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Consensus of Quantum Networks with Continuous-time Markovian Dynamics*

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***Abstract*— In this paper, we investigate the convergence of the state of a quantum network to a consensus (symmetric) state. The state evolution of the quantum network with continuous-time swapping operators can be described by a Lindblad master equation, which also introduces an underlying interaction graph for the network. For a fixed quantum interaction graph, we prove that the state of a quantum network with continuous-time Markovian dynamics converges to a consensus state, with convergence rate given by the smallest nonzero eigenvalue of a matrix serving as the Laplacian of the quantum interaction graph. We show that this convergence rate can be optimized via standard convex programming given a fixed amount of edge weights. For switching quantum interaction graphs, we establish necessary and sufficient conditions for exponential quantum consensus and asymptotic quantum consensus, respectively. The convergence analysis is based on a bridge built between the proposed quantum consensus scheme and classical consensus dynamics, in that quantum consensus of n qubits naturally defines a consensus process on an induced classical graph with 2^{2n} nodes. Existing consensus results on classical networks can thus be adopted to establish the quantum consensus convergence.**

***Index Terms*— quantum control, quantum consensus, quantum network.**

I. Introduction

Quantum technology has been recognized as the second quantum revolution [1] among which quantum information technology is one of the important research areas [2]. Quantum information technology has many important potential applications due to its advantages (e.g., better security for communication and more powerful capability for computation) over traditional information technology [2]. One of the critical tasks required for the advancement of quantum information technology is the development of systematic quantum control theory that can provide a theoretical footing for wide applications of quantum information technology [3], [4], [5].

One of key tasks in quantum control is to stabilize a quantum system to a desired state (or a desired state set) [3], [6]-[12]. For a closed quantum system, its evolution is described by the Schrödinger equation and the Lyapunov methodology is an effective approach for the design of a stabilizing control law [13]. For an open quantum system that interacts with its environment, its evolution can usually be described using an appropriate master equation [14]. Feedback control methods are usually employed to stabilize open quantum systems (see, e.g., [9], [10]). In most of existing results on stabilizing quantum systems, feedback information has been used during the design process (for Lyapunov control of closed quantum systems) or control process (for feedback stabilization of open quantum systems), and the stabilizing methods have not been applied to a quantum network system. In this paper, we investigate a special class of stabilization problems for a quantum network where no feedback information is required during the design or control process.

With recent development in quantum physics and information technology, quantum network systems have yielded significant potential of applications. Consensus in a quantum network [15] can been taken as a special class of stabilization problems of quantum systems, which will be investigated in this paper. It has a close connection to distributed quantum computation, quantum communication and quantum random walks [15]. The subsystems (agents) in a quantum network are quantum systems that should be described by quantum mechanics and the interaction between different agents may involve non-classical correlation (e.g., quantum entanglement [2]). Although different distributed control and optimization methods have been presented for reaching consensus of agents in a classical (non-quantum) network (see, e.g., [16]-[24]), they cannot be straightforwardly extended to the consensus of quantum networks due to their unique quantum characteristics.

Recently, Sepulchre *et al.* [25] generalized consensus algorithms to non-commutative spaces and presented convergence results for quantum stochastic maps. They showed how the Birkhoff theorem can be used to analyze the asymptotic convergence of a quantum system to a fully mixed state. Mazzarella *et al.* [15] extended the consensus framework in the field of distributed control and optimization on classical network systems to quantum networks with discrete-time

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dynamics. They defined four classes of consensus quantum states based on invariance and symmetry properties and presented a quantum generalization of the gossip iteration algorithm for reaching consensus in a quantum network. The quantum gossip iteration algorithm is realized through discrete-time quantum swapping operations between two subsystems on a quantum network and can make the quantum network converge to symmetric states while preserving the expectation of permutation-invariant global observables [26].

