Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

The Evolution of Beliefs over Signed Social Networks

Guodong Shi, Alexandre Proutiere, Mikael Johansson, John S. Baras, Karl H. Johansson

Operations Research
Vol. 64, No. 3, May–June 2016, pp. 585–604
ISSN 0030-364X (print) | ISSN 1526-5463 (online)
http://dx.doi.org/10.1287/opre.2015.1448

The Evolution of Beliefs over Signed Social Networks

Guodong Shi
Research School of Engineering, CECS, The Australian National University, Canberra ACT 0200, Australia,
guodong.shi@anu.edu.au

Alexandre Proutiere, Mikael Johansson
ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden
{alepro@kth.se, mikaelj@kth.se}

John S. Baras
Institute for Systems Research, University of Maryland, College Park, Maryland 20742, baras@umd.edu

Karl H. Johansson
ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden,
kallej@kth.se

We study the evolution of opinions (or beliefs) over a social network modeled as a signed graph. The sign attached to an edge in this graph characterizes whether the corresponding individuals or end nodes are friends (positive links) or enemies (negative links). Pairs of nodes are randomly selected to interact over time, and when two nodes interact, each of them updates its opinion based on the opinion of the other node and the sign of the corresponding link. This model generalizes the DeGroot model to account for negative links: when two adversaries interact, their opinions go in opposite directions. We provide conditions for convergence and divergence in expectation, in mean-square, and in almost sure sense and exhibit phase transition phenomena for these notions of convergence depending on the parameters of the opinion update model and on the structure of the underlying graph. We establish a *no-survivor* theorem, stating that the difference in opinions of any two nodes diverges whenever opinions in the network diverge as a whole. We also prove a *live-or-die* lemma, indicating that almost surely, the opinions either converge to an agreement or diverge. Finally, we extend our analysis to cases where opinions have hard lower and upper limits. In these cases, we study when and how opinions may become asymptotically clustered to the belief boundaries and highlight the crucial influence of (strong or weak) structural balance of the underlying network on this clustering phenomenon.

Keywords: opinion dynamics, signed graph, social networks, opinion clustering.
Subject classifications: networks/graphs: stochastic, theory; planning: community.
Area of review: Special Issue on Information and Decisions in Social and Economic Networks.
History: Received July 2013; revisions received November 2014, August 2015; accepted September 2015. Published online in *Articles in Advance* February 18, 2016.

1. Introduction

1.1. Motivation

We all form opinions about economical, political, and social events that take place in society. These opinions can be binary (e.g., whether one supports a candidate in an election or not) or continuous (to what degree one expects a prosperous future economy). Our opinions are revised when we interact with each other over various social networks. Characterizing the evolution of opinions and understanding the dynamic and asymptotic behavior of the social belief are fundamental challenges in the theoretical study of social networks.

Building a good model on how individuals interact and influence each other is essential for studying opinion dynamics. In interaction models, it is natural that a trusted friend should have a different influence on opinion formation than would a dubious stranger. The observation that sentiment influences opinions can be traced back to the 1940s when Heider (1946) introduced the theory of signed social networks, where each interaction link in the social network is associated with a sign (positive or negative) indicating whether two individuals are friends or enemies. Efforts to understand the structural properties of signed social networks have led to the development of structural balance theory, with seminal contributions by Cartwright and Harary (1956) and Davis (1963, 1967). A fundamental insight from these studies, formalized in Harary's theorem (Harary 1953), is that local structural properties imply hard global constraints on the social network formation.

In this paper, we attempt to model the evolution of opinions in signed social networks when local hostile or antagonistic relations influence the global social belief. The relative strengths and structures of positive and negative relations are shown to have an essential effect on opinion convergence. In some cases, tight conditions for convergence and divergence can be established.

1.2. Related Work

The concept of signed social networks was introduced by Heider (1946). His objective was to formally distinguish between friendly (positive) and hostile (negative) relationships. The notion of structural balance was introduced to understand local interactions and formalize intricate local scenarios (e.g., *two of my friends are enemies*). A number of classical results on social balance was established by Harary (1953), Cartwright and Harary (1956), Davis (1963, 1967), who derived critical conditions on the global structure of the social network which ensure structural balance. Social balance theory has since become an important topic in the study of social networks. On one hand, efforts are made to characterize and compute the degree of balance for real-world large social networks, e.g., Facchetti et al. (2011). On the other hand, dynamical models are proposed for the signs of social links with the aim of describing stable equilibria or establishing asymptotic convergence for the sign patterns, e.g., Galam (1996) (where a signed structure was introduced as a revised Ising model of political coalitions, where two competing world coalitions were shown to have one unique stable formation); Macy et al. (2003) (who verified convergence to structural balances numerically for a Hopfield model); and Marvel et al. (2011) (where a continuous-time dynamical model for the link signs was proposed under which convergence to structural balance was proven).

Opinion dynamics is another longstanding topic in the study of social networks; see Jackson (2008) and Easley and Kleinberg (2010) for recent textbooks. Following the survey Acemoglu and Ozdaglar (2011), we classify opinion evolution models into Bayesian and non-Bayesian updating rules. Their main difference lies in whether each node has access to and acts according to a global model or not. We refer to Banerjee (1992), Bikhchandani et al. (1992), and more recent work (Acemoglu et al. 2011) for Bayesian opinion dynamics. In non-Bayesian models, nodes follow simple local updating strategies. DeGroot's model (DeGroot 1974) is a classical non-Bayesian model of opinion dynamics, where each node updates its belief as a convex combination of its neighbors' beliefs; e.g., DeMarzo et al. (2003), Golub and Jackson (2010), Jadbabaie et al. (2012). Note that DeGroot's model relates to averaging consensus algorithms; e.g., Tsitsiklis (1984), Xiao and Boyd (2004), Boyd et al. (2006), Tahbaz-Salehi and Jadbabaie (2008), Fagnani and Zampieri (2008), Touri and Nedić (2011), Matei et al. (2013). Nonconsensus asymptotic behaviors, e.g., clustering, disagreement, and polarization, have been investigated for linear or nonlinear variations of DeGroot-type update rules, Krause (1997), Blondel et al. (2009, 2010), Dandekar et al. (2013), Shi et al. (2013), Li et al. (2013). Various models from statistical physics have also been applied to study social opinion dynamics; please refer to Castellano et al. (2009) for a survey.

The influence of misbehaving nodes in social networks hase been studied only to some extent. For instance, in Acemoglu et al. (2010), a model of the spread of misinformation in large societies was discussed. There, some individuals are *forceful*, meaning that they influence the beliefs of some of the other individuals they meet but do not change their own opinions. In Acemoglu et al. (2013), the authors studied the propagation of opinion disagreement under DeGroot's model, when some nodes stick to their initial beliefs during the entire evolution. This idea was extended to binary opinion dynamics under the voter model in Yildiz et al. (2013). In Altafini (2012, 2013), the author proposed a linear model for belief dynamics over signed graphs. In Altafini (2013), it was shown that a bipartite agreement, i.e., clustering of opinions, is reached as long as the signed social graph is strongly balanced in the sense of the classical structural balance theory (Cartwright and Harary 1956), which presents an important link between opinion dynamics and structure balance. However, in the model studied in Altafini (2012, 2013), all beliefs converge to a common value, equal to zero, if the graph is not strongly balanced. This behavior seems to be difficult to interpret and justify from real-world observations. A game-theoretical approach for studying the interplay between good and bad players in collaborative networks was introduced in Theodorakopoulos and Baras (2008).

1.3. Contribution

We propose and analyze a new model for belief dynamics over signed social networks. Nodes randomly execute pairwise interactions to update their beliefs. In case of a positive link (representing that the two interacting nodes are friends), the update follows DeGroot's update rule, which drives the two beliefs closer to each other. On the contrary, in case of a negative link (i.e., when the two nodes are adversaries (enemies)), the update increases the difference between the two beliefs. Thus, two opposite types of opinion updates are defined, and the beliefs are driven not only by random node interactions but also by the type of relationship of the interacting nodes. Under this simple attraction-repulsion model for opinions on signed social networks, we establish a number of fundamental results on belief convergence and divergence and study the impact of the parameters of the update rules and of the network structure on the belief dynamics.

Using classical spectral methods, we derive conditions for mean and mean-square convergence and divergence of beliefs. We establish phase transition phenomena for these notions of convergence and study how the thresholds depend on the parameters of the opinion update model and on the structure of the underlying graph. We derive phase transition conditions for almost sure convergence and divergence of beliefs. The proofs are based on what we call the *Triangle lemma*, which characterizes the evolution of the beliefs held by three different nodes. We utilize probabilistic tools such as the Borel–Cantelli lemma, the Martingale convergence theorems, the strong law of large numbers, and sample-path arguments.

We establish two counterintuitive results about the way beliefs evolve: (i) a *no-survivor* theorem, which states that the difference between opinions of any two nodes tends to infinity almost surely (along a subsequence of instants) whenever the difference between the maximum and the minimum beliefs in the network tends to infinity (along a subsequence of instants), and (ii) a *live-or-die* lemma, which demonstrates that almost surely, the opinions either converge to an agreement or diverge. We also show that networks whose positive component includes a hypercube are (essentially, the only) robust networks in the sense that almost sure convergence of beliefs holds irrespective of the number of negative links, their positions in the network, and the strength of the negative update.

The considered model is extended to cases where updates may be asymmetric (in the sense that when two nodes interact, only one of them updates its belief), and where beliefs have hard lower and upper constraints. The latter boundedness constraint adds slight nonlinearity to the belief evolution. It turns out in this case that the classical social network structural balance theory plays a fundamental role in determining the asymptotic formation of opinions:

- If the social network is structurally balanced (strongly balanced or complete and weakly balanced), i.e., the network can be divided into subgroups with positive links inside each subgroup and negative links among different subgroups, then almost surely the beliefs within the same subgroup will be clustered to one of the belief boundaries, when the strength of the negative updates is sufficiently large.
- In the absence of structural balance, and if the positive graph of the social network is connected, then almost surely the belief of each node oscillates between the lower and upper bounds and touches the two belief boundaries an infinite number of times.

For balanced social networks, the boundary clustering results are established based on the almost sure happening of suitable *separation* events; i.e., the node beliefs for a subgroup become group polarized (either larger or smaller than the remaining nodes' beliefs). From this argument such events tend to happen more easily in the presence of small subgroups. As a result, small subgroups contribute to faster clustering of the social beliefs, which is consistent with the study of minority influence in social psychology (Nemeth 1986, Clark and Maass 1990), suggesting that consistent minorities can substantially influence opinions. For unbalanced social networks, the established opinion oscillation contributes to a new type of belief formation that complements polarization, disagreement, and consensus (Dandekar et al. 2013).

1.4. Paper Organization

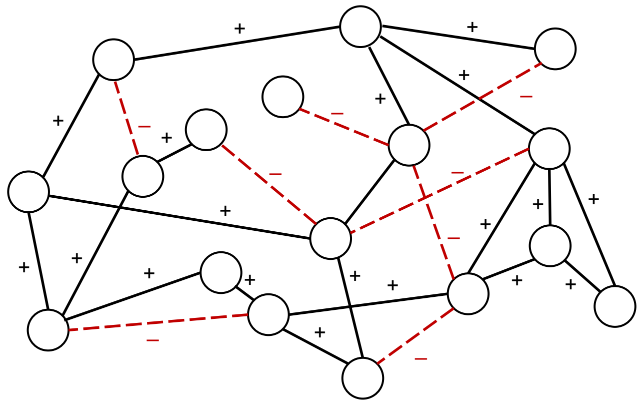
In §2, we present the signed social network model, specify the dynamics along positive and negative links, and define the problem of interest. Section 3 focuses on the mean and mean-square convergence and divergence analysis, and §4 considers convergence and divergence in the almost sure sense. In §5, we study a model with upper and lower belief bounds and asymmetric updates. It is shown how structural balance determines the clustering of opinions. Finally, concluding remarks are given in §6. The proofs of the main statements are in the appendices and some nonessential proofs have been put in the online e-companion (available as supplemental material at http://dx.doi.org/10.1287/opre.2015.1448).

1.5. Notation and Terminology

An undirected graph is denoted by $\mathsf{G} = (\mathsf{V}, \mathsf{E})$. Here $\mathsf{V} = \{1, \ldots, n\}$ is a finite set of vertices (nodes). Each element in E is an unordered pair of two distinct nodes in V called an edge. The edge between nodes $i, j \in \mathsf{V}$ is denoted by $\{i, j\}$. Let $\mathsf{V}_* \subseteq \mathsf{V}$ be a subset of nodes. The *induced* graph of V_* on G, denoted $\mathsf{G}_{\mathsf{V}_*}$, is the graph $(\mathsf{V}_*, \mathsf{E}_{\mathsf{V}_*})$ with $\{u, v\} \in \mathsf{E}_{\mathsf{V}_*}$, $u, v \in \mathsf{V}_*$ if and only if $\{u, v\} \in \mathsf{E}$. A path in G with length k is a sequence of distinct nodes, $v_1 v_2 \ldots v_{k+1}$, such that $\{v_m, v_{m+1}\} \in \mathsf{E}$, $m = 1, \ldots, k$. The length of a shortest path between two nodes i and j is called the distance between the nodes, denoted $d(i, j)$. The greatest length of all shortest paths is called the diameter of the graph, denoted $\mathrm{diam}(\mathsf{G})$. The *degree* matrix of G, denoted $\mathrm{D}(\mathsf{G})$, is the diagonal matrix $\mathrm{diag}(d_1, \ldots, d_n)$ with d_i denoting the number of nodes sharing an edge with i, $i \in \mathsf{V}$. The *adjacency* matrix $\mathrm{A}(\mathsf{G})$ is the symmetric $n \times n$ matrix such that $[\mathrm{A}(\mathsf{G})]_{ij} = 1$ if $\{i, j\} \in \mathsf{E}$ and $[\mathrm{A}(\mathsf{G})]_{ij} = 0$ otherwise. The matrix $\mathrm{L}(\mathsf{G}) := \mathrm{D}(\mathsf{G}) - \mathrm{A}(\mathsf{G})$ is called the *Laplacian* of G. Two graphs containing the same number of vertices are called *isomorphic* if they are identical, subject to a permutation of vertex labels.

All vectors are column vectors and denoted by lowercase letters. Matrices are denoted with uppercase letters. Given a matrix M, M' denotes its transpose and M^k denotes the k-th power of M when it is a square matrix. The ij-entry of a matrix M is denoted $[M]_{ij}$. Given a matrix $M \in \mathbb{R}^{mn}$, the vectorization of M, denoted by $\mathbf{vec}(M)$, is the $mn \times 1$ column vector $([M]_{11}, \ldots, [M]_{m1}, \ldots, [M]_{1n}, \ldots, [M]_{mn})'$. We have $\mathbf{vec}(ABC) = (C' \otimes A)\mathbf{vec}(B)$ for all real matrices A, B, C with ABC well defined. A square matrix M is called a stochastic matrix if all of its entries are nonnegative and the sum of each row of M equals one. A stochastic matrix M is doubly stochastic if M' is also a stochastic matrix. With the universal set prescribed, the complement of a given set S is denoted S^c. The orthogonal complement of a subspace S in a vector space is denoted $S^\perp$. Depending on the argument, $|\cdot|$ stands for the absolute value of a real number, the Euclidean norm of a vector, and the cardinality of a set. Similarly with argument well defined, $\sigma(\cdot)$ represents the σ-algebra of a random variable (vector) or the spectrum of a matrix. The smallest integer no smaller than a given real number a is denoted $\lceil a \rceil$. We use $\mathbb{P}(\cdot)$ to denote the probability, $\mathbb{E}\{\cdot\}$ the expectation, and $\mathbb{V}\{\cdot\}$ the variance of their arguments, respectively.

Figure 1. (Color online) A signed social network.



2. Opinion Dynamics over Signed Social Networks

In this section, we present our model of interaction between nodes in a signed social network, and describe the resulting dynamics of the beliefs held by each node.

