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Controllability of a class of networked passive linear systems

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*Abstract***— In this paper, controllability properties of networks of diffusively coupled linear systems are considered through the controllability Gramian. For a class of passive linear systems, it is shown that the controllability Gramian can be decomposed into two parts. The first part is related to the dynamics of the individual systems whereas the second part is dependent only on the interconnection topology, allowing for a clear interpretation and efficient computation of controllability properties for a class of networked systems. Moreover, a relation between symmetries in the interconnection topology and controllability is given. The results are illustrated by an example.**

I. Introduction

Large-scale interconnected systems appear in fields ranging from technology to nature and include power grids, communication networks and biological or chemical networks (see, e.g., [21], [2] for an overview). Many of these networked systems are subject to external influences, which might either be a control input or disturbance. To analyze the influence of such inputs, the controllability properties of networked systems are considered in this paper.

The study of controllability has a long history [11], [1]. Controllability of networked systems was studied in [22], where subsystems with single-integrator dynamics and diffusive coupling are considered. Further results for such systems are presented in [17], where controllability is related to graph-theoretical properties of the underlying interconnection topology. Extensions and applications of this approach are given in, e.g., [15] and [24]. Whereas the classical notion of controllability (due to Kalman [11]) is considered in these references, a different approach is taken in [13]. In [13], the notion of structural controllability, introduced in [12], is exploited to address controllability properties for networked systems in which the coupling strength is unknown, again considering subsystems with single-integrator dynamics. Structural controllability for interconnected linear systems is studied in [6].

In the current paper, classical controllability properties are considered for networked systems in which the subsystems have higher-dimensional linear dynamics (as opposed to single-integrator dynamics) as in [9]. Also, passivity properties of the subsystems are exploited to gain insight in such controllability properties. Moreover, rather than considering controllability properties directly, the *degree* of controllability will be analyzed by considering the controllability Gramian. Besides providing a more practical characterization of controllability (by characterizing the energy required to control certain states), the controllability Gramian is also instrumental in studying the effects of system noise (through the $\mathcal{H}_2$ norm) and in model order reduction [25]. For networked systems with single-integrator dynamics, some results on the controllability Gramian are given in [5].

For a class of passive subsystems (see [23] for a definition) that is closely related to *lossless* systems, it will be shown that the controllability Gramian of the networked system can be decomposed into two components. The first component is related to the controllability and observability properties of the subsystems, whereas the second component is related to the interconnection topology. This decomposition thus gives insights in the effects of the network topology on controllability properties and, moreover, provides an efficient approach towards the computation of the controllability Gramian of the networked system. Using the controllability Gramian, the effects of (a generalized form of) symmetries in the interconnection topology on controllability properties are analyzed, hereby showing that symmetries lead to an uncontrollable networked system and providing an extension of results in [17]. Finally, it is noted that many results in this paper have direct counterparts in the scope of observability.

The remainder of this paper is organized as follows. In Section II, the problem will be stated. The class of passive systems under analysis will be discussed in Section III, after which controllability of the networked system is analyzed in Section IV. The relation between symmetries in the interconnection topology and controllability is discussed in Section V. The results are illustrated by means of an example in Section VI before drawing conclusions in Section VII.

Notation. The field of real numbers is denoted by $\mathbb{R}$. For a vector $x \in \mathbb{R}^n$, the Euclidian norm is given as $|x| = \sqrt{x^{\mathrm{T}}x}$, whereas $\mathbf{1}$ denotes the column vector of all ones. A symmetric positive (semi-)definite matrix X is denoted as $X \succ 0$ ($X \succcurlyeq 0$). For matrices A and B, $A \otimes B$ represents their Kronecker product [3], which satisfies

$$(A \otimes B)(C \otimes D) = AC \otimes BD, \tag{1}$$

whenever the products AC and BD can be formed.

II. Problem setting

A network of identical subsystems $\mathbf{\Sigma}_i$ is considered, whose linear time-invariant dynamics is given as

$$\mathbf{\Sigma}_i : \begin{cases} \dot{x}_i = Ax_i + Bv_i \\ w_i = Cx_i \end{cases} \tag{2}$$

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with $x_i \in \mathbb{R}^n$, $v_i, w_i \in \mathbb{R}^m$ and $i \in \{1, \ldots, \bar{n}\}$. Throughout this paper, it is assumed that (2) is a minimal realization.

The subsystems $\mathbf{\Sigma}_i$ are interconnected via (linear) diffusive output coupling as

$$v_i = \sum_{j=1, j\neq i}^{\bar{n}} l_{ij}(w_i - w_j) + \sum_{j=1}^{\bar{m}} \gamma_{ij} u_j, \tag{3}$$

with $u_j \in \mathbb{R}^m$, $j \in \{1, \ldots, \bar{m}\}$ the external inputs to the networked system. Furthermore, the constants $l_{ij} \in \mathbb{R}$ characterize the coupling strength between the different subsystems, whereas the parameters $\gamma_{ij} \in \mathbb{R}$ describe the distribution of the external inputs amongst the subsystems.

It is assumed that the coupling strengths l_{ij} satisfy $l_{ij} \leq 0$ and $l_{ij} = l_{ji}$, where $i \neq j$. Consequently, the interconnection (3) can be associated to a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} = \{1, \ldots, \bar{n}\}$ the set of vertices representing the systems $\mathbf{\Sigma}_i$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges, satisfying, for $i \neq j$, $l_{ij} < 0$ if and only if $(i, j) \in \mathcal{E}$.