When continuous-time swapping operations are applied to a quantum network, the swapping operators could define a Lindblad master equation for the dynamical evolution of the quantum network [27], [28]. Under this critical understanding, we investigate the continuous-time analogue of the discrete-time model in [15]. Instead of adopting the contraction mapping argument [25] for the convergence analysis, we constructively build the bridge between the quantum consensus dynamics and classical consensus dynamics, and then establish the convergence results for the considered quantum network. The contribution of this paper is highlighted as follows.

- For fixed quantum interaction graphs, we prove convergence to a consensus state using direct algebraic methods under quantum consensus dynamics. We establish that the convergence rate is governed by the smallest nonzero eigenvalue of a quantum Laplacian as the analogue of Laplancian matrix for classical networks. We also show that this convergence rate can be optimized via standard convex programming given a fixed amount of edge weights.
- For switching quantum interaction graphs, we establish necessary and sufficient conditions for exponential quantum consensus and asymptotic quantum consensus, respectively. By showing that the proposed quantum consensus scheme of n qubits naturally defines a consensus process on an induced classical graph with 2^{2n} nodes, existing consensus results on classical networks are thus adopted to establish these fundamental convergence conditions.

These results illustrate some fundamental possibilities of carrying out quantum network control making use of the various studies from classical networks.

The rest of the paper is organized as follows. Section II presents the preliminaries including relevant concepts in graph theory and quantum systems. The convergence results are provided in Sections III and IV, respectively, for fixed and switching quantum interaction graphs. Finally Section V concludes this paper.

II. Preliminaries

In this section, we introduce some concepts and theories in graph theory [29] and quantum systems [2].

A. *Graph Theory*

A simple undirected graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ consists of a finite set $\mathrm{V} = \{1, \dots, N\}$ of nodes and an edge set E, where an element $e = \{i, j\} \in \mathrm{E}$ denotes an *edge* between two distinct nodes $i \in \mathrm{V}$ and $j \in \mathrm{V}$. A path between two vertices v_1 and v_k in G is an alternating sequence of distinct nodes $v_1 v_2 \dots v_k$ such that for any $m = 1, \dots, k-1$, there is an edge between v_m and v_{m+1}. We call graph G *connected* if, for every pair of distinct nodes in V, there is a path between them. A subgraph of G associated with node set $\mathrm{V}^\dagger \subseteq \mathrm{V}$, denoted as $\mathrm{G}|_{\mathrm{V}^\dagger}$, is the graph $(\mathrm{V}^\dagger, \mathrm{E}^\dagger)$, where $\{i, j\} \in \mathrm{E}^\dagger$ if and only if $\{i, j\} \in \mathrm{E}$ for $i, j \in \mathrm{E}^\dagger$. A connected component (or just component) of an G is a connected subgraph induced by some $\mathrm{V}^\dagger \subseteq \mathrm{V}$, which is connected to no additional nodes in $\mathrm{V} \setminus \mathrm{V}^\dagger$.

The (weighted) Laplacian of G, denoted $L(\mathrm{G})$, is defined as

$$L(\mathrm{G}) = D(\mathrm{G}) - A(\mathrm{G}),$$

where $A(\mathrm{G})$ is the $N \times N$ matrix given by $[A(\mathrm{G})]_{kj} = [A(\mathrm{G})]_{jk} = a_{kj}$ for some $a_{kj} > 0$ if $\{k, j\} \in \mathrm{E}$ and $[A(\mathrm{G})]_{kj} = 0$ otherwise, and $D(\mathrm{G}) = \mathrm{diag}(d_1, \dots, d_N)$ with $d_k = \sum_{j=1, j \neq k}^{N} [A(\mathrm{G})]_{kj}$. It is well known that $L(\mathrm{G})$ is always positive semi-definite, and it holds that

$$\mathrm{rank}(L(\mathrm{G})) = N - C_*(\mathrm{G})$$

with $C_*(\mathrm{G})$ denoting the number of connected components of G.

