On Multi-Vehicle Rendezvous Under Quantized Communication

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Abstract. A rendezvous problem for a team of autonomous vehicles, which communicate over quantized channels, is analyzed. The paper illustrates how communication topologies based on uniform and logarithmic quantizations influence the performance. Since a logarithmic quantizer in general imposes fewer bits to be communicated compared to a uniform quantizer, the results indicate estimates of lower limits on the amount of information that needs to be exchanged in order for the vehicles to meet. Simulation examples illustrate the results.

1 Introduction

Interplay between coordination and communication is important in many multi-vehicle systems, e.g., car platoons on automated highways [8], formations of autonomous underwater vehicles [1], and multi-robot search-and-rescue missions [7]. Constrained communication between vehicles suggest the deployment of distributed (local) control strategies [5,6]. Not only the communication topology is important, however, but also the amount of data being transmitted. Therefore, in this paper we study multi-vehicle control under quantized communication. The stabilization of linear plants with quantized control has recently been extensively studied, see [3] and references therein. The objective of this paper is to illustrates how communication topologies based on uniform and logarithmic quantizations influence the performance. In particular, we consider a prototype problem on the stabilization of $2n$ first-order systems under various communication and control constraints, which can be interpreted as a rendezvous problem for n vehicles with quantized inter-vehicle communication. Our main results show how the rendezvous problem can be solved under certain communication topologies of uniform and logarithmic quantizations. Since a logarithmic quantizer in general imposes fewer bits to be communicated compared to a uniform quantizer, the results indicate estimates of lower limits on the amount of information that needs to be exchanged in order for the vehicles to meet.

The paper is organized as follows. In Section 2 we introduce a mathematical model for the vehicle team and formalize the rendezvous problem. In Section 3 a control law is designed that solves the rendezvous problem when there is no communication limitation. Based on this controller, we show in Section 4 that the vehicle team is able to rendezvous when the data is transmitted through uniformly quantized channels and certain mixtures of uniformly and logarithmically quantized channels. Simulations are presented in Section 5 to illustrate the behavior of the vehicles.

2 Problem formulation

Consider $n \geq 2$ holonomic vehicles with dynamics described by the following linear system

$$\begin{aligned} x_i^+ &= x_i(t+1) = x_i(t) + u_i(t) \\ y_i^+ &= y_i(t+1) = y_i(t) + v_i(t) \quad i = 1, \dots, n, \end{aligned} \tag{1}$$

where $(x_i, y_i)^T \in \mathbb{R}^2$ is the position of vehicle i with respect to a fixed coordinate system and $(u_i, v_i)^T \in \mathbb{R}^2$ is the control input. The dynamics for the x and y coordinates are independent, so therefore we simply consider the system

$$\mathbf{x}^+ = A\mathbf{x} + B\mathbf{u}, \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{u} = (u_1, \dots, u_n)$, and $A = B$ is the $n \times n$ identity matrix. For any pair of vehicles (i, j), $i < j$, we define an output variable $w_{i,j} = x_i - x_j$. Let $\mathbf{w}$ be a vector collecting all these variables. They can be represented by a complete graph $\mathcal{G} = (V, E)$, with $n = |V|$ vertices and an edge $e \in E$ connecting vertices i and j if and only if $\mathbf{w}$ contains $w_{i,j}$. Note that

$$\mathbf{w} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n. \tag{3}$$

Each sub-vector of $\mathbf{w}$ that also has this property corresponds to a tree graph.

The rendezvous problem is to find a (practically stabilizing) state feedback $\mathbf{u} = g(\mathbf{x})$ such that $\mathbf{w}$ asymptotically converges to a pre-specified neighborhood of the origin. If there is no constraints on g, the rendezvous problem is readily solved also in the decentralized case, for example, by the application of deadbeat control. When g is quantized, the problem is non-trivial.

3 Perfect information: the centralized scenario

Consider a complete graph $\mathcal{G}$ and let $\mathbf{w}$ be the vector of outputs associated with it, as previously described. Since a complete graph with n vertices has $m = n(n-1)/2$ edges, we can introduce a matrix $W \in \mathbb{R}^{m \times n}$ such that $\mathbf{w} = W\mathbf{x}$. The matrix W can be interpreted as the incidence matrix of $\mathcal{G}$. Let us consider an optimal control problem for (2), in which the control input is penalized in order to avoid aggressive solutions such as the deadbeat control. The cost function that should be minimized is

$$J(\mathbf{u}) = \sum_{t=0}^{\infty} \mathbf{w}^T(t)\mathbf{w}(t) + \mathbf{u}^T(t)R\mathbf{u}(t) = \sum_{t=0}^{\infty} \mathbf{x}^T(t)W^TW\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t), \tag{4}$$

where $R = R^T$ is a positive definite matrix. Note that the matrix W^TW is singular, so the Riccati equation associated with the optimal control problem does not admit a unique solution. It is however possible to regularize the problem in the following way. Consider any subgraph of $\mathcal{G}$ that is a tree and denote with $\mathbf{z}$ the vector of outputs associated with it. Introduce $C \in \mathbb{R}^{(n-1)\times n}$ such that $\mathbf{z} = C\mathbf{x}$. Then, there exits a matrix L such that

