2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Multi-agent Robust Consensus -Part I: Convergence Analysis

Guodong Shi and Karl Henrik Johansson

***Abstract*— This part of the paper investigates a robust consensus problem for continuous-time multi-agent systems with time-varying communication graphs and weight functions. Convergence rates are presented explicitly with respect to L_∞ and L_1 norms of disturbances. Sufficient and/or necessary connectivity conditions are obtained for the system to reach robust consensus or integral robust consensus.**

Keywords: Multi-agent systems, Robust consensus, Joint connection, Convergence rate

I. INTRODUCTION

Coordination of multi-agent networks has attracted a significant amount of attention in the past few years, due to its broad applications in various fields of science including physics, engineering, biology, ecology and social science [7], [24], [21], [34], [20]. Concerning the issues of interconnected communication, distributed control design on individual dynamics via neighboring information flow has been shown to ensure collective tasks such as formation, flocking, rendezvous, and aggregation [15], [33], [8], [23], [25].

Central to multi-agent coordination study is the study of consensus, or state agreement, which requires that all the agents achieve the desired relative position and the same velocity. Consensus seeking is extensively studied in the literature, in which connectivity of the communication graph plays a key role, and various connectivity conditions have been used to describe frequently switching topologies in different cases [28], [24], [34], [22], [10]. Both continuous-time and discrete-time models are investigated [17], [18], [10], [33]. Researchers are not only concerned with what connectivity conditions can guarantee consensus, but also with the convergence rate: how fast the network reaches a consensus under certain connectivity assumptions [17], [18], [29]. However, few results have been obtained on the convergence rates for continuous-time multi-agent systems reaching a consensus with general (uniformly, or $[t,\infty)$) joint connectivity assumptions, especially when the network is in a noisy environment. Some exceptions include [11], [13], [14], but a clear quantitative description on how much noise can be dealt with by how much communication is still missing.

The primary aim of this paper is to establish the convergence towards a consensus for first-order, continuous-time nonlinear multi-agent systems when there are disturbances entering communication links or in the dynamics with directed and time-varying interconnection graphs. Borrowing ideas from input-to-state stability (ISS) and integral input-to-state stability (iISS) in the agent [30], [31], we reconsider the simple distributed control laws based on relative state feedback over the neighborhood studied in [21], [33] under noisy circumstances. We present explicit convergence rate estimates with respect to L_∞ and L_1 norms of the disturbances respectively. Several necessary and sufficient connectivity conditions are obtained for the system to reach a robust consensus or an integral robust consensus from this analysis. To the best of our knowledge, the results are the first to show that consensus is reached exponentially in t with uniformly joint connected graphs, while exponentially in the times that the joint graph are connected with $[t,\infty)$-joint connection for the considered systems.

The paper is organized as follows. In section II, some preliminary concepts are introduced. We formulate the robust consensus and integral robust consensus problems and the basic assumptions in section III. We then focus on robust consensus and integral robust consensus in sections IV and V, respectively. A number of sufficient and necessary conditions are presented based on the convergence rate for jointly connected interconnection graphs. Comparisons with existing results are given. Finally, concluding remarks are given in section VI. In part II of this paper [27], an application of the results of this paper to event-triggered control is presented.

II. PRELIMINARIES

Here we introduce some theory and notation on graphs and Dini derivatives.

A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E}$. [3]. An element $e = (i, j) \in \mathcal{E}$ is called an *arc* from node $i \in \mathcal{V}$ to $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_n v_n$ of nodes v_i and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, n$, the sequence is called a (directed) *path* with *length* n, and if $v_0 = v_n$ a (directed) *cycle*. A path from i to j is denoted as $i \to j$, and the length of $i \to j$ is denoted as $|i \to j|$. A digraph without cycles is said to be *acyclic*. $\mathcal{G}$ is said to be *strongly connected* if it contains path $i \to j$ and $j \to i$ for every pair of nodes i and j. If there exists a path from node i to node j, then node j is said to be reachable from node i. In particular, each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. $\mathcal{G}$ is said to be *quasi-strongly connected* (QSC) if $\mathcal{G}$ has a center [5].

This work has been supported in part by the Knut and Alice Wallenberg Foundation and the Swedish Research Council. G. Shi and K. H. Johansson are with ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden. Email: guodongs@kth.se, kallej@ee.kth.se

A function $\gamma : R_{\geq 0} \rightarrow R_{\geq 0}$ is said to be a $\mathcal{K}$-class function if it is continuous, strictly increasing, and $\gamma(0) = 0$. Moreover, a function $\beta : R_{\geq 0} \times R_{\geq 0} \rightarrow R$ is a $\mathcal{KL}$-class function if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s, t) \rightarrow 0$ as $t \rightarrow \infty$ for each fixed $s \geq 0$.

