smoothly carries on leading to similar convergence results. As a result, the majority of the random graph models in the literature (including Markovian random graphs) are then covered. The generalization to mixing of random graphs is in Subsection IV-B.

Finally, for random gossiping algorithms [2], we are able to go one step forward and present almost sure finite-time convergence results. Remarkably enough, the random node update process driven by non-reliable $\langle P_k \rangle$ becomes *essential* for almost sure finite-time convergence. This part of the generalization is in Subsection IV-C.

D. Paper Organization

The remainder of the paper is organized as follows. Section II establishes the convergence results for connectivity-independent graphs. Several classes of connectivity-independent graphs, e.g., directed, bidirectional and acyclic graphs, are investigated, respectively. The proof of Theorem 2 is therefore obtained as a direct consequence. Section III turns to arc-independent graphs and proves Theorem 3 using a matrix argument. Section IV is devoted to some nontrivial generalizations of the main results: A few basic random graph models in the literature are discussed illustrating the applicability of the derived results; the connectivity/arc **-mixing* random graphs are introduced with the corresponding convergence results established; conditions for almost sure finite-time convergence of random gossiping algorithms are presented. Finally some concluding remarks are given in Section V.

The proof of Theorem 1 on the consensus impossibilities is a bit tangential to the a.s. convergence results, and is therefore put in Appendix A.

Notation and Terminologies

A (simple) directed graph $G = (V, E)$, or in short, a digraph, consists of a finite set $V = \{1, \dots, N\}$ of nodes and an arc set E , where an element $e = (i, j) \in E$ denotes an *arc* from node $i \in V$ to $j \in V$ with $i \neq j$. A directed path between two vertices v_1 and v_k in G is a sequence of distinct nodes $v_1 v_2 \dots v_k$ such that for any $m = 1, \dots, k-1$, there is an arc from v_m to v_{m+1} ; $v_1 v_2 \dots v_k$ is called a semi-path if for any $m = 1, \dots, k-1$, either $(v_m, v_{m+1}) \in E$ or $(v_{m+1}, v_m) \in E$. As is usual [33], a graph G is termed *strongly connected* if, for every pair of distinct nodes in V , there is a path from one to the other; *quasi-strongly connected* if there exists a node $v \in V$, namely a *root*, such that there is a path from v to all other nodes (equivalently, the graph contains a directed spanning tree); *weakly connected* if there is a semi-path between any two distinct nodes. Deterministic graphs are denoted by \mathcal{G} with arc set \mathcal{E} ; random graphs are denoted by G with arc set E .

We use \mathbb{N} to denote the set of non-negative integers, and \mathbb{R} denotes real numbers. Probability is denoted as $\mathbf{P}(\cdot)$; expectation of random variables is denoted as $\mathbf{E}[\cdot]$; events are denoted by $\mathcal{A}, \mathcal{B}, \dots$. We also use $|\cdot|$ to represent the cardinality of a finite set, or the absolute value of a real number. A sequence $\{b_k\}_{k=0}^{\infty}$ of real numbers, random variables, or events, is always abbreviated as $\langle b_k \rangle$.

II. CONNECTIVITY-INDEPENDENT GRAPHS

In this section, we present the convergence analysis for connectivity-independent random graph processes. We are going to study some general cases relying on the joint graphs only.

The joint graph [1], [18] of G_k on time interval $[k_1, k_2]$ for $0 \leq k_1 \leq k_2 \leq \infty$ is defined as $G([k_1, k_2]) = (V, \bigcup_{k \in [k_1, k_2]} E_k)$. We introduce the following definition of connectivity for the random graph process.

Definition 4: The random graph process $\langle G_k \rangle$ is termed (i) *uniformly stochastically quasi-strongly connected*, if there exist an integer $B \geq 1$ and $0 < q < 1$ such that the sequence $\langle G([mB, (m+1)B - 1]) \rangle$ is connectivity-independent and

$$\mathbf{P}\left(G([mB, (m+1)B - 1]) \text{ is quasi-strongly connected}\right) \geq q$$

for all m ;

(ii) *infinitely stochastically quasi-strongly connected*, if there exist a (deterministic) sequence $0 = C_0 < \dots < C_m < \dots$ and a constant $0 < q < 1$ such that $\langle G([C_m, C_{m+1}]) \rangle$ is connectivity-independent and

$$\mathbf{P}\left(G([C_m, C_{m+1}]) \text{ is quasi-strongly connected}\right) \geq q$$

for all m .

In the remainder of this section, we first establish consensus conditions for uniformly stochastically quasi-strongly connected graph processes, and the proof of Theorem 2 is obtained as a special case. Next, two special cases, bidirectional and acyclic graph processes, are further investigated, respectively.

A. Uniform Stochastic Connectivity

The following result is for consensus convergence over uniformly stochastically quasi-strongly connected graphs.

Proposition 1: Suppose $\langle G_k \rangle$ is uniformly stochastically quasi-strongly connected with $B \geq 1$ and $q > 0$. Then Algorithm (1) achieves global a.s. consensus if $\sum_{s=0}^{\infty} \bar{P}_s = \infty$, where

$$\bar{P}_s = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_l \in [s(n-1)^2 B, (s+1)(n-1)^2 B], l = 1, \dots, n-1 \right\}.$$

Moreover, we have

$$\mathcal{F}_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{i=0}^{M-1} \log \left(1 - \frac{(q\eta/n)^{(n-1)^2}}{2} \cdot \bar{P}_i \right)^{-1} \geq \log \epsilon^{-2} \right\} \times (n-1)^2 B,$$

where η is the constant defined in the weights rule.

Proof: We first establish a lemma characterizing a useful property of uniformly stochastically quasi-strongly connected graphs.

Lemma 1: Assume that $\langle G_k \rangle$ is uniformly stochastically quasi-strongly connected. Then for any $s = 0, 1, \dots$, we have

$$\mathbf{P} \left(\exists i_0 \in \mathbb{V} \text{ and } \begin{aligned} & \tau_1 < \dots < \tau_{n-1} \in [s(n-1)^2, (s+1)(n-1)^2] \\ & \text{s.t. } i_0 \text{ is a root of } G([\tau_j B, (\tau_j + 1)B - 1]) \\ & \text{for all } j = 1, \dots, n-1 \end{aligned} \geq \left(\frac{q}{n}\right)^{(n-1)^2} \right).$$

Proof: The probability that the graph $G([\tau B, (\tau + 1)B - 1])$ has a root is no less than q for $\tau = s(n-1)^2, \dots, (s+1)(n-1)^2 - 1$. Consequently, for each $G([\tau B, (\tau + 1)B - 1])$, there is a (deterministic) node i_0 (which depends on τ) such that i_0 is a root with probability at least q/n . The $(n-1)^2$ graphs, $G([\tau B, (\tau + 1)B - 1])$, $\tau = s(n-1)^2, \dots, (s+1)(n-1)^2 - 1$ will lead to at least $(n-1)^2$ such roots (with possible repetitions, of course). However, the total number of the nodes is n . Thus, at least one node is counted more than $n-2$ times. Then the desired lemma immediately follows from the definition of connectivity independence. \square

Denote $k_s = s(n-1)^2 B$ for $s \geq 0$. We consider the event studied in Lemma 1, that there exists a common root node i_0 such that i_0 is a root of $G([\tau_j B, (\tau_j + 1)B - 1])$ for $j = 1, \dots, n-1$ with $k_s \leq \tau_j B \leq k_{s+1} - 1$ with a probability no smaller than $(q/n)^{(n-1)^2}$. We first assume that

$$x_{i_0}(k_s) \leq \frac{1}{2}h(k_s) + \frac{1}{2}H(k_s). \quad (3)$$

From a symmetric analysis by establishing the lower bound of $h(k_{s+1})$ (or, directly considering $y(k) = (y_1(k) \dots y_n(k))^T$ with $y_i(k) = -x_i(k)$), it is easy to show that the argument continues to hold under the condition that

$$x_{i_0}(k_s) > \frac{1}{2}h(k_s) + \frac{1}{2}H(k_s).$$

We divide the rest of the proof into four steps.

Step 1: In this step, we bound $x_{i_0}(k)$. With the weights rule in our standing assumption, we have

$$\begin{aligned} & \sum_{j \in N_{i_0}(k_s)} a_{i_0 j}(k_s) x_j(k_s) \\ &= a_{i_0 i_0}(k_s) x_{i_0}(k_s) + \sum_{j \in N_{i_0}(k_s) \setminus \{i_0\}} a_{i_0 j}(k_s) x_j(k_s) \\ &\leq a_{i_0 i_0}(k_s) \left(\frac{1}{2}h(k_s) + \frac{1}{2}H(k_s) \right) + (1 - a_{i_0 i_0}(k_s))H(k_s) \\ &= \frac{a_{i_0 i_0}(k_s)}{2}h(k_s) + \left(1 - \frac{a_{i_0 i_0}(k_s)}{2}\right)H(k_s) \\ &\leq \frac{\eta}{2}h(k_s) + \left(1 - \frac{\eta}{2}\right)H(k_s), \end{aligned} \quad (4)$$

where the last equality holds from the facts that $h(k_s) \leq H(k_s)$ and $a_{i_0 i_0}(k_s) \geq \eta$. Thus, since $\eta < 1$, no matter whether node i_0 takes averaging or sticks to its current state, we obtain

$$x_{i_0}(k_s + 1) \leq \frac{\eta}{2}h(k_s) + \left(1 - \frac{\eta}{2}\right)H(k_s).$$

By recursively applying (4) there holds that for any $\varrho = 0, 1, \dots$,

$$x_{i_0}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + \left(1 - \frac{\eta^\varrho}{2}\right)H(k_s). \quad (5)$$

Step 2: Conditioned that i_0 is a root of $G([\tau_1 B, (\tau_1 + 1)B - 1])$, there will be $i_1 \in \mathbb{V}$, which is different from i_0 , and a time instant $t_1 \in [\tau_1 B, (\tau_1 + 1)B - 1]$, such that $(i_0, i_1) \in E_{\tau_1}^2$. Suppose $t_1 = k_s + \varsigma$. In this step, we bound $x_{i_1}(k)$.

If i_1 takes an averaging update at time step t_1 , we obtain from (5) that

$$\begin{aligned} & x_{i_1}(k_s + \varsigma + 1) \\ &= a_{i_1 i_0}(k_s + \varsigma) x_{i_0}(k_s + \varsigma) \\ &+ \sum_{j \in N_{i_1}(k_s + \varsigma) \setminus \{i_0\}} a_{i_1 j}(k_s + \varsigma) x_j(k_s + \varsigma) \\ &\leq \eta \left[\frac{\eta^\varsigma}{2}h(k_s) + \left(1 - \frac{\eta^\varsigma}{2}\right)H(k_s) \right] + (1 - \eta)H(k_s) \\ &= \frac{\eta^{\varsigma+1}}{2}h(k_s) + \left(1 - \frac{\eta^{\varsigma+1}}{2}\right)H(k_s), \end{aligned} \quad (6)$$

which leads to that for any $\varrho = (\tau_1 + 1)B - k_s, \dots$,

$$x_{i_1}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + \left(1 - \frac{\eta^\varrho}{2}\right)H(k_s). \quad (7)$$

Therefore, we conclude from (5) and (7) that

$$\mathbf{P} \left(x_{i_l}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + \left(1 - \frac{\eta^\varrho}{2}\right)H(k_s), \quad l = 0, 1; \right. \\ \left. \varrho \geq (\tau_1 + 1)B - k_s \right) \geq \left(\frac{q}{n}\right)^{(n-1)^2} \min_{k \in [\tau_1 B, (\tau_1 + 1)B - 1]} P_k.$$

Step 3: We proceed the analysis for the time interval $[\tau_2 B, (\tau_2 + 1)B - 1]$. Similarly, $i_2 \neq i_0, i_1$ can be found

²It should be emphasized that here both i_1 and t_1 could be random. We can however treat the remaining analysis for each sample path ω of $i_1(\omega)$ and $t_1(\omega)$, and the same conclusion will be drawn. Therefore, without loss of generality, we omit the discussion regarding possible randomness in i_1 and t_1 .