2.1. Signed Social Network and Peer Interactions

We consider a social network with $n \geqslant 3$ members, each labeled by a unique integer in $\{1, 2, \ldots, n\}$. The network is represented by an undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ whose node set $\mathsf{V} = \{1, 2, \ldots, n\}$ corresponds to the members and whose edge set E describes potential interactions between the members. Each edge in E is assigned a unique label, either $+$ or $-$. In classical social network theory, a $+$ label indicates a friend relation, whereas a $-$ label indicates an enemy relation (Heider 1946, Cartwright and Harary 1956). The graph G equipped with a sign on each edge is then called a *signed graph*. Let E_{pst} and E_{neg} be the collection of the positive and negative edges, respectively; clearly, $\mathsf{E}_{\text{pst}} \cap \mathsf{E}_{\text{neg}} = \varnothing$ and $\mathsf{E}_{\text{pst}} \cup \mathsf{E}_{\text{neg}} = \mathsf{E}$. We call $\mathsf{G}_{\text{pst}} = (\mathsf{V}, \mathsf{E}_{\text{pst}})$ and $\mathsf{G}_{\text{neg}} = (\mathsf{V}, \mathsf{E}_{\text{neg}})$ the positive and the negative graph, respectively; see Figure 1 for an illustration. Without loss of generality, we adopt the following assumption throughout the paper.

Assumption 1. *The underlying graph* G *is connected, and the negative graph* G_{neg} *is nonempty.*

Actual interactions follow the model introduced in Boyd et al. (2006): each node initiates interactions at the instants of a rate-one Poisson process and at each of these instants picks a node at random to interact with. Under this model, at a given time, at most one node initiates an interaction. This allows us to order interaction events in time and to focus on modeling the node pair selection at interaction times. The node selection process is characterized by an $n \times n$ stochastic matrix $P = [p_{ij}]$, complying with the graph G in the sense that $p_{ij} > 0$ always implies $\{i, j\} \in \mathsf{E}$ for $i \neq j \in \mathsf{V}$. The p_{ij} represents the probability that node i initiates an interaction with node j. The node pair selection is then performed as follows.

Definition 1. At each interaction event $k \geqslant 0$, (i) a node $i \in \mathsf{V}$ is drawn uniformly at random, i.e., with probability $1/n$; (ii) node i picks node j with probability p_{ij}. In this case, we say that the unordered node pair $\{i, j\}$ is selected.

The node pair selection process is assumed to be identically and independently distributed (i.i.d.); i.e., the nodes that initiate an interaction and the selected node pairs are identically distributed and independent over $k \geqslant 0$. Formally, the node selection process can be analyzed using the following probability spaces. Let $(\mathsf{E}, \mathscr{S}, \mu)$ be the probability space, where $\mathscr{S}$ is the discrete σ-algebra on E, and μ is the probability measure defined by $\mu(\{i, j\}) = (p_{ij} + p_{ji})/n$ for all $\{i, j\} \in \mathsf{E}$. The node selection process can then be seen as a random event in the product probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\Omega = \mathsf{E}^{\mathbb{N}} = \{\omega = (\omega_0, \omega_1, \ldots,)\colon \forall k, \omega_k \in \mathsf{E}\}$, $\mathscr{F} = \mathscr{S}^{\mathbb{N}}$, and $\mathbb{P}$ is the product probability measure (uniquely) defined by the following: for any finite subset $K \subset \mathbb{N}$, $\mathbb{P}((\omega_k)_{k \in K}) = \prod_{k \in K} \mu(\omega_k)$ for any $(\omega_k)_{k \in K} \in \mathsf{E}^{|K|}$. For any $k \in \mathbb{N}$, we define the coordinate mapping $G_k\colon \Omega \to \mathsf{E}$ by $G_k(\omega) = \omega_k$, for all $\omega \in \Omega$ (note that $\mathbb{P}(G_k = \omega_k) = \mu(\omega_k)$), and we refer to $(G_k, k = 0, 1, \ldots)$ as the *node pair selection process*. We further refer to $\mathscr{F}_k = \sigma(G_0, \ldots, G_k)$ as the σ-algebra capturing the $(k+1)$ first interactions of the selection process.

2.2. Positive and Negative Dynamics

Each node maintains a scalar real-valued *opinion*, or *belief*, which it updates whenever it interacts with other nodes. We let $x(k) \in \mathbb{R}^n$ denote the vector of the beliefs held by nodes at the interaction event k.

The belief update depends on the relationship between the interacting nodes. Suppose that node pair $\{i, j\}$ is selected at time k. The nodes that are not selected keep their beliefs unchanged, whereas the beliefs held by nodes i and j are updated as follows:

- (*Positive Update*) If $\{i, j\} \in \mathsf{E}_{\text{pst}}$, either node $m \in \{i, j\}$ updates its belief as

$$\begin{aligned} x_m(k+1) &= x_m(k) + \alpha(x_{-m}(k) - x_m(k)) \\ &= (1-\alpha)x_m(k) + \alpha x_{-m}(k), \end{aligned} \tag{1}$$

where $-m \in \{i, j\} \setminus \{m\}$ and $0 \leqslant \alpha \leqslant 1$.

- (*Negative Update*) If $\{i, j\} \in \mathsf{E}_{\text{neg}}$, either node $m \in \{i, j\}$ updates its belief as

$$\begin{aligned} x_m(k+1) &= x_m(k) - \beta(x_{-m}(k) - x_m(k)) \\ &= (1+\beta)x_m(k) - \beta x_{-m}(k), \end{aligned} \tag{2}$$

where $\beta \geqslant 0$.

The positive update is consistent with the classical DeGroot model (DeGroot 1974), where each node iteratively updates its belief as a convex combination of the previous beliefs of itself and of the neighbor with which it interacts. This update naturally reflects *trustful* or *cooperative* relationships. It is sometimes referred to as naïve learning in social networks, under which *wisdom* can be held by the crowds (Golub and Jackson 2010). The positive update tends to drive node beliefs closer to each other and can be thought of as the attraction of the beliefs.

The dynamics on the negative edges, on the other hand, are not yet universally agreed upon in the literature. Considerable efforts have been made to characterize these *mistrustful* or *antagonistic* relationships, which has led to a number of insightful models, e.g., Acemoglu et al. (2010, 2013), Altafini (2012), and Altafini (2013). Our negative update rule enforces belief differences between interacting nodes and is the *opposite* of the attraction of beliefs represented by the positive update.

2.3. Model Rationale

2.3.1. Relation to Non-Bayesian Rules. Our underlying signed graph is a *prescribed* world with fixed trust or mistrustful relations where nodes do not switch their relations. Two nodes holding the same opinion can be enemies and vice versa. This contrasts Krause's model, where trustful relations are state dependent and nodes only interact with nodes that hold similar opinions, i.e., whose beliefs are within a given distance.

In our model, the signed graph classifies the social interactions into two categories, positive and negative, each with its own type of dynamics. Studies of stubborn agents in social network (Acemoglu et al. 2013, Yildiz et al. 2013) also classify nodes into two categories, but stubborn agents do not account for the opinion of their neighbors. Our model is more similar to the one introduced by Altafini (2013), where the author proposed a different update rule for two nodes sharing a negative link. The model in Altafini (2013) is written in continuous time (beliefs evolve along some ODE), but its corresponding discrete-time update across a negative link $\{i, j\} \in \mathsf{E}_{\text{neg}}$ is

$$\begin{aligned} x_m(k+1) &= x_m(k) - \beta(x_{-m}(k) + x_m(k)) \\ &= (1-\beta)x_m(k) - \beta x_{-m}(k), \quad m \in \{i, j\}, \end{aligned} \tag{3}$$

where $\beta \in (0, 1)$ represents the negative strength. This update rule admits the following interpretations:

- Node i attempts to *trick* her negative neighbors j, by flipping the sign of her true belief (i.e., $x_i(k)$ to $-x_i(k)$) before revealing it to j;
- Node i recognizes j as her negative neighbor and upon observing j's true belief, $x_j(k)$, she tries to get closer to the opposite view of j since $x_i(k+1)$ is a convex combination of $x_i(k)$ and $-x_j(k)$.

In both of the two interpretations of the Altafini model, the belief origin must be of some particular significance in the nodes' belief space. This is not the case for our model, where the positive/negative dynamics describe *choices* intended to *keep close to friends* and *keep distance from enemies*. When nodes i and j perform a negative update in our model, if $x_i(k) > x_j(k)$, then $x_i(k+1) > x_i(k)$ and if $x_i(k) < x_j(k)$, then $x_i(k+1) < x_i(k)$. That is, in either case, the node's updated opinion is in a direction away from the opinion of the interacting node (i.e., nodes make an effort to "keep distance from the enemies" and do not assign any special meaning to the belief origin).

Remark 1. The Altafini model (Altafini 2013) and the current work are intended for building theories to opinion dynamics over signed social networks. Indeed nontrivial efforts have been made to model the dynamics of signed social networks themselves (Galam 1996, Macy et al. 2003, Marvel et al. 2011). It is intriguing to ask how opinions and social networks shape each other in the presence of trustful/mistrustful relations, where fundamental difficulty arises in how to properly model such couplings as well as the challenges brought by the couplings.

2.3.2. Relation to Bayesian Rules. Bayesian opinion dynamics assume that there is a global *model* of the world and individuals aim to realize asymptotic learning of the underlying world (Banerjee 1992, Bikhchandani et al. 1992, Acemoglu et al. 2011). It has been shown that a DeGroot update can also serve as a naive learning approach as long as the network somehow contains no dictators (Golub and Jackson 2010).

We argue here our model corresponds to the situation where nodes naively follow the code of keeping distance with enemies and keeping close to friends rather than having interest in some underlying world model. Our definition of the negative dynamics becomes quite natural if one views the DeGroot type of update as the approach of keeping close to friends. This simple yet informative model leads to a number of nontrivial belief formations in terms of convergence or divergence for unconstrained evolution, consensus, clustering, or oscillation under boundedness constraint.

We note that it is an interesting open challenge to find a proper model for Bayesian learning over signed social networks since nodes must learn in the presence of *negative* interactions, on the one hand, and may try to prevent their enemies from asymptotic learning, on the other.

3. Mean and Mean-Square Convergence/Divergence

Let $x(k) = (x_1(k), \ldots, x_n(k))'$, $k = 0, 1, \ldots$ be the (random) vector of beliefs at time k resulting from the node interactions. The initial beliefs $x(0)$, also denoted as x^0, is assumed to be deterministic. In this section, we investigate the mean and mean-square evolution of the beliefs for the considered signed social network. We introduce the following definition.

Definition 2. (i) Belief convergence is achieved in *expectation* if $\lim_{k\to\infty} \mathbb{E}\{x_i(k) - x_j(k)\} = 0$ for all i and j; in *mean square* if $\lim_{k\to\infty} \mathbb{E}\{(x_i(k) - x_j(k))^2\} = 0$ for all i and j.

(ii) Belief divergence is achieved in *expectation* if $\limsup_{k\to\infty} \max_{i,j} |\mathbb{E}\{x_i(k)-x_j(k)\}| = \infty$; in *mean square* if $\limsup_{k\to\infty} \max_{i,j} \mathbb{E}\{(x_i(k)-x_j(k))^2\} = \infty$.

The belief dynamics as described above can be written as

$$x(k+1) = W(k)x(k), \tag{4}$$

where $W(k)$, $k=0,1,\ldots$ are i.i.d. random matrices satisfying

$$\begin{aligned} &\mathbb{P}(W(k) = \mathrm{W}_{ij}^{+} := I - \alpha(e_i - e_j)(e_i-e_j)') = \frac{p_{ij}+p_{ji}}{n}, \\ &\qquad\qquad\qquad\qquad \{i,j\}\in \mathsf{E}_{\mathrm{pst}}, \\ &\mathbb{P}(W(k) = \mathrm{W}_{ij}^{-} := I + \beta(e_i - e_j)(e_i-e_j)') = \frac{p_{ij}+p_{ji}}{n}, \\ &\qquad\qquad\qquad\qquad \{i,j\}\in \mathsf{E}_{\mathrm{neg}}, \end{aligned} \tag{5}$$

and $e_m = (0\ldots 0\ 1\ 0 \ldots 0)'$ is the n-dimensional unit vector whose m-th component is 1. In this section, we use spectral properties of the linear system (4) to study convergence and divergence in mean and mean-square. Our results can be seen as extensions of existing convergence results on deterministic consensus algorithms, e.g., Xiao and Boyd (2004).

3.1. Convergence in Mean

We first provide conditions for convergence and divergence in mean. We then exploit these conditions to establish the existence of a phase transition for convergence when the negative update parameter β increases. These results are illustrated at the end of this subsection. For technical reasons we adopt the following assumption in this subsection.

Assumption 2. *There holds either* (*i*) $p_{ii} \geqslant 1/2$ *for all* $i \in \mathsf{V}$ *or* (*ii*) $P=[p_{ij}]$ *is doubly stochastic with* $n \geqslant 4$.

Generalization to the case when Assumption 2 does not hold is essentially straightforward but under a bit more careful treatment.

3.1.1. Convergence/Divergence Conditions. Denote $P^{\dagger} = (P+P')/n$. We write $P^{\dagger} = P^{\dagger}_{\mathrm{pst}} + P^{\dagger}_{\mathrm{neg}}$, where $P^{\dagger}_{\mathrm{pst}}$ and $P^{\dagger}_{\mathrm{neg}}$ correspond to the positive and negative graphs, respectively. Specifically, $[P^{\dagger}_{\mathrm{pst}}]_{ij} = [P^{\dagger}]_{ij}$ if $\{i,j\}\in \mathsf{E}_{\mathrm{pst}}$ and $[P^{\dagger}_{\mathrm{pst}}]_{ij} = 0$ otherwise, whereas $[P^{\dagger}_{\mathrm{neg}}]_{ij} = [P^{\dagger}]_{ij}$ if $\{i,j\}\in \mathsf{E}_{\mathrm{neg}}$ and $[P^{\dagger}_{\mathrm{neg}}]_{ij} = 0$ otherwise. We further introduce the degree matrix $D^{\dagger}_{\mathrm{pst}} = \mathrm{diag}(d_1^{+}\ldots d_n^{+})$ of the positive graph, where $d_i^{+} = \sum_{j=1,\, j\neq i}^{n} [P^{\dagger}_{\mathrm{pst}}]_{ij}$. Similarly, the degree matrix of the negative graph is defined as $D^{\dagger}_{\mathrm{neg}} = \mathrm{diag}(d_1^{-}\ldots d_n^{-})$ with $d_i^{-} = \sum_{j=1,\, j\neq i}^{n} [P^{\dagger}_{\mathrm{neg}}]_{ij}$. Then $L^{\dagger}_{\mathrm{pst}} = D^{\dagger}_{\mathrm{pst}} - P^{\dagger}_{\mathrm{pst}}$ and $L^{\dagger}_{\mathrm{neg}} = D^{\dagger}_{\mathrm{neg}} - P^{\dagger}_{\mathrm{neg}}$ represent the (weighted) Laplacian matrices of the positive graph $\mathsf{G}_{\mathrm{pst}}$ and negative graph $\mathsf{G}_{\mathrm{neg}}$, respectively. It can be easily deduced from (5) that

$$\mathbb{E}\{W(k)\} = I - \alpha L^{\dagger}_{\mathrm{pst}} + \beta L^{\dagger}_{\mathrm{neg}}. \tag{6}$$

Clearly, $\mathbf{1}'\mathbb{E}\{W(k)\} = \mathbb{E}\{W(k)\}\mathbf{1} = \mathbf{1}$ where $\mathbf{1} = (1\ldots 1)'$ denotes the $n\times 1$ vector of all ones, but $\mathbb{E}\{W(k)\}$ is not necessarily a stochastic matrix since it may contain negative entries.