The upper Dini derivative of a function h is defined as

$$D^+h(t) = \limsup_{s \to 0^+} \frac{h(t+s) - h(t)}{s}$$

The next result is given for the calculation of Dini derivative [4], [33].

Lemma 2.1: Let $V_i(t, x) : R \times R^m \rightarrow R,\ i = 1, \ldots, n$ be C^1 and $V(t, x) = \max_{i=1,\ldots,n} V_i(t, x)$. If $\mathcal{I}(t) = \{i \in \{1, 2, \ldots, n\} : V(t, x(t)) = V_i(t, x(t))\}$ is the set of indices where the maximum is reached at t, then $D^+V(t, x(t)) = \max_{i \in \mathcal{I}(t)} \dot{V}_i(t, x(t))$.

III. Problem Formulation

Consider a multi-agent system with agent set $\mathcal{V} = \{1, \ldots, N\}$, for which the dynamics of each agent is the following first-order integrator:

$$\dot{x}_i = u_i, \quad i = 1, \ldots, N \tag{1}$$

where $x_i \in R$ represents the state of agent i, and u_i is the control input. Let $x = (x_1, \ldots, x_N)^T$.

The communication in the network is modeled as a time-varying graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ with $\sigma : [0, +\infty) \rightarrow \mathcal{Q}$ as a piecewise constant function, where $\mathcal{Q}$ is a finite set indicating all possible graphs. Node j is said to be a *neighbor* of i at time t when there is an arc $(i, j) \in \mathcal{E}_{\sigma(t)}$, and $\mathcal{N}_i(\sigma(t))$ represents the set of agent i's neighbors at time t.

An assumption is given to the variation of the graph.

A1. *(Dwell Time)* There is a lower bound $\tau_D > 0$ between two consecutive switching time instants of $\sigma(t)$.

Denote the joint graph of $\mathcal{G}_{\sigma(t)}$ in time interval $[t_1, t_2)$ with $t_1 < t_2 \leq +\infty$ as $\mathcal{G}([t_1, t_2)) = \cup_{t \in [t_1, t_2)} \mathcal{G}(t) = (\mathcal{V}, \cup_{t \in [t_1, t_2)} \mathcal{E}_{\sigma(t)})$. Then we have the following definition.

Definition 3.1: (i) $\mathcal{G}_{\sigma(t)}$ is said to be *uniformly (jointly) strongly connected* (USC) if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is strongly connected for any $t \geq 0$.

(ii) $\mathcal{G}_{\sigma(t)}$ is said to be *uniformly (jointly) quasi-strongly connected* (UQSC) if there exists a constant $T > 0$ such that $\mathcal{G}([t, t+T))$ is quasi-strongly connected for any $t \geq 0$.

Let continuous function $a_{ij}(x, t) > 0$ be the weight of arc (j, i). The control input for each agent is presented in the following.

$$u_i = \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(x, t)(x_j - x_i) + w_i(t),\ i = 1, \ldots, N \tag{2}$$

where $w_i(t)$ is a disturbance function.

Then the closed loop system is

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i(\sigma(t))} a_{ij}(x, t)(x_j - x_i) + w_i(t),\ i = 1, \ldots, N \tag{3}$$

An assumption is given to each $a_{ij}(x, t)$.

A2. *(Weights Rule)* There are two constants $0 < a_* \leq a^*$ such that $a_* \leq a_{ij}(x, t) \leq a^*,\ x \in R^N, t \in R^+$.

Denote

$$\|z\|_\infty \triangleq \sup\{|z(t)|, t \geq 0\}$$

with $|z(t)| \triangleq \max_i |z_i(t)|$. Then we define $\mathcal{F} \triangleq \{z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^N |\ z(t)$ is continuous except for a set with measure zero with $\|z\|_\infty < \infty\}$. Then denoting An disturbance assumption is given to the regularity of the function $w(t) = (w_1(t), \ldots, w_n(t))^T$ in order to ensure the existence of the solutions of (3).

A3. *(Noise Regularity)* $w(t) \in \mathcal{F}$.

In this paper, we assume that A1–3 always are standing assumptions. With assumptions A1 and A3, the set of discontinuous points for the right hand side of equation (3) has measure zero. Therefore, Caratheodory solutions [2] of (3) exist for arbitrary initial conditions, and they are absolutely continuous functions that satisfy (3) for almost all t on the maximum interval of existence. Furthermore, it is not hard to see that assumption A2 ensures each (Caratheodory) solution of (3) exists on $[t_0, \infty)$ without finite time escape. In the following, each solution of (3) is considered in the sense of Caratheodory without explicit mention.

Consider (3) with initial condition $x(t_0) = x^0 = (x_1(t_0), \ldots, x_N(t_0))^T \in R^N$, $t_0 \geq 0$. Let

$$\hbar(t) = \max_{i \in \mathcal{V}}\{x_i(t)\}, \quad \ell(t) = \min_{i \in \mathcal{V}}\{x_i(t)\}$$

be the maximum and minimum agent state value at time t . Moreover, denote

$$\mathcal{H}(x(t)) = \hbar(t) - \ell(t)$$

Inspired by ISS [30], we present the following definitions.