(because i_0 is a root) such that

$$\begin{aligned} \mathbf{P}\left(x_{i_l}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + \left(1 - \frac{\eta^\varrho}{2}\right)H(k_s), \quad l = 0, 1, 2; \right. \\ \left. \varrho \geq (\tau_2 + 1)B - k_s\right) \\ \geq \left(\frac{q}{n}\right)^{(n-1)^2} \min_{\alpha_1, \alpha_2 \in [\tau_1 B, (\tau_2 + 1)B - 1]} P_{\alpha_1} P_{\alpha_2}. \end{aligned}$$

Continuing for time intervals $[\tau_j B, (\tau_j + 1)B - 1]$ for $j = 3, \dots, n - 1$, bounds for i_3, \dots, i_{n-1} can be established as

$$\begin{aligned} \mathbf{P}\left(x_{i_m}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + \left(1 - \frac{\eta^\varrho}{2}\right)H(k_s), \right. \\ \left. m = 0, \dots, n - 1; \right. \\ \left. \varrho = (\tau_{n-1} + 1)B - k_s, \dots\right) \geq \bar{P}_s \left(\frac{q}{n}\right)^{(n-1)^2}, \end{aligned}$$

with

$$\begin{aligned} \bar{P}_s = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{m=1}^{n-1} P_{\alpha_m} : \right. \\ \left. \alpha_m \in [s(n-1)^2 B, (s+1)(n-1)^2 B), m = 1, \dots, n-1 \right\}. \end{aligned}$$

This leads to

$$\begin{aligned} \mathbf{P}\left(H(k_{s+1}) \leq \frac{\eta^{(n-1)^2}}{2}h(k_s) + \left(1 - \frac{\eta^{(n-1)^2}}{2}\right)H(k_s)\right) \\ \geq \bar{P}_s \left(\frac{q}{n}\right)^{(n-1)^2}, \end{aligned} \quad (8)$$

which implies

$$\mathbf{P}\left(\mathcal{H}(k_{s+1}) \leq \left(1 - \frac{\eta^{(n-1)^2}}{2}\right)\mathcal{H}(k_s)\right) \geq \bar{P}_s \left(\frac{q}{n}\right)^{(n-1)^2}. \quad (9)$$

Therefore, noticing $\mathcal{H}(k+1) \leq \mathcal{H}(k)$ always holds, we obtain from (9) that

$$\mathbf{E}[\mathcal{H}(k_{s+1})] \leq \left(1 - \frac{(q\eta/n)^{(n-1)^2}}{2} \cdot \bar{P}_s\right) \mathbf{E}[\mathcal{H}(k_s)]. \quad (10)$$

Step 4: In light of connectivity-independence and the fact that (9) holds only requiring the connectivity of the union graph of $n-1$ subintervals in $[s(n-1)^2, (s+1)(n-1)^2)$, we further conclude from (10) that³

$$\mathbf{E}[\mathcal{H}(k_M)] \leq \prod_{s=0}^{M-1} \left(1 - \frac{(q\eta/n)^{(n-1)^2}}{2} \cdot \bar{P}_s\right) \mathcal{H}(0) \quad (11)$$

for all $M \geq 1$. Thus, it follows from Lemma 7 that

$$\lim_{M \rightarrow \infty} \mathbf{E}[\mathcal{H}(k_M)] = 0, \quad (12)$$

which yields $\lim_{k \rightarrow \infty} \mathbf{E}[\mathcal{H}(k)] = 0$ since $\mathcal{H}(k)$ is non-increasing. Using Fatou's Lemma, we further obtain

$$0 \leq \mathbf{E}\left[\lim_{k \rightarrow \infty} \mathcal{H}(k)\right] \leq \lim_{k \rightarrow \infty} \mathbf{E}[\mathcal{H}(k)] = 0. \quad (13)$$

³Note that in general recursively applying (10) for $s = 0, 1, \dots$ requires independence of the random graph process $\langle G_k \rangle$ that drives $x(k)$. However, from the proof of Step 2 it is clear that connectivity independence has been enough for granting such a recursive analysis and therefore ensuring (11).

Therefore, the convergence claim of the conclusion holds because (13) implies

$$\mathbf{P}\left(\lim_{k \rightarrow +\infty} \mathcal{H}(k) = 0\right) = 1.$$

Applying Markov's Inequality to (11) leads to

$$\begin{aligned} \mathbf{P}\left(\mathcal{H}(k_M) \geq \epsilon \mathcal{H}(0)\right) &\leq \frac{1}{\epsilon} \cdot \frac{\mathbf{E}[\mathcal{H}(k_M)]}{\mathcal{H}(0)} \\ &\leq \frac{1}{\epsilon} \prod_{s=0}^{M-1} \left(1 - \frac{(q\eta/n)^{(n-1)^2}}{2} \cdot \bar{P}_s\right). \end{aligned}$$

Consequently, we have

$$\begin{aligned} \mathcal{T}_{\text{com}}(\epsilon) &\leq \inf \left\{ M : \sum_{s=0}^{M-1} \log \left(1 - \frac{(q\eta/n)^{(n-1)^2}}{2} \cdot \bar{P}_s\right)^{-1} \right. \\ &\quad \left. \geq \log \epsilon^{-2} \right\} \times (n-1)^2 B. \end{aligned}$$

The desired conclusion follows. \square

Remark 1: Suppose $P_{k+1} \leq P_k$ for all k . Then it is not hard to see that $\sum_{s=0}^{\infty} \bar{P}_s = \infty$ if and only if $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$. Then Theorem 2 holds immediately from Proposition 1.

B. Bidirectional Connections

A digraph $\mathcal{G} = (\mathbf{V}, \mathcal{E})$ is *bidirectional* if for any two nodes i and j , $(i, j) \in \mathcal{E}$ and only if $(j, i) \in \mathcal{E}$. In this subsection, we study bidirectional graphs.

Note that we do not impose an upper bound for the length of the intervals $[C_m, C_{m+1})$ in the definition of infinitely stochastically quasi-strong connectivity, which makes an essential difference from the bounded joint connections.

Proposition 2: Suppose there holds a.s. that G_k is bidirectional for all $k \geq 0$. Assume that $\langle G_k \rangle$ is stochastically infinitely quasi-strongly connected. Then Algorithm (1) achieves global a.s. consensus if $\sum_{s=0}^{\infty} \hat{P}_s = \infty$, where

$$\begin{aligned} \hat{P}_s = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_l \in [C_{s(n-1)}, C_{(s+1)(n-1)}) \right. \\ \left. l = 1, \dots, n-1 \right\}. \end{aligned}$$

and also

$$\begin{aligned} \mathcal{T}_{\text{com}}(\epsilon) &\leq \inf \left\{ C_{s(n-1)} : \sum_{i=0}^{s-1} \log \left(1 - (q\eta)^{(n-1)} \cdot \hat{P}_i\right)^{-1} \right. \\ &\quad \left. \geq \log \epsilon^{-2} \right\}. \end{aligned}$$

Proof: The proof is carried out analyzing a sequence a stopping times (see [35]) of the random graph process. Take a node $i_0 \in \mathbf{V}$ with $x_{i_0}(C_0) = h(C_0)$. Define

$$t_1 = \inf_{k \geq C_0} \left\{ N_{i_0}(k) \setminus \{i_0\} \neq \emptyset \right\}.$$

In other words, t_1 is the first time that i_0 has a neighbor other than itself in the random graph process. Then t_1 is a stopping time of the graph process $\langle G_k \rangle$, and there must hold

$$\mathbf{P}(t_1 < C_1) > q$$

since $\langle G_k \rangle$ is infinitely stochastically quasi-strongly connected. We next introduce

$$V_1 = \left\{ j \neq i_0 : (i_0, j) \in E_{t_1} \right\}.$$

We emphasize that by its definition V_1 is nonempty. Again we proceed in steps for the remaining of the argument.

Step 1: Noticing that $x_{i_0}(t_1) = x_{i_0}(C_0)$ from the structure of Algorithm (5), we conclude that

$$\begin{aligned} x_{i_0}(t_1 + 1) &= \sum_{j \in N_{i_0}(t_1)} a_{i_0j}(t_1)x_j(t_1) \\ &= a_{i_0i_0}(t_1)x_{i_0}(t_1) + \sum_{j \in N_{i_0}(t_1) \setminus \{i_0\}} a_{i_0j}(t_1)x_j(t_1) \\ &\leq a_{i_0i_0}(t_1)h(C_0) + (1 - a_{i_0i_0}(t_1))H(C_0) \\ &\leq \eta h(C_0) + (1 - \eta)H(C_0) \end{aligned} \quad (14)$$

if i_0 updates its state at time $t_1 + 1$. Note that (14) continues to hold even if i_0 fails to update at time $t_1 + 1$. On the other hand, for $i_1 \in V_1$, if i_q successfully updates its state at time $t_1 + 1$, then

$$\begin{aligned} x_{i_1}(t_1 + 1) &= \sum_{j \in N_{i_1}(t_1)} a_{i_1j}(t_1)x_j(t_1) \\ &= a_{i_1i_0}(t_1)x_{i_0}(t_1) + \sum_{j \in N_{i_1}(t_1) \setminus \{i_0\}} a_{i_1j}(t_1)x_j(t_1) \\ &\leq a_{i_1i_0}(t_1)h(C_0) + (1 - a_{i_1i_0}(t_1))H(C_0) \\ &\leq \eta h(C_0) + (1 - \eta)H(C_0), \end{aligned}$$

which gives

$$\mathbf{P}\left(x_{i_1}(t_1 + 1) \leq \eta h(C_0) + (1 - \eta)H(C_0)\right) \geq P_{t_1}. \quad (15)$$

The fact that the random node updates are independent of the graph process as well as among different nodes, allows us to further conclude that

$$\begin{aligned} \mathbf{P}\left(x_m(t_1 + 1) \leq \eta h(C_0) + (1 - \eta)H(C_0), m \in \{i_0\} \cup V_1\right) \\ \geq P_{t_1}^{|V_1|}. \end{aligned}$$

Step 2: We continue to define

$$t_2 = \inf_{k \geq t_1} \left\{ \text{there is an arc between } V \setminus (\{i_0\} \cup V_1) \text{ and } \{i_0\} \cup V_1 \text{ in } G_k \right\}$$

and

$$V_2 = \{j \in V \setminus (\{i_0\} \cup V_1) : \text{there is an arc between } \{i_0\} \cup V_1 \text{ and } j \text{ in } G_{t_2}\}.$$

