

Robust Synchronization of Directed Lur'e Networks with Incremental Nonlinearities

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Abstract—In this paper we deal with robust synchronization problems for directed Lur'e networks subject to incrementally passive nonlinearities and incrementally sector bounded nonlinearities, respectively. By making use of general algebraic connectivities of strongly connected graphs and subgraphs, sufficient synchronization conditions are obtained for diffusively interconnected identical Lur'e systems on both the strongly connected interconnection topology and the topology containing a directed spanning tree. The static feedback gain matrices are determined by the matrices defining the individual agent dynamics and the general algebraic connectivities. The synchronization criteria obtained in the present paper extend those for undirected Lur'e networks in our previous work.

I. INTRODUCTION

As a widespread collective phenomenon in nature, technology and human society, synchronization of complex dynamical networks has attracted a lot of attention from multidisciplinary research communities in the last decade, see e.g. [1], [3], [7], [10], [12] to name just a few. This is due to the fact that information synchronization has potential applications in wide areas such as spatiotemporal planning, cooperative multitasking and formation control [8], [11]. In synchronization problems, certain variables of interest are required to reach an agreement through local interactions. Then a network of interconnected dynamical systems (e.g. smart sensors, unmanned aerial vehicles, satellites) can collaborate with each other to fulfill certain complex tasks. For example, often clock synchronization is a prerequisite in telecommunication.

Synchronization problems for linear multi-agent networks have been extensively studied, see [7], [9] and the references therein. In [3], a passivity-based group coordination framework was proposed, which is especially applicable to undirected nonlinear multi-agent networks. However, without the passivity assumption on each agent in a network, there is no unified approach to handle nonlinear multi-agent networks. In this paper, we consider nonlinear multi-agent networks in which the dynamics of each individual agent is described by a Lur'e system, i.e. a nonlinear system consisting of the negative feedback interconnection of a nominal linear

system with an uncertain static nonlinearity around it, see e.g. [5]. Whereas in stabilization of one single Lur'e system the conditions of passivity and sector boundedness for the uncertain nonlinear function in the negative feedback loop are commonly assumed, in our context of networked Lur'e systems we adopt the stronger assumptions of incremental passivity and incremental sector boundedness. A typical example of many control systems in engineering that satisfy the above conditions is Chua's circuit [6].

In [14], [16], we developed sufficient synchronization conditions for undirected Lur'e networks, in which the Laplacian matrices associated with the network topologies are real symmetric. However, for directed networks, the Laplacian matrices are usually asymmetric and thus are not positive semi-definite anymore. The notation of algebraic connectivity, i.e. the second smallest Laplacian eigenvalue of undirected graphs does not work for directed cases. By employing the general algebraic connectivities of directed graphs, we are enabled to handle directed Lur'e networks and obtain sufficient synchronization conditions for unidirectionally networked Lur'e systems with incrementally passive nonlinearities and incrementally sector bounded nonlinearities, respectively.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries, describes the individual agent dynamics, and formulates the synchronization problems we will study in this paper. Our main results are presented in Sections 3 and 4. We establish sufficient conditions of robustly synchronizing protocols for directed Lur'e networks subject to incrementally passive nonlinearities and incrementally sector bounded nonlinearities, respectively. The paper closes with some concluding remarks and discussions for future research in Section 5.

II. PRELIMINARIES

Let \mathbb{R} denote the field of real numbers. We denote by $\mathbb{R}^+ := [0, \infty)$. $\mathbb{R}^{m \times n}$ denotes the space of m by n real matrices. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscript $(\cdot)^T$ denotes the transpose of a matrix. $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of a given real symmetric matrix. We denote the block diagonal matrix with matrices M_b , $b = 1, 2, \dots, d$, on its diagonal by $\text{diag}(M_1, M_2, \dots, M_d)$. The Kronecker product of matrices M_1 and M_2 is denoted by $M_1 \otimes M_2$. An important property of the Kronecker product is $(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3) \otimes (M_2 M_4)$. We denote by $\mathbf{0}$ and I the zero and the identity matrices, respectively, of compatible dimensions. By $\mathbf{1}_N$ and $\mathbf{0}_N$ we denote the

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column vectors of dimension N with all elements equal to one and zero, respectively. $\|\cdot\|$ denotes the Euclidean norm of a vector.