B. *Quantum Systems*

For an open quantum system, its state can be described by the positive Hermitian density operator (or density matrix) ρ satisfying $\mathrm{tr}\rho = 1$. The evolution of ρ cannot generally be described in terms of a unitary transformation. In many situations, a master equation for $\rho(t)$ is a suitable way to describe the dynamics of an open quantum system. One of the simplest cases is when a Markovian approximation can be applied under the assumption of a short environmental correlation time permitting the neglect of memory effects [14]. For an N-dimensional open quantum system with Markovian dynamics, its state $\rho(t)$ can be described by the following Markovian master equation (for details, see, e.g., [14], [27], [28]):

$$\begin{aligned} \dot{\rho}(t) = \; & -\tfrac{i}{\hbar}[H, \rho(t)] \\ & + \tfrac{1}{2} \sum_{j,k=0}^{N^2-1} \beta_{jk} \{[X_j \rho(t), X_k^\dagger] + [X_j, \rho(t) X_k^\dagger]\}. \end{aligned} \tag{1}$$

Here for an arbitrary operator X, $[X, \rho] = X\rho - \rho X$ is the commutation operator, with $i^2 = -1$, $\hbar$ is reduced Planck constant, H is the Hamiltonian of the system, $\{X_j\}_{j=0}^{N^2-1}$ is a basis for the space of linear bounded operators on the underlying Hilbert space with $X_0 = I$, the coefficient matrix $B = (\beta_{jk})$ is positive semidefinite and physically specifies the relevant relaxation rates. Markovian master equations have been widely used to model quantum systems with external inputs in quantum control [30]-[31], especially for Markovian quantum feedback [5].

Let $\mathcal{H}$ be a two-dimensional Hilbert space over $\mathbb{C}$. Assume that a quantum network is a composite quantum system with n qubits, whose state space is within the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. The state of the quantum network can be described by a self-adjoint, positive semi-definite operator with trace one ρ over $\mathcal{H}^{\otimes m}$. In this paper, we will focus on Markovian master equation in the Lindblad form

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H, \rho(t)] + \sum_{k=1}^{K} \gamma_k \mathfrak{D}[L_k]\rho(t), \tag{2}$$

where

$$\mathfrak{D}[L_k]\rho = L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k.$$

Consider a quantum network of n qubits with index set $\mathsf{V} = \{1, \dots, n\}$. We associate the network with an underlying interaction graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, where each element in E is an unordered pair of two distinct qubits, denoted as $\{j, k\} \in \mathsf{E}$ with $i, j \in \mathsf{V}$. A permutation of the set $\mathsf{V} = \{1, \dots, n\}$ is a bijective map from V onto itself. We denote by π a permutation. The set of all permutations of V forms a group, called the n'th permutation group and denoted by $\mathbf{P} = \{\pi\}$. There are $n!$ elements in $\mathbf{P}$. Given $\pi \in \mathbf{P}$, we define a unitary operator, U_π, over $\mathcal{H}^{\otimes n}$, by

$$U_\pi \big(Q_1 \otimes \cdots \otimes Q_n \big) = Q_{\pi(1)} \otimes \cdots \otimes Q_{\pi(n)},$$

where $Q_i \in \mathcal{H}$ for all $i = 1, \dots, n$. Particularly, a permutation π is called a swapping between j and k if $\pi(j) = k$, $\pi(k) = j$, and $\pi(s) = s, s \in \mathcal{V} \setminus \{j, k\}$. In this case we use π_{jk} with π interchangeably and the corresponding operator U_π is denoted as U_{jk} and called a swapping operator between j and k. When we employ the quantum Gossip interaction presented in [15], the evolution of the quantum network can be described by the following master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_{\{j,k\} \in \mathsf{E}} \alpha_{jk} \Big(U_{jk} \rho U_{jk}^\dagger - \rho \Big), \tag{3}$$

where $\alpha_{jk} > 0$ is a constant marking the weight of edge $\{j, k\}$, and U_{jk} is the swapping operator between j and k.