Again t_2 is a stopping time of the graph process $\langle G_k \rangle$, and by definition there holds

$$\mathbf{P}(t_2 < C_2 | t_1 < C_1) \geq q.$$

Repeating the similar analysis as Step 1 we obtain

$$\begin{aligned} \mathbf{P}\left(x_m(t_2 + 1) \leq \eta^2 h(C_0) + (1 - \eta^2)H(C_0), \right. \\ \left. m \in \{i_0\} \cup V_1 \cup V_2\right) \geq P_{t_1}^{|V_1|} P_{t_2}^{|V_2|}. \end{aligned}$$

Step 3: Continuing the analysis, t_3, \dots, t_{μ_0} and V_3, \dots, V_{μ_0} can be found until

$$V = \{i_0\} \cup \left(\bigcup_{l=1}^{\mu_0} V_l \right)$$

for some $\mu_0 \leq n - 1$, and eventually

$$\begin{aligned} \mathbf{P}\left(x_m(t_{\mu_0} + 1) \leq \eta^{\mu_0} h(C_0) + (1 - \eta^{\mu_0})H(C_0), m \in V\right) \\ \geq \prod_{l=1}^{\mu_0} P_{t_l}^{|V_l|}. \end{aligned}$$

This gives us

$$\mathbf{P}\left(\mathcal{H}(t_{\mu_0} + 1) \leq (1 - \eta^{\mu_0})\mathcal{H}(C_0)\right) \geq \prod_{l=1}^{\mu_0} P_{t_l}^{|V_l|}. \quad (16)$$

Noting that $\mu_0 \leq n - 1$ as well as the fact that

$$\mathbf{P}(t_{\mu_0} < C_{n-1}) \geq q^{n-1}$$

in light of the connectivity independence, we further conclude from (16) that

$$\mathbf{P}\left(\mathcal{H}(C_{n-1}) \leq (1 - \eta^{n-1})\mathcal{H}(C_0)\right) \geq \hat{P}_0 q^{n-1}$$

with $\hat{P}_0 = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_j \in [C_0, C_{n-1}), j = 1, \dots, n - 1 \right\}$.

We can recursively apply the above analysis and bounds of $\mathcal{H}(C_{s(n-1)})$ can be obtained for $s = 1, 2, \dots$ as

$$\mathbf{P}\left(\mathcal{H}(C_{(s+1)(n-1)}) \leq (1 - \eta^{n-1})\mathcal{H}(C_{s(n-1)})\right) \geq \hat{P}_s q^{n-1}.$$

This ensures global a.s. consensus and that

$$\begin{aligned} \mathcal{T}_{\text{com}}(\epsilon) &\leq \inf \left\{ C_{s(n-1)} : \sum_{i=0}^{s-1} \log \left(1 - (q\eta)^{(n-1)} \cdot \hat{P}_i \right)^{-1} \right. \\ &\quad \left. \geq \log \epsilon^{-2} \right\}. \end{aligned}$$

The proof is now complete. \square

Remark 2: If we impose the condition that $P_{k+1} \leq P_k$ for all k in Proposition 2, then global a.s. consensus for Algorithm (1) can be guaranteed by $\sum_{s=0}^{\infty} P_{C_{s(n-1)}}^{n-1} = \infty$.

C. Acyclic Connections

A digraph $\mathcal{G} = (V, \mathcal{E})$ is called *acyclic* if it contains no directed cycle. This subsection focuses on acyclic graphs.

Let \mathcal{G} be an acyclic, quasi-strongly connected digraph. Then it is not hard to see that \mathcal{G} has one and only one root. Denote this root as v_0 and let $v_0 \rightarrow j$ be a path from v_0 to j in \mathcal{G} . Let $|v_0 \rightarrow j|$ represent the length of $v_0 \rightarrow j$ as the number of arcs in this path. We now define a function on the vertices of \mathcal{G} by $\hat{h}(v_0) = 0$ and $\hat{h}(j) = \max\{|v_0 \rightarrow j| : v_0 \rightarrow j \text{ is a path in } \mathcal{G}\}$ for any $j \neq v_0$. Let $d_* = \max_{i \in V} \hat{h}(i)$, which is the maximum path length and therefore is upper bounded by $n - 1$. Then we establish the following lemma indicating that this function \hat{h} is surjective from V to $\{0, \dots, d_*\}$.

Lemma 2: Let \mathcal{G} be an acyclic, quasi-strongly connected digraph. Then $\hat{h}^{-1}(m) = \{i : \hat{h}(i) = m\}$ is nonempty for all $m = 0, \dots, d_*$.

Proof: The conclusion holds for $m = 0$ trivially.

Let us prove the conclusion for $m = 1$ by contradiction. Assume that $\tilde{h}^{-1}(1) = \emptyset$. Then we have $m_0 \doteq \inf_{i \neq v_0} \tilde{h}(i) > 1$. Take a node j_0 with $\tilde{h}(j_0) = m_0$. There exists a (simple) path $v_0 \rightarrow j_0$ in \mathcal{G} with length $m_0 > 1$. Let v_* be the node for which arc (v_*, j_0) is included in $v_0 \rightarrow j_0$. According to the definition of m_0 , we have $\tilde{h}(v_*) \geq m_0$. Suppose $v_0 \rightarrow v_*$ is a path with length $\tilde{h}(v_*)$. Note that, j_0 cannot be included in $v_0 \rightarrow v_*$ because then it would generate a cycle $j_0 \rightarrow v_* \rightarrow j_0$. Consequently, another path $v_0 \rightarrow v_* \rightarrow j_0$ is obtained whose length is $\tilde{h}(v_*) + 1 > m_0$. This contradicts the selection rule of j_0 . Therefore, the conclusion holds for $m = 1$.

Next, we construct another graph \mathcal{G}^* from \mathcal{G} by viewing node set $\{v_0\} \cup \tilde{h}^{-1}(1)$ as a single node in the new graph without changing the arcs. We see that $\tilde{h}^{-1}(2)$ of \mathcal{G} is exactly the same as the node set $\tilde{h}^{-1}(1)$ of \mathcal{G}^* , while the latter is nonempty via previous analysis. Continuing the argument, the conclusion follows. \square

Here comes our main result for acyclic graphs.

Proposition 3: Let $\mathcal{G} = (V, \mathcal{E})$ be an acyclic, quasi-strongly connected digraph and assume that a.s. $E_k \subseteq \mathcal{E}$ for all k . Suppose $\langle G_k \rangle$ is infinitely stochastically quasi-strongly connected. Then Algorithm (1) achieves global a.s. consensus if $\sum_{s=0}^{\infty} [\inf_{C_s \leq \alpha < C_{s+1}} P_\alpha] = \infty$.

Proof: Let v_0 be the unique root node of \mathcal{G} . Based on Lemma 2, $V_i = \tilde{h}^{-1}(i)$ for $i = 0, \dots, \tilde{h}_0$ can be defined satisfying $V_0 = \{v_0\}$ and $V = \bigcup_{i=0}^{d_k} V_i$. Obviously we have $\mathbf{P}(x_{v_0}(k) = x_{v_0}(0), k \geq 0) = 1$ because with probability one, v_0 has no neighbor except itself for all k . We first investigate the nodes in V_1 .

Claim: $\mathbf{P}\left(\lim_{k \rightarrow \infty} |x_m(k) - x_{v_0}(0)| = 0\right) = 1$ for all $m \in V_1$.

Take $v_1 \in V_1$. By our assumption v_0 is the only neighbor of v_1 excluding itself in G_k for all k . Define $t_1 = \inf_{k \geq 0} \{(v_0, v_1) \in E_k\}$. Then $\mathbf{P}(t_1 < C_1) \geq q$. We have

$$\begin{aligned} & \left| \sum_{j \in N_{v_1}(t_1)} a_{v_1 j}(t_1) x_j(t_1) - x_{v_0}(0) \right| \\ &= |a_{v_1 v_0}(t_1) x_{v_0}(t_1) + a_{v_1 v_1}(t_1) x_{v_1}(t_1) - x_{v_0}(0)| \\ &= (1 - a_{v_1 v_0}(t_1)) |x_{v_1}(0) - x_{v_0}(0)| \\ &\leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|, \end{aligned}$$

which yields

$$\mathbf{P}\left(|x_{v_1}(t_1 + 1) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|\right) \geq P_{t_1}.$$

Thus, we obtain

$$\mathbf{P}\left(|x_{v_1}(C_1) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|\right) \geq \tilde{P}_0 q,$$

where $\tilde{P}_0 = \inf_{C_0 \leq \alpha < C_1} P_\alpha$. Repeating the analysis on time interval $[C_m, C_{m+1})$, $m = 1, 2, \dots$, we have

$$\mathbf{P}\left(|x_{v_1}(C_{m+1}) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(C_m) - x_{v_0}(0)|\right) \geq \tilde{P}_m q \quad (17)$$

for all m . Then connectivity independence leads to $\mathbf{P}\left(\lim_{k \rightarrow \infty} |x_{v_1}(k) - x_{v_0}(0)| = 0\right) = 1$. The claim is proved.

Next, we turn to V_2 . The analysis of nodes in V_2 will be carried out via a sample-path argument. Let $\ell \geq 1$.

The above claim we just proved implies that there exists a random variable $T_\ell(\omega) > 0$ which is a.s. finite, such that

$$\mathbf{P}\left(|x_m(k) - x_{v_0}(0)| \leq 1/\ell, k \geq T_\ell, m \in V_1\right) = 1.$$

Take v_2 as an arbitrary node in V_2 . We prove $\mathbf{P}\left(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell\right) = 1$. Note that from the definition of V_2 , the state of node v_2 can only be influenced by nodes in $\{v_0\} \cup V_1$ in the entire time horizon.

Define event $\mathcal{E} = \{|x_{v_2}(k) - x_{v_0}(0)| > 1/\ell, k \geq T_\ell\}$. There will be two cases.