$$\mathbf{w} = \begin{pmatrix} I \\ L \end{pmatrix} \mathbf{z}.$$

Note that $\mathbf{z} = 0 \Rightarrow \mathbf{w} = 0$. Minimizing (4) subject to the dynamics (2) is equivalent to minimizing

$$J(\mathbf{u}) = \sum_{t=0}^{\infty} \mathbf{z}^T(t)(I + L^T L)\mathbf{z}(t) + \mathbf{u}^T(t)R\mathbf{u}(t) \tag{5}$$

subject to the dynamics

$$\mathbf{z}^+ = \mathbf{z} + C\mathbf{u}. \tag{6}$$

It is straightforward to show that the optimal control law is equal to

$$\mathbf{u} = K\mathbf{x},$$

where

$$K = -(R + C^T SC)^{-1} C^T SC$$

and S is the positive definite solution of the algebraic Riccati equation

$$SC(R + C^T SC)^{-1} C^T S = I + L^T L. \tag{7}$$

The following result presents a case when we can solve for K analytically.

Proposition 1. *Let the control penalty matrix in* (4) *be given by $R = rI$, $r > 0$. Then, the state feedback that minimizes* (4) *is given by*

$$K = -\beta\,(nI - Z) \qquad\qquad \beta = \frac{n + \sqrt{n^2 + 4r}}{2r + n^2 + n\sqrt{n^2 + 4r}}. \tag{8}$$

where $Z = \mathbb{1}_{n\times n}$ is the $n \times n$ unit matrix.

Proof. See Appendix. □

The state feedback in Proposition 1 obviously solves the rendezvous problem for the case of perfect communication, because it gives $\mathbf{z}(t+1) = (1 - n\beta)\mathbf{z}(t)$ with $0 < 1 - n\beta < 1$.

Proposition 2. *The state feedback* (8) *yields* (6) *asymptotically stable.*

We would like to avoid that the meeting point in general is the origin, i.e., we would like that K should not stabilize (2) despite that it stabilizes (6).

Proposition 3. *The linear feedback $\mathbf{u} = K\mathbf{x}$ with K as defined in* (8) *is not asymptotically stabilizing* (2).

Proof. The equilibrium points of the closed-loop system

$$\mathbf{x}^+ = (I + KC)\mathbf{x}$$

is given by Ker$(C) = \{\mathbf{x} \in \mathbb{R}^n | x_1 = \cdots = x_n\}$. Hence, the closed-loop system is not asymptotically stable. □

The three propositions above show that the proposed state feedback has nice properties for the rendezvous problem under no communication constraints. Next we show that it is possible to base quantized control that solves the rendezvous problem on the same structure.

4 Quantized information: the distributed scenario

Consider the vehicle dynamics (2) with control $\mathbf{u} = g(\mathbf{x})$ again, but suppose now that g involves quantization resulting from that the position information of each vehicle need to be quantized before communicated to other vehicles. We consider a few different communication topologies for two fundamentally different quantizers, namely, uniform and logarithmic quantizations. Logarithmic quantization is more efficient than uniform quantization from a information theoretic perspective since fewer bits need to be communicated for data not close to the origin. On the other hand, position data is less accurate when far from the origin, so from a control perspective it may be harder to stabilize a system. For all cases, we suppose that the communication delay is negligible compared to the vehicle dynamics.

4.1 Uniform quantization

We assume first that all vehicles can exchange information through communication channels with uniform quantization. A graphical representation for $n = 3$ vehicles is shown in Figure 1. The quantization error for a uniform quantizer is bounded as

$$|q_u(x) - x| \leq \epsilon,$$

where ϵ is the quantization level and $x \in \mathbb{R}$ the quantization input. The quantized feedback control is given by the uniform quantization of the optimal feedback law derived in previous section. Thus the control input u_i for vehicle i is given by

$$u_i = -\beta(n-1)x_i + \beta \sum_{\ell=1, \ell \neq i}^{n} q_u(x_\ell) \qquad i = 1, \ldots, n. \tag{9}$$

We next show that this control law solve the rendezvous problem. Note first that the difference $z_{i,j}$, $i < j$, has the dynamics

$$z_{i,j}^{+} = z_{i,j} - \beta\big((n-1)x_i - (n-1)x_j - q_u(x_j) + q_u(x_i)\big) \qquad i, j = 1, \ldots, n. \tag{10}$$

Theorem 1. *The system* (10) *is practically stable.*

Proof. Consider the Lyapunov function $V(z_{i,j}) = |z_{i,j}|$. Then we have

$$\begin{aligned} \Delta V(z_{i,j}) &= |z_{i,j} - 3\beta z_{i,j}| - |z_{i,j}| + \beta|x_i - q_u(x_i)| + \beta|x_j - q_u(x_j)| \\ &\leq -3\beta|z_{i,j}| + 2\beta\epsilon \qquad\qquad i, j = 1, \ldots, n \text{ and } i < j. \end{aligned}$$

Hence, $\Delta V(z_{i,j}) < 0$ if $|z_{i,j}| > 2\epsilon/3$, since $3\beta \leq 1$. Thus we have asymptotic convergence when $z_{i,j}$ is not too small, i.e., practical stability. By considering $\mathcal{V} = V(z_{1,2}) + V(z_{1,3}) + V(z_{2,3})$, it follows that the uniformly quantized system (10) is practically stable.