III. Fixed Interaction Graph

For simplicity, we first consider the same assumption of $H = 0$ as that in [15] for the quantum network. Let the density operator $\rho(t)$ denote the state of the network at time t. The evolution of $\rho(t)$ is described in the following Lindblad master equation

$$\frac{d\rho}{dt} = \sum_{\{j,k\} \in \mathsf{E}} \alpha_{jk} \Big(U_{jk} \rho U_{jk}^\dagger - \rho \Big), \tag{4}$$

Without loss of generality we assume the initial time is $t_0 = 0$ and the initial state is denoted as $\rho(0) = \rho_0$. Note that along the Lindblad master equation (4), $\rho(t)$ will be preserved as positive, Hermitian, and with trace one, as long as $\rho(0)$ defines a proper density operator. While the convergence conditions intended to derive in the paper are irrelevant with these properties held by density operators. Therefore, throughout the rest of the paper, we assume that $\rho(t) \in \mathbb{C}^{n \times n}$ lies in the general space $\mathbb{C}^{n \times n}$.

A. Convergence to quantum consensus

Define an operator over the density operators of $\mathcal{H}^{\otimes n}$, $\mathscr{P}_*$, by

$$\mathscr{P}_*(\rho) = \frac{1}{n!} \sum_{\pi \in \mathbf{P}} U_\pi \rho U_\pi^\dagger.$$

The state $\rho_* = \mathscr{P}_*(\rho_0)$ has been defined as a quantum consensus state in [15]. We consider that the system reaches the quantum consensus state and have the following convergence result.

Theorem 1: Suppose G is connected. Then System (4) achieves a quantum consensus in the sense that $\lim_{t \to \infty} \rho(t) = \rho_*$ with $\rho_* = \mathscr{P}_*(\rho_0)$.

Proof: We complete the proof in three steps.

Step 1. We now proceed to prove that $\rho(t)$ converges to a limit when t tends to infinity.

Using our knowledge of linear algebra [32], we have the following fact. Given a matrix $M \in \mathbb{C}^{mn}$, the vectorization of M, denoted by $\mathbf{vec}(M)$, is the $mn \times 1$ column vector $([M]_{11}, \dots, [M]_{m1}, [M]_{12}, \dots, [M]_{m2}, \dots, [M]_{1n}, \dots, [M]_{mn})'$. We have $\mathbf{vec}(ABC) = (C^T \otimes A)\mathbf{vec}(B)$ for all matrices A, B, C with ABC well defined, where $\otimes$ stands for the Kronecker product.

Under the standard computational basis, we write the system (4) in the following vector form:

$$\begin{aligned} &\tfrac{d}{dt}\mathbf{vec}(\rho(t)) = \\ &\qquad \textstyle\sum_{\{j,k\} \in \mathsf{E}} \alpha_{jk} \Big(U_{jk} \otimes U_{jk} - I_{2^n} \otimes I_{2^n} \Big) \mathbf{vec}(\rho(t)). \end{aligned} \tag{5}$$

Note that (5) defines a linear time-invariant system. Denote $M_{jk} := U_{jk} \otimes U_{jk} - I_{2^n} \otimes I_{2^n}$. It is straightforward to see that

$$\begin{aligned} U_{jk} \otimes U_{jk} \mathbf{1}_{2^{2n}} &= (U_{jk} \otimes U_{jk})(\mathbf{1}_{2^n} \otimes \mathbf{1}_{2^n}) \\ &= (\mathbf{1}_{2^n} \otimes \mathbf{1}_{2^n}) = \mathbf{1}_{2^{2n}}. \end{aligned} \tag{6}$$

In other words, each $U_{jk} \otimes U_{jk}$ is a row stochastic matrix with nonnegative entries. To proceed with the proof, we present the Geršgorin disc Theorem as follows.