Definition 3.2: (i) System (3) achieves a global robust consensus (GRC) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x^0), t) + \gamma(\|w\|_\infty) \tag{4}$$

for all $w \in \mathcal{F}$ and initial conditions $x(t_0) = x^0$.

(ii) System (3) achieves a global integral robust consensus (GIRC) if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x^0), t) + \int_0^t \gamma(|w(s)|) ds, \tag{5}$$

for all $w \in \mathcal{F}$ and initial conditions $x(t_0) = x^0$.

Remark 3.1: In this paper, $|\cdot|$ denotes the maximum norm. All the results obtained hold if $|\cdot|$ denotes the Euclidean norm.

We define global consensus and global asymptotic consensus in the following way.

Definition 3.3: (i) A global consensus (GC) is achieved for system (3) if

$$\lim_{t \to \infty} \mathcal{H}(x(t)) = 0$$

for any initial condition $x(t_0) = x^0$.

(ii) Assume that $\mathcal{F}_0 \subseteq \mathcal{F}$. Then a global asymptotic consensus (GAC) with respect to $\mathcal{F}_0$ is achieved for system (3) if $\forall w \in \mathcal{F}_0$, $\forall \varepsilon > 0$, $\forall c > 0$, $\exists T > 0$ such that $\forall t_0 \geq 0$,

$$\mathcal{H}(x^0) \leq c \Rightarrow \mathcal{H}(x(t)) \leq \varepsilon, \forall t \geq t_0 + T.$$

IV. Robust Consensus

In this section, we study the robust consensus for system (3) with uniformly jointly quasi-strongly connected (UQSC) topology. With out loss of generality, we assume $N \geq 2$ in the sequel.

A. Main Result

Before we present the main result on GRC, we present several useful lemmas.

Lemma 4.1: The following inequalities hold:

$$D^+\hbar(t)| \leq |w(t)|; \quad D^+\ell(t) \geq -|w(t)|$$

for all $t \geq 0$.

Proof: We just focus on the proof of $D^+\hbar(t) \leq |w(t)|$, and the other inequality can be proved in the same way.

Let $\mathcal{I}(t)$ represent the set containing all the agents that reach the maximum in the definition of $\hbar(t)$ at time t, i.e., $\mathcal{I}(t) \triangleq \{i|\ x_i(t) = \hbar(t)\}$. Then according to Lemma 2.1, we obtain

$$\begin{aligned} D^+\hbar(t) &= \max_{i\in\mathcal{I}(t)} \dot{x}_i(t) \\ &= \max_{i\in\mathcal{I}(t)} [\sum_{j\in N_i(\sigma(t))} a_{ij}(x,t)(x_j - x_i) + w_i(t)] \\ &\leq \max_{i\in\mathcal{I}(t)} w_i(t) \\ &\leq |w(t)|. \end{aligned} \tag{6}$$

Then we complete the proof. □

Lemma 4.2: Suppose $\mathcal{G}_{\sigma(t)}$ is UQSC. Then there exists a center i_0 from which there is a path $i_0 \to i$ for all $i \in \mathcal{V}$ in $\mathcal{G}([t, t+\hat{T}))$ with $\hat{T} \triangleq T+2\tau_D$, and each arc of $i_0 \to i$ exists in a time interval with length τ_D at least during $[t, t+\hat{T})$.

Proof: Denote t_1 as the first moment when the interaction topology switches within $[t, t+\hat{T})$ (suppose there are switchings without loss of generality).

If $t_1 \geq t + \tau_D$, then, there exists a center i_0 from which there is a path $i_0 \to i$ for all $i \in \mathcal{V}$ in $\mathcal{G}([t, t+T))$ since $\mathcal{G}([t, t+T))$ is QSC, and moreover, each arc of path $i_0 \to i$ stays there for at least the dwell time τ_D during $[t, t+T+\tau_D)$ due to the definition of τ_D.

On the other hand, if $t_1 < t+\tau_D$, we have $t_1+T+\tau_D < t+\hat{T}$. Then, for any $i \in \mathcal{V}_F$, there is also a center i_0 from which there is a path $i_0 \to i$ for all $i \in \mathcal{V}$ in $\mathcal{G}([t_1, t_1+T))$, each arc of which exists for at least τ_D during $[t_1, t_1+T+\tau_D)$. This completes the proof. □

Based on Lemma 4.2, for all $i = 1, 2, \dots,$ we can define $f(i) = \{j|j$ is a center in $\mathcal{G}([(i-1)\hat{T}, i\hat{T}))$ satisfying the condition of Lemma 4.2$\}$. Therefore, we get a set-valued function $f : \mathbb{Z}^+ \to 2^{\{1,\dots,N\}}$, where $2^{\{1,\dots,N\}}$ represent the set containing all the subsets of $\{1, \dots, N\}$. Moreover, one has $f(i) \neq \emptyset, i = 1, 2, \dots$. The following lemma is given to establish an important property for this set-valued function.