Introduce $y_i(k) = x_i(k) - \sum_{s=1}^{n} x_s(k)/n$ and let $y(k) = (y_1(k)\ldots y_n(k))'$. Define $U := \mathbf{1}\mathbf{1}'/n$ and note that $y(k) = (I-U)x(k)$; furthermore, $(I-U)W(k) = W(k)(I-U) = W(k) - U$ for all possible realizations of $W(k)$. Hence, the evolution of $\mathbb{E}\{y(k)\}$ is linear:

$$\begin{aligned} \mathbb{E}\{y(k+1)\} &= \mathbb{E}\{(I-U)W(k)x(k)\} \\ &= \mathbb{E}\{(I-U)W(k)(I-U)x(k)\} \\ &= (\mathbb{E}\{W(k)\} - U)\,\mathbb{E}\{y(k)\}. \end{aligned}$$

The following elementary inequalities

$$\begin{aligned} &|\mathbb{E}\{x_i(k)-x_j(k)\}| \leqslant |\mathbb{E}\{y_i(k)\}| + |\mathbb{E}\{y_j(k)\}|, \\ &|\mathbb{E}\{y_i(k)\}| \leqslant \frac{1}{n}\,\mathbb{E}\sum_{s=1}^{n} |x_i(k) - x_s(k)| \end{aligned} \tag{7}$$

imply that belief convergence in expectation is equivalent to $\lim_{k\to\infty} |\mathbb{E}\{y(k)\}| = 0$, and belief divergence is equivalent to $\limsup_{k\to\infty} |\mathbb{E}\{y(k)\}| = \infty$. Belief convergence or divergence is hence determined by the spectral radius of $\mathbb{E}\{W(k)\} - U$.

With Assumption 2, there always holds that

$$d_i^{+} = \sum_{j=1,\, j\neq i}^{n} [P^{\dagger}_{\mathrm{pst}}]_{ij} \leqslant \sum_{j=1,\, j\neq i}^{n} (p_{ij}+p_{ji})/n \leqslant 1/2.$$

As a result, Geršhgorin's Circle Theorem (see, e.g., Theorem 6.1.1 in Horn and Johnson 1985) guarantees that each eigenvalue of $I - \alpha L^{\dagger}_{\mathrm{pst}}$ is nonnegative. It then follows that each eigenvalue of $I - \alpha L^{\dagger}_{\mathrm{pst}} - U$ is nonnegative since $L^{\dagger}_{\mathrm{pst}}U = UL^{\dagger}_{\mathrm{pst}} = 0$ and the two matrices $I - \alpha L^{\dagger}_{\mathrm{pst}}$ and U share the same eigenvector $\mathbf{1}$ for eigenvalue one. Moreover, it is well known in algebraic graph theory that $L^{\dagger}_{\mathrm{pst}}$ and $L^{\dagger}_{\mathrm{neg}}$ are positive semidefinite matrices. As a result, Weyl's inequality (see Theorem 4.3.1 in Horn and Johnson 1985) further ensures that each eigenvalue of $\mathbb{E}\{W(k)\} - U$ is also nonnegative. To summarize, we have shown the following:

Proposition 1. *Let Assumption 2 hold. Belief convergence is achieved in expectation for all initial values if* $\lambda_{\max}(I - \alpha L^{\dagger}_{\mathrm{pst}} + \beta L^{\dagger}_{\mathrm{neg}} - U) < 1$; *belief divergence is achieved in expectation for almost all initial values if* $\lambda_{\max}(I - \alpha L^{\dagger}_{\mathrm{pst}} + \beta L^{\dagger}_{\mathrm{neg}} - U) > 1$.

In the above proposition and what follows, $\lambda_{\max}(M)$ denotes the largest eigenvalue of the real symmetric matrix M; by "almost all initial conditions," we mean that the property holds for any initial condition $y(0)$ except if $y(0)$ is perfectly orthogonal to the eigenspace of $\mathbb{E}\{W(k)\} - U$ corresponding to its maximal eigenvalue $\lambda_{\max}(I - \alpha L^{\dagger}_{\mathrm{pst}} + \beta L^{\dagger}_{\mathrm{neg}} - U)$. Hence the set of initial conditions where the property does not hold has zero Lebesgue measure.

The Courant-Fischer Theorem (see Theorem 4.2.11 in Horn and Johnson 1985) implies

$$\lambda_{\max}(I-\alpha L^{\dagger}_{\mathrm{pst}}+\beta L^{\dagger}_{\mathrm{neg}}-U)$$
$$=\sup_{|z|=1} z'(I-\alpha L^{\dagger}_{\mathrm{pst}}+\beta L^{\dagger}_{\mathrm{neg}}-U)z$$
$$=1+\sup_{|z|=1}\Bigg[-\alpha\sum_{\{i,j\}\in\mathsf{E}_{\mathrm{pst}}}[P^{\dagger}]_{ij}(z_i-z_j)^2 +\beta\sum_{\{i,j\}\in\mathsf{E}_{\mathrm{neg}}}[P^{\dagger}]_{ij}(z_i-z_j)^2-\frac{1}{n}\bigg(\sum_{i=1}^{n}z_i\bigg)^2\Bigg]. \quad (8)$$

We see from (8) that the influence of $\mathsf{G}_{\mathrm{pst}}$ and $\mathsf{G}_{\mathrm{neg}}$ on the belief convergence/divergence in mean are *separated*: links in $\mathsf{E}_{\mathrm{pst}}$ contribute to belief convergence, whereas links in $\mathsf{E}_{\mathrm{neg}}$ contribute to belief divergence. As will be shown later on, this separation property no longer holds for mean-square convergence, and there may be a nontrivial correlation between the influence of $\mathsf{E}_{\mathrm{pst}}$ and that of $\mathsf{E}_{\mathrm{neg}}$.

3.1.2. Phase Transition. Next we study the impact of update parameters α and β on the convergence in expectation. Define $f(\alpha,\beta):=\lambda_{\max}(I-\alpha L^{\dagger}_{\mathrm{pst}}+\beta L^{\dagger}_{\mathrm{neg}}-U)$. The function f has the following properties under Assumption 2:

(i) (*Convexity*) Since both $L^{\dagger}_{\mathrm{pst}}$ and $L^{\dagger}_{\mathrm{neg}}$ are symmetric, $f(\alpha,\beta)$ is the spectral norm of $I-\alpha L^{\dagger}_{\mathrm{pst}}+\beta L^{\dagger}_{\mathrm{neg}}-U$. Because every matrix norm is convex, we have

$$f(\gamma(\alpha_1,\beta_1)+(1-\gamma(\alpha_2,\beta_2)) \leqslant \gamma f(\alpha_1,\beta_1)+(1-\gamma)f(\alpha_2,\beta_2) \quad (9)$$

for all $\gamma\in[0,1]$ and $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{R}$. This implies that $f(\alpha,\beta)$ is convex in (α,β).

(ii) (*Monotonicity*) From (8), $f(\alpha,\beta)$ is nonincreasing in α for fixed β and nondecreasing in β for fixed α. As a result, setting $\alpha=1$ provides the *fastest* convergence whenever belief convergence in expectation is achieved (for a given fixed β). Note that when $\alpha=1$, when two nodes interact, they simply switch their beliefs.

When $\mathsf{G}_{\mathrm{pst}}$ is connected, the second smallest eigenvalue of $L^{\dagger}_{\mathrm{pst}}$, denoted by $\lambda_2(L^{\dagger}_{\mathrm{pst}})$, is positive. We can readily see that $f(\alpha,0)=1-\alpha\lambda_2(L^{\dagger}_{\mathrm{pst}})<1$. From (8), we also have $f(\alpha,\beta)\to\infty$ as $\beta\to\infty$ provided that $\mathsf{G}_{\mathrm{neg}}$ is nonempty. Combining these observations with the monotonicity of f, we conclude the following:

Proposition 2. *Assume that $\mathsf{G}_{\mathrm{pst}}$ is connected and let Assumption 2 hold. Then for any fixed $\alpha\in(0,1]$, there exists a threshold value $\beta_\star>0$ (that depends on α) such that*

(i) *Belief convergence in expectation is achieved for all initial values if $0\leqslant\beta<\beta_\star$;*

(ii) *Belief divergence in expectation is achieved for almost all initial values if $\beta>\beta_\star$.*

We remark that belief divergence can only happen for almost all initial values since if the initial beliefs of all the nodes are identical, they do not evolve over time.

3.1.3. Examples. An interesting question is to determine how the phase transition threshold $\beta_\star$ scales with the network size. Answering this question seems challenging. However, there are networks for which we can characterize $\beta_\star$ exactly. Next we derive explicit expressions for $\beta_\star$ when G is a complete graph or a ring graph. These two topologies represent the most dense and almost the most sparse structures for a connected network.

Example 1 (Complete Graph). Let $\mathsf{G}=\mathsf{K}_n$, where K_n is the complete graph with n nodes, and consider the node pair selection matrix $P=(\mathbf{1}\mathbf{1}'-I)/(n-1)$. Let $\mathrm{L}(\mathsf{K}_n)=nI-\mathbf{1}\mathbf{1}'$ be the Laplacian of K_n. Then $\mathrm{L}(\mathsf{K}_n)$ has eigenvalue 0 with multiplicity 1 and eigenvalue n with multiplicity $n-1$. Define $\mathrm{L}(\mathsf{G}_{\mathrm{neg}})$ as the standard Laplacian of $\mathsf{G}_{\mathrm{neg}}$. Observe that

$$I-\alpha L^{\dagger}_{\mathrm{pst}}+\beta L^{\dagger}_{\mathrm{neg}}-U$$
$$=I-\alpha(L^{\dagger}_{\mathrm{pst}}+L^{\dagger}_{\mathrm{neg}})+(\alpha+\beta)L^{\dagger}_{\mathrm{neg}}-U$$
$$=I-\frac{2\alpha}{n(n-1)}\mathrm{L}(\mathsf{K}_n)+\frac{2(\alpha+\beta)}{n(n-1)}\mathrm{L}(\mathsf{G}_{\mathrm{neg}})-U. \quad (10)$$

Also note that $\mathrm{L}(\mathsf{G}_{\mathrm{neg}})\mathrm{L}(\mathsf{K}_n)=\mathrm{L}(\mathsf{K}_n)\mathrm{L}(\mathsf{G}_{\mathrm{neg}})=n\mathrm{L}(\mathsf{G}_{\mathrm{neg}})$. From these observations, we can then readily conclude the following:

$$\beta_\star=\frac{n\alpha}{\lambda_{\max}(\mathrm{L}(\mathsf{G}_{\mathrm{neg}}))}-\alpha. \quad (11)$$

Example 2 (Erdős-Rényi Negative Graph Over Complete Graph). Let $\mathsf{G}=\mathsf{K}_n$ with $P=(\mathbf{1}\mathbf{1}'-I)/(n-1)$. Let $\mathsf{G}_{\mathrm{neg}}$ be the Erdős-Rényi random graph (Erdős and Rényi 1960) where for any $i,j\in\mathsf{V}$, $\{i,j\}\in\mathsf{E}_{\mathrm{neg}}$ with probability p (independently of other links). Note that since $\mathsf{G}_{\mathrm{neg}}$ is a random subgraph, the function $f(\alpha,\beta)$ becomes a random variable, and we denote by $\mathbf{P}$ the probability measure related to the randomness of the graph in Erdős-Rényi's model. Spectral theory for random graphs (Ding and Jiang 2010) suggests that

$$\frac{\lambda_{\max}(\mathrm{L}(\mathsf{G}_{\mathrm{neg}}))}{pn}\to 1, \quad \text{as } n\to\infty \quad (12)$$

in probability. Now for fixed p, we deduce from (11) and (12) that the threshold $\beta_\star$ converges, as n grows large, to α/p in probability. Now let us fix the update parameters α and β and investigate the impact of the probability p on the convergence in mean.

• If $p<\alpha/(\alpha+\beta)$, we show that $\mathbf{P}[f(\alpha,\beta)<1]\to 1$, when $n\to\infty$, i.e., when the network is large, we likely achieve convergence in mean. Let $\epsilon<\alpha/((\alpha+\beta)p)-1$. It follows from (12) that

$$\mathbf{P}(f(\alpha,\beta)<1)$$
$$=\mathbf{P}\bigg(1-\frac{2\alpha}{n(n-1)}n+\frac{2(\alpha+\beta)}{n(n-1)}\lambda_{\max}(\mathrm{L}(\mathsf{G}_{\mathrm{neg}}))<1\bigg)$$

$$
\begin{aligned}
&= \mathbf{P}((\alpha+\beta)\lambda_{\max}(\mathsf{L}(\mathsf{G}_{\text{neg}})) < \alpha n)\\
&= \mathbf{P}\left(\frac{\lambda_{\max}(\mathsf{L}(\mathsf{G}_{\text{neg}}))}{pn} < \frac{\alpha}{(\alpha+\beta)p}\right)\\
&\geqslant \mathbf{P}\left(\left|\frac{\lambda_{\max}(\mathsf{L}(\mathsf{G}_{\text{neg}}))}{pn} - 1\right| < \epsilon\right) \to 1, \quad \text{as } n\to\infty. \qquad (13)
\end{aligned}
$$

• If $p > \alpha/(\alpha+\beta)$, we similarly establish that $\mathbf{P}(f(\alpha,\beta) > 1) \to 1$, when $n \to \infty$, i.e., when the network is large, we observe divergence in mean with high probability.

Hence we have a sharp phase transition between convergence and divergence in mean when the proportion of negative links p increases and goes above the threshold $p_\star = \alpha/(\alpha+\beta)$.

Example 3 (Ring Graph). Denote R_n as the ring graph with n nodes. Let $\mathsf{A}(\mathsf{R}_n)$ and $\mathsf{L}(\mathsf{R}_n)$ be the adjacency and Laplacian matrices of R_n, respectively. Let the underlying graph $\mathsf{G} = \mathsf{R}_n$ with only one negative link (if one has more than two negative links, it is easy to see that divergence in expectation is achieved irrespective of $\beta > 0$). Take $P = \mathsf{A}(\mathsf{R}_n)/2$. We know that $\mathsf{L}(\mathsf{R}_n)$ has eigenvalues $2 - 2\cos(2\pi k/n)$, $0 \leqslant k \leqslant n/2$. Applying Weyl's inequality we obtain $f(\alpha,\beta) \geqslant 1 + (\beta-\alpha)/n$. We conclude that $\beta_\star < \alpha$, irrespective of n.