$\square$

4.2 Logarithmic quantization

In order to reduce the number of bits communicated over each channel, we try to substitute uniform quantizers with logarithmic quantizers. This would allow to express data far from the origin with fewer

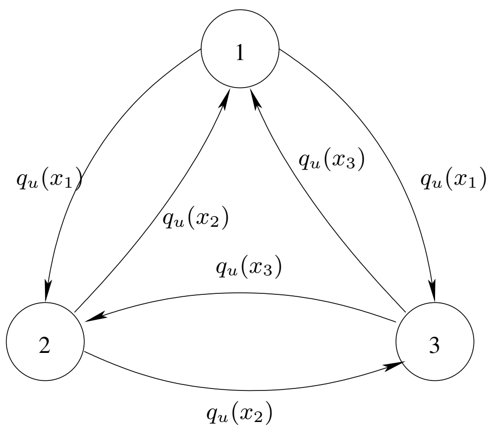

Fig. 1. Directed graph that represents uniformly quantized communication between $n = 3$ vehicles.

bits. The quantization error for the logarithmic quantizer depends on the amplitude of the input signal as

$$|q_\ell(x) - x| \leq \delta|x|,$$

where δ is a positive parameter.

In the literature (e.g., [2]), logarithmic quantizers have been considered for the stabilization of linear systems. The main point of using a logarithmic quantizer is that if the state is far from the origin we do not need a very precise value of the state (i.e., $\delta|x|$ large) while the state will be high accuracy when it is close to the origin. In the rendezvous problem we are interested in the convergence of the n vehicles to a neighborhood of any point such that $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n$. Therefore, we will logarithmically quantize differences between positions of vehicle pairs, since we want these differences to be small. We limit the following discussion to $n = 3$ vehicles.

By replacing some of the uniform quantizers in Figure 1 by logarithmic quantizers, we investigate control structures that require less communication. Figure 2 shows three considered cases, where the edges represent uniform quantization of the state $\mathbf{x}_i$. The remaining channels are based on logarithmic quantizers. As an illustration, Figure 3 shows the communication topology for Case 1 of Figure 2. Figure 3 indicates also the data that each vehicles receive after a transmission. For each case of Figure 2, the rendezvous problem is solved by the following corresponding control law:

Case 1:

$$\begin{aligned} u_1 &= -2\beta q_\ell(q_u(x_1) - x_3) + \beta q_\ell(q_u(x_2) - x_3) \\ u_2 &= \beta q_\ell(q_u(x_1) - x_3) - 2\beta q_\ell(q_u(x_2) - x_3) \\ u_3 &= -\beta\big(x_3 - q_u(x_1) + x_3 - q_u(x_2)\big) \end{aligned} \tag{11}$$

Case 2:

$$\begin{aligned} u_1 &= -\beta q_\ell(q_u(x_1) - x_2) - \beta q_\ell(q_u(x_1) - x_3) \\ u_2 &= -2\beta\big(x_2 - q_u(x_1)\big) - \beta q_\ell(q_u(x_1) - x_3) \\ u_3 &= 2\beta\big(q_u(x_1) - x_3\big) - \beta q_\ell(q_u(x_1) - x_2) \end{aligned} \tag{12}$$

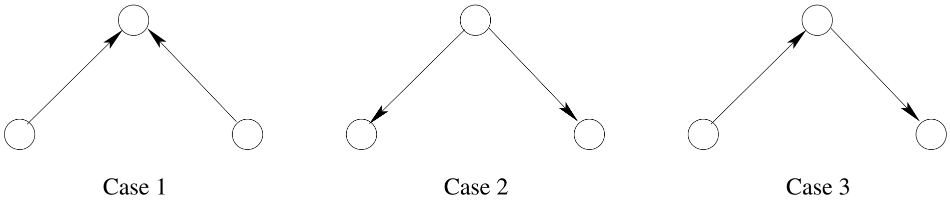

Fig. 2. All trees represented by the incidence matrix C^T for $n = 3$.

Case 3:

$$\begin{aligned} u_1 &= -2\beta q_\ell(q_u(x_1) - x_2) - \beta q_\ell(q_u(x_2) - x_3) \\ u_2 &= -\beta\big(x_2 - q_u(x_1)\big) - \beta q_\ell(q_u(x_2) - x_3) \\ u_3 &= -2\beta\big(x_3 - q_u(x_2)\big) + \beta q_\ell(q_u(x_1) - x_2) \end{aligned} \tag{13}$$

This is summarized in the following result.