- (i) Since v_2 can only possibly connect to nodes in $\{v_0\} \cup V_1$, there must hold

$$|x_{v_2}(s) - x_{v_0}(0)| \leq 1/\ell, s \geq k$$

if $|x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell$ for $k \geq T_\ell$. This is equivalent to that

$$\mathbf{P}\left(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell | \mathcal{E}^c\right) = 1.$$

- (ii) By our assumption, $\langle G_k \rangle$ is infinitely stochastically quasi-strongly connected, and $\sum_{s=0}^{\infty} \tilde{P}_s = \infty$ with $\tilde{P}_s = \inf_{C_s \leq \alpha < C_{s+1}} P_\alpha$. Moreover, whenever $G([C_m, C_{m+1}))$ is quasi-strongly connected, there must be a time instant between $[C_m, C_{m+1})$ at which there is an arc from $\{v_0\} \cup V_1$ to v_2 . Therefore, applying the Borel-Cantelli lemma we see that a.s. there is an infinite sequence of (random) time instants

$$t_1 < t_2 < \dots < t_m < \dots$$

where at each time t_m , there is at least one arc from $\{v_0\} \cup V_1$ to v_2 and v_2 successfully updates its state. Then we obtain for all $t_m \geq T_\ell$ that

$$\begin{aligned} & |x_{v_2}(t_m + 1) - x_{v_0}(0)| \\ &= \left| \sum_{j \in N_{v_2}(t_m)} a_{v_2 j}(t_m) x_j(t_m) - x_{v_0}(0) \right| \\ &= \sum_{j \in N_{v_2}(t_m): j \in \{v_0\} \cup V_1} a_{v_2 j}(t_m) |x_j(t_m) - x_{v_0}(0)| \\ &\quad + a_{v_2 v_2}(t_m) |x_{v_2}(t_m) - x_{v_0}(0)| \\ &\leq [1 - a_{v_2 v_2}(t_m)] \frac{1}{\ell} + a_{v_2 v_2}(t_m) |x_{v_2}(t_m) - x_{v_0}(0)| \\ &\leq [1 - \eta] \frac{1}{\ell} + \eta |x_{v_2}(t_m) - x_{v_0}(0)|, \end{aligned}$$

which reads

$$|x_{v_2}(t_m + 1) - x_{v_0}(0)| - \frac{1}{\ell} \leq \eta \left(|x_{v_2}(t_m) - x_{v_0}(0)| - \frac{1}{\ell} \right) \quad (18)$$

for all $t_m \geq T_\ell$. Noticing the definition of t_m , this has already proved that

$$\mathbf{P}\left(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell | \mathcal{E}\right) = 1.$$

As a result, we can now conclude that

$$\mathbf{P}\left(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell\right) = 1.$$

Now that ℓ is chosen arbitrarily and v_2 is an arbitrary node in V_2 , we further obtain

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} |x_m(k) - x_{v_0}(0)| = 0\right) = 1, \quad m \in V_2.$$

Finally, a recursive analysis for node sets V_3, \dots, V_{d^*} eventually gives $\mathbf{P}(\lim_{k \rightarrow \infty} |x_m(k) - x_{v_0}(0)| = 0) = 1$ for all $m \in V$. The desired conclusion thus follows. \square

Remark 3: Proposition 3 immediately implies the following interesting observations when there holds $E_k \subseteq \mathcal{E}$ for all k for some acyclic, quasi-strongly connected digraph $\mathcal{G} = (V, \mathcal{E})$.

- (i). If $\langle G_k \rangle$ is infinitely stochastically quasi-strongly connected, and $P_{k+1} \leq P_k$ for all k , then Algorithm (1) achieves global a.s. consensus when $\sum_{m=0}^{\infty} P_{C_m} = \infty$.
- (ii). Suppose $\langle G_k \rangle$ is uniformly stochastically quasi-strongly connected with respect to $B \geq 1$. Let either $B = 1$ or $P_{k+1} \leq P_k$, $k \geq 0$. Then Algorithm (1) achieves global a.s. consensus if and only if $\sum_{k=0}^{\infty} P_k = \infty$.

III. ARC-INDEPENDENT GRAPHS

In this section, we investigate the convergence with an arc-independent random graph process and prove Theorem 3. We present the convergence analysis using a stochastic matrix argument.

Let $e_i = (0 \dots 1 \dots 0)^T$ be the $n \times 1$ unit vector with the i th component equal to one. Denote $\bar{a}_i(k) = (\bar{a}_{i1}(k) \dots \bar{a}_{in}(k))^T$ as an $n \times 1$ vector with $\bar{a}_{ij}(k) = a_{ij}(k)$ if $j \in N_i(k)$, and $\bar{a}_{ij}(k) = 0$ otherwise. Let $\chi_i(k)$, $i = 1, \dots, n$, $k \in \mathbb{N}$ be a sequence of independent Bernoulli random variables satisfying $\mathbf{E}[\chi_i(k)] = P_k$. Denote $W(k) = (w_1(k) \dots w_n(k))^T \in \mathbb{R}^{n \times n}$ as a random matrix with

$$w_i(k) = \chi_i(k)\bar{a}_i(k) + (1 - \chi_i(k))e_i, \quad i = 1, \dots, n. \quad (19)$$

Algorithm (1) can be written into a compact form:

$$x(k+1) = W(k)x(k). \quad (20)$$

In the remainder of this section, we first establish several useful lemmas on the product of stochastic matrices, and then the proof of Theorem 3 is presented.

A. Stochastic Matrices

A finite square matrix $M = [M_{ij}] \in \mathbb{R}^{n \times n}$ is called *stochastic* if $M_{ij} \geq 0$ for all i, j and $\sum_{j=1}^n M_{ij} = 1$ for all i [32]. For a stochastic matrix M , introduce

$$\delta(M) = \max_j \max_{\alpha, \beta} |M_{\alpha j} - M_{\beta j}|$$

and

$$\lambda(M) = 1 - \min_{\alpha, \beta} \sum_j \min\{M_{\alpha j}, M_{\beta j}\}.$$

The $\lambda(M)$ is often called the Dobrushin's ergodicity coefficient [50]. Note that $\delta(M)$ describes how much the weights assigned by two nodes α, β to a node j can differ, and $\lambda(M) = 0$ if and only if for each pair of nodes α and β , for each node j either α or β has no weight to j . If $\lambda(M) < 1$, M is called a *scrambling* matrix. A scrambling matrix does

not have two orthogonal rows, which means that any two rows share at least one column, where both of them have strictly positive elements. The following lemma can be found in [16].

Lemma 3: For any k ($k \geq 1$) stochastic matrices M_1, \dots, M_k , there holds

$$\delta(M_1 M_2 \dots M_k) \leq \prod_{i=1}^k \lambda(M_i).$$

Let $M = [M_{ij}] \in \mathbb{R}^{n \times n}$ be a matrix with nonnegative entries. We can associate a unique digraph $\mathcal{G}_M = (V, \mathcal{E}_M)$ with M on node set $V = \{1, \dots, n\}$ such that $(j, i) \in \mathcal{E}_M$ if and only if $M_{ij} > 0$. We call \mathcal{G}_M the *induced graph* of M .

The following lemma is given on the induced graphs of products of stochastic matrices.

Lemma 4: Let M_1, \dots, M_k be $k \geq 1$ stochastic matrices with positive diagonal entries. Then $(\bigcup_{i=1}^k \mathcal{G}_{M_i}) \subseteq \mathcal{G}_{M_1 \dots M_k}$.

Proof: We prove the case $k = 2$, and the conclusion will then follow by induction.

Denote $[M_1]_{ij}$, $[M_2]_{ij}$ and $[M_2 M_1]_{ij}$ as the ij -entries of M_1 , M_2 and $M_2 M_1$, respectively. Note that, we have

$$[M_2 M_1]_{ij} = \sum_{m=1}^n [M_2]_{im} [M_1]_{mj} \geq [M_2]_{ii} [M_1]_{ij} + [M_2]_{ij} [M_1]_{jj}.$$

The desired conclusion follows immediately from the fact that $[M_2]_{ii}, [M_1]_{jj} > 0$. \square

The following lemma helps in determining whether a product of stochastic matrices is a scrambling matrix.

Lemma 5: Let M_1, \dots, M_{n-1} be $n-1$ stochastic matrices with positive diagonal entries. Suppose there exists a node $i_0 \in V$ such that i_0 is a root of \mathcal{G}_{M_τ} for all $\tau = 1, \dots, n-1$. Then $M_{n-1} \dots M_1$ is a scrambling matrix.

Proof: Since i_0 is a root of \mathcal{G}_{M_1} , at least one node i_1 exists different from i_0 such that $(i_0, i_1) \in \mathcal{E}_{M_1}$. This immediately implies $[M_1]_{i_1 i_0} > 0$ according to the definition of induced graph.

Further, according to Lemma 4, we have $[M_2 M_1]_{i_1 i_0} > 0$ resulting from $[M_1]_{i_1 i_0} > 0$. Since i_0 is also a root of \mathcal{G}_{M_2} , there must be a node i_2 different with i_0 and i_1 such that there is at least one arc leaving from $\{i_0, i_1\}$ entering i_2 in \mathcal{G}_{M_2} . There will be two cases.

- (i) When $(i_0, i_2) \in \mathcal{E}_{M_2}$, there holds $[M_2]_{i_2 i_0} > 0$. This implies $[M_2 M_1]_{i_2 i_0} > 0$ based on Lemma 4.
- (ii) When $(i_1, i_2) \in \mathcal{E}_{M_2}$, there holds $[M_2]_{i_2 i_1} > 0$. In this case we have

$$[M_2 M_1]_{i_2 i_0} = \sum_{\tau=1}^n [M_2]_{i_2 \tau} [M_1]_{\tau i_0} \geq [M_2]_{i_2 i_1} [M_1]_{i_1 i_0} > 0.$$

We recursively carry out the above analysis, and then i_3, \dots, i_{n-1} can be found with $V = \{i_0, \dots, i_{n-1}\}$ such that

$$[M_{n-1} \dots M_1]_{i_m i_0} > 0 \quad (21)$$

for all $m = 0, 1, \dots, n-1$. According to the definition of $\delta(\cdot)$, (21) implies

$$\lambda(M_{n-1} \dots M_1) \leq 1 - \min_{m=0, \dots, n-1} [M_{n-1} \dots M_1]_{i_m i_0} < 1.$$

The desired lemma follows. \square

B. Proof of Theorem 3: Convergence

This subsection presents the proof of the conclusion on a.s. consensus in Theorem 3. We only need to show the sufficiency part. Note that global a.s. consensus of (1) is equivalent to

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} \delta(W(k) \cdots W(0)) = 0\right) = 1.$$

We define $\Psi_k = \mathbf{I}_{\sum_{i=1}^n \chi_i^k}$ with \mathbf{I} representing the indicator function. In other words,

$$\Psi_k = \begin{cases} 1, & \text{if at least one } \chi_i(k) = 1 \text{ at } k; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have $\Psi_k = 1$ with probability $1 - (1 - P_k)^n$ and $\Psi_k = 0$ with probability $(1 - P_k)^n$. Moreover, Ψ_0, Ψ_1, \dots are independent. Applying the Borel-Cantelli Lemma to $\langle \chi_i(k) \rangle, i = 1, \dots, n$ and invoking the independence Ψ_k we conclude the following property of Ψ_k .

Lemma 6: $\mathbf{P}(\Psi_k = 1 \text{ for infinitely many } k) = 1$ if and only if $\sum_{k=0}^{\infty} P_k = \infty$.

Noting the fact that

$$1 - ny \leq (1 - y)^n$$

for all $y \in [0, 1]$ and $n \geq 1$, there holds for all k that

$$1 - (1 - P_k)^n \leq nP_k.$$

As a result, we obtain

$$\mathbf{P}(\chi_i(k) = 1 \mid \Psi_k = 1) = \frac{P_k}{1 - (1 - P_k)^n} \geq \frac{P_k}{nP_k} = \frac{1}{n} \quad (22)$$

for all i and k .