3.2. Mean-Square Convergence

We now turn our attention to the analysis of the mean-square convergence and divergence. Define:

$$
\begin{aligned}
\mathbb{E}\{|y(k)|^2\} &= \mathbb{E}\{x(k)'(I-U)x(k)\}\\
&= x(0)'\mathbb{E}\{W(0)\ldots W(k-1)(I-U)\\
&\qquad \cdot W(k-1)\ldots W(0)\}x(0). \qquad (14)
\end{aligned}
$$

Again based on inequalities (7), we see that belief convergence in mean square is equivalent to $\lim_{k\to\infty}\mathbb{E}\{|y(k)|^2\} = 0$, and belief divergence to $\limsup_{k\to\infty}\mathbb{E}\{|y(k)|^2\} = \infty$. Define

$$
\Phi(k) = \begin{cases} \mathbb{E}\{W(0)\ldots W(k-1)(I-U) & \\ \quad \cdot W(k-1)\ldots W(0)\}, & k \geqslant 1,\\ I-U, & k = 0. \end{cases} \qquad (15)
$$

Then $\Phi(k)$ evolves as a linear dynamical system (Fagnani and Zampieri 2008):

$$
\begin{aligned}
\Phi(k) &= \mathbb{E}\{W(0)\ldots W(k-1)(I-U)W(k-1)\ldots W(0)\}\\
&= \mathbb{E}\{W(0)(I-U)W(1)\ldots W(k-1)(I-U)\\
&\qquad \cdot W(k-1)\ldots W(1)(I-U)W(0)\}\\
&= \mathbb{E}\{(W(k)-U)\Phi(k-1)(W(k)-U)\}, \qquad (16)
\end{aligned}
$$

where in the second equality we have used the facts that $(I-U)^2 = I-U$ and $(I-U)W(k) = W(k)(I-U) = W(k)-U$ for all possible realizations of $W(k)$, and the third equality is due to $W(k)$ and $W(0)$ are i.i.d. We can rewrite (16) using an equivalent vector form:

$$
\mathbf{vec}(\Phi(k)) = \Theta\mathbf{vec}(\Phi(k-1)), \qquad (17)
$$

where Θ is the matrix in $\mathbb{R}^{n^2\times n^2}$ given by

$$
\begin{aligned}
\Theta &= \mathbb{E}\{(W(0)-U)\otimes(W(0)-U)\}\\
&= \sum_{\{i,j\}\in\mathsf{G}_{\text{pst}}} [P^\dagger]_{ij}((\mathsf{W}_{ij}^+ - U)\otimes(\mathsf{W}_{ij}^+ - U))\\
&\quad + \sum_{\{i,j\}\in\mathsf{G}_{\text{neg}}} [P^\dagger]_{ij}((\mathsf{W}_{ij}^- - U)\otimes(\mathsf{W}_{ij}^- - U)).
\end{aligned}
$$

Let S_λ be the eigenspace corresponding to an eigenvalue λ of Θ. Define

$$
\lambda_\star := \max\{\lambda\in\sigma(\Theta)\colon \mathbf{vec}(I-U)\notin S_\lambda^\perp\},
$$

which denotes the spectral radius of Θ restricted to the smallest invariant subspace containing $\mathbf{vec}(I-U)$; i.e., $\mathrm{S} := \mathrm{span}\{\Theta^k\mathbf{vec}(I-U), k = 0,1,\ldots\}$. Then mean-square belief convergence/divergence is fully determined by $\lambda_\star$: convergence in mean square for all initial conditions is achieved if $\lambda_\star < 1$, and divergence for almost all initial conditions is achieved if $\lambda_\star > 1$.

Observing that $\lambda \leqslant 1$ for every $\lambda\in\sigma(\mathsf{W}_{ij}^+)$ and $\lambda \geqslant 1$ for every $\lambda\in\sigma(\mathsf{W}_{ij}^-)$, we can also conclude that each link in E_{pst} contributes positively to $\lambda_{\max}(\Theta)$ and each link in E_{neg} contributes negatively to $\lambda_{\max}(\Theta)$. However, unlike in the case of the analysis of convergence in expectation, although $\lambda_\star$ defines a precise threshold for the phase-transition between mean-square convergence and divergence, it is difficult to determine the influence E_{pst} and E_{neg} have on $\lambda_\star$. The reason is that they are coupled in a nontrivial manner for the invariant subspace S. Nevertheless, we are still able to propose the following conditions for mean-square belief convergence and divergence:

Proposition 3. *Belief convergence is achieved for all initial values in mean square if*

$$
\lambda_{\max}(I - 2\alpha(1-\alpha)L_{\text{pst}}^\dagger + 2\beta(1+\beta)L_{\text{neg}}^\dagger - U) < 1;
$$

belief divergence is achieved in mean square for almost all initial values if $\lambda_{\max}(I - \alpha L_{\text{pst}}^\dagger + \beta L_{\text{neg}}^\dagger - U) > 1$ *or* $\lambda_{\min}(I - 2\alpha(1-\alpha)L_{\text{pst}}^\dagger + 2\beta(1+\beta)L_{\text{neg}}^\dagger - U) > 1$.

The condition $\lambda_{\max}(I - \alpha L_{\text{pst}}^\dagger + \beta L_{\text{neg}}^\dagger - U)$ is sufficient for mean square divergence, in view of Proposition 1 and that $\mathcal{L}_1$ divergence implies $\mathcal{L}_p$ divergence for all $p \geqslant 1$. The other conditions are essentially consistent with the upper and lower bounds of $\lambda_\star$ established in Proposition 4.4 of Fagnani and Zampieri (2008). Proposition 3 is a consequence of Lemma 3 (see appendix).

4. Almost Sure Convergence/Divergence

In this section, we explore the almost sure convergence of beliefs in signed social networks. We introduce the following definition.

Definition 3. Belief convergence is achieved *almost surely* (a.s.) if $\mathbb{P}(\lim_{k\to\infty}|x_i(k)-x_j(k)|=0)=1$ for all i and j; Belief divergence is achieved *almost surely* if $\mathbb{P}(\limsup_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty)=1$.

Basic probability theory tells us that mean-square belief convergence implies belief convergence in expectation (mean convergence); similarly, belief divergence in expectation implies belief divergence in mean square. However, in general there is no direct connection between almost sure convergence/divergence and mean or mean-square convergence/divergence. Finally observe that, a priori, it is not clear that either a.s. convergence or a.s. divergence should be achieved.

Although the analysis of the convergence of beliefs in mean and square-mean mainly relied on spectral arguments, we need more involved probabilistic methods (e.g., sample-path arguments, martingale convergence theorems) to study almost sure convergence or divergence. We first establish two insightful properties of the belief evolutions: (i) the *no-survivor* property, stating that in case of almost sure divergence, the difference between the beliefs of any two nodes in the network tends to infinity (along a subsequence of instants); (ii) the *live-or-die* property, which essentially states that the maximum difference between the beliefs of any two nodes either grows to infinity or vanishes to zero. We then show a zero-one law and a phase transition of almost sure convergence/divergence. Finally, we investigate the robustness of networks against negative links. More specifically, we show that when the graph G_{pst} of positive links contains a hypercube, and when the positive updates are truly averaging, i.e., $\alpha=1/2$, then almost sure belief convergence is reached in finite time, irrespective of the number of negative links, their positions in the network, and the negative update parameter β. We believe that these are the only networks enjoying this strong robustness property.

4.1. The No-Survivor Theorem

The following theorem establishes that in the case of almost sure divergence, there is no pair of nodes that can *survive* this divergence: for any two nodes, the difference in their beliefs grow arbitrarily large.

Theorem 1 (No-Survivor). *Fix the initial condition and assume almost sure belief divergence. Then*

$$\mathbb{P}(\limsup_{k\to\infty}|x_i(k)-x_j(k)|=\infty)=1 \textit{ for all } i\neq j\in\mathsf{V}.$$

Observe that the above result only holds for the almost sure divergence. We may easily build simple network examples where we have belief divergence in expectation (or mean square), but where some node pairs survive, in the sense that the difference in their beliefs vanishes (or at least bounded). The no-survivor theorem indicates that to check almost sure divergence, we may just observe the evolution of beliefs held at two arbitrary nodes in the network.

4.2. The Live-or-Die Lemma and Zero-One Laws

Next we further classify the ways beliefs can evolve. Specifically, we study the following events: for any initial beliefs x^0,

$$\mathcal{C}_{x^0}\doteq\{\limsup_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=0\},$$
$$\mathcal{D}_{x^0}\doteq\{\limsup_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty\},$$
$$\mathcal{C}^*_{x^0}\doteq\{\liminf_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=0\},$$
$$\mathcal{D}^*_{x^0}\doteq\{\liminf_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty\},$$

and

$$\mathcal{C}\doteq\{\limsup_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=0 \text{ for all } x^0\in\mathbb{R}^n\},$$
$$\mathcal{D}\doteq\{\exists \text{ (deterministic) } x^0\in\mathbb{R}^n,\ \text{s.t. } \limsup_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty\}.$$

We establish that the maximum difference between the beliefs of any two nodes either goes to ∞, or to zero. This result is referred to as *live-or-die* lemma:

Lemma 1 (Live-or-Die). *Let* $\alpha\in(0,1)$ *and* $\beta>0$. *Suppose* G_{pst} *is connected. Then* (i) $\mathbb{P}(\mathcal{C}_{x^0})+\mathbb{P}(\mathcal{D}_{x^0})=1$; (ii) $\mathbb{P}(\mathcal{C}^*_{x^0})+\mathbb{P}(\mathcal{D}^*_{x^0})=1$.

As a consequence, almost surely, one of the following events happens:

$$\{\lim_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=0\};$$
$$\{\lim_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty\};$$
$$\{\liminf_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=0;\ \limsup_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty\}.$$

The live-or-die lemma deals with events where the initial beliefs have been fixed. We may prove stronger results on the probabilities of events that hold for *any* initial condition, e.g., $\mathcal{C}$, or for at least one initial condition, e.g., $\mathcal{D}$:

Theorem 2 (Zero-One Law). *Let* $\alpha\in[0,1]$ *and* $\beta>0$. *Both* $\mathcal{C}$ *and* $\mathcal{D}$ *are trivial events* (*i.e., each of them occurs with probability equal to either* 1 *or* 0) *and* $\mathbb{P}(\mathcal{C})+\mathbb{P}(\mathcal{D})=1$.

To prove this result, we show that $\mathcal{C}$ is a tail event, and hence trivial in view of Kolmogorov's zero-one law (the same kind of argument has been used by Tahbaz-Salehi and Jadbabaie 2008 and Touri and Nedić 2011). From the live-or-die lemma, we then simply deduce that $\mathcal{D}$ is also a trivial event. Note that $\mathcal{C}_{x^0}$ and $\mathcal{D}_{x^0}$ may not be trivial events. In fact, we can build examples where $\mathbb{P}(\mathcal{C}_{x^0}) = 1/2$ and $\mathbb{P}(\mathcal{D}_{x^0}) = 1/2$. The detailed proof of Theorem 2 has been put in the e-companion of this paper.

4.3. Phase Transition

As for the convergence in expectation, for fixed positive update parameter α, we are able to establish the existence of thresholds for the value β of the negative update parameter, which characterizes the almost sure belief convergence and divergence.

Theorem 3 (Phase Transition). *Suppose* G_{pst} *is connected. Fix* $\alpha \in (0,1)$ *with* $\alpha \neq 1/2$. *Then*

(i) *there exists* $\beta^\natural(\alpha) > 0$ *such that* $\mathbb{P}(\mathcal{C}) = 1$ *if* $0 \leqslant \beta < \beta^\natural$;

(ii) *there exists* $\beta^\sharp(\alpha) > 0$ *such that*

$$\mathbb{P}(\liminf_{k\to\infty} \max_{i,j} |x_i(k) - x_j(k)| = \infty) = 1$$

for almost all initial values if $\beta > \beta^\sharp$.

It should be observed that the divergence condition in (ii) is stronger than our notion of almost sure belief divergence (the maximum belief difference between two nodes diverges almost surely to ∞). Also note that $\beta^\natural \leqslant \beta^\sharp$, and we were not able to show that the gap between these two thresholds vanishes (as in the case of belief convergence in expectation or mean-square).

4.4. Robustness to Negative Links: The Hypercube

We have seen in Theorem 3 that when $\alpha \neq 1/2$, one single negative link is capable of driving the network beliefs to almost sure divergence as long as β is sufficiently large. The following result shows that the evolution of the beliefs can be robust against negative links. This is the case when nodes can reach an agreement in finite time. In what follows, we provide conditions on α and the structure of the graph under which finite time belief convergence is reached.

Proposition 4. *Suppose there exist an integer* $T \geqslant 1$ *and a finite sequence of node pairs* $\{i_s, j_s\} \in \mathsf{G}_{\text{pst}}$, $s = 1, 2, \ldots, T$ *such that* $\mathrm{W}^+_{i_T j_T} \ldots \mathrm{W}^+_{i_1 j_1} = U$. *Then* $\mathbb{P}(\mathcal{C}) = 1$ *for all* $\beta \geqslant 0$.

Proposition 4 is a direct consequence of the Borel-Cantelli Lemma. If there is a finite sequence of node pairs $\{i_s, j_s\} \in \mathsf{G}_{\text{pst}}$, $s = 1, 2, \ldots, T$ such that $\mathrm{W}^+_{i_T j_T} \cdots \mathrm{W}^+_{i_1 j_1} = U$, then

$$\mathbb{P}(W(k+T)\ldots W(k+1) = U) \geqslant \left(\frac{p_*}{n}\right)^T,$$

for all $k \geqslant 0$, where $p_* = \min\{p_{ij} + p_{ji}\colon \{i,j\} \in \mathsf{E}\}$. Noting that $UW(k) = W(k)U = U$ for all possible realizations of $W(k)$, the Borel-Cantelli Lemma guarantees that

$$\mathbb{P}(\lim_{k\to\infty} W(k)\ldots W(0) = U) = 1$$

for all $\beta \geqslant 0$, or equivalently, $\mathbb{P}(\mathcal{C}) = 1$ for all $\beta \geqslant 0$. This proves Proposition 4.

The existence of such finite sequence of node pairs under which the beliefs of the nodes in the network reach a common value in finite time is crucial (we believe that this condition is actually necessary) to ensure that the influence of G_{neg} vanishes. It seems challenging to know whether this is at all possible. As it turns out, the structure of the positive graph plays a fundamental role. To see that, we first provide some definitions.

Definition 4. Let $\mathsf{G}_1 = (\mathsf{V}_1, \mathsf{E}_1)$ and $\mathsf{G}_2 = (\mathsf{V}_2, \mathsf{E}_2)$ be a pair of graphs. The Cartesian product of G_1 and G_2, denoted by $\mathsf{G}_1 \,\square\, \mathsf{G}_2$, is defined by

(i) the vertex set of $\mathsf{G}_1 \,\square\, \mathsf{G}_2$ is $\mathsf{V}_1 \times \mathsf{V}_2$, where $\mathsf{V}_1 \times \mathsf{V}_2$ is the Cartesian product of V_1 and V_2;

(ii) for any two vertices $(v_1, v_2), (u_1, u_2) \in \mathsf{V}_1 \times \mathsf{V}_2$, there is an edge between them in $\mathsf{G}_1 \,\square\, \mathsf{G}_2$ if and only if either $v_1 = u_1$ and $\{v_2, u_2\} \in \mathsf{E}_2$ or $v_2 = u_2$ and $\{v_1, u_1\} \in \mathsf{E}_1$.

Let K_2 be the complete graph with two nodes. The m-dimensional hypercube H^m is then defined as

$$\mathsf{H}^m = \underbrace{\mathsf{K}_2 \,\square\, \mathsf{K}_2 \ldots \square\, \mathsf{K}_2}_{m \text{ times}}.$$

An illustration of hypercubes is in Figure 2.

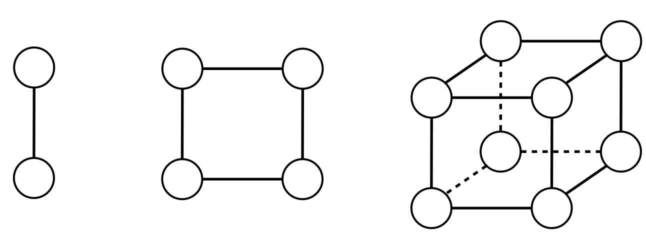
The following result provides sufficient conditions to achieve finite-time convergence, whose proof can be found in the E-companion.

Proposition 5. *If* $\alpha = 1/2$, $n = 2^m$ *for some integer* $m > 0$, *and* G_{pst} *has a subgraph isomorphic with an m-dimensional hypercube, then there exists a sequence of* $(n \log_2 n)/2$ *node pairs* $\{i_s, j_s\} \in \mathsf{G}_{\text{pst}}$, $s = 1, \ldots, (n\log_2 n)/2$ *such that* $\mathrm{W}^+_{i_{(n\log_2 n)/2} j_{(n\log_2 n)/2}} \cdots \mathrm{W}^+_{i_1 j_1} = U$.

Next we derive necessary conditions for finite time convergence. Let us first recall the following definition.

Definition 5. Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a graph. A *matching* of G is a set of pairwise nonadjacent edges in the sense that no two edges share a common vertex. A *perfect matching* of G is a matching that matches all vertices.

Figure 2. The hypercubes H^1, H^2, and H^3.