Theorem 2. *Let C^T be the incidence matrix corresponding to Cases 1–3 with $z = Cx$. Then the closed-loop system $\mathbf{z}^+ = \mathbf{z}+\mathbf{u}$, with the control $\mathbf{u} = (u_1, u_2, u_3)^T$ as given in* (11)–(13)*, is practically stable.*

Proof. See Appendix. □

5 Simulations

Figures 4(a)-4(c) and Figures 6(a)-6(c) show the trajectories of rendezvous coordination of three vehicles starting from two different initial conditions. Dashed lines correspond to vehicles communicating perfectly. Solid lines show the trajectories when the information exchanged among the vehicles is quantized with logarithmic quantizers (the uniform quantization error is negligible in the simulations). The three plots (a)–(c) refer to the three different topologies for shown in Figure 2. Figures 5(a)–5(b) and Figures 7(a)–7(b) show how the performance decreases when quantization is influencing the communication. Here solid line shows the output variable $\mathbf{z}$ for the vehicles when the communication is perfect. Such cases represent an lower bound on the speed of convergence. Dashed, dash-dotted and dotted line show the output variable ‡ for the three cases of logarithmic quantized channels. Note that Case 1 has slower convergence than the others. This is probably due to that in this case two vehicles base their control action only on logarithmic quantized information. To compare Case 2 and Case 3, a more precise analysis is needed to understand if one configuration is preferable.

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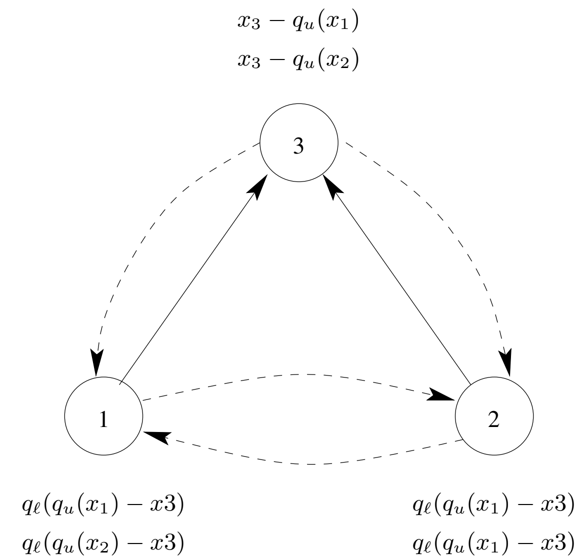

Fig. 3. Data known by $n = 3$ vehicles after the communication through uniform (in dashed line) and logarithmic (in solid line) quantizers.

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Appendix

Proof Proposition 1

Let L be the matrix such that $W = (I, L^T)^T C$ as we defined in Section 3. The matrix W^T is the incidence matrix of the complete oriented graph $\mathcal{G}$. Since the graph is complete, there is an edge between any vertex i and j with $i \neq j$ and the degree[1] of each vertex is $n-1$ where n is number of vertices. The matrix W^TW is the Laplacian matrix of the complete graph $\mathcal{G}$, as defined for example in [4]. The Laplacian matrix W^TW, since the graph is complete, is such that

$$(W^TW)_{ij} = \begin{cases} n-1 = \deg(v_i) & \text{if } i = j \\ -1 & \text{otherwise.} \end{cases}$$

Thus we can write the Laplacian as $W^TW = nI - Z$, with $Z = \mathbb{1}_{n\times n}$. We have then

$$nC^T(I + L^TL)C = nW^TW = n^2I - nZ,$$

[1] The degree of a vertex of an oriented graph is the total number of edges arriving and departing from that vertex

We now prove that the $S = \alpha n(I + L^T L)$, with $\alpha > 0$ is a solution of the Riccati equation (7). If we substitute we get that

$$\alpha n(I + L^T L)C(R + \alpha C^T n(I + L^T L)C)^{-1} C^T \alpha n(I + L^T L) = I + L^T L.$$

Multiplying on the right and the left the previous expression with C^T and C respectively and since $nC^T(I + L^T L)C = n^2 I - nZ$, we get

$$\alpha(n^2 I - nZ)\big(rI + \alpha(n^2 I - nZ)\big)^{-1} \alpha(n^2 I - nZ) = n^2 I - nZ.$$

The matrix $n^2 I - nZ$ is a circulant matrix. Any circulant matrix can be diagonalized by the following matrix

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega^1 & \dots & \omega^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

where $\omega = \exp(2\pi j/n)$. Then, $n^2 I - nZ = F^H \Delta F$ where F^H is the Hermitian of F and $\Delta = \text{diag}\,(Fc)$, where $c = (n^2 - n, -n, \dots, -n)^T$. Since also $rI + \alpha(n^2 I - nZ)$ is circulant, we can write $rI + \alpha(n^2 I - nZ) = F^H \Gamma F$ with

$$\Gamma = \text{diag}\big(F(r + \alpha(n^2 - n), -\alpha n, \dots, -\alpha n)^T\big)$$

The Riccati equation then becomes

$$\alpha n^2 I - nZ(rI + \alpha n^2 I - nZ)^{-1} \alpha n^2 I - nZ - n^2 I - nZ = \alpha^2 F^H \Delta F F^H \Gamma^{-1} F F^H \Delta F - F^H \Delta F.$$