Lemma 1: (pp. 344, [32]) Let $A = [a_{jk}] \in \mathbb{C}^{n \times n}$. Then all eigenvalues of A are located in the union of n discs

$$\bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} : |z - a_{ii}| \le \sum_{j=1, j \neq i}^{n} |a_{jk}| \Big\}.$$

Using Lemma 1, we know that each Geršgorin disc of matrix M_{jk} is contained in the closed left half plane. This in turn leads to the fact that the Geršgorin disc of

$$L := -\sum_{\{j,k\}\in\mathcal{E}} \alpha_{jk} M_{jk}$$

is within the closed right half plane. Moreover, apparently each M_{jk} is symmetric, and thus all nonzero eigenvalues of L are positive real numbers. Therefore, denoting the smallest eigenvalue other than zero of L as $\lambda_2(L)$, and letting S_0 be the eigenspace corresponding to eigenvalue zero for the matrix L, we know that there exists a constant $c_0 > 0$ such that

$$\Big|\mathbf{vec}(\rho(t)), P_{\mathrm{S}_0}(\mathbf{vec}(\rho_0))\Big| \le c_0 e^{-\lambda_2(L)t}, \quad t \ge 0,$$

where P_{S_0} is the projection onto S_0. This proves the convergence of $\mathbf{vec}(\rho(t))$, and thus the convergence of $\rho(t)$.

Step 2. In this step, we establish some properties of S_0.

The following equation holds:

$$\begin{aligned}\mathrm{S}_0 &= \Big\{\mathbf{vec}(z) : \sum_{\{j,k\}\in\mathcal{E}} \alpha_{jk}\big(U_{jk} z U_{jk}^{\dagger} - z\big) = 0\Big\}\\ &\overset{a)}{=} \Big\{\mathbf{vec}(z) : U_{jk} z U_{jk}^{\dagger} = z, \{j,k\} \in \mathsf{E}\Big\}\\ &\overset{b)}{=} \Big\{\mathbf{vec}(z) : U_{\pi} z U_{\pi}^{\dagger} = z, \pi \in \mathbf{P}\Big\}\\ &\overset{c)}{=} \Big\{\mathbf{vec}(z) : \mathscr{P}_*(z) = z\Big\}. \end{aligned} \tag{7}$$

Here $a)$ is based on Lemma 5.2 in [33], $b)$ holds from the fact that G is a connected graph so that the swapping permutations along each edge consist of a generating set of the group $\mathbf{P}$ (cf. Proposition 8 and Lemma 1 of [15]). Regarding equality $c)$, on one hand it is straightforward that $\Big\{\mathbf{vec}(z) : U_{\pi} z U_{\pi}^{\dagger} = z, \pi \in \mathbf{P}\Big\} \subseteq \Big\{\mathbf{vec}(z) : \mathscr{P}_*(z) = z\Big\}$. On the other hand if $\mathscr{P}_*(z) = z$, then

$$U_{\pi} z U_{\pi}^{\dagger} = U_{\pi}\big(\mathscr{P}_*(z)\big)U_{\pi}^{\dagger} = \mathscr{P}_*(z) = z$$

since $\pi\mathbf{P} = \mathbf{P}$ for any $\pi \in \mathbf{P}$. Thus we also have $\Big\{\mathbf{vec}(z) : \mathscr{P}_*(z) = z\Big\} \subseteq \Big\{\mathbf{vec}(z) : U_{\pi} z U_{\pi}^{\dagger} = z, \pi \in \mathbf{P}\Big\}$. This proves (7).

Step 3. In this step, we show that the limit of $\rho(t)$ must be the given ρ_*.

From the property of S_0 established in Eq. (7), it must hold that

$$\lim_{t\to\infty} \|\mathscr{P}_*(\rho(t)) - \rho(t)\| = 0. \tag{8}$$

Again noting that $\pi\mathbf{P} = \mathbf{P}$ for any $\pi \in \mathbf{P}$, we have $\mathscr{P}_*\big(\rho(t)\big) = \mathscr{P}_*\big(U_{jk}\rho(t)U_{jk}^{\dagger}\big)$. This observation gives us

$$\begin{aligned}&\tfrac{d}{dt}\mathscr{P}_*\big(\rho(t)\big)\\ &\quad = \textstyle\sum_{\{j,k\}\in\mathsf{E}} \alpha_{jk}\Big(\mathscr{P}_*\big(U_{jk}\rho(t)U_{jk}^{\dagger}\big) - \mathscr{P}_*\big(\rho(t)\big)\Big)\\ &\quad \equiv 0. \end{aligned} \tag{9}$$