PROPOSITION 6. *If there exist an integer $T \geqslant 1$ and a sequence of node pairs $\{i_s, j_s\} \in \mathsf{G}_{\rm pst}$, $s = 1, 2, \ldots, T$ such that $\mathrm{W}^+_{i_T j_T} \cdots \mathrm{W}^+_{i_1 j_1} = U$, then $\alpha = 1/2$, $n = 2^m$, and $\mathsf{G}_{\rm pst}$ has a perfect matching.*

In fact, in the proof of Proposition 6 (see e-companion), we show that if $\mathrm{W}^+_{i_T j_T} \ldots \mathrm{W}^+_{i_1 j_1} = U$, then a subset of

$$\{\{i_1, j_1\}, \ldots, \{i_T, j_T\}\}$$

forms a perfect matching of $\mathsf{G}_{\rm pst}$.

We have seen that the belief dynamics and convergence can be robust against negative links, but this robustness comes at the expense of strong conditions on the number of the nodes and the structure of the positive graph.

5. Belief Clustering and Structural Balance

So far we have studied the belief dynamics when the node interactions are symmetric and the values of beliefs are unconstrained. The results illustrate that often either convergence or divergence can be predicted for the social network beliefs. Although this symmetric and unconstrained belief update rule is plausible for ideal social network models, in reality these assumptions might not hold. That is, when $\{i, j\}$ is selected, it might happen that only one of the two nodes in i and j updates its belief; there might be a hard constraint on beliefs: $x_i(k) \in [-A, A]$ for all i and k and for some $A > 0$.

In this section, we consider the following model for the updates of the beliefs. Define

$$\mathcal{P}_A(z) = \begin{cases} -A, & \text{if } z < -A; \\ z, & \text{if } z \in [-A, A]; \\ A, & \text{if } z > A. \end{cases} \tag{18}$$

Let $a, b, c > 0$ be three positive real numbers such that $a + b + c = 1$, and define the function θ: $\mathsf{E} \to \mathbb{R}$ so that $\theta(\{i, j\}) = \alpha$ if $\{i, j\} \in \mathsf{E}_{\rm pst}$ and $\theta(\{i, j\}) = -\beta$ if $\{i, j\} \in \mathsf{E}_{\rm neg}$. Assume that node i interacts with node j at time k. Nodes i and j update their beliefs as

[*Asymmetric and Constrained Belief Evolution*]

$$\begin{aligned} &x_i(k+1) = \mathcal{P}_A((1-\theta)x_i(k) + \theta x_j(k)) \quad \text{and} \\ &x_j(k+1) = x_j(k), \quad \text{with probability } a; \\ &x_j(k+1) = \mathcal{P}_A((1-\theta)x_j(k) + \theta x_i(k)) \quad \text{and} \\ &x_i(k+1) = x_i(k), \quad \text{with probability } b; \\ &x_m(k+1) = \mathcal{P}_A((1-\theta)x_m(k) + \theta x_{-m}(k)), \\ &\qquad\qquad m \in \{i, j\}, \quad \text{with probability } c. \end{aligned} \tag{19}$$

Enforcing the belief within the interval $[-A, A]$ can be viewed as a social member's decision based on her fundamental model of the world. With asymmetric and constrained belief evolution, the dynamics become essentially nonlinear, which brings new challenges in the analysis. We continue to use $\mathbb{P}$ to denote the overall probability measure capturing the randomness of the updates in the asymmetric constrained model.

5.1. Balanced Graphs and Clustering

We introduce the notion of *balance* for signed graphs, for which we refer to Wasserman and Faust (1994) for a comprehensive discussion.

DEFINITION 6. Let $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ be a signed graph. Then

(i) G is *weakly balanced* if there is an integer $k \geqslant 2$ and a partition of $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2 \ldots \cup \mathsf{V}_k$, where $\mathsf{V}_1, \ldots, \mathsf{V}_k$ are nonempty and mutually disjoint, such that any edge between different V_i's is negative, and any edge within each V_i is positive.

(ii) G is *strongly balanced* if it is weakly balanced with $k = 2$.

Harary's balance theorem states that a signed graph G is strongly balanced if and only if there is no cycle with an odd number of negative edges in G (Cartwright and Harary 1956), whereas G is weakly balanced if and only if no cycle has exactly one negative edge in G (Davis 1967).

It turned out that with certain balance of the underlying graph, clustering arises for the social network beliefs. We make the following definition.

DEFINITION 7. (i) Let G be strongly balanced subject to partition $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2$. Then almost sure boundary belief clustering for the initial value x^0 is achieved if there are two random variables $B^\dagger_1(x^0)$ and $B^\dagger_2(x^0)$, both taking values in $\{-A, A\}$, such that

$$\mathbb{P}\Big(\lim_{k\to\infty} x_i(k) = B^\dagger_1(x^0),\ i \in \mathsf{V}_1; \\ \lim_{k\to\infty} x_i(k) = B^\dagger_2(x^0),\ i \in \mathsf{V}_2\Big) = 1. \tag{20}$$

(ii) Let G be weakly balanced subject to partition $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2 \ldots \cup \mathsf{V}_m$ for some $m \geqslant 2$. Then almost sure boundary belief clustering for the initial value x^0 is achieved if there are there are m random variables, $B^\sharp_1(x^0), \ldots, B^\sharp_m(x^0)$, each of which takes values in $\{-A, A\}$, such that

$$\mathbb{P}\Big(\lim_{k\to\infty} x_i(k) = B^\sharp_j(x^0),\ i \in \mathsf{V}_j,\ j = 1, \ldots, m\Big) = 1. \tag{21}$$

In the case of strongly balanced graphs, we can show that beliefs are asymptotically clustered when β is large enough, as stated in the following theorem.

THEOREM 4. *Assume that G is strongly balanced under partition $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2$ and that $\mathsf{G}_{\mathsf{V}_1}$ and $\mathsf{G}_{\mathsf{V}_2}$ are connected. For any $\alpha \in (0, 1)\backslash\{1/2\}$, when β is sufficiently large, for almost all initial values x^0, almost sure boundary belief clustering is achieved under the update rule (19).*

In fact, there holds $B^\dagger_1(x^0) + B^\dagger_2(x^0) = 0$ almost surely in the above boundary belief clustering for strongly balanced social networks. Theorem 4 states that for strongly balanced social networks, beliefs are eventually polarized to the two opinion boundaries.

The analysis of belief dynamics in weakly balanced graphs is more involved, and we restrict our attention to complete graphs. In social networks, this case means that everyone knows everyone else—which constitutes a suitable model for certain social groups of small sizes (a classroom, a sport team, or the United Nations;, see Easley and Kleinberg 2010). As stated in the following theorem, for weakly balanced complete graphs, beliefs are again clustered.

Theorem 5. *Assume that* G *is a complete and weakly balanced graph under the partition* $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2 \ldots \cup \mathsf{V}_m$ *with* $m \geqslant 2$. *Further assume that* $\mathsf{G}_{\mathsf{V}_j}, j = 1, \ldots, m$ *are connected. For any* $\alpha \in (0,1)\backslash\{1/2\}$, *when* β *is sufficiently large, almost sure boundary belief clustering is achieved for almost all initial values under* (19).

Remark 2. Under the model (3), it can be shown (see Altafini 2013, Shi et al. 2015)

(i) if G is strongly balanced and $\beta \in (0,1)$, then there are two values $z_1(x^0)$ and $z_2(x^0)$ such that

$$\mathbb{P}\Big(\lim_{k\to\infty} x_i(k) = z_1(x^0),\, i \in \mathsf{V}_1,\, \lim_{k\to\infty} x_i(k) = z_2(x^0), i \in \mathsf{V}_2\Big) = 1. \tag{22}$$

(ii) if G is not strongly balanced (i.e., even if it is weakly balanced) and $\beta \in (0,1)$, then

$$\mathbb{P}\Big(\lim_{k\to\infty} x_i(k) = 0,\ i \in \mathsf{V}\Big) = 1, \tag{23}$$

where the impact of the initial beliefs is entirely erased from the asymptotic limit.

Our Theorem 4 appears to be similar to (22), but the clustering in Theorem 4 is due to fundamentally different reasons: along with the strong balance of the social network, it is the nonlinearity in the constrained update ($\mathcal{P}_A(\cdot)$) and the sufficiently large β that makes the boundary clustering arise in Theorem 4. In contrast, (22) is resulted from the crucial condition that $\beta \in (0,1)$. Under the Altafini model (3), even when β is sufficiently large, it is easy to see that the boundary clustering in Theorem 5 can never happen for weakly balanced graphs.

The distribution of the clustering limits established in Theorems 4 and 5 relies on the initial value. In this way, the initial beliefs make an impact on the final belief limit, which is either A or $-A$. The boundary clustering is due to the hard boundaries of the beliefs as well as the negative updates (ironically, the larger the better), whose mechanism is fundamentally different with the opinion clustering phenomena resulted from missing of connectivity in Krause types of models (Krause 1997, Blondel et al. 2009, 2010, Li et al. 2013) or nonlinear bias in the opinion evolution (Dandekar et al. 2013).

Remark 3. Note that in the considered asymmetric and constrained belief evolution, we take symmetric belief boundaries $[-A, A]$ just for simplifying the discussion. Theorems 4 and 5 continue to hold if the belief boundaries are chosen to be $[A, B]$ for arbitrary $-\infty < A < B < \infty$.[1] Letting $A = 0, B = 1$, our boundary clustering results in Theorems 4 and 5 are then formally consistent with the belief polarization result, Theorem 3, in Dandekar et al. (2013). It is worth mentioning that Theorem 3 in Dandekar et al. (2013) relies on a type of strong balance (the two-island assumption) and that the initial beliefs should be separated, whereas Theorems 4 and 5 hold for almost arbitrary initial values.

The proof of Theorems 4 and 5 is obtained by establishing the almost sure happening of suitable *separation* events, i.e., the node beliefs for a subnetwork become group polarized (either larger or smaller than the remaining nodes' beliefs). From the analysis it is clear that such events tend to happen more easily for small subnetworks in the partition of (strongly or weakly) balanced social networks. On the other hand, boundary belief clustering follows quickly after the separation event, even in the presence of large subgroups. For a large subgroup, the boundary clustering to a consensus for its members is more a consequence of the "push" by the already separated small subgroups rather than the trustful interactions therein. This means relatively small subgroups contribute to faster occurrence of the clustering of the entire social network beliefs. Therefore, these results are in strong consistency with the research of minority influence in social psychology (Nemeth 1986, Clark and Maass 1990), which suggests that consistent minorities can substantially modify people's private attitudes and opinions.

5.2. When Balance is Missing

Since the boundary constraint only restricts the negative update, similar to Theorem 3, for sufficiently small β, almost sure state consensus can be guaranteed when the positive graph $\mathsf{G}_{\mathrm{pst}}$ is connected.

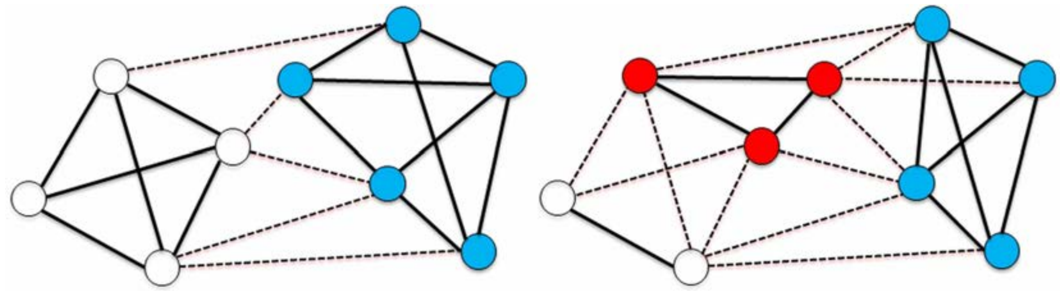
In absence of any balance property for the underlying graph, belief clustering may not happen. However, we can establish that when the positive graph is connected, then clustering cannot be achieved when β is large enough. In fact, the belief of a given node touches the two boundaries $-A$ and A an infinite number of times. Note that if the positive graph is connected, then the graph cannot be balanced.

Theorem 6. *Assume that the positive graph* $\mathsf{G}_{\mathrm{pst}}$ *is connected. For any* $\alpha \in (0,1)\backslash\{1/2\}$, *when* β *is sufficiently large, for almost all initial beliefs, under* (19), *we have for all* $i \in \mathsf{V}$,

$$\mathbb{P}\Big(\liminf_{k\to\infty} x_i(k) = -A,\ \limsup_{k\to\infty} x_i(k) = A\Big) = 1. \tag{24}$$

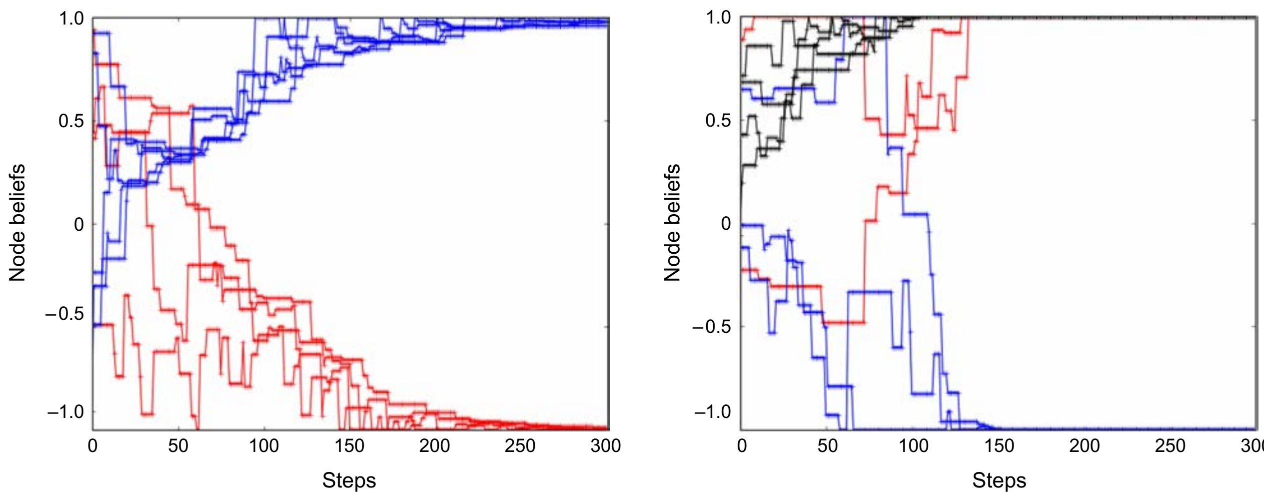
Theorem 6 suggests a new class of collective formation for the social beliefs beyond consensus, disagreement, or clustering studied in the literature.

Figure 3. (Color online) Strongly balanced (left) and weakly balanced (right) social graphs.



Notes. The negative links are shadowed. Nodes within the same subgraph in the balance partition are marked with the same color.

Figure 4. (Color online) The evolution of beliefs for strongly balanced (left) and weakly balanced (right) graphs.



Note. The beliefs of nodes within the same subgraph in the balance partition are marked with the same color.

Remark 4. The condition that β being sufficiently large in Theorems 4 and 5 is just a technical assumption ensuring almost sure boundary clustering. Practically, one can often encounter such clustering even for a small β, as illustrated in the coming numerical examples. On the other hand, Theorem 6 relies crucially on a large β, whereas a small β leads to belief consensus even in the presence of the negative edges.

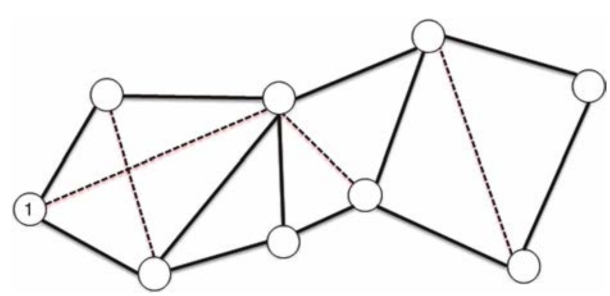
5.3. Numerical Examples

We now provide a few numerical examples to illustrate the results established in this section. We take $A = 1$ so that the node beliefs are restricted to the interval $[-1, 1]$. We take $\alpha = 1/3$ for the positive dynamics and $a = b = c = 1/3$ for the random asymmetric updates. The pair selection process is given by that when a node i is drawn, it will choose one of its neighbors with equal probability $1/\deg(i)$, where $\deg(i)$ is the degree of node i in the underlying graph G.