Since F is a unitary matrix, we can simplify this to

$$\alpha^2 \Delta \Gamma^{-1} \Delta - \Delta = 0.$$

We have n equations:

$$\frac{\alpha^2 \big(n^2 - n - n \sum_{i=1}^{n-1} \omega^k i\big)^2}{r + \alpha(n^2 - n) - \alpha n \sum_{i=1}^{n-1} \omega^k i} = n^2 - n + \sum_{i=1}^{n-1} -n\omega^k i, \qquad k = 0, 1, \dots, n-1$$

The first equation is satisfied for any α while the last $n - 1$ are satisfied for

$$\alpha^2 n^2 - \alpha n^2 - r = 0 \qquad \alpha = \frac{1}{2} \pm \frac{1}{2n}\sqrt{n^2 + 4r}$$

With $\alpha = \frac{1}{2} + \frac{1}{2n}\sqrt{n^2 + 4r} > 0$, $S = \alpha C^T C$ is a solution of the Riccati equation. The resulting K is then computed as

$$K = \alpha^2 F^H \Gamma^{-1} \Delta F$$

which is the following circulant matrix

$$K = \beta \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{pmatrix}$$

with $\beta = (n + \sqrt{n^2 + 4r})/(2r + n^2 + n\sqrt{n^2 + 4r})$. $\square$

Proof of Theorem 2

Consider the Lyapunov function $V(z_{i,j}) = |z_{i,j}|$. Let $\mathcal{V}(\mathbf{z}) = V(z_{1,2}) + V(z_{1,3}) + V(z_{2,3})$. In order to prove practical stability we will consider $\Delta\mathcal{V}(\mathbf{z}) = \mathcal{V}(\mathbf{z}^{+}) - \mathcal{V}(\mathbf{z})$. We have the following cases.

Case 1: The differences $z_{i,j}$ for $i < j$ are such that

$$\begin{aligned}
z_{1,2}^{+} &= z_{1,2} + 3\beta q_{\ell}(x_3 - q_u(x1)) - 3\beta q_{\ell}(x_3 - q_u(x_1)) \\
z_{1,3}^{+} &= z_{1,3} + 2\beta q_{\ell}(x_3 - q_u(x1)) - \beta q_{\ell}(x_e - q_u(x_2)) + 2\beta x_3 - \beta q_u(x_1) - \beta q_u(x_2) \\
z_{2,3}^{+} &= z_{2,3} - \beta q_{\ell}(x_3 - q_u(x_1)) + 2\beta q_{\ell}(x_3 - q_u(x_2)) + 2\beta x_3 - \beta q_u(x_1) - \beta q_u(x_2).
\end{aligned}$$

After some calculations we obtain

$$\Delta\mathcal{V}(\mathbf{z}) \leq -3\beta(|z_{1,2}| + |z_{1,3}| + |z_{2,3}|) + 12\beta\delta\epsilon + 6\beta\epsilon + 6\beta\delta|z_{1,3}| + 3\beta\delta\big(|z_{1,2}| + |z_{2,3}|\big).$$

Case 2: The differences $z_{i,j}$ for $i < j$ are such that

$$\begin{aligned}
z_{1,2}^{+} &= z_{1,2} - 2\beta q_{\ell}(q_u(x_1) - x_2) - \beta q_{\ell}(q_u(x_1) - x_3) + 2\beta q_{\ell}(x_2 - q_u(x_1)) + \beta q_{\ell}(q_u(x_1) - x_3) \\
z_{1,3}^{+} &= z_{1,3} - \beta q_{\ell}(q_u(x_1) - x_2) - \beta q_{\ell}(q_u(x_1) - x_3) + 2\beta(x_3 - 2q_u(x_1) + \beta q_{\ell}(q_u(x_1) - x_2) \\
z_{2,3}^{+} &= z_{2,3} - 2\beta\big(x_2 - q_u(x_1)\big) - \beta q_{\ell}(q_u(x_1) - x_3) + 2\beta\big(x_3 - q_u(x_1)\big) + \beta q_{\ell}(q_u(x_1) + x_2).
\end{aligned}$$

After some calculations we obtain

$$\Delta\mathcal{V}(\mathbf{z}) \leq -3\beta(|z_{1,2}| + |z_{1,3}| + |z_{2,3}|) + 4\beta\delta\epsilon + 6\beta\epsilon + 2\beta\delta(|z_{1,2}| + |z_{1,3}|).$$