Next, we introduce the following sequence (known as the Bernoulli sequence of $\langle \Psi_k \rangle$):

$$\zeta_1 < \dots < \zeta_m < \zeta_{m+1} < \dots, \quad (23)$$

where ζ_m is the m 'th time when $\Psi_k = 1$. The ζ_m are stopping times of the sequence $\langle \Psi_k \rangle$, and Lemma 5 ensures that each ζ_m is almost surely finite when $\sum_{k=0}^{\infty} P_k = \infty$. For any $(i, j) \in \mathcal{E}^\dagger$, there hold for all $m = 1, 2, \dots$ that

$$\begin{aligned} & \mathbf{P}((i, j) \in G_{W(\zeta_m)}) \\ & \stackrel{a)}{=} \mathbf{P}(\chi_j(\zeta_m) = 1 \text{ and } (i, j) \in G_{\zeta_m}) \\ & \stackrel{b)}{=} \mathbf{P}(\chi_j(\zeta_m) = 1) \cdot \mathbf{P}((i, j) \in G_{\zeta_m}) \\ & \stackrel{c)}{\geq} \frac{q_*}{n}, \end{aligned} \quad (24)$$

where $a)$ is from the structure of the algorithm, $b)$ is from the independence between the random graph process and the $\chi_i(k)$, and $c)$ holds from (22).

Denote $Q_1 = W(\zeta_{|\mathcal{E}^\dagger|}) \cdots W(\zeta_2)W(\zeta_1)$, where $|\mathcal{E}^\dagger|$ represents the number of elements in \mathcal{E}^\dagger . From (24), we can pick up $(i_\tau, j_\tau), \tau = 1, \dots, |\mathcal{E}^\dagger|$ as all the arcs in \mathcal{E}^\dagger , and the arc-independence (cf., the ζ_m are stopping times of i.i.d. random process $\langle \Psi_k \rangle$, and the random graph process $\langle G_k \rangle$ is independent with the Ψ_k) leads to

$$\mathbf{P}\left((i_\tau, j_\tau) \in G_{W(\zeta_\tau)}, \tau = 1, \dots, |\mathcal{E}^\dagger|\right) \geq \left(\frac{q_*}{n}\right)^{|\mathcal{E}^\dagger|},$$

which yields

$$\mathbf{P}\left(\mathcal{G}^\dagger \subseteq G_{Q_1}\right) \geq \mathbf{P}\left(\mathcal{G}^\dagger \subseteq \left(\bigcup_{\tau=1}^{|\mathcal{E}^\dagger|} G_{W(\zeta_\tau)}\right)\right) \geq \left(\frac{q_*}{n}\right)^{|\mathcal{E}^\dagger|}$$

according to Lemma 4.

We continue to define $Q_m = W(\zeta_{m|\mathcal{E}^\dagger|}) \cdots W(\zeta_{(m-1)|\mathcal{E}^\dagger|+1})$ for $m = 2, 3, \dots$, and similarly

$$\mathbf{P}\left(\mathcal{G}^\dagger \subseteq G_{Q_m}\right) \geq \left(\frac{q_*}{n}\right)^{|\mathcal{E}^\dagger|}$$

for all m . Because \mathcal{G}^\dagger is quasi-strongly connected, Lemma 5 yields

$$\mathbf{P}\left(\lambda(Q_{n-1} \cdots Q_1) < 1\right) \geq \left(\frac{q_*}{n}\right)^{(n-1)|\mathcal{E}^\dagger|}. \quad (25)$$

Moreover, $Q_{n-1} \cdots Q_1$ represents a product of $(n-1)|\mathcal{E}^\dagger|$ stochastic matrices, each of which satisfies the weights rule. Therefore, there holds

$$[Q_{n-1} \cdots Q_1]_{ij} \geq \eta^{(n-1)|\mathcal{E}^\dagger|} \quad (26)$$

for every nonzero entry $[Q_{n-1} \cdots Q_1]_{ij}$ of $Q_{n-1} \cdots Q_1$.

We see from (25) and (26) that

$$\mathbf{P}\left(\lambda(Q_{n-1} \cdots Q_1) \leq 1 - \eta^{(n-1)|\mathcal{E}^\dagger|}\right) \geq \left(\frac{q_*}{n}\right)^{(n-1)|\mathcal{E}^\dagger|}.$$

Denoting $U_m = Q_{m(n-1)} \cdots Q_{(m-1)(n-1)+1}$ for $m = 1, 2, \dots$, we can now further conclude for all $m = 1, 2, \dots$ that

$$\mathbf{P}\left(\lambda(U_m) \leq 1 - \eta^{(n-1)|\mathcal{E}^\dagger|}\right) \geq \left(\frac{q_*}{n}\right)^{(n-1)|\mathcal{E}^\dagger|}.$$

Thus, based on Lemma 3, we have

$$\lim_{m \rightarrow \infty} \mathbf{E}\left[\delta(U_m \cdots U_1)\right] \leq \lim_{m \rightarrow \infty} \mathbf{E}\left[\prod_{\tau=1}^m \lambda(U_m)\right] = 0, \quad (27)$$

Then from Fatou's Lemma we have

$$\mathbf{P}\left(\lim_{m \rightarrow \infty} \delta(U_m \cdots U_1) = 0\right) = 1.$$

which leads to

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} \delta(W(k) \cdots W(0)) = 0\right) = 1$$

since $W(k) = I_n$ for any $k \notin \{\zeta_1, \zeta_2, \dots\}$ (I_n is the identity matrix). Hence, we have proved global a.s. consensus for Algorithm (1).

C. Proof of Theorem 3: Computation Time

In this subsection, we establish the upper bound of $\mathcal{T}_{\text{com}}(\epsilon)$ given in Theorem 3.

Denote $\Theta_k = W(k-1) \dots W(0)$. For all i, j and τ , there holds

$$\begin{aligned} & |x_i(k) - x_j(k)| \\ &= \left| \sum_{\alpha=1}^n [\Theta_k]_{i\alpha} x_\alpha(0) - \sum_{\alpha=1}^n [\Theta_k]_{j\alpha} x_\alpha(0) \right| \\ &= \left| \sum_{\alpha=1}^n [\Theta_k]_{i\alpha} (x_\alpha(0) - x_\tau(0)) \right. \\ &\quad \left. - \sum_{\alpha=1}^n [\Theta_k]_{j\alpha} (x_\alpha(0) - x_\tau(0)) \right| \\ &\leq \sum_{\alpha=1}^n |[\Theta_k]_{i\alpha} - [\Theta_k]_{j\alpha}| \cdot \max_{\alpha} |x_\alpha(0) - x_\tau(0)| \\ &\leq n\delta(\Theta_k) \cdot \max_{\alpha} |x_\alpha(0) - x_\tau(0)|. \end{aligned}$$

Therefore, we obtain that for all $k \geq 1$,

$$\mathcal{H}(k) \leq n\delta(\Theta_k)\mathcal{H}(0). \quad (28)$$

Introduce

$$\zeta_k := \sum_{i=0}^{k-1} \Psi_i,$$

for $k = 0, 1, \dots$. Then there holds

$$\zeta_k = \max \{m : \zeta_m \leq k-1\}.$$

Denote $E_0 = (n-1)|\mathcal{E}^\dagger|$. Then according to Lemma 3 and applying Markov's Inequality, (28) implies

$$\begin{aligned} \mathbf{P}\left(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\right) &\leq \mathbf{P}\left(\delta(\Theta_k) \geq \frac{\epsilon}{n}\right) \\ &\leq \frac{n}{\epsilon} \mathbf{E}\left[\delta(\Theta_k)\right] \\ &\leq \frac{n}{\epsilon} \mathbf{E}\left[\lambda_{\lfloor \frac{\zeta_k}{E_0} \rfloor} \dots \lambda_1\right], \end{aligned} \quad (29)$$

where $E_0 = (n-1)|\mathcal{E}^\dagger|$, and by definition $\lambda_m = \lambda(U_m)$ with U_m introduced in the previous subsection. Here $\lfloor z \rfloor$ represents the largest integer no greater than z .

From (25) we conclude that

$$\mathbf{E}[\lambda_m] \leq 1 - \left(\frac{\eta q_*}{n}\right)^{E_0}$$

for all m . This further yields

$$\begin{aligned} \mathbf{E}\left[\lambda_{\lfloor \frac{\zeta_k}{E_0} \rfloor} \dots \lambda_1\right] &= \mathbf{E}\left[\mathbf{E}\left[\lambda_{\lfloor \frac{\zeta_k}{E_0} \rfloor} \dots \lambda_1 \mid \zeta_k\right]\right] \\ &\leq \left(1 - \left(\frac{\eta q_*}{n}\right)^{E_0}\right)^{\mathbf{E}\left[\lfloor \frac{\zeta_k}{E_0} \rfloor\right]} \\ &\leq \left(1 - \left(\frac{\eta q_*}{n}\right)^{E_0}\right)^{\sum_{i=0}^{k-1} \mathbf{E}[\Psi_i]/E_0-1} \\ &= \left(1 - \left(\frac{\eta q_*}{n}\right)^{E_0}\right)^{\sum_{i=0}^{k-1} [1-(1-P_i)^n]/E_0-1}. \end{aligned} \quad (30)$$

Combining (29) and (30), we obtain

$$\mathbf{P}\left(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\right) \leq \frac{n}{\epsilon} \left(1 - \left(\frac{\eta q_*}{n}\right)^{E_0}\right)^{\sum_{i=0}^{k-1} [1-(1-P_i)^n]/E_0-1},$$

based on which the desired upper bound of $\mathcal{T}_{\text{com}}(\epsilon)$ follows immediately from some simple algebra.

IV. EXAMPLES, MIXING, AND GOSSIPING

In this section, we first investigate a few basic random graph models from the literature discussing the applicability of the previously established results. Next, we extend the concepts of connectivity-independence and arc-independence to **-mixing* of random graphs, and establish the corresponding a.s. consensus convergence result. An immediate advantage of this extension is that the majority of random graph models is covered by **-mixing* random graphs. Finally, we look further into random gossiping algorithms and present results on a.s. finite-time consensus for this important class of random algorithms.

A. Examples

We discuss a few random graph examples and show the applicability of the connectivity and arc independence. Apparently all independent random graph processes are immediately connectivity independent, but not necessarily arc independent.

Let \mathbf{K}_n denote the complete digraph with n nodes, i.e., $(i, j) \in \mathbf{K}_n$ for all $i \neq j \in \mathbf{V}$.

Example 1 (Erdős-Rényi): The random graph process $\langle G_k \rangle$ is a sequence of i.i.d. bidirectional or directed Erdős-Rényi random graphs. A bidirectional Erdős-Rényi graph is a random graph over node set \mathbf{V} such that independently with a probability p , there is a bidirectional edge between every pair of nodes in \mathbf{V} . A directed Erdős-Rényi graph is defined in that independently with a probability p , there is a directed arc (i, j) for every ordered pair (i, j) of nodes in \mathbf{V} . Consensus dynamics over Erdős-Rényi random graphs were studied in [21] and [22].

Both the directed and bidirectional Erdős-Rényi graph processes are arc-independent with respect to the basic graph \mathbf{K}_n . Note that arc appearance in a fixed bidirectional Erdős-Rényi random graph G_k is not independent, since $(i, j) \in E_k$ implies $(j, i) \in E_k$.

Example 2 (Random Link Failure): Let $\mathcal{G}^u = (\mathbf{V}, \mathcal{E}^u)$ be an underlying digraph representing fundamentally who can possibly talk to whom in the network. Independently at different time instants and among different arcs in \mathcal{E}^u , $(i, j) \in E_k$ with a given probability $p_{ij} > 0$. In this way we obtain an i.i.d. random graph process $\langle G_k \rangle$. This model describes independent random failure in node communications, and consensus algorithm with random link failure is studied in [24] and [26]–[28]. Then we see $\langle G_k \rangle$ given by this random link failure process is arc-independent with respect to the basic graph \mathcal{G}^u .