Lemma 4.3: For any $t = 1, 2, \dots,$ there exists $k_0 \in \{1, 2, \dots, N\}$ such that $k_0 \in f(i)$ for as many as at least $N-1$ $i's$ during $i \in [t, t+(N-2)N]$.

Proof: Suppose $k \in f(i)$ for less than $N-1$ times (i.e. less than or equal $N-2$) during $[t, t+(N-2)N]$ for all $k \in \{1, 2, \dots, N\}$. Then, the total number of the elements of all the preimages of f on interval $i \in [t, t+(N-2)N]$ is no larger than $(N-2)N$. However, on the other hand, there are at least $(N-2)N+1$ elements (counting times for the same node) belonging to $f(i)$ during $i \in [t, t+(N-2)N]$ since $f(j) \neq \emptyset, j = 1, 2, \dots$. Then we get the contradiction and the conclusion is proved. □

Suppose $w \in \mathcal{F}$. Then we have the main result on GRC, which is consistent with the study for robust consensus for discrete-time dynamics in [11], [12].

Theorem 4.1: System (3) achieves a global robust consensus (GRC) if and only if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proof: (Sufficiency.) Assume that the initial time is $t_0 = 0$ for simplicity. We will estimate the convergence rate for $\mathcal{H}(x(t))$ during time interval $t \in [sK_0, (s+1)K_0]$, where $s = 0, 1, 2, \dots$ and $K_0 = (N-1)^2\hat{T}$.

Based on Lemma 4.1, one has

$$\hbar(t) \leq \hbar(sK_0) + \|w\|_\infty K_0; \quad \ell(t) \geq \ell(sK_0) - \|w\|_\infty K_0 \tag{7}$$

for any $t \in [sK_0, (s+1)K_0]$.

We divide the rest of the proof into 3 Steps.

Step 1. Lemma 4.3 shows that during $[sK_0, (s+1)K_0]$, in all $(N-1)^2$ subintervals defined by $[j\hat{T}, (j+1)\hat{T}), j = s(N-1)^2, s(N-1)^2+1, \dots, (s+1)(N-1)^2-1$, the joint graphs on at least $N-1$ of them share a common center, k_0. We denote the $N-1$ subintervals with k_0 being a center of the joint graph as $[j_m\hat{T}, (j_m+1)\hat{T}), m = 1, 2, \dots, N-1$. In this step, we show estimations for $x_{k_0}(t)$ on time interval $[sK_0, (s+1)K_0]$.

Without less of generality, we assume that

$$x_{k_0}(sK_0) \leq \frac{\ell(sK_0) + \hbar(sK_0)}{2} \tag{8}$$

because the other condition with $x_{k_0}(sK_0) \geq \frac{\ell(sK_0)+\hbar(sK_0)}{2}$ can be proved similarly. Then with (7), we obtain

$$\begin{aligned} \frac{d}{dt}x_{k_0}(t) \leq& -(N-1)a^*x_{k_0}(t) + (N-1)a^*(\hbar(sK_0)+ \\ & \|w\|_\infty K_0) + \|w\|_\infty \end{aligned} \tag{9}$$

for all $t \in [sK_0, (s+1)K_0]$. As a result, by (8), we have

$$\begin{aligned} x_{k_0}(t) \leq& e^{-(N-1)a^*(t-sK_0)}x_{k_0}(sK_0) + (1- \\ & e^{-(N-1)a^*(t-sK_0)}). \\ & \frac{(N-1)a^*(\hbar(sK_0) + \|w\|_\infty K_0) + \|w\|_\infty}{(N-1)a^*} \\ \leq& \alpha_0\ell(sK_0) + (1-\alpha_0)\hbar(sK_0) + \lambda_0\|w\|_\infty \end{aligned} \tag{10}$$

for all $t \in [sK_0, (s+1)K_0]$, where $\alpha_0 \triangleq \frac{1}{2}e^{-(N-1)a^*K_0}$ and $\lambda_0 \triangleq K_0 + \frac{1}{(N-1)a^*}$.

Step 2. Based on Lemma 4.2, since k_0 is a center in $\mathcal{G}([j_1\hat{T}, (j_1+1)\hat{T}))$, there exist a node $k_1 \in \mathcal{N}$ different from k_0 and a moment $\hat{t}_1$ such that $(k_0, k_1) \in \mathcal{G}_{\sigma(t)}$ for $t \in [\hat{t}_1, \hat{t}_1+\tau_D) \subseteq [j_1\hat{T}, (j_1+1)\hat{T})$. Then in this step, we estimate $x_{k_1}(t)$ on time interval $[\hat{t}_1, (s+1)K_0]$.