In this paper, the interconnection topology of a network of unidirectionally interconnected dynamical systems is denoted by a directed graph \mathcal{G} that consists of a finite, nonempty node set $\mathcal{V} = \{1, 2, \dots, N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. We assume that the graph \mathcal{G} is simple, i.e., it does not contain any self-loop (i, i) , $\forall i = 1, 2, \dots, N$, and there is at most one edge between any pair of ordered nodes. If $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$, $j \neq i$, then the graph is undirected. A directed path from node i_0 to node i_l is a sequence of directed edges of the form (i_{p-1}, i_p) , $p = 1, \dots, l$. The graph \mathcal{G} is strongly connected if there is a directed path from any node to all the distinct nodes. A graph contains a directed spanning tree if there exists a node called the root that has directed paths to all the other nodes in the graph. The adjacency matrix \mathcal{A} associated with the graph \mathcal{G} is defined as $[\mathcal{A}]_{ij} = a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $[\mathcal{A}]_{ij} = 0$ otherwise, where a_{ij} is the edge weight of (j, i) . Then the in-degree of node i is given by $d_i = \sum_{j=1}^N a_{ij}$. Denote $\mathcal{D} := \text{diag}(d_1, d_2, \dots, d_N)$ as the in-degree matrix of the graph \mathcal{G} . The Laplacian matrix of the graph \mathcal{G} is defined by $\mathcal{L} := \mathcal{D} - \mathcal{A}$. According to the Gershgorin circle theorem, the real parts of all the eigenvalues of \mathcal{L} are nonnegative. It is well known that $\mathcal{L}\mathbf{1}_N = \mathbf{0}_N$, i.e., $\mathbf{1}_N$ is an eigenvector associated with the Laplacian eigenvalue 0. Furthermore, zero is a simple Laplacian eigenvalue if and only if the graph \mathcal{G} contains a directed spanning tree. \mathcal{L} is irreducible if and only if the graph \mathcal{G} is strongly connected.

In this paper, we consider a directed multi-agent network of $N (\geq 2)$ nonlinear dynamical systems described by the following identical Lur'e systems (see Fig. 1)

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + Ez_i \\ y_i = Cx_i \\ z_i = -\phi(y_i, t) \end{cases}, \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ and $y_i(t) \in \mathbb{R}^s$ are the state to be synchronized, the diffusive coupling input and the output of the i th agent, respectively. The equation $z_i(t) = -\phi(y_i(t), t)$ represents a time-varying, memoryless, nonlinear negative feedback loop. The function $\phi(\cdot, t)$ from $\mathbb{R}^s \times \mathbb{R}^+$ to \mathbb{R}^s is uncertain and can be any function from a set to be specified later. A , B , C and E are given, constant system matrices of compatible dimensions. The interconnection topology among these agents is represented by the directed graph \mathcal{G} .

In this paper, the agents (1) in a network are assumed to be

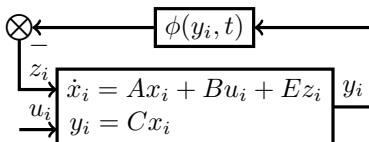


Fig. 1. Lur'e System

interconnected by means of the following distributed static protocol

$$u_i = F \sum_{j=1}^N a_{ij}(x_i - x_j), \quad i = 1, 2, \dots, N, \quad (2)$$

where $F \in \mathbb{R}^{m \times n}$ is a common feedback gain matrix to be determined later, $\mathcal{A} = [a_{ij}]$ is the adjacency matrix of the graph \mathcal{G} . By interconnecting (1) and (2) we get the Lur'e dynamical network

$$\begin{aligned} \dot{x}_i &= Ax_i + BF \sum_{j=1}^N a_{ij}(x_i - x_j) - E\phi(Cx_i, t), \\ & i = 1, 2, \dots, N. \end{aligned} \quad (3)$$

Definition 1: The network (3) of agents (1) with the protocol (2) is robustly synchronized if $x_i(t) - x_j(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$, $\forall i, j = 1, 2, \dots, N$, for all initial conditions and all $\phi(\cdot, t)$ from a given function set.

In the next two sections, we present sufficient conditions that guarantee robust synchronization for the network (3) when $\phi(\cdot, t)$ is incrementally passive and incrementally sector bounded, respectively.

III. INCREMENTAL PASSIVITY

In this section, we assume the function $\phi(\cdot, t)$ to belong to the set of all incrementally passive functions:

$$(y_1 - y_2)^T (\phi(y_1, t) - \phi(y_2, t)) \geq 0 \quad (4)$$

for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$.