Case 3: The differences $z_{i,j}$ for $i < j$ are such that

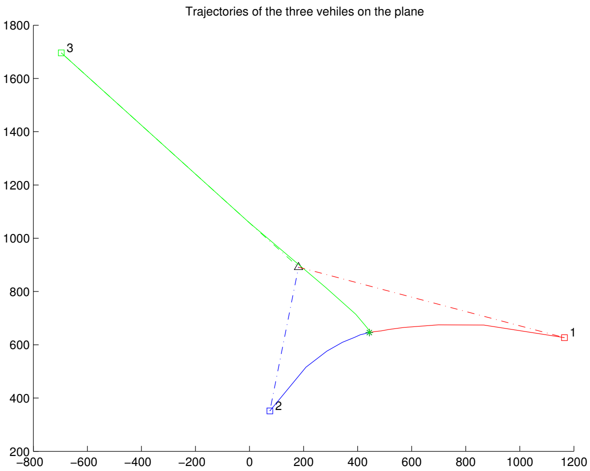
$$\begin{aligned}
z_{1,2}^{+} &= z_{1,2} - 2\beta q\ell(q_u(x_1) - x_2) + \beta(x_2 - q_u(x_1)) \\
z_{1,3}^{+} &= z_{1,3} - 3\beta q_{\ell}(q_u(x_1 - x_2)) - \beta q_{\ell}(q_u(x_2) - x_3) + 2\beta\big(x_3 - q_u(x_2)\big) \\
z_{2,3}^{+} &= z_{2,3} - \beta\big(x_2 - q_u(x_1)\big) - \beta q_l(q_u(x_2) - x_3) + 2\beta\big(x_3 - q_u()x_2\big) - \beta q_{\ell}(q_u(x_1) - x_2).
\end{aligned}$$

After some calculations we obtain

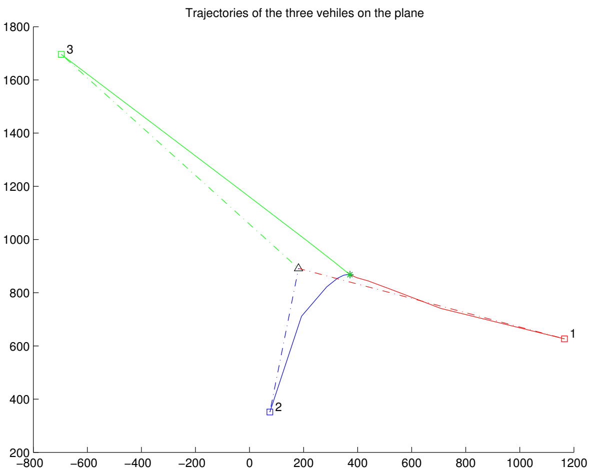
$$\Delta\mathcal{V}(\mathbf{z}) \leq -3\beta(|z_{1,2}| + |z_{1,3}| + |z_{2,3}|) + 8\beta\delta\epsilon + 9\beta\epsilon + 6\beta\delta|z_{1,2}| + 2\beta\delta|z_{2,3}|.$$

Hence, the system , in all three cases, is practically stable. By choosing the quantization parameters $\epsilon > 0$ and $\delta > 0$ sufficiently small, we can thus force y to converge to any neighborhood of the origin.

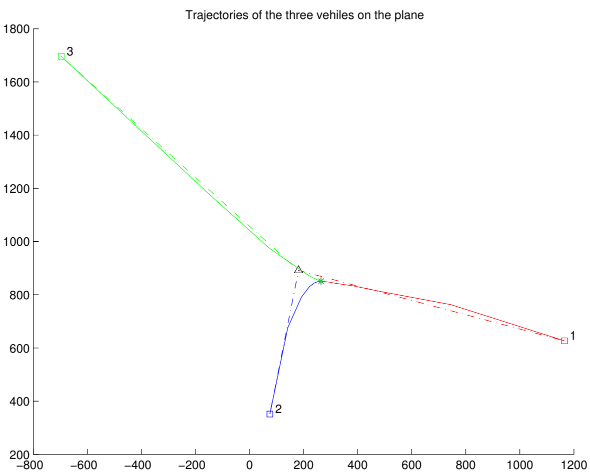
□



(a) Communication topology as in Case 1.

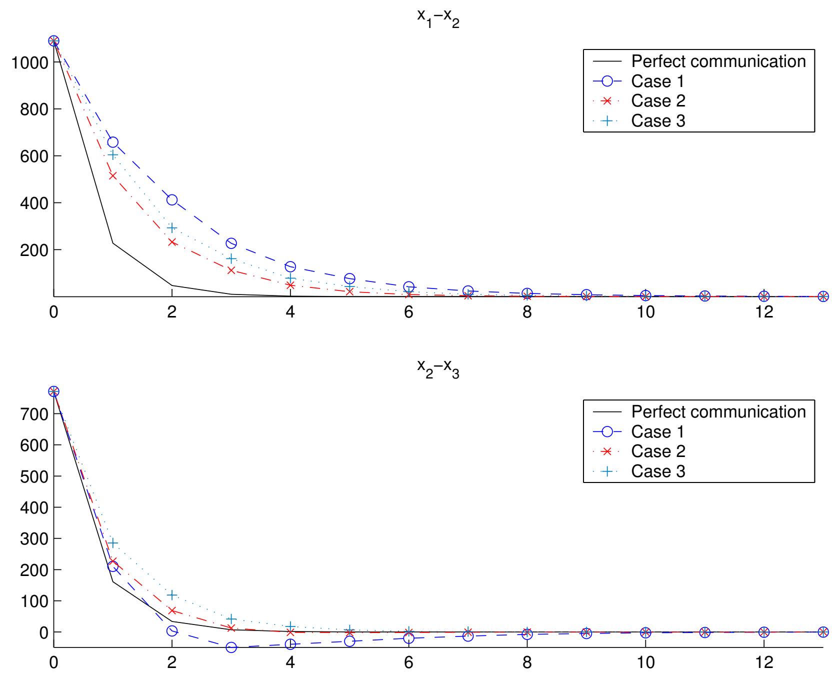


(b) Communication topology as in Case 2.