Finally, combining Eq. (8) and Eq. (9) it becomes clear that $\lim_{t\to\infty}\rho(t) = \mathscr{P}_*(\rho(0)) = \mathscr{P}_*(\rho_0)$. This completes the proof. ■

Remark 1: As shown in [15], $\mathscr{P}_*(\rho_0)$ is the quantum analogue of initial average for classical consensus seeking. The state $\mathscr{P}_*(\rho_0)$ is a symmetric state which is invariant under any swapping operation, and thus every qubit in $\mathscr{P}_*(\rho_0)$ contains exactly the same information.

Remark 2: In classical consensus dynamics, the graph Laplacian plays an essential role [16]. The matrix $L := -\sum_{\{j,k\}\in\mathcal{E}} \alpha_{jk} M_{jk}$ plays the same role here for the considered quantum network to reach a consensus state, which can thus be viewed as the Laplacian of the quantum network. In fact, L admits some similar properties as a classical Laplacian: all off-diagonal entries are non-negative with zero sum along each row. The difference comes from that when the interaction graph G is connected, the multiplicity of the zero eigenvalue of L is no longer one.

Remark 3: Consider

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_{\{j,k\}\in\mathsf{E}} \alpha_{jk}\Big(U_{jk}\rho U_{jk}^{\dagger} - \rho\Big). \tag{10}$$

Assume that H is a time-independent Hermitian Hamiltonian. Introduce $\tilde{\rho}(t) = e^{iHt/\hbar}\rho(t)e^{-iHt/\hbar}$. Some simple calculation leads to the fact that the evolution of $\tilde{\rho}(t)$ satisfies

$$\frac{d\tilde{\rho}}{dt} = \sum_{\{j,k\}\in\mathsf{E}} \alpha_{jk}\Big(U_{jk}\tilde{\rho}U_{jk}^{\dagger} - \tilde{\rho}\Big) \tag{11}$$

if $[H, U_{\pi}] = 0$ for all $\pi \in \mathbf{P}$. In this case, plugging the result in Theorem 1 we have

$$\lim_{t\to\infty}\Big(\rho(t) - e^{-iHt/\hbar}\rho_* e^{iHt/\hbar}\Big) = 0 \tag{12}$$

along (10).

B. Convergence Rate Optimization

It is clear from the proof of Theorem 1 that consensus is reached exponentially for the system (4), and the convergence speed is determined by the smallest eigenvalue other than zero of L, i.e., $\lambda_2(L)$. As a continuous-time and quantum analogue of [17], we consider how to optimally distribute a certain amount, say $q > 0$, of edge weights onto the edges so that the fastest convergence rate can be reached. Given a connected interaction G, with respect to the decision variables $\big\{\alpha_{jk}, \{j,k\} \in \mathsf{E}\big\}$ we formulate the following fastest quantum consensus problem (FQCP):

$$\begin{aligned} &\text{maximize} \quad \lambda_2(L)\\ &\text{subject to} \quad \sum_{\{i,k\}\in\mathsf{E}} \alpha_{jk} \le q. \end{aligned} \tag{13}$$

Denote $m = \dim(\mathrm{S}_0)$ and take an orthonormal basis of S_0: $\xi_1, \dots, \xi_m$. Denote

$$J := qI - L + \sum_{j=1}^{m} \xi_j \xi_j^T.$$

Since all eigenvalues of L are nonnegative and are no larger than q in light of Lemma 1, we conclude that

$$\lambda_2(L) = q - \lambda_{\max}(J) = q - \big\|qI - L + \sum_{j=1}^{m} \xi_j \xi_j^T\big\|_2. \quad (14)$$

Noting that every matrix norm is a convex function, we see from (14) that $\lambda_2(L)$ is a concave function of L. Therefore, the considered FQCP problem is standard convex programming, and thus can be solved efficiently.