Before moving on, we will give the definition of general algebraic connectivity for strongly connected directed graphs. For an undirected graph with Laplacian matrix \mathcal{L} , we know that its algebraic connectivity, i.e. the second smallest Laplacian eigenvalue is given by (see e.g. [4])

$$\lambda_2(\mathcal{L}) = \min_{x^T \mathbf{1}_N = 0, x \neq \mathbf{0}_N} \frac{x^T \mathcal{L} x}{x^T x}.$$

However, for a directed graph, its Laplacian matrix is probably not symmetric and thus the definition above is not suitable. The general algebraic connectivity for strongly connected directed graphs is given below.

Definition 2: For a strongly connected directed graph with Laplacian matrix \mathcal{L} , the general algebraic connectivity is defined to be the real number

$$c = \min_{x^T \xi = 0, x \neq \mathbf{0}_N} \frac{x^T (\Xi \mathcal{L} + \mathcal{L}^T \Xi) x}{2x^T \Xi x},$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_N)^T$, $\Xi = \text{diag}(\xi_1, \xi_2, \dots, \xi_N)$, $\xi^T \mathcal{L} = \mathbf{0}_N^T$ with $\xi_i > 0$, $\forall i = 1, 2, \dots, N$, and $\sum_{i=1}^N \xi_i = 1$.

We have $c > 0$, see Corollary 2 in [13].

Theorem 1: Assume that the graph \mathcal{G} is strongly connected and $\phi(\cdot, t)$ satisfies (4) for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$. If there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a positive real number k such that

$$PA^T + AP - 2kcBB^T < \mathbf{0}, \quad (5)$$

and

$$E = PC^T, \quad (6)$$

where c is the general algebraic connectivity of \mathcal{G} , then the Lur'e network (3), where $F := -kB^T P^{-1}$, is robustly synchronized for all incrementally passive $\phi(\cdot, t)$.

Proof. Let $\bar{x} = \sum_{j=1}^N \xi_j x_j$, where ξ_j is given in Definition 2. Denote $e_i = x_i - \bar{x}$, $i = 1, 2, \dots, N$, and $e = [e_1^T, e_2^T, \dots, e_N^T]^T$. Then we get

$$\begin{aligned} \dot{e}_i &= Ae_i + BF \sum_{j=1}^N a_{ij}(e_i - e_j) - E\phi(Cx_i, t) \\ &\quad - BF \sum_{j=1}^N \xi_j \sum_{k=1}^N a_{jk}(e_j - e_k) + E \sum_{j=1}^N \xi_j \phi(Cx_j, t) \\ &= Ae_i + BF \sum_{j=1}^N a_{ij}(e_i - e_j) - E\phi(Cx_i, t) \\ &\quad + E \sum_{j=1}^N \xi_j \phi(Cx_j, t), \quad i = 1, 2, \dots, N, \end{aligned} \quad (7)$$

where the second equality holds due to the fact that

$$BF \sum_{j=1}^N \xi_j \sum_{k=1}^N a_{jk}(e_j - e_k) = (\xi^T \mathcal{L} \otimes BF) e = \mathbf{0}_n.$$

It is obvious that $x_i - x_j = \mathbf{0}$, $\forall i, j = 1, 2, \dots, N$ if and only if $e = \mathbf{0}_{Nn}$. Therefore, robust synchronization of $x = [x_1^T, x_2^T, \dots, x_N^T]^T$ is equivalent to global asymptotic stability of e . Choose the Lyapunov function candidate $V_1(e) = \sum_{i=1}^N \xi_i e_i^T P^{-1} e_i$, where $P > \mathbf{0}$ satisfies (5) and (6). The time derivative of $V_1(e)$ along the trajectories of (7) is given by

$$\begin{aligned} \dot{V}_1(e) &= 2 \sum_{i=1}^N \xi_i e_i^T P^{-1} \dot{e}_i \\ &= 2 \sum_{i=1}^N \xi_i e_i^T P^{-1} \left[Ae_i + BF \sum_{j=1}^N a_{ij}(e_i - e_j) \right. \\ &\quad \left. - E\phi(Cx_i, t) + E \sum_{j=1}^N \xi_j \phi(Cx_j, t) \right] \\ &= 2e^T (\Xi \otimes P^{-1}A + \Xi \mathcal{L} \otimes P^{-1}BF) e \\ &\quad - 2 \sum_{i=1}^N \xi_i e_i^T P^{-1}E \left(\phi(Cx_i, t) - \sum_{j=1}^N \xi_j \phi(Cx_j, t) \right) \\ &= 2e^T (\Xi \otimes P^{-1}A + \Xi \mathcal{L} \otimes P^{-1}BF) e \\ &\quad - 2 \sum_{i=1}^N \xi_i e_i^T C^T (\phi(Cx_i, t) - \phi(C\bar{x}, t)) \\ &= e^T [\Xi \otimes (P^{-1}A + A^T P^{-1}) \\ &\quad - k(\Xi \mathcal{L} + \mathcal{L}^T \Xi) \otimes P^{-1}BB^T P^{-1}] e \\ &\quad - 2 \sum_{i=1}^N \xi_i (Cx_i - C\bar{x})^T (\phi(Cx_i, t) - \phi(C\bar{x}, t)) \end{aligned}$$