First of all we select two social graphs, one strongly balanced and the other weakly balanced, as shown in Figure 3. We take $\beta = 0.2$ and randomly select the nodes' initial values. It is observed that the boundary clustering phenomena established in Theorems 4 and 5 practically show up in every run of the random belief updates. We plot one of their typical sample paths in Figure 4, respectively, for the strongly balanced and weakly balanced graphs in Figure 3. In fact one can see that the clustering is achieved in around 300 steps.

Next, we select a social graph that is neither strongly nor weakly balanced, as in Figure 5. In Figure 6, we plot one of the typical sample paths of the random belief evolution with $\beta = 0.2$, where clearly belief consensus is achieved.

Figure 5. (Color online) A social network that is neither strongly nor weakly balanced.



Note. The negative links are dashed.

Figure 6. (Color online) The social network beliefs tend to a consensus with $\beta = 0.2$.

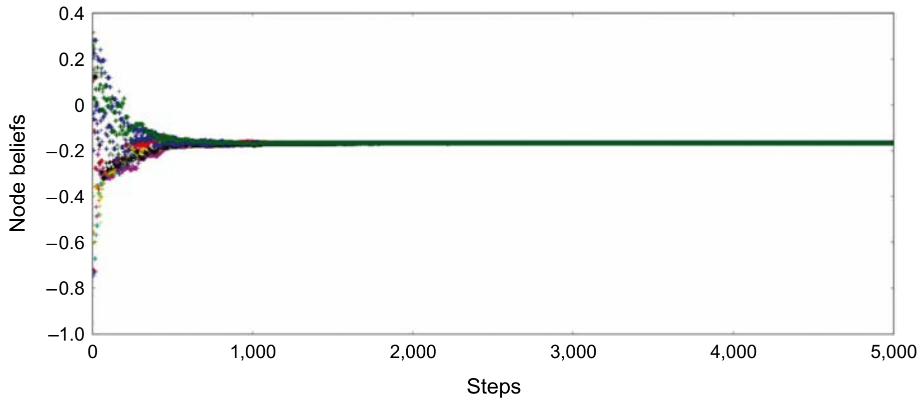
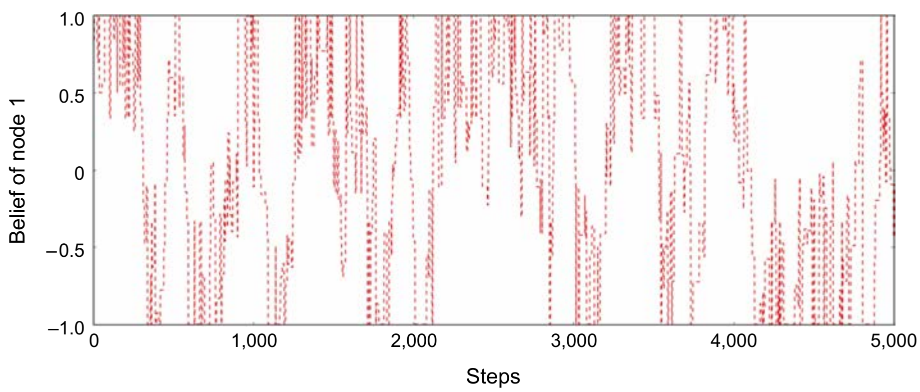


Figure 7. (Color online) The belief oscillation for a particular node with $\beta = 7$.



In Figure 7, we plot one of the typical sample paths of the random belief evolution for a selected node with $\beta = 7$, where the node belief alternatively touches the two boundaries -1 and 1 in the plotted 5,000 steps.

These numerical results are consistent with the results in Theorems 4–6.

6. Conclusions

The evolution of opinions over signed social networks was studied. Each link marking interpersonal interaction in the network was associated with a sign indicating friend or enemy relations. The dynamics of opinions were defined along positive and negative links, respectively. We have presented a comprehensive analysis to the belief convergence and divergence under various modes: in expectation, in mean-square, and almost surely. Phase transitions were established with sharp thresholds for the mean and mean-square convergence. In the almost sure sense, some surprising results were presented. When opinions have hard lower and upper bounds with asymmetric updates, the classical structure balance properties were shown to play a key role in the belief clustering. We believe that these results have largely extended our understanding to how trustful and antagonistic relations shape social opinions.

Some interesting directions for future research include the following topics. Intuitively, there is some natural coupling between the structure dynamics and the opinion evolution for signed networks. How this coupling determines the formation of the social structure is an interesting question bridging the studies on the dynamics of signed graphs (e.g., Marvel et al. 2011) and the opinion dynamics on signed social networks (e.g., Altafini 2012, 2013). It will also be interesting to ask what might be a proper model, and what the role of structure balance is, for Bayesian opinion evolution on signed social networks (e.g., Bikhchandani et al. 1992).

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2015.1448.

Acknowledgments

The authors thank the area editor, the associate editor, and the anonymous reviewers for their valuable comments and suggestions, which helped us improve the presentation of the paper considerably. The authors also gratefully thank Dr. Shuangshuang Fu for her generous help in preparing the numerical examples. This work has been supported in part by the Knut and Alice Wallenberg Foundation, the Swedish Research Council, and KTH Strategic Researsch Area–The Next Generation. The fourth author was supported by AFOSR MURI FA-9550-10-1-0573, NSF CNS-1035655, the Maryland Procurement Office contract H98230-14-C-0127, and Tage Erlander Guest Professorship in Sweden.

Appendix A. The Triangle Lemma

We establish a key technical lemma on the relative beliefs of three nodes in the network in the presence of at least one link among the three nodes. Denote $J_{ab}(k) := |x_a(k) - x_b(k)|$ for $a, b \in \mathsf{V}$ and $k \geqslant 0$. The proof of this lemma can be found in the e-companion.

Lemma 2. *Let i_0, i_1, i_2 be three different nodes in V. Suppose $\{i_0, i_1\} \in \mathsf{E}$. There exist a positive number $\delta > 0$ and an integer $Z > 0$ such that*

(i) *there is a sequence of Z successive node pairs leading to $J_{i_1 i_2}(Z) \geqslant \delta J_{i_0 i_1}(0)$;*

(ii) *there is a sequence of Z successive node pairs leading to $J_{i_1 i_2}(Z) \geqslant \delta J_{i_0 i_2}(0)$.*

Here δ and Z are absolute constants in the sense that they do not depend on i_0, i_1, or i_2 nor on the values held at these nodes.

Appendix B. Proof of Theorem 1

Introduce

$$X_{\min}(k) = \min_{i\in\mathsf{V}} x_i(k); \quad X_{\max}(k) = \max_{i\in\mathsf{V}} x_i(k).$$

We define $\mathcal{X}(k) = X_{\max}(k) - X_{\min}(k)$. Suppose belief divergence is achieved almost surely. Take a constant N_0 such that $N_0 > \mathcal{X}(0)$. Then almost surely

$$K_1 := \inf_k \{\mathcal{X}(k) \geqslant N_0\}$$

is a finite number. Then K_1 is a *stopping time* for the node pair selection process $G_k, k = 0, 1, 2, \ldots$ since

$$\{K_1 = k\} \in \sigma(G_0, \ldots, G_{k-1})$$

for all $k = 1, 2, \ldots$ becausse $\mathcal{X}(k)$ is, indeed, a function of $G_0, \ldots, G_{k-1}$. Strong Markov Property leads to $G_{K_1}, G_{K_1+1}, \ldots$ are independent of $\mathcal{F}_{K_1-1}$, and they are i.i.d. with the same distribution as G_0 (e.g., Theorem 4.1.3 in Durrett 2010).

Now take two different (deterministic) nodes i_0 and j_0. Since $\mathcal{X}(K_1) \geqslant N_0$, there must be two different (random) nodes i_* and j_* satisfying $x_{i_*}(K_1) < x_{j_*}(K_1)$ with $J_{i_* j_*}(K_1) \geqslant N_0$. We make the following claim.

Claim. There exist a positive number $\delta_0 > 0$ and an integer $Z_0 > 0$ (δ_0 and Z_0 are deterministic constants) such that we can always select a sequence of node pairs for time steps $K_1, K_1+1, K_1 + Z_0 - 1$ that guarantees $J_{i_0 j_0}(K_1 + Z_0) \geqslant \delta_0 N_0$.

First of all, note that i_* and j_* are independent with G_{K_1}, $G_{K_1+1}, \ldots,$ since $i_*, j_* \in \mathcal{F}_{K_1-1}$. Therefore, we can treat i_* and j_* as deterministic and prove the claim for all choices of such i_* and j_* (because we can always carry out the analysis conditioned on different events $\{i_* = i, j_* = j\}$, $i, j \in \mathsf{V}$). We proceed the proof recursively taking advantage of the Triangle Lemma.

Suppose $\{i_0, j_0\} = \{i_*, j_*\}$, the claim holds trivially. Now suppose $i_0 \notin \{i_*, j_*\}$. Either $J_{i_0 i_*}(K_1) \geqslant N_0/2$ or $J_{i_0 j_*}(K_1) \geqslant N_0/2$ must hold. Without loss of generality we assume $J_{i_0 i_*}(K_1) \geqslant N_0/2$. Since G is connected, there is a path $i_0 i_1 \ldots i_\tau j_0$ in G with $\tau \leqslant n-2$.

Based on Lemma 2, there exist $\delta > 0$ and integer $Z > 0$ such that a selection of node pair sequence for $K_1, K_1+1, \ldots, K_1 + Z - 1$ leads to

$$J_{i_0 i_1}(K_1 + Z) \geqslant \delta J_{i_0 i_*}(K_1) \geqslant \frac{\delta N_0}{2}$$

since $\{i_0, i_1\} \in \mathsf{E}$. Applying recursively the Triangle Lemma based on $\{i_1, i_2\}, \ldots, \{i_\tau, j_0\} \in \mathsf{E}$, we see that a selection of node pair sequence for $K_1, K_1+1, \ldots, K_1 + (\tau+1)Z - 1$ will give us

$$J_{i_0 j_0}(K_1 + (\tau+1)Z) \geqslant \delta^{\tau+1} J_{i_0 i_*}(K_1) \geqslant \frac{\delta^{\tau+1} N_0}{2}.$$

Since $\tau \leqslant n-2$, the claim always holds for $\delta_0 = \delta^{n-1}/2$ and $Z_0 = (n-1)Z$, independently of i_* and j_*.

Therefore, denoting $p_* = \min\{p_{ij} + p_{ji} \colon \{i, j\} \in \mathsf{E}\}$, the claim we just proved yields that

$$\mathbb{P}\left(J_{i_0 j_0}(K_1 + (n-1)Z) \geqslant \frac{\delta^{n-1} N_0}{2}\right) \geqslant \left(\frac{p_*}{n}\right)^{(n-1)Z}. \tag{B1}$$

We proceed the analysis by recursively defining

$$K_{m+1} := \inf\{k \geqslant K_m + Z_0 \colon \mathcal{X}(k) \geqslant N_0\}, \quad m = 1, 2, \ldots.$$

Given that belief divergence is achieved, K_m is finite for all $m \geqslant 1$ almost surely. Thus,

$$\mathbb{P}\left(J_{i_0 j_0}(K_m + Z_0) \geqslant \frac{\delta^{n-1} N_0}{2}\right) \geqslant \left(\frac{p_*}{n}\right)^{Z_0}, \tag{B2}$$

for all $m = 1, 2, \ldots$. Moreover, the node pair sequence

$$G_{K_1}, \ldots, G_{K_1+Z_0-1}; \ldots\ldots; G_{K_m}, \ldots, G_{K_m+Z_0-1}; \ldots\ldots$$

are independent and have the same distribution as G_0 (This is because $\mathcal{F}_{K_1} \subseteq \mathcal{F}_{K_1+1} \subseteq \cdots \subseteq \mathcal{F}_{K_1+Z_0-1} \subseteq \mathcal{F}_{K_2} \ldots$ (see Theorem 4.1.4 in Durrett 2010)).

Therefore, we can finally invoke the second Borel-Cantelli Lemma (cf. Theorem 2.3.6 in Durrett 2010) to conclude that almost surely, there exists an infinite subsequence K_{m_s}, $s = 1, 2, \ldots,$ satisfying

$$J_{i_0 j_0}(K_{m_s} + Z_0) \geqslant \frac{\delta^{n-1} N_0}{2}, \quad s = 1, 2, \ldots, \tag{B3}$$

conditioned on that belief divergence is achieved. Since δ is a constant and N_0 is arbitrarily chosen, (B3) is equivalent to $\mathbb{P}(\limsup_{k\to\infty} |x_{i_0}(k) - x_{j_0}(k)| = \infty) = 1$, which completes the proof.

Appendix C. Proof of Lemma 1

(i) It suffices to show that $\mathbb{P}(\limsup_{k\to\infty} \mathcal{X}(k) \in [a_*, b_*]) = 0$ for all $0 < a_* < b_*$. We prove the statement by contradiction. Suppose $\mathbb{P}(\limsup_{k\to\infty} \mathcal{X}(k) \in [a_*, b_*]) = p > 0$ for some $0 < a_* < b_*$.

Take $0 < \varepsilon < 1$ and define $a = a_*(1-\varepsilon)$, $b = b_*(1+\varepsilon)$. We introduce

$$T_1 := \inf_k \{\mathcal{X}(k) \in [a, b]\}.$$

Then T_1 is finite with probability at least p. T_1 is a stopping time. $G_{T_1}, G_{T_1+1}, \ldots$ are independent of $\mathcal{F}_{T_1-1}$, and they are i.i.d. with the same distribution as G_0.

Now since $\mathsf{G}_{\rm neg}$ is nonempty, we take a link $\{i_\star, j_\star\} \in \mathsf{E}_{\rm neg}$. Repeating the same analysis as the proof of Theorem 1, the following statement holds true conditioned on that T_1 is finite: there exist a positive number $\delta_0 > 0$ and an integer $Z_0 > 0$ (δ_0 and Z_0 are deterministic constants) such that we can always select a sequence of node pairs for time steps T_1, T_1+1, T_1+Z_0-1 which guarantees $J_{i_\star j_\star}(T_1+Z_0) \geqslant \delta_0 a$.

Here δ_0 and Z_0 follow from the same definition in the proof of Theorem 1. Take

$$m_0 = \left\lceil \log_{2\beta+1} \frac{2b}{\delta_0 a} \right\rceil$$

and let $\{i_\star, j_\star\}$ be selected for $T_1+Z_0, \ldots, T_1+Z_0+m_0-1$. Then noting that $\{i_\star, j_\star\} \in \mathsf{E}_{\rm neg}$, the choice of m_0 and the fact that $J_{i_\star j_\star}(s+1) = (2\beta+1)J_{i_\star j_\star}(s)$, $s = T_1+Z_0, \ldots, T_1+Z_0+m_0-1$ lead to

$$\begin{aligned}\mathfrak{X}(T_1+Z_0+m_0) &\geqslant J_{i_\star j_\star}(T_1+Z_0+m_0)\\ &\geqslant (2\beta+1)^{m_0}\delta_0 a \geqslant 2b \geqslant 2b_*.\end{aligned}$$

We have proved that

$$\mathbb{P}(\mathfrak{X}(T_1+Z_0+m_0) \geqslant 2b_* \mid T_1 < \infty) \geqslant \left(\frac{p_*}{n}\right)^{Z_0+m_0}. \tag{C1}$$

Similarly, we proceed the analysis by recursively defining

$$T_{m+1} := \inf\{k \geqslant T_m + Z_0 + m_0\colon \mathfrak{X}(k) \in [a, b]\}, \quad m = 1, 2, \ldots.$$

Given $\mathbb{P}(\limsup_{k\to\infty} \mathfrak{X}(k) \in [a_*, b_*]) = p$, T_m is finite for all $m \geqslant 1$ with probability at least p. Thus, there holds

$$\mathbb{P}(\mathfrak{X}(T_m+Z_0+m_0) \geqslant 2b_* \mid T_m < \infty) \geqslant \left(\frac{p_*}{n}\right)^{Z_0+m_0}, \quad m = 1, 2, \ldots. \tag{C2}$$

The independence of

$$G_{T_1}, \ldots, G_{T_1+Z_0+m_0-1}; \ldots\ldots; G_{T_m}, \ldots, G_{T_m+Z_0+m_0-1}; \ldots\ldots$$

once again allows us to invoke the Borel-Cantelli Lemma to conclude that almost surely, there exists an infinite subsequence T_{m_s}, $s = 1, 2, \ldots$, satisfying

$$\mathfrak{X}(T_{m_s}+Z_0+m_0) \geqslant 2b_*, \quad s = 1, 2, \ldots, \tag{C3}$$

given that T_m, $m = 1, 2\ldots$, are finite. In other words, we have obtained that

$$\mathbb{P}\left(\limsup_{k\to\infty} \mathfrak{X}(k) \geqslant 2b_* \,\middle|\, \limsup_{k\to\infty} \mathfrak{X}(k) \in [a_*, b_*]\right) = 1, \tag{C4}$$

which is impossible, and the first part of the theorem has been proved.