There are two cases.

- There exists a moment $\bar{t}_1 \in [\hat{t}_1, \hat{t}_1 + \tau_D]$ such that

$$x_{k_1}(\bar{t}_1) \leq x_{k_0}(\bar{t}_1). \quad (11)$$

- Otherwise, we have $x_{k_1}(t) \geq x_{k_0}(t), t \in [\hat{t}_1, \hat{t}_1 + \tau_D)$. Thus, one has

$$\frac{d}{dt}x_{k_1}(t) \leq -((N-2)a^* + a_*)x_{k_1}(t) + (N-2)a^* (\hbar(sK_0) + \|w\|_\infty K_0) + a_*(\alpha_0 \ell(sK_0) + (1-\alpha_0)\hbar(sK_0) + \lambda_0\|w\|_\infty) + \|w\|_\infty$$

for $t \in [\hat{t}_1, \hat{t}_1 + \tau_D)$. Therefore, denoting $m_0 = e^{-((N-2)a^* + a_*)\tau_D}$, we have

$$x_{k_1}(\hat{t}_1 + \tau_D) \leq \zeta\alpha_0\ell(sK_0) + (1-\zeta\alpha_0)\hbar(sK_0) + (\lambda_0 + \lambda_1)\|w\|_\infty \quad (12)$$

where

$$\zeta \triangleq \frac{(1-m_0)a_*}{(N-2)a^* + a_*},\ \lambda_1 \triangleq K_0 + \frac{(1-m_0)(1-a_*K_0)}{(N-2)a^* + a_*}.$$

Noting the fact that $\zeta < 1$ and (9) also holds for $x_{k_1}(t)$, both (11) and (12) lead to

$$x_{k_1}(t) \leq \alpha_1\ell(sK_0) + (1-\alpha_1)\hbar(sK_0) + \lambda_0\|w\|_\infty + \hat{\lambda}_0\|w\|_\infty,$$

for $t \in [(j_1+1)\hat{T}, (s+1)K_0]$, where $\alpha_1 = \hat{\zeta} \cdot \alpha_0$ with $\hat{\zeta} = e^{-(N-1)a^*K_0}\zeta$, and $\hat{\lambda}_0 = \lambda_0 + \lambda_1$.

Step 3. We proceed similar analysis on time intervals $[j_m\hat{T}, (j_m+1)\hat{T})$ for $m = 3, \ldots, N-1$, and estimations for nodes $k_m, m = 3, \ldots, N-1$ can be given by

$$x_{k_m}(t) \leq \alpha_m\ell(sK_0) + (1-\alpha_m)\hbar(sK_0) + \lambda_0\|w\|_\infty + m\hat{\lambda}_0\|w\|_\infty,$$

for $t \in [(j_m+1)\hat{T}, (s+1)K_0]$, where $\alpha_m = \hat{\zeta}^m \cdot \alpha_0, m = 2, \ldots, N-1$. Moreover, every two k_m's are distinct.

Therefore, noting the fact that $\alpha_0 < \alpha_1 < \cdots < \alpha_{N-1} < 1$, we obtain

$$\hbar((s+1)K_0) \leq \alpha_{N-1}\ell(sK_0) + (1-\alpha_{N-1})\hbar(sK_0) + \lambda_0\|w\|_\infty + (N-1)\hat{\lambda}_0\|w\|_\infty, \quad (13)$$

which implies

$$\mathcal{H}(x((s+1)K_0)) \leq (1-\alpha_{N-1})\mathcal{H}(x(sK_0)) + \gamma_0\|w\|_\infty,$$

where $\gamma_0 \triangleq \lambda_0 + (N-1)\hat{\lambda}_0 + K_0$. Because it holds that $0 < \alpha_{N-1} = \hat{\zeta}^{N-1} \cdot \alpha_0 < 1$, we obtain

$$\mathcal{H}(x(nK_0)) \leq (1-\alpha_{N-1})^n\mathcal{H}(x^0) + \frac{\gamma_0}{\alpha_{N-1}} \cdot \|w\|_\infty$$

for any $n = 0, 1, \ldots$. The global robust consensus is therefore obtained by

$$\beta(\mathcal{H}(x^0), t) = (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor}\mathcal{H}(x^0),$$

and

$$\gamma(\|w\|_\infty) = (2K_0 + \frac{\gamma_0}{\alpha_{N-1}}) \cdot \|w\|_\infty,$$

where $\lfloor \frac{t}{K_0} \rfloor$ denotes the largest integer no greater than $\frac{t}{K_0}$.