$$\begin{aligned} &\leq e^T [\Xi \otimes (P^{-1}A + A^T P^{-1}) - 2kc\Xi \otimes P^{-1}BB^T P^{-1}] e \\ &= e^T [\Xi \otimes (P^{-1}A + A^T P^{-1} - 2kcP^{-1}BB^T P^{-1})] e, \end{aligned}$$

which is negative definite. The fourth equality holds since we have $\sum_{i=1}^N \xi_i e_i^T = \mathbf{0}_n^T$. Thus the system (7) is globally asymptotically stable, i.e., the Lur'e network (3) is robustly synchronized. This completes the proof. \square

Remark 1: Note that there exists a solution pair (P, k) with $P > \mathbf{0}$ and $k > 0$ to the LMI (5) if and only if (A, B) is stabilizable. The equality condition (6) is commonly used to deal with passive nonlinearities, see e.g. [5]. Its feasibility can be only checked numerically when the matrices C and E are known.

The strong connectedness assumption is not suitable to many practical circumstances. Inspired by [13], we are able to analyze robust synchronization of the Lur'e network (3) assuming that the graph \mathcal{G} contains a directed spanning tree, which is a much more flexible condition. Before moving on, some preliminaries are presented below.

Assume that the graph \mathcal{G} contains a directed spanning tree and has $p(\geq 2)$ strongly connected subgraphs. Then the Laplacian matrix \mathcal{L} of the graph \mathcal{G} can be written in its Frobenius normal form (see e.g. [2])

$$L = \begin{bmatrix} L_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ L_{21} & L_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ L_{p1} & L_{p2} & \cdots & L_{pp} \end{bmatrix}, \quad (8)$$

where $L_{qq} \in \mathbb{R}^{n_q \times n_q}$ is irreducible, $q = 1, 2, \dots, p$. Let $L_{qq} = L_q + D_q$, where L_q is the Laplacian matrix associated with the q th strongly connected subgraph in the graph \mathcal{G} . Obviously, L_{11} is associated with the strongly connected subgraph which is composed of all the roots in this graph and $D_1 = \mathbf{0}$. It is easy to see that the diagonal matrices $D_q \geq \mathbf{0}$ and $D_q \neq \mathbf{0}$, $q = 2, \dots, p$. Intuitively, there is information flow from the first subgraph to the others. In addition, $p = 1$ if and only if \mathcal{G} is a strongly connected graph.

Remark 2: Note that \mathcal{L} and L describe the same graph \mathcal{G} . To obtain L , we can relabel the nodes in the graph. Of course, this does not change the topology structure and thus our following analysis based on L holds for \mathcal{L} .

Lemma 1: [13] Let

$$M = \begin{bmatrix} M_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ M_{21} & M_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ M_{p1} & M_{p2} & \cdots & M_{pp} \end{bmatrix},$$

where $M_{qq} \in \mathbb{R}^{n_q \times n_q}$, n_q is a positive integer, $q = 1, \dots, p$. If there exist positive definite diagonal matrices $Q_q \in \mathbb{R}^{n_q \times n_q}$ such that $Q_q M_{qq} + M_{qq}^T Q_q < \mathbf{0}$, then there exists a positive definite diagonal matrix $\delta = \text{diag}(\delta_1 I_{n_1}, \dots, \delta_p I_{n_p})$ with positive real numbers δ_q such that $\delta Q M + M^T Q \delta < \mathbf{0}$, where $Q = \text{diag}(Q_1, \dots, Q_p)$.

Below we give the definition of general algebraic connectivity of the strongly connected subgraphs.