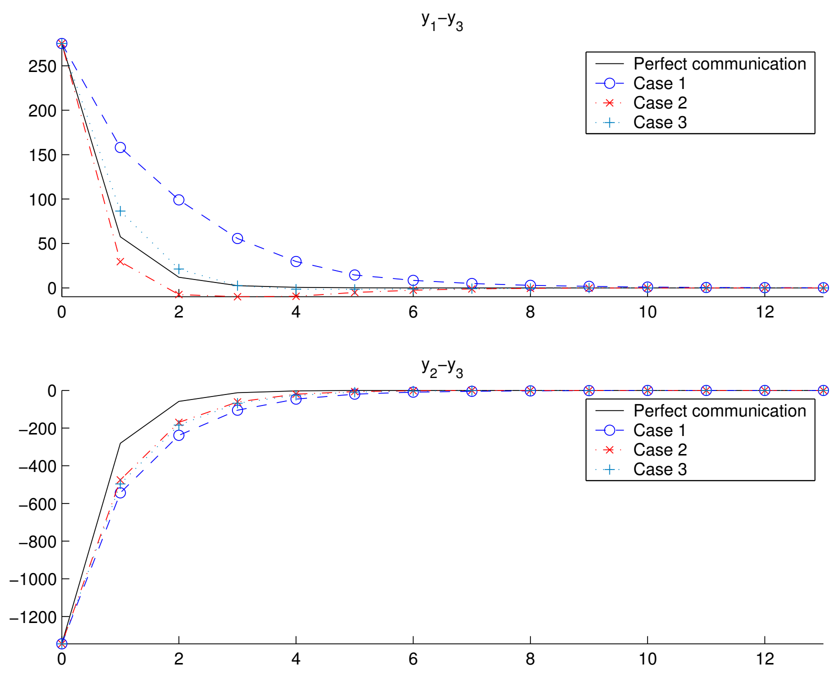


(c) Communication topology as in Case 3.

Fig. 4. Trajectories for three vehicles for the three different topologies of figure 2. In the simulations we assumed the uniform quantization error equal to zero.

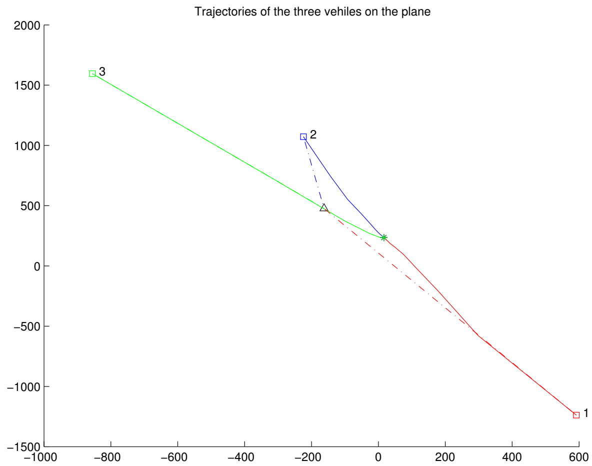


(a) Differences $x_1 - x_2$ and $x_2 - x_3$ for the three vehicles corresponding to the trajectories of Figures 4(a)– 4(c).

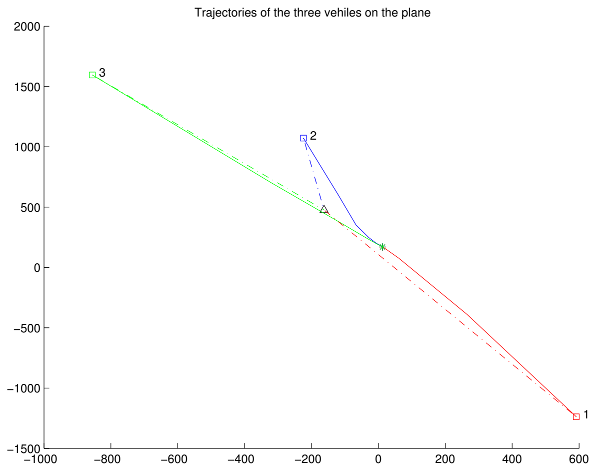


(b) Differences $y_1 - y_2$ and $y_2 - y_3$ for the three vehicles corresponding to the trajectories of Figures 4(a)– 4(c).