After defining $l_{ii} = -\sum_{j=1, j\neq i}^{\bar{n}} l_{ij}$, the coupling strengths l_{ij} can be collected in the matrix L as $L = \{l_{ij}\}$. This matrix is known as the (weighted) graph Laplacian and satisfies $L = L^{\mathrm{T}} \succcurlyeq 0$ and $L\mathbf{1} = 0$ [8]. Then, by introducing Γ as $\Gamma = \{\gamma_{ij}\}$, the interconnection (3) can be written as

$$v = -(L \otimes I)w + (\Gamma \otimes I)u \tag{4}$$

with $v^{\mathrm{T}} = [\, v_1^{\mathrm{T}} \ \ldots \ v_{\bar{n}}^{\mathrm{T}} \,]$ and $w^{\mathrm{T}} = [\, w_1^{\mathrm{T}} \ \ldots \ w_{\bar{n}}^{\mathrm{T}} \,]$. By combining the subsystem dynamics (2) with the interconnection topology (4), the networked system is given by

$$\mathbf{\Sigma}\colon\ \dot{x} = (I \otimes A - L \otimes BC)x + (\Gamma \otimes B)u = \bar{A}x + \bar{B}u, \tag{5}$$

with $x^{\mathrm{T}} = [\, x_1^{\mathrm{T}} \ \ldots \ x_{\bar{n}}^{\mathrm{T}} \,] \in \mathbb{R}^{\bar{n}n}$ and $u^{\mathrm{T}} = [\, u_1^{\mathrm{T}} \ \ldots \ u_{\bar{m}}^{\mathrm{T}} \,] \in \mathbb{R}^{\bar{m}m}$. Here, $\bar{A}$ and $\bar{B}$ are defined as $\bar{A} := I \otimes A - L \otimes BC$ and $\bar{B} := \Gamma \otimes B$, respectively.

In this paper, stability and controllability properties of the networked system $\mathbf{\Sigma}$ as in (5) are of interest. In particular, the controllability Gramian $\bar{P}$ will be considered, as the Gramian provides (for asymptotically stable $\mathbf{\Sigma}$) a full characterization of the *degree* of controllability. Namely, when $\bar{P} \succ 0$, the controllability Gramian satisfies

$$L_o(x_0) := \inf_{u \in \mathcal{L}_2^m((-\infty, 0])} \int_{-\infty}^{0} |u(t)|^2 \, \mathrm{d}t = x_0^{\mathrm{T}} \bar{P}^{-1} x_0, \tag{6}$$

where $u(\cdot)$ is an input that steers (5) from $x(-\infty) = 0$ to $x(0) = x_0$ [7]. Also, it is well-known (see, e.g., [25]) that, when $\bar{A}$ is Hurwitz, the controllability Gramian of (5) can be obtained as the unique solution of the Lyapunov equation

$$\bar{A}\bar{P} + \bar{P}\bar{A}^{\mathrm{T}} + \bar{B}\bar{B}^{\mathrm{T}} = 0. \tag{7}$$

Of course, the controllability properties of (5) can directly be obtained by computing $\bar{P}$ from (7). However, due to the potentially large size of the network and the state-space dimension of the subsystems $\mathbf{\Sigma}_i$, evaluation of (7) might be numerically infeasible. More importantly, this direct computation does not yield any insights in the structure of the controllability Gramian. Therefore, in this paper, it is analyzed to which extent the controllability Gramian $\bar{P}$ can be related to the controllability properties of the subsystems (2) and the interconnection structure (4).

III. Passive and lossless systems

It will be shown that, for a class of systems, the controllability Gramian $\bar{P}$ allows for an insightful decomposition in which properties of the subsystems (2) and interconnection (3) can be considered separately. In particular, (a class of) passive systems will be analyzed as in the following definition (see, e.g., [23], [4]).

Definition 1: A system $\mathbf{\Sigma}_i$ as in (2) is said to be passive if there exists a differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying $V \geq 0$ and a constant $\varepsilon \geq 0$ such that

$$\dot{V}(x_i) := \frac{\partial V}{\partial x_i}(x_i)\dot{x}_i \leq v_i^{\mathrm{T}} w_i - \varepsilon |w_i|^2 \tag{8}$$

holds along trajectories of (2). If $\varepsilon > 0$, the system $\mathbf{\Sigma}_i$ is said to be output strictly passive. If (8) holds with equality and $\varepsilon = 0$, then $\mathbf{\Sigma}_i$ is said to be lossless.

Furthermore, when $\mathbf{\Sigma}_i$ as in (2) is asymptotically stable, its controllability and observability Gramian can be introduced. These Gramians are denoted as P and Q, respectively, and are the unique solutions of the Lyapunov equations

$$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} = 0, \tag{9}$$

$$A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C = 0, \tag{10}$$

see, e.g., [25]. Here, it is noted that asymptotic stability and minimality of (2) guarantee that the solutions of (9) and (10) are positive definite, i.e., $P \succ 0$ and $Q \succ 0$.

The following lemma is closely related to [25, Theorem 8.3], and is therefore stated without proof.

Lemma 1: Consider the asymptotically stable system $\mathbf{\Sigma}_i$ as in (2) and let P and Q denote its controllability and observability Gramian, respectively. If P and Q satisfy $PQ = \sigma^2 I$ for some $\sigma > 0$, then there exists a unitary matrix U such that $\sigma B^{\mathrm{T}} = UCP$.