Definition 3: [13] For a graph containing a directed spanning tree with its Laplacian matrix in the form of (8), the general algebraic connectivity of the q th strongly connected subgraph ($2 \leq q \leq p$) is defined to be the real number

$$c_q = \min_{x \neq \mathbf{0}_{n_q}} \frac{x^T (\Xi_q L_{qq} + L_{qq}^T \Xi_q) x}{2x^T \Xi_q x} \\ = \lambda_{\min} \left(\frac{1}{2} \sqrt{\Xi_q}^{-1} (\Xi_q L_{qq} + L_{qq}^T \Xi_q) \sqrt{\Xi_q}^{-1} \right),$$

where $\Xi_q = \text{diag}(\xi_{q1}, \dots, \xi_{qn_q})$, $\xi_q = (\xi_{q1}, \dots, \xi_{qn_q})^T$, $\xi_q^T L_q = \mathbf{0}_{n_q}^T$ with $\xi_{qr} > 0$, $\forall r = 1, \dots, n_q$, and $\sum_{r=1}^{n_q} \xi_{qr} = 1$, $\sqrt{\Xi_q} = \text{diag}(\sqrt{\xi_{q1}}, \dots, \sqrt{\xi_{qn_q}})$.

Denote the general algebraic connectivity of the first strongly connected subgraph as c_1 , see Definition 2. We have $c_q > 0$, $\forall q = 1, \dots, p$, see Lemma 14 in [13]. Now it is ready to discuss the directed spanning tree case.

Theorem 2: Assume that the graph \mathcal{G} contains a directed spanning tree and $\phi(\cdot, t)$ satisfies (4) for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$. If there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a positive real number k such that

$$PA^T + AP - 2k \min_{1 \leq q \leq p} \{c_q\} BB^T < \mathbf{0}, \quad (9)$$

and

$$E = PC^T, \quad (10)$$

then the Lur'e network (3), where $F := -kBP^{-1}$, is robustly synchronized for all incrementally passive $\phi(\cdot, t)$.

Proof. By Theorem 1 and (9)-(10), the agents on the first strongly connected subgraph are robustly synchronized. Then we can collapse their closed-loop dynamics to that of one single system which, in particular, can be written as

$$\dot{s} = As - E\phi(Cs, t) + f,$$

where $s(t) \in \mathbb{R}^n$, $f(t) \rightarrow \mathbf{0}_n$ as $t \rightarrow \infty$. In this case, we neglect the rows corresponding to the first n_1 nodes and collapse the Laplacian matrix L to get

$$\hat{L} = \begin{bmatrix} \hat{l}_{21} & L_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{l}_{p1} & L_{p2} & \cdots & L_{pp} \end{bmatrix} \in \mathbb{R}^{(N-n_1) \times (N-n_1+1)},$$

where $[\hat{l}_{21}^T, \dots, \hat{l}_{p1}^T]^T = [L_{21}^T, \dots, L_{p1}^T]^T \mathbf{1}_{N-n_1}$. Below we will show that under conditions (9) and (10), the rest agents converge to $s(t)$ as well.

We rewrite the dynamics of the rest agents in the Lur'e network (3) as the following form

$$\dot{\hat{x}} = (I_{N-n_1} \otimes A) \hat{x} + \left(\hat{L} \otimes BF \right) \begin{bmatrix} s \\ \hat{x} \end{bmatrix} - (I_{N-n_1} \otimes E) \hat{\Phi},$$

where $\hat{x} = [x_{n_1+1}^T, \dots, x_N^T]^T$, $\hat{\Phi} = [\phi(Cx_{n_1+1}, t)^T, \dots, \phi(Cx_N, t)^T]^T$. Define the synchronization errors as $\hat{e}_i = x_i - s$, $i = n_1 + 1, \dots, N$. Denote $\hat{e} = [\hat{e}_{n_1+1}^T, \dots, \hat{e}_N^T]^T$. The dynamics of the synchronization errors is given by

$$\dot{\hat{e}} = \left(I_{N-n_1} \otimes A + \tilde{L} \otimes BF \right) \hat{e} - \mathbf{1}_{N-n_1} \otimes f$$

$$- (I_{N-n_1} \otimes E) \left(\hat{\Phi} - \mathbf{1}_{N-n_1} \otimes \phi(Cs, t) \right), \quad (11)$$

where

$$\tilde{L} = \begin{bmatrix} L_{22} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ L_{p2} & \cdots & L_{pp} \end{bmatrix}.$$

Consider the Lyapunov function candidate

$$V_2(\hat{e}) = \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1} \hat{e}_i,$$

where $P > \mathbf{0}$ satisfies (9) and (10), $\xi_{q,i-N_{q-1}}$ is given in Definition 3, $N_{q-1} = \sum_{r=1}^{q-1} n_r$, δ_q is a positive real number to be determined later. The time derivative of $V_2(\hat{e})$ along the trajectories of (11) is given by (12), where $\tilde{e}_q^T = [\hat{e}_{N_{q-1}+1}^T, \dots, \hat{e}_{N_{q-1}+n_q}^T]^T$.