(Necessity.) Assume that $\mathcal{G}_{\sigma(t)}$ is not UQSC. Then for any $T_* > 0$ there exists $t_* \geq 0$ such that $\mathcal{G}([t_*, t_* + T_*))$ is not quasi-strongly connected. Thus, there exists two distinct nodes i and j such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, where $\mathcal{V}_1 = \{$nodes from which i is reachable in $\mathcal{G}([t_*, t_* + T_*))\}$ and $\mathcal{V}_2 = \{$nodes from which j is reachable in $\mathcal{G}([t_*, t_* + T_*))\}$. Taking $w_i(t) \equiv 0, i \in \mathcal{V}_1$ and $w_i(t) \equiv 1, i \in \mathcal{V}_2$ when $t \in [t_*, t_* + T_*]$. Let initial condition $t_0 = t_*$ with $x_i(t_*) = 0, \forall i \in \mathcal{V}$. Then it is not hard to find that $\mathcal{H}(x(t_* + T_*)) = T_*$. Therefore, the global robust consensus cannot be achieved since T_* can be arbitrarily large. □

Remark 4.1: If we assume that $w(t) \equiv 0$ in (3), we obtain

$$\mathcal{H}(x(t)) \leq (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor}\mathcal{H}(x^0) \leq (1-\alpha_{N-1})^{\frac{t}{K_0} - 1}\mathcal{H}(x^0) = \frac{1}{1-\alpha_{N-1}} \cdot e^{t \cdot \frac{\ln(1-\alpha_{N-1})}{K_0}}\mathcal{H}(x^0),$$

which implies that the multi-agent network will reach a consensus exponentially.

B. Consensus with Noise

We define a set

$$\mathcal{F}_1 \triangleq \{z(t) \in \mathcal{F}_\infty : \lim_{t\to\infty} z(t) = 0\}.$$

Then the following conclusion holds on GC.

Proposition 4.1: System (3) achieves a GC for any $w \in \mathcal{F}_1$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proof: Let $w_0 \in \mathcal{F}_1$ be a fixed function. Then, $\forall \varepsilon > 0$, $\exists T(\varepsilon) > 0$ such that $|w_0(t)| < \gamma^{-1}(\varepsilon), \forall t \geq T(\varepsilon)$. Thus, applying Theorem 4.1 on system (3) with $t_0 = T(\varepsilon)$, we obtain

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x(T(\varepsilon))), t - T(\varepsilon)) + \varepsilon. \quad (14)$$

Then since ε can be arbitrarily small, the global consensus follows immediately by taking $t \to \infty$ in (14). □

Let $\mathcal{F}_1^0 \subseteq \mathcal{F}_1$ be a subset with $\lim_{t\to\infty}\sup_{z\in\mathcal{F}_1^0}|z(t)| = 0$. Then the following result is given on GAC.

Proposition 4.2: System (3) achieves a GAC with respect to $\mathcal{F}_1^0$ if and only if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proof: The necessity part follows by the same argument as the proof of Theorem 4.1. We focus on the sufficiency part. $\forall \varepsilon > 0$, $\exists \tilde{T}(\varepsilon) > 0$ such that $|w(t)| < \gamma^{-1}(\frac{\varepsilon}{2}), \forall t \geq \tilde{T}(\varepsilon), \forall w \in \mathcal{F}_1^0$. Denoting $\omega^* = \sup_{t\in[t_0,\tilde{T}]}\{\sup_{z\in\mathcal{F}_1^0}|z(t)|\}$, there will be two cases.

- When $t_0 < \tilde{T}(\varepsilon)$, one has $\forall t \geq t_0$,

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x(\tilde{T}(\varepsilon))), t - \tilde{T}(\varepsilon)) + \frac{\varepsilon}{2} \leq \beta(\beta(\mathcal{H}(x^0) + \gamma(\omega^*), 0), t - \tilde{T}(\varepsilon)) + \frac{\varepsilon}{2}.$$

  Furthermore, $\forall c > 0$, $\exists T_1(c, \tilde{T}(\varepsilon)) > 0$ such that

$$\beta(\beta(c + \gamma(\omega^*), 0), t - \tilde{T}(\varepsilon)) \leq \frac{\varepsilon}{2}, \forall t > T_1,$$

- When $t_0 \geq \tilde{T}(\varepsilon)$, one has $\forall t \geq t_0$,

$$\mathcal{H}(x(t)) \leq \beta(\mathcal{H}(x^0), t - t_0) + \frac{\varepsilon}{2}. \quad (15)$$

Then $\forall c > 0$, $\exists T_2(c) > 0$ such that $\beta(\mathcal{H}(x^0), t - t_0) < \frac{\varepsilon}{2}, \forall t > T_2$.

Taking $T = \max\{T_1, T_2\}$, we obtain

$$\mathcal{H}(x^0) \leq c \Rightarrow \mathcal{H}(x(t)) \leq \varepsilon, \forall t \geq t_0 + T, \forall w \in \mathcal{F}_1^0.$$

This completes the proof. □

Remark 4.2: When $w(t) \equiv 0, t \geq t_0$, Proposition 4.2 is consistent with the main result, that is, Theorem 3.8 in [33].