Example 3 (Markovian Random Graph): The random graph process $\langle G_k \rangle$ is generated by a Markov chain whose state space is (or, a subset of) \mathcal{G} . Such examples were studied in [19] and [29]. Since Markovian random graphs introduce local dependence in time, in general it will be difficult for $\langle G_k \rangle$ to be either connectivity or arc independent. However, if the Markov chain is defined on the subset of quasi-strongly connected digraphs in \mathcal{G} , then connectivity-independence holds trivially for $\langle G_k \rangle$.

Example 4 (Random Gossiping): Independently at each time step, a pair of nodes are randomly selected for averaging their states, called *gossiping*. Comprehensive analysis for random gossiping algorithms is established in [2], and gossiping algorithm also serves as a model for social belief propagation [10], [11]. An independent random gossiping process $\langle G_k \rangle$ is arc-independent and trivially connectivity-independent (the probability of G_k being quasi-strongly connected equals zero if $n > 2$, and one if $n = 2$). We can consider the random joint graphs $G([mT, (m+1)T - 1])$ over time intervals $[mT, (m+1)T - 1]$ for $m \in \mathbb{N}$, in which $\langle G([mT, (m+1)T - 1]) \rangle$ is both connectivity and arc independent, and for sufficiently large integer T the condition of Proposition 1 will be satisfied.

B. Connectivity-Mixing and Arc-Mixing Random Graphs

In the previous subsection, we have illustrated that the concepts of connectivity/arc independence can be applied to many independent random graph models, but become restrictive with non-independent, e.g., Markovian random graph processes. We now propose an alternative method from *mixing* theory, which, inherits the spirit of a network version of Borel-Cantelli lemma in the presence of connectivity/arc independence (and many our analysis techniques smoothly apply), however allows for possible weakening of the independence assumption.

1) *Arc/Connectivity *-Mixing:* First we recall some basic concepts and results from mixing of random processes (see [47] for a comprehensive survey). Let $\langle X_k \rangle$ be a sequence of random variables, where each X_k is taken from a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the ease of presentation we continue to use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote the product probability space where $(X_k, k \in \mathbb{N})$ lies in, the existence of which is guaranteed by the Kolmogorov's extension theorem [35]. The σ -algebra generated by X_a, \dots, X_b is denoted as \mathcal{F}_a^b for $0 \leq a \leq b \leq \infty$. The tail σ -algebra of $\langle X_k \rangle$, denoted \mathcal{T} , is defined by $\mathcal{T} = \bigcap_{m=0}^{\infty} \mathcal{F}_m^{\infty}$. Events belonging to \mathcal{T} are called *tail events*. For a sequence of events $\langle \mathcal{A}_k \rangle$ with $\mathcal{A}_k \in \mathcal{F}$, we define $I_{\mathcal{A}_k}$ as their indicators, i.e., $I_{\mathcal{A}_k}(\omega) = 1$ if $\omega \in \mathcal{A}_k$ and $I_{\mathcal{A}_k}(\omega) = 0$ otherwise for all $k \in \mathbb{N}$. Then $\langle I_{\mathcal{A}_k} \rangle$ is a sequence of random variables over $(\Omega, \mathcal{F}, \mathbb{P})$. We quote the following definition from [46].⁴

Definition 5: A sequence of random variables $\langle X_k \rangle$ is called **-mixing* if there exists a non-increasing sequence of real numbers $\langle f_m \rangle$ with $\lim_{m \rightarrow \infty} f_m = 0$, such that there holds

$$|\mathbb{P}(\mathcal{A} \cap \mathcal{B}) - \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})| \leq f_m \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$$

for all $\mathcal{A} \in \mathcal{F}_0^t$, $\mathcal{B} \in \mathcal{F}_{t+m}^{\infty}$, and $t \in \mathbb{N}$. A sequence of events $\langle \mathcal{A}_k \rangle$ is **-mixing* if $\langle I_{\mathcal{A}_k} \rangle$ is **-mixing*.

Associated with $\langle G_k \rangle$, we let $\langle C_k \rangle$ be the sequence of connectivity events defined by

$$C_k := \left\{ \omega : G_k(\omega) \text{ is quasi-strongly connected} \right\}, \quad k \in \mathbb{N}.$$

⁴There are various types of mixing [47]. The **-mixing* originated in [46] we quote in this paper corresponds to the ψ -mixing in [47].

Similarly, take a deterministic digraph $\mathcal{G}^\dagger = (\mathbb{V}, \mathcal{E}^\dagger)$ with $\mathcal{E}^\dagger = \{(i_\tau, j_\tau) : \tau = 1, \dots, |\mathcal{E}^\dagger|\}$. Denote

$$\mathcal{E}_k(\tau) := \{(i_\tau, j_\tau) \in G_k\}$$

for $\tau = 1, \dots, |\mathcal{E}^\dagger|$ and $k \in \mathbb{N}$. We can now introduce the following concepts of **-mixing* for the random graph process $\langle G_k \rangle$.

Definition 6: A random graph process $\langle G_k \rangle$ is termed

- (i) *connectivity *-mixing* if $\langle C_k \rangle$ is **-mixing*;
- (ii) *arc *-mixing* with respect to $\mathcal{G}^\dagger = (\mathbb{V}, \mathcal{E}^\dagger)$ if $\langle \mathcal{E}_k(\tau_k) \rangle$ is **-mixing* for any deterministic sequence $\langle \tau_k \rangle$ with each τ_k taking value from $\{1, \dots, |\mathcal{E}^\dagger|\}$.

2) *Consensus for *-Mixing Random Graphs:* We now present our main consensus convergence result for random graph processes that are connectivity or arc **-mixing*.

Theorem 4: Let $\langle G_k \rangle$ be a random graph process equipped with a probability measure \mathbb{P} .

- (i) Assume that $P_{k+1} \leq P_k$. Suppose there exist an integer $B \geq 1$ and a constant $0 < q \leq 1$ such that
 - $\langle G([mB, (m+1)B - 1]) \rangle$ is connectivity **-mixing*;
 - $\mathbb{P}(G([mB, (m+1)B - 1]) \text{ is quasi-strongly connected}) \geq q$ for all k .

Then Algorithm (1) achieves global a.s. consensus if $\sum_{s=0}^{\infty} P_k^{n-1} = \infty$.

- (ii) Assume that $\langle G_k \rangle$ is arc **-mixing* with respect to \mathcal{G}^\dagger . Suppose there is a constant $0 < q \leq 1$ such that $\mathbb{P}((i, j) \in G_k) \geq q$ for all $(i, j) \in \mathcal{E}^\dagger$ and all $k \in \mathbb{N}$. Then Algorithm (1) achieves global a.s. consensus if and only if $\sum_{s=0}^{\infty} P_k = \infty$.

This theorem is proved following the same lines as the proofs of Proposition 1 and Theorem 3 for connectivity-independent and arc-independent graphs, where in the argument whenever necessary, we replace the standard Borel-Cantelli Lemma for a sequence of independent events with the corresponding Borel-Cantelli Lemma for **-mixing* sequence of events established in [46]. The proof of Theorem 4 is in Appendix B.

On the other hand, the bounds of $\mathcal{T}_{\text{com}}(\epsilon)$ established in Proposition 1 and Theorem 3 no longer apply to the **-mixing* random graphs. The reason is that the recursion of $\mathbf{E}[\mathcal{H}(k)]$ with arc/connectivity independence fails to stand under **-mixing*, and therefore we cannot obtain the bounds of $\mathcal{T}_{\text{com}}(\epsilon)$ from the Markov's inequality. We however would like to point out that making use of the Strong Law of Large Numbers for **-mixing* sequence of random variables (see [46, Th. 1]), one can still build upper bounds for $\mathcal{T}_{\text{com}}(\epsilon)$ under the **-mixing* conditions in Theorem 4. Apparently in this case $\mathcal{T}_{\text{com}}(\epsilon)$ relies not only on the presence of **-mixing*, but also on the speed of **-mixing*, i.e., the f_m in Definition 5.

3) *An Illustrative Example:* It is known that for a strictly stationary, finite-state Markov chain, it is **-mixing* if and only if it is irreducible and aperiodic (see [47, Th. 3.1]). Then it becomes immediate that Theorem 4 is applicable to a large class of Markovian random graph processes $\langle G_k \rangle$. We give an example below.

Example 5 (Random Walk on Graphs): We define the random graph process $\langle G_k \rangle$ as follows. Let $\mathcal{G}^1 = (\mathbb{V}, \mathcal{E}^1)$ be a strongly connected digraph for node interactions. Let $\langle s_k \rangle$ be

an irreducible and aperiodic Markov chain with state space V and state transition matrix $P = [P_{ij}]$. Then $\langle s_k \rangle$ gives a random walk on the graph \mathcal{G}_P , where \mathcal{G}_P is the induced graph of matrix P . The random graph G_k is defined as $E_k = \{j : (j, i) \in \mathcal{E}^I\}$ if $s_k = i$. Algorithms of this type have been used in networked sub-gradient algorithms for convex optimization [19].

Let $\langle s_k \rangle$ be initialized at its invariant distribution and take $B_0 = |\mathcal{E}^I|$. It is then straightforward to conclude that $\langle G([mB_0, (m+1)B_0 - 1]) \rangle$ is connectivity $*$ -mixing and there exists $q_0 > 0$ such that $\mathbb{P}(G([mB, (m+1)B - 1]) \text{ is quasi-strongly connected}) \geq q_0$ for all k . Moreover, $\langle G([mB_0, (m+1)B_0 - 1]) \rangle$ is also arc $*$ -mixing with respect to \mathcal{G}^I and there also exists $q'_0 > 0$ such that $\mathbb{P}((i, j) \in E([mB, (m+1)B - 1])) \geq q'_0$ for all $(i, j) \in \mathcal{E}^I$ and for all k . Therefore, Theorem 4 is directly applicable.

C. Random Gossiping

Random gossiping is an important model in engineering (computer-to-computer, wireless communication) [2] and social networks [10]. The applicability of the established results to random gossiping algorithms has been discussed in Example 4. In standard random gossip algorithms, interactions always happen pairwise and the two interacting nodes average their states during their interaction. This simple nature of random gossiping certainly admits deeper results beyond the previous discussions. In this subsection, we make a further investigation to random gossiping algorithms.

1) *Gossiping Algorithms*: Let $S = [S_{ij}]$ be an $n \times n$ stochastic matrix. A gossip algorithm is given by a node pair selection process [2], [10].

Definition 7: Independently at each time $k \geq 0$, (i) a node $i \in V$ is drawn with probability $1/n$; (ii) the selected node i picks the pair $\{i, j\}$ with probability S_{ij} .

This random pair selection process generates a sequence of i.i.d. random graphs $\langle G_k \rangle$, where

$$E_k = \{(i, j), (j, i)\}$$

when node pair $\{i, j\}$ is selected. When $E_k = \{(i, j), (j, i)\}$, Algorithm (1) under this gossiping interaction reads as

- (i) $x_i(k+1) = x_i(k)/2 + x_j(k)/2$ with probability P_k , and $x_i(k+1) = x_i(k)$ with probability $1 - P_k$;
- (ii) Independent with node i , $x_j(k+1) = x_i(k)/2 + x_j(k)/2$ with probability P_k , and $x_j(k+1) = x_j(k)$ with probability $1 - P_k$;
- (iii) $x_m(k+1) = x_m(k)$ for all $m \notin \{i, j\}$.