Remark 1: The eigenvalues of the product PQ equal the squared Hankel singular values of (2). The condition $PQ = \sigma^2 I$ in Lemma 1 thus implies that all Hankel singular values are identical (and equal σ). Systems satisfying this property are closely related to so-called *all-pass* systems, whose frequency response function is characterized by a constant magnitude. Details can be found in [7]. ◁

The following lemma relates the conditions in the statement of Lemma 1 to passivity as in Definition 1.

Lemma 2: Consider the conditions in the statement of Lemma 1. Then, the unitary matrix U in Lemma 1 satisfies $U = I$ if and only if $\mathbf{\Sigma}_i$ in (2) is passive.

Proof: Necessity of passivity follows directly by using $V(x_i) = \frac{1}{2\sigma} x_i^{\mathrm{T}} Q x_i$ as a candidate storage function. Namely, the differentiation of V along trajectories of (2) yields

$$\dot{V}(x_i) = -\tfrac{1}{2\sigma} x_i^{\mathrm{T}} C^{\mathrm{T}} C x_i + v_i^{\mathrm{T}} U w_i \leq v_i^{\mathrm{T}} U w_i, \tag{11}$$

where (10) and the property $B^{\mathrm{T}}Q = \sigma UC$ (which follows from Lemma 1 and the property $PQ = \sigma^2 I$) is used. Thus, V is a storage function for the supply rate $s(v_i, w_i) = v_i^{\mathrm{T}} U w_i$, which reduces to the supply rate for passivity when $U = I$.

To prove sufficiency, it is assumed that $\mathbf{\Sigma}_i$ is passive. By the Kalman-Yakubovich-Popov lemma [4], there exists a storage function $V(x_i) = x_i^{\mathrm{T}} K x_i$, where K satisfies

$$A^{\mathrm{T}}K + KA \prec 0, \quad KB = C^{\mathrm{T}}. \tag{12}$$

As (2) is a minimal realization, $K = K^{\mathrm{T}}$ is positive definite, see [23]. Combining the equality in (12) with the property $B^{\mathrm{T}}Q = \sigma UC$ leads to $B^{\mathrm{T}}Q = \sigma UB^{\mathrm{T}}K$. This yields

$$QBB^{\mathrm{T}}Q = \sigma^2 KBB^{\mathrm{T}}K, \tag{13}$$

where the property $U^{\mathrm{T}}U = I$ is used. In the remainder of the proof it will be assumed, without loss of generality, that the coordinates are chosen such that B is given as $B = [\,I\ 0\,]^{\mathrm{T}}$. After partitioning Q and K accordingly as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathrm{T}} & K_{22} \end{bmatrix}, \tag{14}$$

it can be concluded from (13) that $Q_{11}Q_{11} = \sigma^2 K_{11}K_{11}$, where it is noted that Q_{11} and K_{11} are positive definite. Consequently, the equality $Q_{11} = \sigma K_{11}$ holds (see also [10, Theorem 7.2.6]). Rewriting the quality $B^{\mathrm{T}}Q = \sigma UB^{\mathrm{T}}K$ using $B = [\,I\ 0\,]^{\mathrm{T}}$ and the partitioning (14) gives

$$\left[\, Q_{11}\ Q_{12} \,\right] = \sigma U \left[\, K_{11}\ K_{12} \,\right], \tag{15}$$

such that the result $Q_{11} = \sigma K_{11}$ and positive definiteness of K_{11} imply $U = I$, which proves sufficiency. ■

As can be concluded from Lemmas 1 and 2, the properties of having identical Hankel singular values and passivity are closely related. In the next section, systems $\mathbf{\Sigma}_i$ will be considered which satisfy both properties.

Assumption 1: The systems $\mathbf{\Sigma}_i$ as in (2) are asymptotically stable, minimal, passive as in Definition 1 and satisfy $PQ = \sigma^2 I$ for some $\sigma > 0$, where P and Q are the Gramians as in (9) and (10), respectively.

Remark 2: Systems $\mathbf{\Sigma}_i$ satisfying Assumption 1 in fact satisfy the stronger property of output strict passivity as in Definition 1. This follows from (11) in the proof of Lemma 1, for $U = I$, such that ε as in (8) satisfies $\varepsilon = \frac{1}{2\sigma} > 0$. ◁

Systems satisfying Assumption 1 allow for a realization with an insightful physical interpretation. Namely, by introducing a decomposition of the product AP in (9) as $AP = \tilde{J} + S$ with $\tilde{J} = -\tilde{J}^{\mathrm{T}}$ and $S = S^{\mathrm{T}}$ and by exploiting the property $PQ = \sigma^2 I$, it can be shown that $S = -\frac{1}{2}BB^{\mathrm{T}}$ and that (2) can be written as

$$\mathbf{\Sigma}_i : \begin{cases} \dot{x}_i = \left(J - \frac{1}{2\sigma}BB^{\mathrm{T}}\right)\left(\frac{1}{\sigma}Q\right)x_i + Bv_i, \\ w_i = B^{\mathrm{T}}\left(\frac{1}{\sigma}Q\right)x_i. \end{cases} \tag{16}$$

Here, Lemma 2 is used to relate the input and output matrices and J is given as $J = \frac{1}{\sigma}\tilde{J}$. The form (16) represents a *port-Hamiltonian* system (see, e.g., [20]), allowing for a physical interpretation. Namely, the Hamiltonian $H(x_i) = \frac{1}{2}x_i^{\mathrm{T}}\left(\frac{1}{\sigma}Q\right)x_i$ represents the total energy stored in the system, whereas the product $u^{\mathrm{T}}y$ gives the power supplied to the system. In fact, the system $\mathbf{\Sigma}_i$ can be considered as the feedback interconnection of the *lossless* passive system

$$\mathbf{\Sigma}_i^l : \begin{cases} \dot{x}_i = J\left(\frac{1}{\sigma}Q\right)x_i + Bv_i^l, \\ w_i = B^{\mathrm{T}}\left(\frac{1}{\sigma}Q\right)x_i. \end{cases} \tag{17}$$

and the static feedback $v_i^l = -\frac{1}{2\sigma}w_i + v_i$. Lossless systems are energy-conserving (i.e., they do not dissipate energy internally) and present a generalization of Hamiltonian systems, which model many laws of physics. Examples include undamped mechanical systems and electronic circuits without resistive elements. Furthermore, systems including dissipative elements can be approximated by large lossless systems [18].