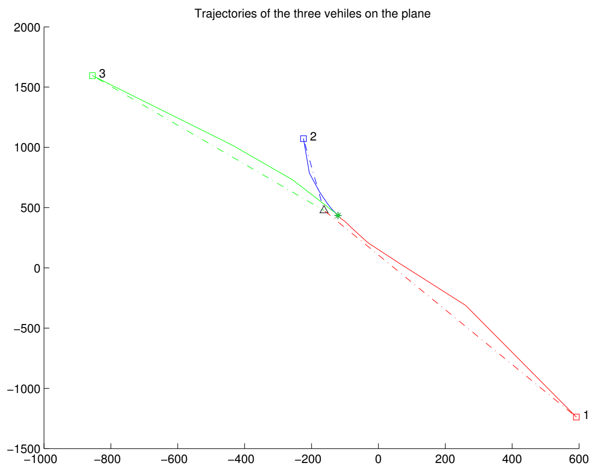
Fig. 5. Performance comparison of the difference communication topologies.



(a) Communication topology as in Case 1.

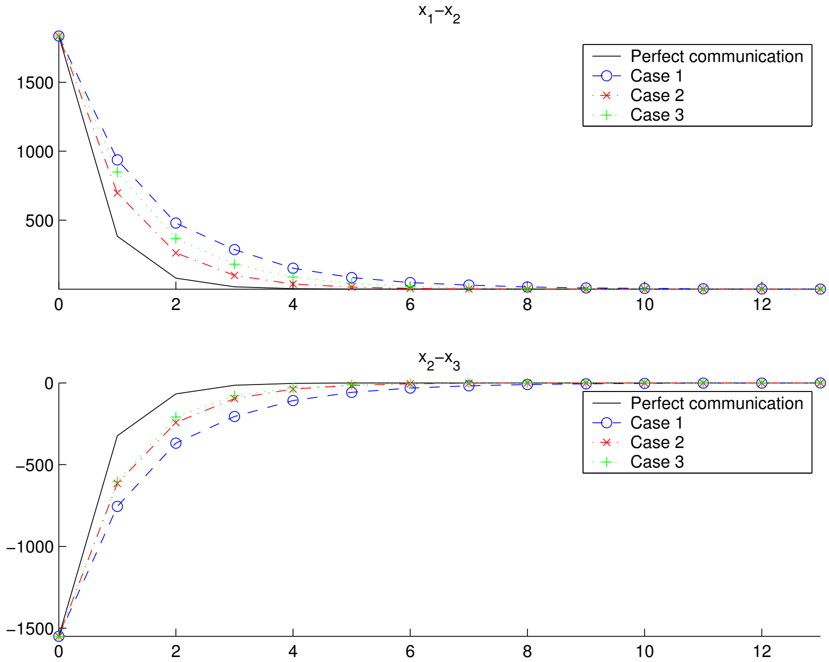


(b) Communication topology as in Case 2.

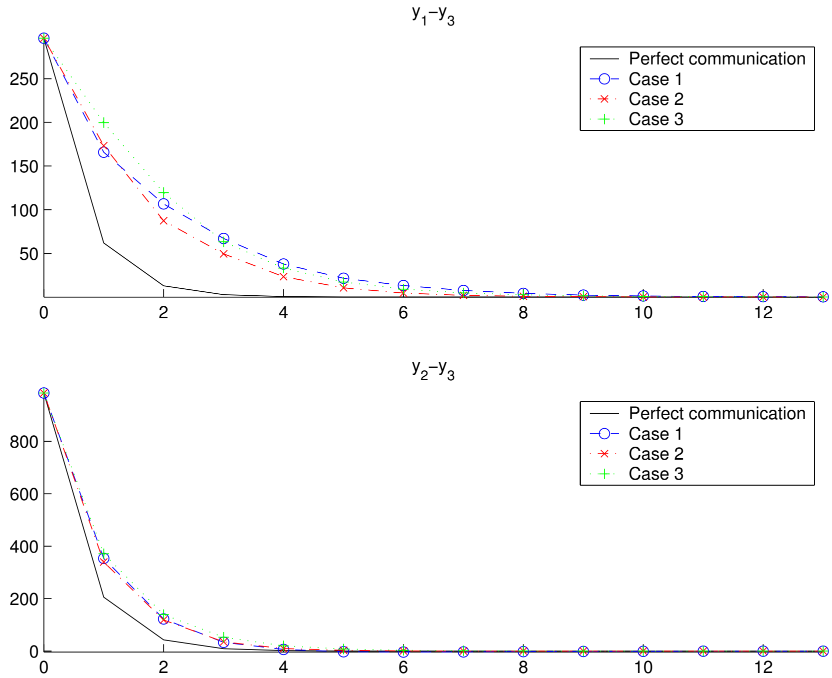


(c) Communication topology as in Case 3.

Fig. 6. Trajectories for three vehicles for the three different topologies of figure 2. In the simulations we assumed the uniform quantization error equal to zero.



(a) Differences $x_1 - x_2$ and $x_2 - x_3$ for the three vehicles corresponding to the trajectories of Figures 6(a)– 6(c).



(b) Differences $y_1 - y_2$ and $y_2 - y_3$ for the three vehicles corresponding to the trajectories of Figures 6(a)– 6(c).

Fig. 7. Performance comparison of the difference communication topologies.