This model generalizes the standard gossiping algorithm in [2] with $\langle P_k \rangle$ characterizing possible communication failure in every gossiping pairs. Let the induced graph of S , \mathcal{G}_S , be quasi-strongly connected. We can apply Theorem 3 and directly conclude that if and only if $\sum_{k=0}^{\infty} P_k = \infty$, the above random gossiping algorithm converges almost surely. On the other hand, considering the following random graph process

$$\langle G([m|\mathcal{E}_S|, (m+1)|\mathcal{E}_S| - 1]) \rangle,$$

there holds

$$\mathbb{P}\left(G([m|\mathcal{E}_S|, (m+1)|\mathcal{E}_S| - 1]) \text{ is quasi-strongly connected}\right) \geq (S_*/n)^{|\mathcal{E}_S|}$$

with $S_* = \min\{S_{ij} + S_{ji} : S_{ij} + S_{ji} > 0\}$. Therefore, Theorem 2 is also applicable to random gossiping process.

This structure of E_k in this random gossiping process however allows us to go beyond the asymptotic consensus analysis. We define a.s. finite-time consensus in the following.

Definition 8: Global finite-time a.s. consensus is achieved for Algorithm (1) if for any initial value x^0 , there exists a random time $K_{x^0}(\omega) \in \mathbb{N}$ which is a.s. finite such that $\mathbf{P}_{x^0}(\mathcal{H}(K_{x^0}) = 0) = 1$. The finite-time computation time $\mathcal{T}_{\text{com}}^f$ is defined as

$$\mathcal{T}_{\text{com}}^f = \sup_{x^0 \in \mathbb{R}^n} \min_k \{k \in \mathbb{N} : \mathcal{H}(k) = 0\}.$$

Note that $\mathcal{T}_{\text{com}}^f$ is by definition a random variable, in contrast to $\mathcal{T}_{\text{com}}(\epsilon)$ which is a constant.

2) *Almost Sure Finite-Time Convergence*: For finite-time convergence of random gossiping process, we present the following result.

Theorem 5: Let $\langle G_k \rangle$ be a random gossiping process defined by stochastic matrix $S = [S_{ij}]$. Suppose $\mathcal{G}_S = K_n$. Suppose $n = 2^m + r$ with $0 \leq r < 2^m$.

(i). Let $0 < P_* < 1/2$ be a constant such that $P_* \leq P_k \leq 1 - P_*$ for all k and suppose $\langle P_k \rangle$ is monotonic. Then global a.s. finite-time consensus is achieved if $\sum_{k=0}^{\infty} [P_k(1 - P_k)]^{2^{N_0}} = \infty$ with $N_0 = r + m(n+r)/2$. Moreover, in this case there holds

$$\mathbf{E}[\mathcal{T}_{\text{com}}^f] \leq N_0 \left(\frac{n}{P_*^2 S_*}\right)^{N_0}.$$

(ii). Let $P_k = 1$ for all k . Then global a.s. finite-time consensus is achieved if and only if there is an integer $m > 0$ such that $n = 2^m$. In fact,

- a) When $n = 2^m$, there holds

$$\mathbf{E}[\mathcal{T}_{\text{com}}^f] \leq N(n/S_*)^N$$

with $N = (n \log_2 n)/2$;

- b) When n is not some power of two (i.e., $r > 0$), we have $\mathbf{P}_{x^0}(\mathcal{H}(k) > 0, k \in \mathbb{N}) = 0$ for almost all $x^0 \in \mathbb{R}^n$ under standard Lebesgue measure.

The proof of Theorem 5 is in Appendix C. The proof is obtained by incorporating the results on finite-time convergence of deterministic gossip algorithms [44] with the Borel-Cantelli Lemma for independent sequence of events. Therefore, Theorem 5 continues to hold for a large class of Markovian gossiping processes (e.g., in Example 5, node i only selects a node j from its neighbors for gossiping when $s_k = i$), based on the same $*$ -mixing analysis as Theorem 4.

It is worth emphasizing from Theorem 5 that, interestingly enough, the random update process on one hand decelerates the asymptotic convergence of the gossiping algorithm, but on the other hand makes finite time convergence in general possible (cf., without random node updates, i.e., $P_k = 1$ for all k , a.s. finite-time convergence exists only if n is some power of two, which is a rather strong restriction to the network).

V. CONCLUSIONS

This paper investigated standard consensus algorithms coupled with randomized individual node decision-making over stochastically time-varying graphs. Each node determined its update by a sequence of Bernoulli trials with time-varying probabilities. The central aim of this work was to investigate the relation between the required level of independence of the graph process and the overall convergence. We consequently introduced connectivity-independence and arc-independence for the random graph processes. In light of the Borel-Cantelli lemma, a universal impossibility theorem showed that a.s. consensus cannot be achieved unless the sum of the success probability sequence diverges. Then a series of sufficiency conditions were given for the network to reach a global a.s. consensus under various connectivity assumptions. We established sharp threshold of almost sure consensus obeying the zero-one law. Convergence rates are established by lower and upper bounds of the ϵ -computation time. We also generalized the concepts of connectivity/arc independence to their analogues from the $*$ -mixing point of view, which covers the majority of random graph models in the literature such as Erdős-Rényi, gossiping, random walk, and Markovian random graphs. Under the $*$ -mixing setting, our convergence analysis continued to stand and almost sure consensus conditions were established. Finally, we further investigated almost sure finite-time convergence of random gossiping algorithms. Surprisingly, the random node update process plays a key role in making a.s. finite-time convergence possible.

 APPENDIX A
 PROOF OF THEOREM 1

The proof relies on a few preliminary lemmas. First of all, the following lemma is well-known (see [49, Th. 15.5, p. 300]).

Lemma 7: Suppose $0 \leq b_k < 1$ for all k . Then $\sum_{k=0}^{\infty} b_k = \infty$ if and only if $\prod_{k=0}^{\infty} (1 - b_k) = 0$.

Lemma 8: Suppose there exists a constant a_* such that $a_{ii}(k) \geq a_* > 1/2$ for all i and k . Then for all $k \geq 0$, there always holds $\mathcal{H}(k+1) \geq (2a_* - 1)\mathcal{H}(k)$.

Proof: Let $x_m(k) = h(k)$ for some $m \in V$. Then we have

$$\begin{aligned} & \sum_{j \in N_m(k)} a_{mj}(k)x_j(k) \\ &= a_{mm}(k)x_m(k) + \sum_{j \in N_m(k) \setminus \{m\}} a_{mj}(k)x_j(k) \\ &\leq a_{mm}(k)h(k) + (1 - a_{mm}(k))H(k) \\ &\leq a_*h(k) + (1 - a_*)H(k), \end{aligned}$$

which implies

$$h(k+1) \leq a_*h(k) + (1 - a_*)H(k). \quad (31)$$

A symmetric argument leads to

$$H(k+1) \geq (1 - a_*)h(k) + a_*H(k). \quad (32)$$

Based on (31) and (32), we obtain

$$\begin{aligned} & \mathcal{H}(k+1) \\ &= H(k+1) - h(k+1) \\ &\geq (1 - a_*)h(k) + a_*H(k) - [a_*h(k) + (1 - a_*)H(k)] \\ &\geq (2a_* - 1)\mathcal{H}(k). \end{aligned} \quad (33)$$

The desired conclusion follows. \square

Let \mathcal{S}_0 be a set of stochastic matrices. Consider the following (deterministic) distributed averaging algorithms

$$x(k+1) = W_k x(k), \quad (34)$$

where each $W_k \in \mathcal{S}_0$ for all k . We define

$$\begin{aligned} \mathcal{Z}_0 &\doteq \left\{ z \in \mathbb{R}^n : \exists W_0, \dots, W_s \in \mathcal{S}_0, s \geq 0 \right. \\ &\quad \left. \text{s.t. } W_s \cdots W_0 z \in \text{span}\{\mathbf{1}\} \right\}. \end{aligned}$$

Let $\mu(\cdot)$ represent the standard Lebesgue measure on \mathbb{R}^n . The following lemma from [44] holds for the finite-time convergence of averaging algorithm (34).

Lemma 9: Suppose \mathcal{S}_0 is a set with at most countable elements. Then either $\mathcal{Z}_0 = \mathbb{R}^n$ or $\mu(\mathcal{Z}_0) = 0$. In fact, if $\mathcal{Z}_0 \neq \mathbb{R}^n$, then \mathcal{Z}_0 is a union of at most countably many linear spaces whose dimensions are no larger than $n - 1$.

We are now ready to present the detailed proof of Theorem 1.

Proof of (i): Let $\mathcal{H}(0) > 0$. Then there exist at least two nodes, say i and j with different initial conditions. Considering the probability that both node i and node j remain constant: From Algorithm (1), we have

$$\begin{aligned} & \mathbf{P}\left(x_i(k+1) = x_i(k), \text{ and } x_j(k+1) = x_j(k) \text{ } k = 0, 1, \dots\right) \\ &\geq \left[\prod_{k=0}^{\infty} (1 - P_k)\right]^2 \doteq r_0^2, \end{aligned}$$

if $\sum_{k=0}^{\infty} P_k < \infty$, where $0 < r_0 = \prod_{k=0}^{\infty} (1 - P_k) < 1$ according to Lemma 7. The necessity claim holds.

Next, let $\mathcal{H}(0) > 0$ and denote by i and j two nodes satisfying $x_i(0) = h(0)$ and $x_j(0) = H(0)$, respectively. Notice that when both of them never make a state update, they will remain at all times with minimum (resp. maximum) states, because the other nodes cannot go further by making convex combinations. Hence,

$$\mathbf{P}\left(\mathcal{H}(k) > \epsilon \mathcal{H}(0)\right) \geq \mathbf{P}\left(\mathcal{H}(k) \geq \mathcal{H}(0)\right) \geq \prod_{h=0}^{k-1} (1 - P_h)^2.$$

Then the lower bound of the ϵ -computation can be easily obtained.

Proof of (ii): We rewrite Algorithm (1) into

$$x(k+1) = A_k x_k \quad (35)$$

with $A_k = [a_{ij}(k)]$ being a random matrix taking value from the set of stochastic matrices. Now that $\{a_{ij}(k) : i, j \in V, k \in \mathbb{N}\}$ contains at most countably many elements, applying Lemma 9 we conclude that one of the following two cases must happen:

a^\dagger) There exists a constant $K \geq 0$ under which a sample path $A_k(\omega)$ satisfies

$$\text{rank}(A_K(\omega) \dots A_0(\omega)) = 1.$$

This implies that $\mathcal{H}(K)(\omega) = 0$ for all initial value x^0 .

b^\dagger) For almost all initial value x^0 and all ω , $\mathcal{H}(k)(\omega) > 0$ for all $k \in \mathbb{N}$.

The First Borel-Cantelli Lemma [35] ensures that every node updates its state only for some finite times when $\sum_{k=0}^{\infty} P_k < \infty$. Therefore, it becomes immediate to see that the cases a^\dagger) and b^\dagger) correspond to cases a) and b), respectively.