IV. Controllability of Networked Systems

In this section, properties of the controllability Gramian $\bar{P}$ of the networked system $\mathbf{\Sigma}$ will be discussed. As asymptotic stability of the networked system is required in order to define the Gramian, the following lemma is stated.

Lemma 3: Consider the networked system $\mathbf{\Sigma}$ as in (5) and assume that the subsystems $\mathbf{\Sigma}_i$ are passive for some $\varepsilon \geq 0$ and that (2) is a minimal realization. Then, the networked system $\mathbf{\Sigma}$ is asymptotically stable if $\varepsilon > 0$.

Proof: The proof follows from [14]. In particular, by passivity, there exists a function V satisfying (8) for all $i \in \{1, \ldots, \bar{n}\}$. Due to minimality of $\mathbf{\Sigma}_i$, this function is positive definite (see [23]). Then, the time-differentiation of the composite function $\bar{V}(x) = \sum_{i=1}^{\bar{n}} V(x_i)$ leads to

$$\dot{\bar{V}}(x) \leq \sum_{i=1}^{\bar{n}} v_i^{\mathrm{T}} w_i - \varepsilon |w_i|^2 = v^{\mathrm{T}} w - \varepsilon |w|^2. \tag{18}$$

The substitution of (4), for $u = 0$, in (18) gives

$$\dot{\bar{V}}(x) \leq -w^{\mathrm{T}}((\varepsilon I + L) \otimes I)w \leq 0. \tag{19}$$

For $\varepsilon > 0$, the matrix $(\varepsilon I + L) \otimes I$ is positive definite (as $L = L^{\mathrm{T}} \succcurlyeq 0$). Then, asymptotic stability follows from observability of $\mathbf{\Sigma}_i$ via LaSalle's invariance principle. ■

Now, the following theorem can be stated, which shows that, the controllability Gramian can be written in a convenient form for systems satisfying Assumption 1.

Theorem 4: Consider the networked system $\mathbf{\Sigma}$ as in (5), where the subsystems $\mathbf{\Sigma}_i$ as in (2) satisfy Assumption 1. Then, $\mathbf{\Sigma}$ is asymptotically stable and the controllability Gramian of $\mathbf{\Sigma}$ can be written as

$$\bar{P} = \Xi \otimes P, \tag{20}$$

with P the controllability Gramian of the subsystems $\mathbf{\Sigma}_i$ and $\Xi = \Xi^{\mathrm{T}}$ the unique solution of

$$\left(\tfrac{1}{2}I + \sigma L\right)\Xi + \Xi\left(\tfrac{1}{2}I + \sigma L\right)^{\mathrm{T}} - \Gamma\Gamma^{\mathrm{T}} = 0. \tag{21}$$

Proof: Asymptotic stability of $\mathbf{\Sigma}_i$ follows directly from Remark 2 and Lemma 3. Consequently, the controllability Gramian $\bar{P}$ is uniquely characterized by the Lyapunov equation (7). Thus, if $\Xi \otimes P$ is shown to satisfy (7), it is in fact the controllability Gramian $\bar{P}$. Therefore, (20) is substituted in (7), along with the definitions of $\bar{A}$ and $\bar{B}$, leading to

$$(I \otimes A - L \otimes BC)(\Xi \otimes P) + (\Xi \otimes P)(I \otimes A - L \otimes BC)^{\mathrm{T}} + (\Gamma \otimes B)(\Gamma \otimes B)^{\mathrm{T}} = 0. \tag{22}$$

Then, exploiting the property (1) yields

$$\Xi \otimes (AP + PA^{\mathrm{T}}) - L\Xi \otimes BCP - \Xi L^{\mathrm{T}} \otimes PC^{\mathrm{T}}B^{\mathrm{T}} + \Gamma\Gamma^{\mathrm{T}} \otimes BB^{\mathrm{T}} = 0. \tag{23}$$

At this point, it is recalled that Assumption 1 implies, by Lemma 1 and Lemma 2, that $CP = \sigma B^{\mathrm{T}}$. The substitution of this result in (23) yields

$$-\Xi \otimes BB^{\mathrm{T}} - \sigma L\Xi \otimes BB^{\mathrm{T}} - \sigma \Xi L^{\mathrm{T}} \otimes BB^{\mathrm{T}} + \Gamma\Gamma^{\mathrm{T}} \otimes BB^{\mathrm{T}} = 0, \tag{24}$$

where the leftmost term is obtained by applying (9). Grouping terms in (24) and multiplication with -1 leads to

$$\left(\Xi + \sigma L\Xi + \sigma\Xi L^{\mathrm{T}} - \Gamma\Gamma^{\mathrm{T}}\right) \otimes BB^{\mathrm{T}} = 0. \tag{25}$$