C. Discussions: USC Graph

Note that, if $\mathcal{G}_{\sigma(t)}$ is USC, k_0 will be the center of joint graphs on $N-1$ subintervals $[t, t+\hat{T}), \ldots, [t+(N-2)\hat{T}, t+(N-1)\hat{T})$. Therefore, we can replace K_0 by $K_* \triangleq (N-1)\hat{T}$ in the proof of Theorem 4.1, and then the GRC inequality can also be given by $\beta_*(\mathcal{H}(x^0), t) = (1-\alpha_{N-1}^*)^{\lfloor \frac{t}{K_*} \rfloor}\mathcal{H}(x^0)$ and $\gamma_*(\|w\|_\infty) = (2K_* + \frac{\gamma_0^*}{\alpha_{N-1}^*}) \cdot \|w\|_\infty$, where α_{N-1}^* and γ_0^* are defined by replacing K_0 by K_* in the definition of α_{N-1} and γ_0, respectively.

Moreover, if $w(t) \equiv 0$ in (3), we have

$$\mathcal{H}(x(t)) \leq \frac{1}{1-\alpha_{N-1}^*} \cdot e^{t \cdot \frac{\ln(1-\alpha_{N-1}^*)}{K_*}} \mathcal{H}(x^0). \quad (16)$$

Then a faster convergence rate is achieved since we have that $\alpha_{N-1}^* < \alpha_{N-1}$ and $K_* < K_0$.

V. Integral Robust Consensus

In this section, we study the global integral robust consensus for system (3). We present the main results in the first subsection, and then the detailed proofs are given.

A. Main Results

Previous discussions show that UQSC is sufficient and necessary for GRC. However, it is not true in regard to GIRC. The following conclusion shows that UQSC is still sufficient to ensure GIRC.

Theorem 5.1: System (3) achieves a GIRC if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Furthermore, we present two sufficient and necessary conditions for GIRC under more restrictive communications.

Theorem 5.2: Suppose $\mathcal{G}_{\sigma(t)}$ is undirected for any $t \geq 0$. Then System (3) achieves a GIRC if and only if $\mathcal{G}([t, \infty))$ is QSC for any $t \geq 0$.

Theorem 5.3: Suppose $\mathcal{G}([0, +\infty))$ is acyclic. Then System (3) achieves a GIRC if and only if $\mathcal{G}([t, \infty))$ is QSC for any $t \geq 0$.

Furthermore, we define another set

$$\mathcal{F}_2 \triangleq \{z \in \mathcal{F} | \int_0^\infty |z(t)| dt < \infty\}$$

Then we present the following conclusions on GC and GAC.

Proposition 5.1: Let $\mathcal{F}_2^0 \subseteq \mathcal{F}_2$ be a subset with $\int_0^\infty \sup_{z \in \mathcal{F}_2^0} |z(t)| dt < \infty$. Then System (3) achieves a GAC with respect to $\mathcal{F}_2^0$ if $\mathcal{G}_{\sigma(t)}$ is UQSC.

Proposition 5.2: Assume that either $\mathcal{G}_{\sigma(t)}$ being undirected for any $t \geq 0$, or $\mathcal{G}([0, +\infty))$ being acyclic. Then system (3) achieves a GC for all $w \in \mathcal{F}_2$ if and only if $\mathcal{G}([t, \infty))$ is QSC for any $t \geq 0$.

B. Proofs

In this subsection, we prove the various statements which are given in previous subsection. We will use the same notations for parameters $\alpha_0, \ldots, \alpha_{N-1}, \zeta, \hat{\zeta}$, which are defined in the proof of Theorem 4.1 in the following.

Proof of Theorem 5.1 Denote $\theta_s = \int_{sK_0}^{(s+1)K_0} |w(t)| dt, s = 0, 1, 2, \ldots$. Based on Lemma 4.1, one has

$$\hbar(t) \leq \hbar(s\hat{T}) + \theta_s; \quad \ell(t) \geq \ell(s\hat{T}) - \theta_s \quad (17)$$

for any $t \in [sK_0, (s+1)K_0]$.

We divide the rest of the proof into 3 Steps.

Step 1. There is also a node k_0 which is a center of joint graphs on $N-1$ time intervals $[j_m\hat{T}, (j_m+1)\hat{T}) \subseteq [sK_0, (s+1)K_0], m = 1, 2, \ldots, N-1$. Then we give estimations for $x_{k_0}(t)$ on time interval $[sK_0, (s+1)K_0]$ in condition that $x_{k_0}(sK_0) \leq \frac{\ell(sK_0)+\hbar(sK_0)}{2}$ in this step.