Proof of (iii): Let $\mathcal{H}(0) > 0$. There holds

$$\begin{aligned} \sum_{k=0}^{\infty} P_k < \infty &\Leftrightarrow \prod_{k=0}^{\infty} (1 - P_k) > 0 \\ &\Leftrightarrow \prod_{k=0}^{\infty} (1 - P_k)^n > 0 \\ &\Leftrightarrow \sum_{k=0}^{\infty} (1 - (1 - P_k)^n) < \infty, \end{aligned} \quad (36)$$

where the last equivalence is obtained by taking $b_k = 1 - (1 - P_k)^n$ in Lemma 7. From the definition of Algorithm (1) we see that

$$\begin{aligned} &\mathbf{P}(\mathcal{H}(k+1) < \mathcal{H}(k)) \\ &\leq \mathbf{P}(\text{at least one node takes averaging at time } k) \\ &= 1 - (1 - P_k)^n. \end{aligned} \quad (37)$$

In light of Lemma 8, and applying the First Borel-Cantelli Lemma [35] on (37), it follows immediately that

$$\begin{aligned} &\mathbf{P}\left(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0\right) \\ &\leq \mathbf{P}(\mathcal{H}(k+1) < \mathcal{H}(k) \text{ for infinitely many } k) = 0. \end{aligned} \quad (38)$$

The desired conclusion follows.

We have now completed the proof. \square

APPENDIX B PROOF OF THEOREM 4

The proof is based on the following Borel-Cantelli lemma for $*$ -mixing sequence of events.

Lemma 10 [46, Lemma 6]: Let $\langle \mathcal{A}_k \rangle$ be a sequence of events which is $*$ -mixing. Then $\sum_{k=0}^{\infty} \mathbb{P}(\mathcal{A}_k) = \infty$ implies $\mathbf{P}(\limsup_{k \rightarrow \infty} \mathcal{A}_k) = 1$.

(i). Let $k_s = s(n-1)^2 B$. We introduce the following sequence of events $\langle \mathcal{E}_s \rangle$:

$$\mathcal{H}_s := \left\{ \mathcal{H}(k_{s+1}) \leq \left(1 - \frac{\eta^{(n-1)^2}}{2}\right) \mathcal{H}(k_s) \right\}$$

for $s = 0, 1, \dots$. We also define the sequence of event $\langle \mathcal{B}_m \rangle$:

$$\mathcal{B}_m := \left\{ \mathcal{G}([mB, (m+1)B-1]) \text{ is quasi-strongly connected} \right\}$$

for $m = 0, 1, \dots$. The proof of Proposition 1 implies

$$\mathcal{H}_s \supseteq \mathcal{Z}_s, \quad (39)$$

where by definition

$$\begin{aligned} \mathcal{Z}_s &:= \left(\bigcap_{m=0}^{(n-1)^2 B-1} \mathcal{B}_{s(n-1)^2 B+m} \right) \\ &\quad \cap \left\{ \text{the } i_1, \dots, i_{n-1} \in \mathbb{V} \text{ in turn update their states} \right\}. \end{aligned}$$

Noticing the facts that $\langle \mathcal{B}_m \rangle$ is $*$ -mixing, and the node updates are independent of the graph process, node states and other nodes, $\langle \mathcal{Z}_s \rangle$ is also $*$ -mixing. Moreover, with the monotonicity of $\langle P_k \rangle$, there holds

$$\mathbf{P}(\mathcal{Z}_s) \geq (q/n)^{(n-1)^2} P_{s(n-1)^2 B}^{n-1} \quad (40)$$

from the same argument as we obtain (9).

Therefore, we can apply Lemma 10 to $\langle \mathcal{Z}_s \rangle$, and conclude from (39) and (40) that if $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$, then

$$\mathbf{P}(\limsup_{s \rightarrow \infty} \mathcal{H}_s) \geq \mathbf{P}(\limsup_{s \rightarrow \infty} \mathcal{Z}_s) = 1. \quad (41)$$

Observing that the event $\limsup_{s \rightarrow \infty} \mathcal{H}_s$ means the \mathcal{H}_s happen for infinitely many s , and that $\mathcal{H}(k+1) \leq \mathcal{H}(k)$ always holds true, (41) immediately leads to global a.s. consensus for Algorithm (1). We have now completed the proof for connectivity $*$ -mixing random graphs.

(ii). Again, the proof is obtained by adapting the proof of Theorem 3 to the case of arc $*$ -mixing graphs. All we need is to adopting the above Borel-Cantelli argument after (25), instead of the original one using Fatou's Lemma. The details are therefore omitted.

We have now completed the proof. \square

APPENDIX C PROOF OF THEOREM 5

The proof is built upon some preliminary results on deterministic finite-time gossiping [44] (for a more comprehensive treatment we hereby refer to [45]). Introduce the following two sets of stochastic matrices:

$$\mathfrak{M}_n := \left\{ I_n - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j \in \mathbb{V} \right\},$$

and

$$\mathfrak{M}_n^\dagger := \left\{ I_n - \frac{e_i(e_i - e_j)^T}{2} : i, j \in \mathbb{V} \right\}.$$

The matrices in \mathfrak{M}_n and \mathfrak{M}_n^\dagger represent the network state transition matrix for the realizations of successful node pair interactions [2].

(i). We recall the following conclusion established in [44].

Lemma 11: Suppose $n = 2^m + r$ with $m \geq 0$ and $0 \leq r < 2^m$. Then there are $N_0 = r + m(n+r)/2$ matrices $M_1, \dots, M_{N_0} \in \mathfrak{M}_n \cup \mathfrak{M}_n^\dagger$ such that $\text{rank}(M_{N_0} \dots M_1) = 1$.

Now let M_1, \dots, M_{N_0} be the N_0 matrices defined in Lemma 11 from the set $\mathfrak{M}_n \cup \mathfrak{M}_n^\dagger$ with

$$\text{rank}(M_{N_0} \dots M_1) = 1.$$

Each M_k corresponds to a pair of nodes $\{i_k, j_k\}$. The probability of a given pair of nodes selected at time k is no smaller than S_*/n from the definition of the gossiping process. Depending on $M_k \in \mathfrak{M}_n$ or $M_k \in \mathfrak{M}_n^\dagger$, there are two cases: M_k is

realized when both the two nodes $\{i_k, j_k\}$ successfully update their states; M_k is realized when only one node (i_k or j_k) successfully updates its state. For either of the two cases we call the selected node pair $\{i_k, j_k\}$ *realizes* M_k . Then at time k , M_k is realized with probability either P_k^2 or $P_k(1 - P_k)$ given that $\{i_k, j_k\}$ is selected.

Consider the following events:

$$\mathcal{E}_s := \left\{ (i_k, j_k) \in E_{sN_0+k-1} \text{ and } M_k \text{ is realized,} \right. \\ \left. k = 1, \dots, N_0 \right\}, \quad s \in \mathbb{N}.$$

Note that $\text{rank}(M_{N_0} \dots M_1)$ implies that for all initial value x^0 , any \mathcal{E}_{s-1} happening will lead to $\mathcal{H}(sN_0) = 0$.

Suppose that $\langle P_k \rangle$ is non-increasing and without loss of generality we let $P_0 < 1$. Then there holds for all k that

$$P_k(1 - P_k) \geq (1 - P_0)P_k \geq (1 - P_0)P_k^2,$$

and $\sum_{k=0}^{\infty} [P_k(1 - P_k)]^{2N_0} = \infty$ leads to $\sum_{k=0}^{\infty} P_k^{2N_0} = \infty$. This gives us

$$\mathbf{P}(\mathcal{E}_{s-1}) \geq \left((1 - P_0)(S_*/n)P_{sN_0}^2 \right)^{N_0}. \quad (42)$$

Thus, invoking the Borel-Cantelli lemma, a.s. finite-time consensus is achieved for Algorithm (5) if $\sum_{k=0}^{\infty} [P_k(1 - P_k)]^{2N_0} = \infty$.

The case when $\langle P_k \rangle$ is non-decreasing holds from a symmetric argument, whose details are therefore omitted. This concludes the proof of a.s. finite-time convergence.

Next, letting $P_* \leq P_k \leq 1 - P_*$ for all k , we establish the upper bound for $\mathbf{E}[\mathcal{I}_{\text{com}}^f]$. Note that there always holds

$$\mathcal{I}_{\text{com}}^f \leq N_0 \inf \left\{ s \geq 1 : \mathcal{E}_{s-1} \text{ happens} \right\}.$$

As a result, we conclude that

$$\mathbf{E}[\mathcal{I}_{\text{com}}^f] \leq \sum_{k=1}^{\infty} N_0 k (1 - p)^{k-1} p = N_0 \sum_{k=0}^{\infty} (1 - p)^k = N_0/p,$$

where $p = (P_*^2 S_*/n)^{N_0}$ is a lower bound for the probability of \mathcal{E}_s . This proves the proposed upper bound for $\mathbf{E}[\mathcal{I}_{\text{com}}^f]$.

(ii). The proof requires the following conclusion from [44].

Lemma 12: (i) If $n = 2^m$, then there are $N = \frac{n \log_2 n}{2}$ matrices M_1, \dots, M_N with each $M_i \in \mathfrak{M}_n$ such that $M_N \dots M_1 = \mathbf{1}\mathbf{1}^T/n$.

(ii) If n is not some power of two, then for almost all initial value x^0 , there holds

$$M_t \dots M_1 x^0 \notin \text{span}\{\mathbf{1}\}$$

for all $M_1, \dots, M_t \in \mathfrak{M}_n$ and for all $t \in \mathbb{N}$.

Proof of a): Let $n = 2^m$ for some integer m . Let M_1, \dots, M_N be the $N = \frac{n \log_2 n}{2}$ matrices defined in Lemma 12 from the set \mathfrak{M}_n with $M_N \dots M_1 = \mathbf{1}\mathbf{1}^T/n$. Similarly, each M_k corresponds to a pair of nodes $\{i_k, j_k\}$, and the probability of a given pair of nodes selected at time k is no smaller than S_*/n from the definition of the gossiping process. Consider the following events:

$$\mathcal{D}_s := \left\{ (i_k, j_k) \in E_{sN+k-1}, \quad k = 1, \dots, N \right\}, \quad s \in \mathbb{N}.$$

Since $P_k = 1$ for all k , the fact that $M_N \dots M_1 = \mathbf{1}\mathbf{1}^T/n$ implies that for all initial value x^0 , any \mathcal{D}_{s-1} happening will lead to $\mathcal{H}(sN) = 0$. Therefore, a.s. finite-time consensus follows immediately from the Borel-Cantelli lemma. Moreover, there always holds

$$\mathcal{I}_{\text{com}}^f \leq N \inf \left\{ s \geq 1 : \mathcal{D}_{s-1} \text{ happens} \right\}.$$

Similarly, with $p_* = (S_*/n)^N$ being a lower bound for the probability of \mathcal{D}_s , we obtain

$$\mathbf{E}[\mathcal{I}_{\text{com}}^f] \leq N/p_*,$$

This proves the proposed upper bound for $\mathbf{E}[\mathcal{I}_{\text{com}}^f]$.

Proof of b): The desired conclusion follows directly from Lemma 12.(ii), which indeed reveals the impossibility of finite-time consensus for every sample path for almost all initial values.

We have now completed the proof of the theorem. \square

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