Here, the matrix BB^{T} contains at least one non-zero element (minimality of $\mathbf{\Sigma}_i$ ensures that the pathological case $B = 0$ is excluded). However, by (21), the left-hand-side of the Kronecker product (25) equals zero. Consequently, (22) holds and the controllability Gramian $\bar{P}$ satisfies (20). ■

The characterization of the controllability Gramian in Theorem 4 provides several advantages. In particular, the result (20) gives a characterization of $\bar{P}$ in which the influence of the subsystems and the interconnection topology can be considered separately. Namely, P is the controllability Gramian of the subsystems $\mathbf{\Sigma}_i$, which can be obtained by analyzing the subsystems only and is independent of the interconnection topology. Next, the matrix Ξ directly depends, through (21), on the graph Laplacian L and the Hankel singular value σ of $\mathbf{\Sigma}_i$. Thus, Ξ can be obtained without explicitly taking the dynamics of the subsystems into account. As a result, (20) allows for a direct analysis of the influence of the interconnection topology on the controllability Gramian $\bar{P}$.

Besides providing insights in the structure of the controllability Gramian $\bar{P}$, Theorem 4 also enables a numerically efficient approach towards the computation of $\bar{P}$. Rather than computing $\bar{P}$ directly through (7), the result (20) suggest the computation of P and Ξ through the Lyapunov equations (9) and (21), respectively. As the latter two Lyapunov equations are of significantly smaller dimension than (7), this gives a large computational advantage.

Remark 3: The Lyapunov equation (21) can be associated to a linear system with system matrix $-(\frac{1}{2}I + \sigma L)$ and input matrix Γ. As the controllability properties of this system are determined by the pair (L, Γ), existing results on the relation between controllability and graph-theoretical properties (through L) as given in [17], [15], [24] apply. ◁

Remark 4: Even though it is assumed that the interconnection (4) is such that the matrix L is the graph Laplacian, the result in Theorem 4 can be shown to hold for arbitrary matrix L (as long as asymptotic stability of $\mathbf{\Sigma}$ is guaranteed).

Specifically, by replacing L by $L + D$, where $D = \mathrm{diag}\{d_i\}$ is a diagonal matrix, the result also holds when the graph $\mathcal{G}$ contains self-loops (with weights d_i). In light of the interpretation given below (17), this might also be interpreted as the coupling of *non-identical subsystems* in which the lossless part $\mathbf{\Sigma}_i^l$ as in (17) remains unchanged. ◁

Remark 5: As σ and L appear as a product in (21), it is clear that a change in the magnitude of the Hankel singular value σ has the same effect as a change in the coupling strength (i.e., replacing L by κL for some gain $\kappa > 0$). Next, let Ξ_σ denote the solution of (21) for a given σ and assume the graph is connected (such that the zero eigenvalue of L has multiplicity one). Then, it can be shown that

$$\lim_{\sigma\to\infty} \Xi_\sigma = \left(\frac{\mathbf{1}^{\mathrm{T}}\Gamma\Gamma^{\mathrm{T}}\mathbf{1}}{\bar{n}^2}\right)\mathbf{1}\mathbf{1}^{\mathrm{T}} \tag{26}$$

holds, indicating that for large Hankel singular value (or, equivalently, strong coupling) the only practically controllable direction corresponds to the case in which the subsystems $\mathbf{\Sigma}_i$ have the same trajectories (when $\Gamma^{\mathrm{T}}\mathbf{1} \neq 0$). ◁

Theorem 4 requires the conditions in Assumption 1 to guarantee the existence of the decomposition (20). However, these conditions are, to some extent, also necessary.

Lemma 5: Let there exist a matrix $\tilde{\Xi} = \tilde{\Xi}^{\mathrm{T}}$ such that the controllability Gramian $\bar{P}$ of the asymptotically stable networked system $\mathbf{\Sigma}$ as in (5) can be written as $\bar{P} = \tilde{\Xi} \otimes P$, with P the controllability Gramian of the asymptotically stable controllable subsystems $\mathbf{\Sigma}_i$ as in (2). Moreover, assume that the systems are single-input single-output ($m = 1$) and that Γ is chosen such that $L\Gamma \neq 0$. Then, one of the following relations hold:

1) $\tilde{\Xi} = \Gamma\Gamma^{\mathrm{T}}$ and $C = 0$, with C the output matrix of $\mathbf{\Sigma}_i$;
2) There exists a parameter $\tilde{\sigma} > 0$ such that $\tilde{\sigma}B^{\mathrm{T}} = CP$, $PQ = \tilde{\sigma}^2 I$ and where $\tilde{\Xi}$ satisfies

$$\left(\tfrac{1}{2}I + \tilde{\sigma}L\right)\tilde{\Xi} + \tilde{\Xi}\left(\tfrac{1}{2}I + \tilde{\sigma}L\right)^{\mathrm{T}} - \Gamma\Gamma^{\mathrm{T}} = 0. \tag{27}$$

Here, Q denotes the observability Gramian of $\mathbf{\Sigma}_i$.