(ii) It suffices to show that $\mathbb{P}(\liminf_{k\to\infty} \mathfrak{X}(k) \in [a_*, b_*]) = 0$ for all $0 < a_* < b_*$. The proof is again by contradiction. Assume that $\mathbb{P}(\liminf_{k\to\infty} \mathfrak{X}(k) \in [a_*, b_*]) = q > 0$. Let a, b, and $T_1 := \inf_k\{\mathfrak{X}(k) \in [a, b]\}$ as defined earlier. T_1 is finite with probability at least q.

Let $\ell_0 \in \mathsf{V}$ satisfying $x_{\ell_0}(T_1) = X_{\min}(T_1)$. There is a path from ℓ_0 to every other node in the network since $\mathsf{G}_{\rm pst}$ is connected. We introduce

$$\mathsf{V}_t^\dagger := \{J\colon d(\ell_0, j) = t \text{ in } \mathsf{G}_{\rm pst}\}, \quad t = 0, \ldots, \mathrm{diam}(\mathsf{G}_{\rm pst})$$

as a partition of V. We relabel the nodes in $\mathsf{V}\backslash\{\ell_0\}$ in the following manner.

$$\ell_s \in \mathsf{V}_1^\dagger, s = 1, \ldots, |\mathsf{V}_1^\dagger|;$$
$$\ell_s \in \mathsf{V}_2^\dagger, s = |\mathsf{V}_1^\dagger|+1, \ldots, |\mathsf{V}_1^\dagger|+|\mathsf{V}_2^\dagger|;$$
$$\ldots\ldots$$
$$\ell_s \in \mathsf{V}_{\mathrm{diam}(\mathsf{G}_{\rm pst})}^\dagger, s = \sum_{t=1}^{\mathrm{diam}(\mathsf{G}_{\rm pst})-1} |\mathsf{V}_t^\dagger|, \ldots, n-1.$$

Then the definition of $\mathsf{V}_t^\dagger$ and the connectivity of $\mathsf{G}_{\rm pst}$ allow us to select a sequence of node pairs in the form of

$$G_{T_1+s} = \{\ell_\rho, \ell_{s+1}\}, \quad \{\ell_\rho, \ell_{s+1}\} \in \mathsf{E}_{\rm pst} \text{ with } \rho \leqslant s,$$

for $s = 0, \ldots, n-2$. Next we give an estimation for $\mathfrak{X}$ under the selected sequence of node pairs.

- Since $\{\ell_0, \ell_1\}$ is selected at time T_1, we have

$$\begin{aligned}x_{\ell_0}(T_1+1) &= (1-\alpha)x_{\ell_0}(T_1) + \alpha x_{\ell_1}(T_1)\\ &\leqslant (1-\alpha)X_{\min}(T_1) + \alpha X_{\max}(T_1);\\ x_{\ell_1}(T_1+1) &= (1-\alpha)x_{\ell_1}(T_1) + \alpha x_{\ell_0}(T_1)\\ &\leqslant (1-\alpha)X_{\max}(T_1) + \alpha X_{\min}(T_1).\end{aligned} \tag{C5}$$

This leads to $x_{\ell_s}(T_1+1) \leqslant (1-\alpha_*)X_{\min}(T_1) + \alpha_* X_{\max}(T_1)$, $s = 0, 1$, where $\alpha_* = \max\{\alpha, 1-\alpha\}$.

- Note that $X_{\max}(T_1+1) = X_{\max}(T_1)$, and that either $\{\ell_0, \ell_2\}$ or $\{\ell_1, \ell_2\}$ is selected at time T_1+1. We deduce

$$\begin{aligned}x_{\ell_s}(T_1+2) &\leqslant (1-\alpha)[(1-\alpha_*)X_{\min}(T_1) + \alpha_* X_{\max}(T_1)]\\ &\quad + \alpha X_{\max}(T_1)\\ &\leqslant (1-\alpha_*)^2 X_{\min}(T_1)\\ &\quad + (1-(1-\alpha_*)^2)X_{\max}(T_1), \quad s = 0, 1;\\ x_{\ell_2}(T_1+2) &\leqslant \alpha[(1-\alpha_*)X_{\min}(T_1) + \alpha_* X_{\max}(T_1)]\\ &\quad + (1-\alpha)X_{\max}(T_1)\\ &\leqslant (1-\alpha_*)^2 X_{\min}(T_1)\\ &\quad + (1-(1-\alpha_*)^2)X_{\max}(T_1),\end{aligned} \tag{C6}$$

Thus we obtain $x_{\ell_s}(T_1+2) \leqslant (1-\alpha_*)^2 X_{\min}(T_1) + (1-(1-\alpha_*)^2) \cdot X_{\max}(T_1)$, $s = 0, 1, 2$.

- We carry on the analysis recursively and finally get

$$\begin{aligned}x_{\ell_s}(T_1+n-1) &\leqslant (1-\alpha_*)^{n-1} X_{\min}(T_1)\\ &\quad + (1-(1-\alpha_*)^{n-1})X_{\max}(T_1),\end{aligned}$$

for $s=0,1,2,\ldots,n-1$. Equivalently,

$$X_{\max}(T_1+n-1)\leqslant(1-\alpha_*)^{n-1}X_{\min}(T_1)+(1-(1-\alpha_*)^{n-1})X_{\max}(T_1). \tag{C7}$$

We conclude that

$$\begin{aligned}\mathcal{X}(T_1+n-1)&=X_{\max}(T_1+n-1)-X_{\min}(T_1+n-1)\\&=X_{\max}(T_1+n-1)-X_{\min}(T_1)\\&\leqslant r_0\mathcal{X}(T_1),\end{aligned}\tag{C8}$$

where $r_0=1-(1-\alpha_*)^{n-1}$ is a constant in $(0,1)$.

With the above analysis taking

$$L_0=\left\lceil \log_{r_0}\frac{a}{2b}\right\rceil,$$

and selecting the given pair sequence periodically for L_0 rounds, we obtain

$$\mathcal{X}(T_1+(n-1)L_0)\leqslant r_0^{L_0}\mathcal{X}(T_1)\leqslant\frac{a}{2b}\cdot b=\frac{a}{2}<\frac{a_*}{2}. \tag{C9}$$

In light of (C9) and the selection of the node pair sequence, we have obtained that

$$\mathbb{P}\left(\mathcal{X}(T_1+(n-1)L_0)\leqslant\frac{a_*}{2}\right)\geqslant\left(\frac{p_*}{n}\right)^{(n-1)L_0} \tag{C10}$$

given that T_1 is finite. We repeat the above argument for T_{m+1}, $m=2,3\ldots$. The Borel-Cantelli Lemma then implies

$$\mathbb{P}\left(\liminf_{k\to\infty}\mathcal{X}(k)\leqslant\frac{a_*}{2}\,\middle|\,\liminf_{k\to\infty}\mathcal{X}(k)\in[a_*,b_*]\right)=1, \tag{C11}$$

which is impossible and completes the proof.

Appendix D. Proof of Theorem 3

Theorem 3 is a direct consequence of the following lemmas.

Lemma 3. *Suppose G_{pst} is connected. Then for every fixed $\alpha\in(0,1)$, we have $\mathbb{P}(\mathcal{C})=1$ for all $0\leqslant\beta<\beta^{\natural}$ with*

$$\beta^{\natural}:=\sup\left\{\beta\colon\beta(1+\beta)<\frac{\lambda_2(L_{\text{pst}}^{\dagger})}{\lambda_{\max}(L_{\text{neg}}^{\dagger})}\alpha(1-\alpha)\right\}.$$

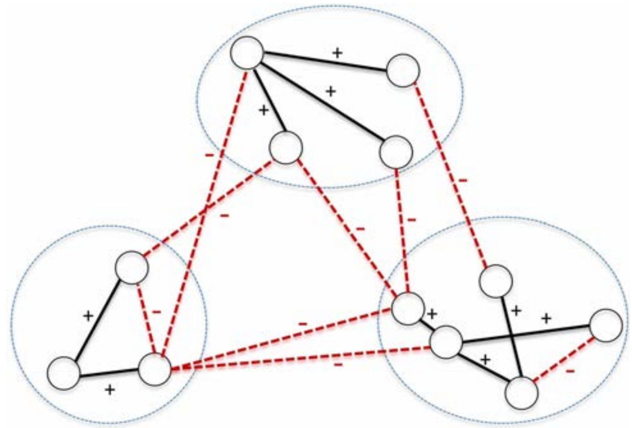
Lemma 4. *Suppose $\alpha\in[0,1]$ with $\alpha\neq1/2$. There exists a constant $\beta^{\sharp}>0$ such that*

$$\mathbb{P}\left(\liminf_{k\to\infty}\max_{i,j}|x_i(k)-x_j(k)|=\infty\right)=1$$

for almost all initial beliefs when $\beta>\beta^{\sharp}$.

Lemma 3 is proved utilizing a martingale convergence theorem, whereas Lemma 4 is established in view of Kolmogorov's strong law of large numbers. Their detailed proofs can be found in the e-companion.

Figure E.1. (Color online) There is a unique partition of G into subgraphs following the connected components of G_{pst}.



Note. Viewing each subgraph as a super node, the graph is connected and has only negative edges.

Appendix E. Proof of Theorem 4

We first state and prove intermediate lemmas that will be useful for the proofs of Theorems 4–6.

Lemma 5. *Assume that $\alpha\in(0,1)$. Let $i_1\ldots i_k$ be a path in the positive graph; i.e., $\{i_s,i_{s+1}\}\in\mathsf{G}_{\text{pst}}$, $s=1,\ldots,k-1$. Take a node $i_*\in\{i_1,\ldots,i_k\}$. Then for any $\varepsilon>0$, there always exists an integer $Z_\star(\varepsilon)\geqslant1$, such that we can select a sequence of node pairs from $\{i_s,i_{s+1}\}$, $s=1,\ldots,k-1$ under asymmetric updates, which guarantees*

$$J_{i_*i_s}(Z_\star)\leqslant2A\varepsilon,\quad s\in\{1,\ldots,k\}$$

for all initial condition $x_{i_s}(0)$, $s=1,\ldots,k$.

Proof. The proof is easy and an appropriate sequence of node pairs can be built just observing that $J_{i_*i_s}\leqslant2A$ for all $s\in\{1,\ldots,k\}$. □

Lemma 6. *Fix $\alpha\in(0,1)$ with $\alpha\neq1/2$. Under belief dynamics (19), there exist an integer $Z_0\geqslant1$ and a constant $\vartheta_0>0$ such that*

$$\mathbb{P}\left(\exists\{i_*,j_*\}\in\mathsf{G}_{\text{neg}}\ \text{s.t.}\ J_{i_*j_*}(Z_0)\geqslant\frac{1}{2n}\mathcal{X}(0)\right)\geqslant\vartheta_0. \tag{E1}$$

Proof. We can always uniquely divide V into $m_0\geqslant1$ mutually disjoint sets $V_1,\ldots,V_{m_0}$ such that $\mathsf{G}_{\text{pst}}(V_k)$, $k=1,\ldots,m_0$ are connected graphs, where $\mathsf{G}_{\text{pst}}(V_k)$ is the induced graph of G_{pst} by node set V_k. The idea is to treat each $\mathsf{G}_{\text{pst}}(V_k)$ as a *super node* (an illustration of this partition is shown in Figure E.1). Since G is connected and G_{neg} is nonempty, these super nodes form a connected graph whose edges are negative.

One can readily show that there exist two distinct nodes $\eta_1,\eta_2\in\mathsf{V}$ with $\eta_i\in V_{\nu_i}$, $i=1,2$ (V_{ν_1} and V_{ν_2} can be the same, of course) such that there is at least one *negative* edge between V_{ν_1} and V_{ν_2} and such that

$$J_{\eta_1\eta_2}(0)\geqslant\frac{1}{m_0}\mathcal{X}(0). \tag{E2}$$

Now select $v_1 \in V_{\nu_1}$ and $v_2 \in V_{\nu_2}$ such that $\{v_1, v_2\} \in \mathsf{E}_{\rm neg}$. In view of Lemma 5 and observing that asymmetric updates happen with a strictly positive probability, we can always find $\vartheta_0 > 0$ and $Z_0 \geqslant 1$ (both functions of (α, n, a, b, c)) such that

$$\mathbb{P}\left(x_{\nu_i}(Z_0) = x_{\nu_i}(0),\ J_{v_i\nu_i}(Z_0) \leqslant \frac{1}{4n}\mathfrak{X}(0),\ i=1,2\right) \geqslant \vartheta_0, \tag{E3}$$

(because $\mathsf{G}_{\rm pst}(V_{\nu_i}), i=1,2$ are connected graphs). (E1) follows from (E2) and (E3) since $m_0 \leqslant n$. □

LEMMA 7. *Fix $\alpha \in (0,1)$ with $\alpha \neq 1/2$. Under belief dynamics (19), there exists $\beta^\diamond(\alpha) > 0$ such that $\mathbb{P}(\limsup_{k\to\infty} \mathfrak{X}(k) = 2A) = 1$ for almost all initial beliefs if $\beta > \beta^\diamond$.*

PROOF. In view of Lemma 6, we have

$$\mathbb{P}\left(\mathfrak{X}(Z_0+t) \geqslant \min\left\{\frac{(\beta+1)^t}{2n}\mathfrak{X}(0), 2A\right\}\right) \geqslant \left(\frac{cp_*}{n}\right)^t \vartheta_0, \quad t = 0,1,\ldots. \tag{E4}$$

We can conclude that

$$\mathbb{P}\left(\limsup_{k\to\infty}\mathfrak{X}(k) = 2A\right) + \mathbb{P}\left(\limsup_{k\to\infty}\mathfrak{X}(k) = 0\right) = 1 \tag{E5}$$

as long as $\beta > 0$ using the same argument as that used in the proof of statement (i) in Lemma 1.