IV. Switching Interaction Graph

In this section, we continue to discuss the case with switching interaction graphs. Let Ω denote the set of all undirected graphs over node set $\mathsf{V} = \{1, \dots, n\}$. Let $\sigma(\cdot) : [0, \infty) \mapsto \Omega$ be a piecewise constant function. The obtained time-varying graph is then denoted as $\mathsf{G}_{\sigma(t)} = (\mathsf{V}, \mathsf{E}_{\sigma(t)})$. We assume that there is a constant $\tau_D > 0$ as a lower bound between any two consecutive switching instants of $\sigma(t)$.

The corresponding state evolution of the considered quantum networks under switching interaction graph then becomes

$$\frac{d\rho}{dt} = \sum_{\{j,k\} \in \mathsf{E}_{\sigma(t)}} \alpha_{jk} \Big(U_{jk} \rho U_{jk}^{\dagger} - \rho \Big). \quad (15)$$

Here $\alpha_{jk} > 0$ is again the weight of edge $\{j, k\}$. The following analysis shows that the generalization to time-varying weight $\alpha_{jk}(t)$ is straightforward.

We consider the state evolution of the system (15) with initial time $t_0 \geq 0$ and the initial state $\rho(0) = \rho(t_0)$. We have the following two convergence results.

Theorem 2: The system (15) achieves a global exponential quantum consensus, i.e.,

$$\Big\| \rho(t) - \mathscr{P}_*(\rho(t_0)) \Big\| \leq C(\rho(t_0)) e^{-\gamma(t - t_0)}, \quad t \geq t_0$$

for all initial state $\rho(t_0)$ and initial time $t_0 \geq 0$ with $C(\rho(t_0)) > 0$ (which may depend on the initial state $\rho(t_0)$) and $\gamma > 0$ (which does not depend on $\rho(t_0)$), if and only if there exists a constant $T > 0$ such that $\mathsf{G}([t, t+T)) := (\mathsf{V}, \bigcup_{t \in [t, t+T)} \mathsf{E}_{\sigma(t)})$ is connected for all $t \geq 0$.

Theorem 3: The system (15) achieves a global quantum consensus, i.e.,

$$\lim_{t \to \infty} \rho(t) = \mathscr{P}_*(\rho(t_0))$$

for all initial state $\rho(t_0)$ and all initial time $t_0 \geq 0$, if and only if $\mathsf{G}([t, \infty)) := (\mathsf{V}, \bigcup_{t \in [t, \infty)} \mathsf{E}_{\sigma(t)})$ is connected for all $t \geq 0$.

The proofs of the above two theorems are established on the critical understanding that the quantum consensus of n qubits defines a consensus process on an induced classical graph with 2^{2n} nodes. Under vectorization, the system (15) is equivalent to the following vector form:

$$\frac{d}{dt} \mathbf{vec}(\rho(t)) = -L(\sigma(t)) \mathbf{vec}(\rho(t)), \quad (16)$$

where by definition

$$L(\sigma(t)) := \sum_{\{j,k\} \in \mathsf{E}_{\sigma(t)}} \alpha_{jk} \Big(I_{2^n} \otimes I_{2^n} - U_{jk} \otimes U_{jk} \Big)$$

is a 2^{2n} by 2^{2n} matrix. Note that the system (16) defines a classical consensus over a switching graph. We introduce the following definition.

Definition 1: The induced graph of $L(\sigma(t))$ is defined as the graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, where $\mathcal{V} = \{1, \dots, 2^{2n}\}$ and $\{r, s\} \in \mathcal{E}_{\sigma(t)}, r \neq s \in \mathcal{V}$ if and only if $[L(\sigma(t))]_{rs} \neq 0$.

The following two lemmas establish some fundamental connections between the underlying quantum network $\mathsf{G}_{\sigma(t)}$ and the induced graph $\mathcal{G}_{\sigma(t)}$.

Lemma 2: Let $T > 0$ be a constant. Then $\mathcal{G}([t, t+T))$ has exactly $m = \dim\big(\{\mathbf{vec}(z) : \mathscr{P}_*(z) = z\}\big)$ connected components if $\mathsf{G}([t, t+T))$ is connected.