Proof: By asymptotic stability of $\mathbf{\Sigma}$, the controllability Gramian $\bar{P}$ is given as the unique solution of (7). The substitution of $\bar{P} = \tilde{\Xi} \otimes P$ in (7) leads to

$$L\tilde{\Xi} \otimes BCP + \tilde{\Xi}L^{\mathrm{T}} \otimes PC^{\mathrm{T}}B^{\mathrm{T}} = (\Gamma\Gamma^{\mathrm{T}} - \tilde{\Xi}) \otimes BB^{\mathrm{T}}, \tag{28}$$

where the definitions of $\bar{A}$ and $\bar{B}$ as well as the Lyapunov equation (9) are used. It is remarked that the matrices $L\tilde{\Xi}$, $\tilde{\Xi}L^{\mathrm{T}}$ and $\Gamma\Gamma^{\mathrm{T}} - \tilde{\Xi}$ are all of the same dimension. Let α_{ij} denote the ij-th entry of $L\tilde{\Xi}$, denoted as $\alpha_{ij} := (L\tilde{\Xi})_{ij}$. Moreover, set $\beta_{ij} := (\Gamma\Gamma^{\mathrm{T}} - \tilde{\Xi})_{ij}$. Then, the ij-th block of the Kronecker product (28) reads

$$\alpha_{ij}BCP + \alpha_{ji}PC^{\mathrm{T}}B^{\mathrm{T}} = \beta_{ij}BB^{\mathrm{T}}, \tag{29}$$

where the property $\alpha_{ji} = (\tilde{\Xi}L^{\mathrm{T}})_{ij}$ is used. Clearly, in order to satisfy (28), (29) has to hold for all indices ij. Two distinct solutions of (29) can be found, which are discussed next.

A first solution to (29) is obtained when $\beta_{ij} = 0$ for all ij, which translates to $\tilde{\Xi} = \Gamma\Gamma^{\mathrm{T}}$. Then, it follows from the assumption $L\Gamma \neq 0$ that there exists indices ij such that $\alpha_{ij} = (L\tilde{\Xi})_{ij} = (L\Gamma\Gamma^{\mathrm{T}})_{ij}$ is nonzero. In this case, as $B \neq 0$ and $P \succ 0$, it follows from (29) with $\beta_{ij} = 0$ that $C = 0$, thus providing the first relation.

To obtain the second solution, it is assumed that there exists indices ij such that $\beta_{ij} \neq 0$. Here, it is noted that

$\beta_{ij} = \beta_{ji}$ due to symmetry of the matrix $\Gamma\Gamma^{\mathrm{T}} - \tilde{\Xi}$. Then, summing the equation for the block ij as in (29) and the corresponding equation for the block ji leads to

$$(\alpha_{ij}+\alpha_{ji})BCP+(\alpha_{ji}+\alpha_{ij})PC^{\mathrm{T}}B^{\mathrm{T}}=(\beta_{ij}+\beta_{ji})BB^{\mathrm{T}}, \tag{30}$$

where $\alpha_{ij} + \alpha_{ji}$ is necessarily nonzero for (30) to hold. Then, a parameter $\tilde{\sigma}$ can be defined as $\tilde{\sigma} = \frac{\beta_{ij}+\beta_{ji}}{2(\alpha_{ij}+\alpha_{ji})}$. It is remarked that $\tilde{\sigma}$ is constant (i.e., independent of the indices ij), as no consistent solution exists otherwise. Thus, the application of $\tilde{\sigma}$ in (30) yields

$$BCP + PC^{\mathrm{T}}B^{\mathrm{T}} = 2\tilde{\sigma}BB^{\mathrm{T}}, \tag{31}$$

which has a solution $\tilde{\sigma}B^{\mathrm{T}} = CP$. This solution is unique when the systems $\mathbf{\Sigma}_i$ are single-input single-output, as assumed in the statement of this lemma. To prove the relation $PQ = \tilde{\sigma}^2 I$, the observability Lyapunov equation (10) is considered. Exploiting $\tilde{\sigma}B^{\mathrm{T}} = CP$ in (10) gives

$$A^{\mathrm{T}}Q + QA + \tilde{\sigma}^2 P^{-1}BB^{\mathrm{T}}P^{-1} = 0, \tag{32}$$

where it is noted that P^{-1} exists due to the assumption of controllability of $\mathbf{\Sigma}_i$, implying $P \succ 0$. Due to asymptotic stability, the solution of (32) is unique, and a comparison with (9) shows that $Q = \tilde{\sigma}^2 P^{-1}$, proving the desired result. Finally, (27) follows as in the proof of Theorem 4. ■

The first condition in Lemma 5 corresponds to the case when there is no interconnection between the systems $\mathbf{\Sigma}_i$ (due to $C = 0$). As a result, controllability properties are determined only by the distribution of the external input u amongst the subsystems through the input matrix Γ. The second relation is more relevant and corresponds, for single-input single-output subsystems, to Assumption 1, indicating that systems satisfying this assumption are in fact the only systems for which a decomposition of the form $\bar{P} = \tilde{\Xi} \otimes P$ holds.

V. Symmetries in networked systems

The controllability Gramian $\bar{P}$ provides a full characterization of controllability of the networked system $\mathbf{\Sigma}$ and has, by Theorem 4, an insightful structure when the subsystems satisfy Assumption 1. For example, $\bar{P}$ can be used to directly find the controllable subspace $\mathcal{X}_c$ of the network (5). The next theorem shows that (a generalized form of) symmetry in the interconnection topology directly implies that $\mathbf{\Sigma}$ is uncontrollable (i.e., $\mathcal{X}_c$ is a proper subset of $\mathbb{R}^{n\bar{n}}$).

Theorem 6: Consider the asymptotically stable networked system $\mathbf{\Sigma}$ as in (5). If there exists a non-identity permutation matrix S and a matrix X such that

$$(I - S)L = X(I - S), \tag{33}$$

$$(I - S)\Gamma = 0, \tag{34}$$

then the networked system is not fully controllable. In particular, the orthogonal complement of $\mathcal{X}_c$ satisfies

$$\mathrm{range}((I - S)^{\mathrm{T}} \otimes I) \subseteq \mathcal{X}_c^{\perp}. \tag{35}$$

Proof: In order to prove the theorem, the Lyapunov controllability equation (7) is pre- and post-multiplied by

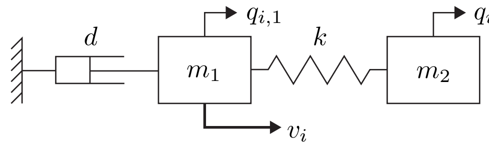

Fig. 1. Two-mass mechanical subsystem $\mathbf{\Sigma}_i$.