We see that

$$\frac{d}{dt} x_{k_0}(t) \leq (N-1)a^*(\hbar(sK_0) + \theta_s - x_{k_0}(t)) + |w(t)|$$

for $t \in [sK_0, (s+1)K_0]$, which implies

$$\begin{aligned} x_{k_0}(t) &\leq \alpha_0 \ell(sK_0) + (1-\alpha_0)\hbar(sK_0) + \theta_s \\ &\quad + \int_{sK_0}^t e^{-(N-1)a^*(t-\tau)} |w(\tau)| d\tau \\ &\leq \alpha_0 \ell(sK_0) + (1-\alpha_0)\hbar(sK_0) + 2\theta_s \end{aligned}$$

for $t \in [sK_0, (s+1)K_0]$. *Step 2.* In this step, we also estimate $x_{k_1}(t)$ on time interval $[\hat{t}_1, (s+1)K_0]$ with $(k_0, k_1) \in \mathcal{G}_{\sigma(t)}$ for $t \in [\hat{t}_1, \hat{t}_1 + \tau_D) \subseteq [j_1\hat{T}, (j_1+1)\hat{T})$. There are two cases.

- There exists a moment $\tilde{t}_1 \in [\hat{t}_1, \hat{t}_1 + \tau_D]$ such that

$$x_{k_1}(\tilde{t}_1) \leq x_{k_0}(\tilde{t}_1) \leq \alpha_0 \ell(sK_0) + (1-\alpha_0)\hbar(sK_0) + 2\theta_s. \quad (18)$$

- Otherwise, we have $x_{k_1}(t) \geq x_{k_0}(t), t \in [\hat{t}_1, \hat{t}_1 + \tau_D)$. Similarly one has

$$x_{k_1}(\hat{t}_1 + \tau_D) \leq \zeta\alpha_0 \ell(sK_0) + (1-\zeta\alpha_0)\hbar(sK_0) + 3\theta_s$$

Thus, we have

$$x_{k_1}(t) \leq \alpha_1 \ell(sK_0) + (1-\alpha_1)\hbar(sK_0) + 4\theta_s \quad (19)$$

with $t \in [(j_1+1)\hat{T}, (s+1)K_0]$ for both of the two cases.

Step 3. Proceeding similar analysis on time intervals $[j_m\hat{T}, (j_m+1)\hat{T})$ for $m = 3, \ldots, N-1$, estimations for distinct nodes $k_m, m = 3, \ldots, N-1$ can be given by

$$x_{k_m}(t) \leq \alpha_m \ell(sK_0) + (1-\alpha_m)\hbar(sK_0) + 4m\theta_s, \quad (20)$$

for $t \in [(j_m+1)\hat{T}, (s+1)K_0]$, which implies

$$\mathcal{H}(x((s+1)K_0)) \leq (1-\alpha_{N-1})\mathcal{H}(x(sK_0)) + (4N-3)\theta_s.$$

Because it holds that $0 < \alpha_{N-1} = \hat{\zeta}^{N-1} \cdot \alpha_0 < 1$, we obtain

$$\mathcal{H}(x(nK_0)) \leq (1-\alpha_{N-1})^n \mathcal{H}(x^0) + (4N-3) \sum_{j=0}^{n-1} (1-\alpha_{N-1})^{n-1-j} \theta_j \quad (21)$$

for any $n = 0, 1, 2, \dots$. Therefore, we obtain

$$\mathcal{H}(x(t)) \leq (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor}\mathcal{H}(x^0) + (4N-3) \int_0^t (1-\alpha_{N-1})^{\lfloor \frac{t}{K_0} \rfloor - g(\tau)} |w(\tau)| d\tau \qquad (22)$$

where

$$g(\tau) = \begin{cases} i+1, & \tau \in [iK_0, (i+1)K_0), i = 0, \dots, \lfloor \frac{t}{K_0} \rfloor - 1 \\ \lfloor \frac{t}{K_0} \rfloor, & \tau \in [\lfloor \frac{t}{K_0} \rfloor \cdot K_0, t] \end{cases} \qquad (23)$$

Then it is straightforward to see the global integral robust consensus holds. □

Proof of Theorem 5.2 See [26].

Proof of Theorem 5.3 See [26].

Proofs of Propositions 5.1 and 5.2 Note that, suppose $\{b_j, j = 1, 2, \dots\}$ is a sequence with $\sum_{j=1}^{\infty} |b_j| < \infty$ and $0 < a < 1$, then we have $\lim_{n\to\infty} \sum_{j=1}^{n} a^{n-j} b_j = 0$. Then the conclusions hold immediately from the GIRC estimations. □

Remark 5.1: When $\mathcal{G}_{\sigma(t)}$ is USC, similar estimation as (22) can also be given by similar form with K_* and α^*_{N-1}.

VI. CONCLUSIONS

This paper focused on the multi-agent consensus problem in a noisy environment. Our central aim was to draw a clear picture, on "how much connectivity is required on how much uncertainties" for the networks to agree asymptotically. The ideas of input-to-state stability and integral input-to-state stability inspired us to build the definitions of robust consensus and integral robust consensus. Sufficient and necessary connectivity conditions were given, and convergence rates are shown explicitly.

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