Lemma 3: Suppose both $\mathsf{G}([T_1, T_2))$ and $\mathsf{G}([T_3, T_4))$ are connected for some $T_1 < T_2 \leq T_3 < T_4$. Then there is a partition of the node set $\mathcal{V} = \bigcup_{k=1}^{m} \mathcal{V}_k$, such that

(i) $\big(\mathcal{G}([T_1, T_2))\big)|_{\mathcal{V}_k}$ forms a connected component for each $k = 1, \dots, m$;

(ii) $\big(\mathcal{G}([T_3, T_4))\big)|_{\mathcal{V}_k}$ also forms a connected component for each $k = 1, \dots, m$.

The detailed proofs of Lemmas 2 and 3 can refer to the extended version of this paper [34]. Based on the understanding established in Lemmas 2 and 3, the necessity statements in Theorems 2 and 3 follow from the same idea and analysis as the properties of classical consensus dynamics (e.g., see the necessity proof of Theorem 4.1 and Theorem 5.2 in [24]).

In light of Lemma 2, denoting

$$T_* := \inf_t \big\{ \mathsf{G}([0, t)) \text{ is connected} \big\},$$

there is a partition of the node set $\mathcal{V} = \bigcup_{k=1}^{m} \mathcal{V}_k$, such that $\big(\mathcal{G}([0, T_*))\big)|_{\mathcal{V}_k}$ forms a connected component for each $k = 1, \dots, m$. Moreover, with Lemma 3, we can further conclude that

$$[L(\sigma(t))]_{rs} = 0,$$

whenever $r \in \mathcal{V}$ and $s \in \mathcal{V}$ belong to different $\mathcal{V}_k$'s. As a result, there is a permutation matrix $W \in \mathbb{R}^{2^{2n} \times 2^{2n}}$ defining a change of indices of $\mathcal{V}$, such that the system (16) can be rewritten according to $W\mathbf{vec}(\rho(t))$ and $L_k(\sigma(t))$ where $L_k(\sigma(t))$ is the Laplacian of the subgraph $\mathcal{G}_{\sigma(t)}|_{\mathcal{V}_k}$ corresponding to node set $\mathcal{V}_k$. We further write $W\mathbf{vec}(\rho(t)) = (y_1^T(t) \dots y_m^T(t))^T$ where $y_k(t)$ is the vector corresponds to the states of nodes in $\mathcal{V}_k$. It is clear that the system (16) defines m completely decoupled classical consensus processes:

$$\frac{d}{dt} y_k(t) = -L_k(\sigma(t)) y_k(t), \quad k = 1, \dots, m. \quad (17)$$

Using the above fact, the proof of the sufficiency claim can be completed. We refer to [34] for the detailed proof.

V. Conclusions

We investigated the convergence of the state of a quantum network with n qubits to a consensus state through continuous-time swapping operators. For fixed quantum interaction graphs, we proved convergence of the network to a consensus state, in which each qubit holds exactly the same state. The convergence rate is determined by the smallest nonzero eigenvalue of a quantum Laplacian matrix associated with the interaction graph. We showed that this convergence rate can be optimized via standard convex programming. For switching quantum interaction graphs, we established necessary and sufficient conditions for exponential quantum consensus and asymptotic quantum consensus, respectively. We revealed that quantum consensus in a network consisting of n qubits naturally defines a consensus process on an induced classical graph with 2^{2n} nodes. With understandings about how the two graphs are related, existing consensus results on classical networks were adopted to establish the quantum consensus convergence results. The results illustrated the possibility of making use of existing distributed control and networking techniques to large-scale quantum systems. Although we assumed that each subsystem in the quantum network is a qubit system, it is straightforward to extend the results to a quantum network consisting of finite-dimensional quantum subsystems. The consensus problem of a quantum network in this paper can be taken as a special class of stabilization problems in quantum control [4]-[10] where the control actions are realized by swapping operators. It is also worth investigating consensus algorithms for other consensus states in quantum networks and developing control methods for stabilizing the states of quantum networks.

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