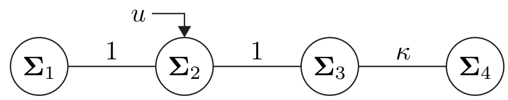

Fig. 2. Networked system $\mathbf{\Sigma}$ of size $\bar{n} = 4$ of subsystems $\mathbf{\Sigma}_i$.

$(I - S) \otimes I$ and $(I - S)^{\mathrm{T}} \otimes I$, respectively, leading to

$$0 = \big(I \otimes A - X \otimes BC\big)((I-S)\otimes I)\bar{P}((I-S)^{\mathrm{T}}\otimes I) + ((I-S)\otimes I)\bar{P}((I-S)^{\mathrm{T}}\otimes I)\big(I \otimes A - X \otimes BC\big)^{\mathrm{T}}. \tag{36}$$

Here, the relations (33) and (34) as well as the Kronecker product property (1) are used. Next, by exploiting the singular value decomposition of $(I - S)$ in the relation (33), the eigenvalues of X can be characterized. This characterization can be shown to imply stability of the matrix $(I \otimes A - X \otimes BC)$, as appears in (36). Consequently, (36) is a Lyapunov equation with a unique solution, which reads

$$((I - S) \otimes I)\bar{P}((I - S)^{\mathrm{T}} \otimes I) = 0. \tag{37}$$

It is well-known (see, e.g., [1]), that $\mathcal{X}_c = \mathrm{range}(\bar{P})$, such that the orthogonal complement satisfies $\mathcal{X}_c^{\perp} = \mathrm{null}(\bar{P}^{\mathrm{T}}) = \mathrm{null}(\bar{P})$. Symmetry of $\bar{P}$ in (37) implies that $\bar{P}((I - S)^{\mathrm{T}} \otimes I) = 0$, such that the range of $(I - S)^{\mathrm{T}} \otimes I$ forms part of the null space of $\bar{P}$, which proves the result (35). ■

An important subclass of the conditions in the statement of Theorem 6 is obtained when $X = L$ in (33), leading to

$$SL = LS. \tag{38}$$

Condition (38) represents a graph automorphism [8] and characterizes symmetry properties of a graph $\mathcal{G}$. Such symmetries are studied in the context of controllability in [17], where it is shown that (when combined to condition (34)) symmetry implies uncontrollability. Theorem 6 thus generalizes these results in two aspects. Firstly, (33) represents a relaxation with respect to (38), and, secondly, systems with higher-order internal dynamics are considered in this paper.

Remark 6: In [16], it is shown that, for autonomous networks, the condition (33) implies the existence of a invariant manifold corresponding to a partially synchronized state. The additional condition (34) in Theorem 6 basically implies that this manifold is independent of the input signal, providing a link between controllability and (partial) synchronization. ◁

VI. Illustrative example

To illustrate the results of Section IV, a network of mechanical systems as in Figure 1 is considered. After choosing the state components as $x_i = [\, m_1\dot{q}_{i,1} \;\; m_2\dot{q}_{i,2} \;\; q_{i,1} - q_{i,2} \,]^{\mathrm{T}}$

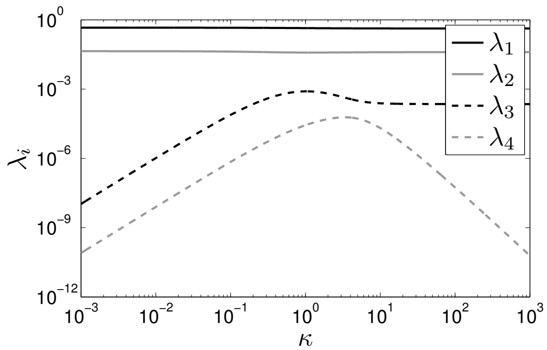

Fig. 3. Eigenvalues $\lambda_i(\Xi)$ for varying coupling strength κ.

and output $w_i = \dot{q}_{i,1}$, the dynamics of $\mathbf{\Sigma}_i$ can be written in the form (16), where $\sigma = (2d)^{-1}$ and

$$J = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \ \tfrac{1}{\sigma}Q = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & k \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (39)$$

Four subsystems $\mathbf{\Sigma}_i$ are coupled as in (3) according to a path graph, see Figure 2. Here, the coupling strengths are given as $l_{12} = l_{21} = l_{23} = l_{32} = 1$ and $l_{34} = l_{43} = \kappa$, whereas an external input is applied to the second system, such that $\Gamma = [\,0\ 1\ 0\ 0\,]^{\mathrm{T}}$. As $\mathbf{\Sigma}_i$ as in (39) satisfies Assumption 1, the result of Theorem 4 applies and the controllability Gramian $\bar{P}$ is given as, for $d = 1$ ($\sigma = \frac{1}{2}$) and $\kappa = 1$,

$$\bar{P} = \Xi \otimes P = \frac{1}{10} \begin{bmatrix} 0.478 & 0.956 & 0.382 & 0.149 \\ 0.956 & 3.920 & 0.804 & 0.214 \\ 0.382 & 0.804 & 0.306 & 0.116 \\ 0.149 & 0.214 & 0.116 & 0.058 \end{bmatrix} \otimes P, \quad (40)$$