With (E4), we have

$$\mathbb{P}\left(\mathfrak{X}(Z_0+1) \geqslant \frac{\beta+1}{2n}\mathfrak{X}(0)\right) \geqslant \frac{cp_*}{n}\vartheta_0 \tag{E6}$$

conditioned on $\mathfrak{X}(0) \leqslant 4An/(1+\beta)$. Therefore, we can invoke exactly the same argument as that used in the proof of Lemma 4 to conclude that there exists $\beta^\diamond(\alpha) > 0$ such that

$$\mathbb{P}\left(\limsup_{k\to\infty}\mathfrak{X}(k) \geqslant 4An/(1+\beta)\right) = 1 \tag{E7}$$

for all $\beta > \beta^\diamond(\alpha)$. Combining (E5) and (E7), we get the desired result. □

LEMMA 8. *Assume that the graph is strongly balanced under partition $\mathsf{V} = \mathsf{V}_1 \cup \mathsf{V}_2$ and that $\mathsf{G}_{\mathsf{V}_1}$ and $\mathsf{G}_{\mathsf{V}_2}$ are connected. Let $\alpha \in (0,1)\backslash\{1/2\}$. Fix the initial beliefs x^0. Then under belief dynamics (19), there are two random variables, $B_1^\dagger(x^0), B_2^\dagger(x^0)$ both taking value in $\{-A, A\}$, such that*

$$\mathbb{P}\left(\lim_{k\to\infty} x_i(k) = B_1^\dagger,\ i \in \mathsf{V}_1;\ \lim_{k\to\infty} x_i(k) = B_2^\dagger,\ i\in\mathsf{V}_2 \,\middle|\, \mathscr{E}_{\rm sep}(\epsilon)\right) = 1 \tag{E8}$$

for all $\epsilon > 0$, where by definition, $\mathscr{E}_{\rm sep}(\epsilon)$ is the ϵ-separation event

$$\mathscr{E}_{\rm sep}(\epsilon) := \left\{\limsup_{k\to\infty}\max_{i\in\mathsf{V}_1, j\in\mathsf{V}_2}|x_i(k) - x_j(k)| \geqslant \epsilon\right\}.$$

PROOF. Suppose $x_{i_1}(0) - x_{i_2}(0) \geqslant \epsilon > 0$ for $i_1 \in \mathsf{V}_1$ and $i_2 \in \mathsf{V}_2$. By assumption, $\mathsf{G}_{\mathsf{V}_1}$ and $\mathsf{G}_{\mathsf{V}_2}$ are connected. Thus, from Lemma 5, there exist an integer $Z_1 \geqslant 1$ and a constant $\bar{p}$ (both depending on $\epsilon, n, \alpha, a, b$) such that

$$\min_{i\in\mathsf{V}_1} x_i(Z_1) - \max_{i\in\mathsf{V}_2} x_i(Z_1) \geqslant \frac{\epsilon}{2} \tag{E9}$$

happens with probability at least $\bar{p}$. Intuitively, Equation (E9) characterizes the event where the beliefs in the two sets V_1 and V_2 are *completely separated*. Since all edges between the two sets are negative, conditioned on event (E9), it is then straightforward to see that almost surely we have $\lim_{k\to\infty} x_i(k) = A,\ i\in\mathsf{V}_1$ and $\lim_{k\to\infty} x_i(k) = -A,\ i \in \mathsf{V}_2$.

Given $\mathscr{E}_{\rm sep}(\epsilon)$, $\{\exists i_1 \in \mathsf{V}_1, i_2\in\mathsf{V}_2$ s.t. $x_{i_1}(k) - x_{i_2}(k) \geqslant \epsilon$ for infinitely many $k\}$ is an almost sure event. Based on our previous discussion and by a simple stopping time argument, the Borel-Cantelli Lemma implies that the complete separation event happens almost surely given $\mathscr{E}_{\rm sep}(\epsilon)$. This completes the proof. □

LEMMA 9. *Assume that the graph is strongly balanced under partition $\mathsf{V} = \mathsf{V}_1\cup\mathsf{V}_2$ and that $\mathsf{G}_{\mathsf{V}_1}$ and $\mathsf{G}_{\mathsf{V}_2}$ are connected. Suppose $\alpha\in(0,1)\backslash\{1/2\}$. Then under dynamics (19), there exists β sufficiently large such that $\mathbb{P}(\mathscr{E}_{\rm sep}(A/2)) = 1$ for almost all initial beliefs.*

PROOF. Let us first focus on a fixed time instant k. Suppose $x_i(k) - x_j(k) \geqslant A$ for some $i, j \in \mathsf{V}$. If i and j belong to different sets V_1 and V_2, we already have $\max_{i\in\mathsf{V}_1, j\in\mathsf{V}_2}|x_i(k) - x_j(k)| \geqslant A$. Otherwise, say $i, j\in\mathsf{V}_1$. There must be another node $l \in \mathsf{V}_2$. We have $\max_{i\in\mathsf{V}_1, j\in\mathsf{V}_2}|x_i(k) - x_j(k)| \geqslant A/2$ since either $|x_i(k) - x_l(k)| \geqslant A/2$ or $|x_j(k) - x_l(k)| \geqslant A/2$ must hold. Therefore, we conclude that

$$\mathfrak{X}(k) \geqslant A \implies \max_{i\in\mathsf{V}_1, j\in\mathsf{V}_2}|x_i(k) - x_j(k)| \geqslant A/2. \tag{E10}$$

Then the desired conclusion follows directly from Lemma 7. □

Theorem 4 is a direct consequence of Lemmas 8 and 9.

Appendix F. Proof of Theorem 5

The proof is similar to that of Theorem 4. We just provide the main arguments.

First by Lemma 7 we have $\mathbb{P}(\limsup_{k\to\infty}\mathfrak{X}(k) = 2A) = 1$ for almost all initial values with sufficiently large β. Then as for (E10), we have

$$\mathfrak{X}(k) \geqslant A \implies \max_{i\in\mathsf{V}_s, j\in\mathsf{V}_t, s\neq t\in\{1,\ldots,m\}}|x_i(k) - x_j(k)| \geqslant \frac{A}{m}, \tag{F1}$$

where $m \geqslant 2$ comes from the definition of weak balance. Therefore, introducing

$$\mathscr{E}^*_{\rm sep}(\epsilon) := \left\{\limsup_{k\to\infty}\max_{i\in\mathsf{V}_s, j\in\mathsf{V}_t, s\neq t\in\{1,\ldots,m\}}|x_i(k)-x_j(k)| \geqslant \epsilon\right\},$$

we can show that $\mathbb{P}(\mathscr{E}^*_{\rm sep}(A/m)) = 1$ for almost all initial beliefs, for sufficiently large β.

Next, suppose there exist a constant $\eta > 0$ and two node sets V_{i_1} and V_{i_2} with $i_1, i_2 \in \{1,\ldots,m\}$ such that the complete separation event

$$\min_{i\in\mathsf{V}_{i_1}} x_i(k) - \max_{i\in\mathsf{V}_{i_2}} x_i(k) \geqslant \eta \tag{F2}$$

happens. Recall that the underlying graph is complete. Then if $(\beta+1)\eta \geqslant 2A$, we can always select $Z_* := |\mathsf{V}_{i_1}| + |\mathsf{V}_{i_2}|$ negative edges between nodes in the sets V_{i_1} and V_{i_2} so that after the corresponding updates

$$x_i(k+Z_*) = A, \quad i \in \mathsf{V}_{i_1}, \\ x_i(k+Z_*) = -A, \quad i \in \mathsf{V}_{i_2}. \tag{F3}$$

One can easily see that we can continue to build the (finite) sequence of edges for updates such that nodes in V_k will hold the same belief in $\{-A, A\}$, for all $k = 1, \ldots, m$. After this sequence of updates, the beliefs held at the various nodes remain unchanged (two nodes with the same belief cannot influence each other, even in presence of a negative link; and two nodes with different beliefs are necessarily enemies). To summarize, conditioned on the complete separation event (F2), we can select a sequence of node pairs under which belief clustering is reached, and this clustering state is an *absorbing* state.

Finally, the Borel-Cantelli Lemma and $\mathbb{P}(\mathscr{E}^*_{\text{sep}}(A/m)) = 1$ guarantee that almost surely the complete separation event (F2) happens an infinite number of times if $\eta = A/2m$ in view of Lemma 5. The end of the proof is then done as in that of Theorem 4.

Appendix G. Proof of Theorem 6

Again the result is obtained by combining Lemmas 5 and 7 with Borel-Cantelli lemma.

Endnote

1. This further confirms that in our model, the origin of the belief space has no special meaning at all, in contrast to the model of Altafini (2013).

References

Acemoglu D, Ozdaglar A (2011) Opinion dynamics and learning in social networks. *Dynamic Games Appl.* 1:3–49.
Acemoglu D, Ozdaglar A, ParandehGheibi A (2010) Spread of (mis)information in social networks. *Games Econom. Behav.* 70:194–227.
Acemoglu D, Como G, Fagnani F, Ozdaglar A (2013) Opinion fluctuations and disagreement in social networks. *Math. Oper. Res.* 38(1):1–27.
Acemoglu D, Dahleh MA, Lobel I, Ozdaglar A (2011) Bayesian learning in social networks. *Rev. Econom. Stud.* 78:1201–1236.
Altafini C (2012) Dynamics of opinion forming in structurally balanced social networks. *PLoS ONE* 7(6):e38135.
Altafini C (2013) Consensus problems on networks with antagonistic interactions. *IEEE Trans. Automatic Control* 58:935–946.
Banerjee A (1992) A simple model of herd behavior. *Quart. J. Econom.* 107:797–817.
Bikhchandani S, Hirshleifer D, Welch I (1992) A theory of fads, fashion, custom, and cultural change as information cascades. *J. Political Econom.* 100:992–1026.
Boyd S, Ghosh A, Prabhakar B, Shah D (2006) Randomized gossip algorithms. *IEEE Trans. Inf. Theory* 52:2508–2530.
Blondel VD, Hendrickx JM, Tsitsiklis JN (2009) On Krause's multi-agent consensus model with state-dependent connectivity. *IEEE Trans. Autom. Control* 54:2586–2597.
Blondel VD, Hendrickx JM, Tsitsiklis JN (2010) Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *SIAM J. Control Optim.* 48(8):5214–5240.
Castellano C, Fortunato S, Loreto V (2009) Statistical physics of social dynamics. *Rev. Modern Phys.* 81(2):591–646.
Cartwright D, Harary F (1956) Structural balance: A generalization of Heider's theory. *Psychol Rev.* 63:277–293.
Clark RD, Maass A (1990) The effects of majority size on minority influence. *Eur. J. Soc. Psych.* 20(2):99–117.
Dandekara P, Goel A, Lee DT (2013) Biased assimilation, homophily, and the dynamics of polarization. *Proc. Natl. Acad. Sci.* 110(15): 5791–5796.
Davis JA (1963) Structural balance, mechanical solidarity, and interpersonal relations. *Amer. J. Sociol.* 68:444–462.
Davis JA (1967) Clustering and structural balance in graphs. *Human Relations* 20:181–187.
DeGroot MH (1974) Reaching a consensus. *J. Amer. Statist. Assoc.* 69:118–121.
DeMarzo PM, Vayanos D, Zwiebel J (2003) Persuasion bias, social influence, and unidimensional opinions. *Quart. J. Econom.* 118: 909–968.
Ding X, Jiang T (2010) Spectral distributions of adjacency and Laplacian matrices of random graphs. *Ann. Appl. Probab.* 20:2086–2117.
Durrett R (2010) *Probability Theory: Theory and Examples*, 4th ed. (Cambridge University Press, New York).
Easley D, Kleinberg J (2010) *Networks, Crowds, and Markets: Reasoning About a Highly Connected World* (Cambridge University Press, New York).
Erdős P, Rényi A (1960) On the evolution of random graphs. *Publications Math. Institute Hungarian Acad. Sci.* 5:17–61.
Facchetti G, Iacono G, Altafini C (2011) Computing global structural balance in large-scale signed social networks. *Proc. Natl. Acad. Sci.* 108(52):20953–20958.
Fagnani F, Zampieri S (2008) Randomized consensus algorithms over large scale networks. *IEEE J. Selected Areas Comm.* 26:634–649.
Galam S (1996) Fragmentation versus stability in bimodal coalitions. *Physica A* 230:174–188.
Golub B, Jackson MO (2010) Naive learning in social networks and the wisdom of crowds. *Amer. Econom. J.: Microeconomics* 2(1): 112–149.
Harary F (1953) On the notion of balance of a signed graph. *Michigan Math. J.* 2(2):143–146.
Heider F (1946) Attitudes and cognitive organization. *J. Psychol* 21: 107–112.
Horn RA, Johnson CR (1985) *Matrix Analysis* (Cambridge University Press, New York).
Jackson MO (2008) *Social and Economic Networks*, 1st ed. (Princeton University Press, Princeton, NJ).
Jadbabaie A, Molavi P, Sandroni A, Tahbaz-Salehi A (2012) Non-Bayesian social learning. *Games Econom. Behav.* 76:210–225.
Krause U (1997) Soziale dynamiken mit vielen interakteuren. Eine problemskizze. *Proc. Modellierung Simul. von Dynamiken mit vielen interagierenden Akteuren* 37–51.
Li L, Scaglione A, Swami A, Zhao Q (2013) Consensus, polarization and clustering of opinions in social networks. *IEEE J. Selected Areas Comm.* 31:1072–1083.
Macy MW, Kitts JA, Flache A, Benard S (2003) Polarization in dynamic networks: A Hopfield model of emergent structure. *Dynam. Soc. Network Modeling Anal.* 162–173.
Marvel SA, Kleinberg J, Kleinberg RD, Strogatz SH (2011) Continuous-time model of structural balance. *Proc. Natl. Acad. Sci.* 108(5): 1751–1752.
Matei I, Baras JS, Somarakis C (2013) Convergence results for the linear consensus problem under Markovian random graphs. *SIAM J. Control Optim.* 51(2):1574–1591.
Nemeth CJ (1986) Differential contributions of majority and minority influence. *Psych. Rev.* 93(1):23–32.
Shi G, Johansson M, Johansson KH (2013) How agreement and disagreement evolve over random dynamic networks. *IEEE J. Selected Areas Comm.* 31:1061–1071.
Shi G, Li B, Johansson M, Johansson KH (2014) When do gossip algorithms converge in finite time? *The 21st Internat. Sympos. Math. Theory of Networks and Systems* (*MTNS*) (Groningen, Netherlands).

Shi G, Proutiere A, Johansson M, Baras JS, Johansson KH (2015) Emergent behaviors over signed random dynamical networks: State-flipping model. *IEEE Trans. Control of Network Systems* 2:142–153.

Tahbaz-Salehi A, Jadbabaie A (2008) A necessary and sufficient condition for consensus over random networks. *IEEE Trans. Autom. Control* 53:791–795.

Theodorakopoulos G, Baras JS (2008) Game theoretic modeling of malicious users in collaborative networks. *IEEE J. Selected Areas Comm.* 26:1317–1327.

Touri B, Nedić A (2011) On ergodicity, infinite flow and consensus in random models. *IEEE Trans. Automatic Control* 56:1593–1605.

Tsitsiklis JN (1984) Problems in decentralized decision making and computation. Unpublished doctoral dissertation, Massachusetts Institute of Technology, Cambridge, MA.

Wasserman S, Faust K (1994) *Social Network Analysis: Methods and Applications, Structural Analysis in the Social Sciences* (Cambridge University Press, New York).

Xiao L, Boyd S (2004) Fast linear iterations for distributed averaging. *Systems Control Lett.* 53:65–78.

Yildiz E, Acemoglu D, Ozdaglar A, Saberi A, Scaglione A (2013) Binary opinion dynamics with stubborn agents. *ACM Trans. Econ. Comp.* 1(4):Article No. 19.

Guodong Shi is a lecturer and future engineering leadership fellow at the Research School of Engineering, The Australian National University. His research interests include social opinion dynamics, distributed control systems, and quantum networking and decisions.

Alexandre Proutiere is an associate professor at the School of Electrical Engineering at KTH Royal Institute of Technology. His research interests are in probability, statistical inference, and optimization with applications to communication networks and multi-agent systems.

Mikael Johansson is a professor at the School of Electrical Engineering at KTH Royal Institute of Technology. His research focuses on large-scale and distributed optimization with applications in control, communications, bioinformatics and sports.

John S. Baras is the Lockheed Martin endowed Chair Professor with the Institute for Systems Research and the Department of Electrical and Computer Engineering at the University of Maryland College Park. His research interests include dynamical systems, control theory, optimization, decision theory, game theory, communication systems, computing systems.

Karl Henrik Johansson is a professor at the School of Electrical Engineering at KTH Royal Institute of Technology and director of the ACCESS Linnaeus Center. His research interest is in networked control systems with applications in transportation, energy, and automation networks.