with $P = \sigma^2 Q^{-1}$ and Q as in (39). Because of the partitioning (40), the effect of the coupling strength κ can be assessed by analyzing Ξ only. In particular, the eigenvalues of $\bar{P}$ are given as the products $\lambda_i(\Xi)\lambda_j(P)$, with $\lambda_i(\Xi)$, $i \in \{1, \ldots, \bar{n}\}$ and $\lambda_j(P)$, $j \in \{1, \ldots, n\}$ the eigenvalues of Ξ and P, respectively [3]. Therefore, the eigenvalues $\lambda_i(\Xi)$ for varying κ are shown in Figure 3. Here, it can be seen that the networked system becomes (practically) uncontrollable for small or large values of κ. Namely, for small κ, subsystem $\mathbf{\Sigma}_4$ becomes almost uncoupled from the network, making it very hard to control. Moreover, the remaining networked system with subsystems $\mathbf{\Sigma}_1$ to $\mathbf{\Sigma}_3$ is symmetric with respect to the input location, such that the difference between subsystems $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_3$ is very hard to control as well. On the other hand, for large κ, the subsystems $\mathbf{\Sigma}_3$ and $\mathbf{\Sigma}_4$ are so strongly coupled that they are very hard to control independently.

VII. Conclusions

In this paper, controllability properties networked passive linear systems are analyzed through the controllability Gramian. It is shown that, for a class of passive subsystems, the Gramian can be decomposed into two parts, which are related to the subsystems and the interconnection topology, respectively. Moreover, a relation between (a generalized form of) network symmetry and controllability is presented.

Future work will focus on the use of these results in the scope of model reduction for networked systems as in [19].

References

[1] P.J. Antsaklis and A.N. Michel. *Linear systems*. Birkhäuser, Boston, USA, 2006.

[2] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang. Complex networks: structure and dynamics. *Physics Reports*, 424(4-5):175–308, 2006.

[3] J.W. Brewer. Kronecker products and matrix calculus in system theory. *IEEE Transactions on Circuits and Systems*, CAS-25(9):772–781, 1978.

[4] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative systems analysis and control: theory and applications*. Communications and Control Engineering Series. Springer-Verlag, London, Great Britain, second edition, 2007.

[5] A. Chapman and M. Mesbahi. System theoretic aspects of influenced consensus: Single input case. *IEEE Transactions on Automatic Control*, 57(6):1505–1511, 2012.

[6] E.J. Davison. Connectability and structural controllability of composite systems. *Automatica*, 13(2):109–123, 1977.

[7] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds. *International Journal of Control*, 39(6):1115–1193, 1984.

[8] C. Godsil and C. Royle. *Algebraic graph theory*, volume 207 of *Graduate Text in Mathematics*. Springer-Verlag, New York, USA, 2001.

[9] S. Hara, T. Hayakawa, and H. Sugata. Stability analysis of linear systems with generalized frequency variables and its applications to formation control. In *Proceedings of the 46th IEEE Conference on Decision and Control, New Orleans, USA*, pages 1459–1466, 2007.

[10] R.A. Horn and C.R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, United Kingdom, 1990.

[11] R.E. Kalman. Mathematical description of linear dynamical systems. *Journal of the Society for Industrial & Applied Mathematics, Series A: Control*, 1(2):152–192, 1963.

[12] C.-T. Lin. Structural controllability. *IEEE Transactions on Automatic Control*, AC-19(3):201–208, 1974.

[13] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási. Controllability of complex networks. *Nature*, 473(7346):167–173, 2011.

[14] P.J. Moylan and D.J. Hill. Stability criteria for large-scale systems. *IEEE Transactions on Automatic Control*, AC-23(2):143–149, 1978.

[15] G. Parlangeli and G. Notarstefano. On the reachability and observability of path and cycle graphs. *IEEE Transactions on Automatic Control*, 57(3):743–748, 2012.

[16] A.Y. Pogromsky. A partial synchronization theorem. *Chaos*, 18(3):037107, 2008.

[17] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[18] H. Sandberg, J. Delvenne, and J.C. Doyle. On lossless approximations, the fluctuation-dissipation theorem, and limitations of measurements. *IEEE Transactions on Automatic Control*, 56(2):293–308, 2011.

[19] H. Sandberg and R.M. Murray. Model reduction of interconnected linear systems. *Optimal Control Applications and Methods*, 30(3):225–245, 2009.

[20] A. van der Schaft. Port-Hamiltonian systems: an introductory survey. In *Proceedings of the International Congress of Mathematicians, Madrid, Spain*, volume III, pages 1339–1365, 2006.

[21] S.H. Strogatz. Exploring complex networks. *Nature*, 410(6825):268–276, 2001.

[22] H.G. Tanner. On the controllability of nearest neighbor interconnections. In *Proceedings of the 43rd IEEE Conference on Decision and Control, Atlantis, Bahamas*, volume 3, pages 2467–2472, 2004.

[23] J.C. Willems. Dissipative dynamical systems part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, 1972.

[24] S. Zhang, M.K. Camlibel, and M. Cao. Controllability of diffusively-coupled multi-agent systems with general and distance regular coupling topologies. In *Proceedings of the joint 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, USA*, pages 759–764, 2011.

[25] K. Zhou, J.C. Doyle, and K. Glover. *Robust and optimal control*. Prentice Hall, Upper Saddle River, USA, 1996.