Let

$$M_{qq} = I_{n_q} \otimes P^{-1} A - k L_{qq} \otimes P^{-1} B B^T P^{-1},$$

$$M_{qj} = -k L_{qj} \otimes P^{-1} B B^T P^{-1}, \quad Q_q = \Xi_q \otimes I_n.$$

Since (9) holds, using the same argument in the proof of Theorem 1, we have $Q_q M_{qq} + M_{qq}^T Q_q < \mathbf{0}$, $q = 2, \dots, p$. By Lemma 1, there exist positive real numbers δ_q , $q = 2, \dots, p$, such that the first sum after the inequality in (12) is negative definite. We note that there always exists a positive real number α (sufficiently small) such that

$$\dot{V}_2(\hat{e}) \leq -\alpha \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1} \hat{e}_i \\ - 2 \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1} f.$$

For an arbitrary $\beta > 0$ and sufficiently large t , $f(t)$ is small enough such that if $\|\hat{e}_i(t)\| \geq \beta$, then there exists a positive real number γ such that

$$\dot{V}_2(\hat{e}) \leq -\gamma \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1} \hat{e}_i.$$

This implies that for sufficiently large t , $\|\hat{e}_i(t)\| < \beta$. Therefore, $\lim_{t \rightarrow \infty} \|\hat{e}_i\| = 0$. The proof is completed. \square

IV. INCREMENTAL SECTOR BOUNDEDNESS

In this section, we assume that $\phi(\cdot, t)$ is given by incrementally sector bounded functions within sector $[S_1, S_2]$, where $S_1, S_2 \in \mathbb{R}^{s \times s}$ satisfy $\mathbf{0} \leq S_1 < S_2$:

$$[z_1 - z_2 - S_1(y_1 - y_2)]^T [z_1 - z_2 - S_2(y_1 - y_2)] \leq 0 \quad (13)$$

for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$, where $z_1 = \phi(y_1, t)$ and $z_2 = \phi(y_2, t)$.

Theorem 3: Assume that the graph \mathcal{G} is strongly connected and $\phi(\cdot, t)$ satisfies (13) for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$.

$$\begin{aligned}
\dot{V}_2(\hat{e}) &= 2 \sum_{q=2}^p \delta_q \left[\tilde{e}_q^T (\Xi_q \otimes P^{-1}A + \Xi_q L_{qq} \otimes P^{-1}BF) \tilde{e}_q + \sum_{j=2}^{q-1} \tilde{e}_q^T (\Xi_q L_{qj} \otimes P^{-1}BF) \tilde{e}_j \right] \\
&\quad - 2 \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} [(Cx_i - Cs)^T (\phi(Cx_i, t) - \phi(Cs, t)) + \hat{e}_i^T P^{-1}f] \\
&\leq \sum_{q=2}^p \delta_q \left\{ \tilde{e}_q^T \left[(\Xi_q \times I_n) (I_{n_q} \otimes P^{-1}A - kL_{qq} \otimes P^{-1}BB^T P^{-1}) + (I_{n_q} \otimes P^{-1}A - kL_{qq} \otimes P^{-1}BB^T P^{-1})^T \right. \right. \\
&\quad \left. \left. (\Xi_q \times I_n) \tilde{e}_q - \sum_{j=2}^{q-1} \tilde{e}_j^T \left[(\Xi_q \otimes I_n) (kL_{qj} \otimes P^{-1}BB^T P^{-1}) + (kL_{qj} \otimes P^{-1}BB^T P^{-1})^T (\Xi_q \otimes I_n) \right] \tilde{e}_j \right\} \right. \\
&\quad \left. - 2 \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1}f \right. \tag{12}
\end{aligned}$$

If there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and two positive real numbers k, ρ such that

$$\left[\begin{array}{c|c} P(A - \frac{1}{2}E(S_1 + S_2)C)^T & \\ \hline + (A - \frac{1}{2}E(S_1 + S_2)C)P & PC^T \\ + \frac{1}{2}\rho EE^T - 2kcBB^T & \\ \hline CP & 2\rho(S_2 - S_1)^{-2} \end{array} \right] < \mathbf{0}, \tag{14}$$

then the Lur'e network (3), where $F := -kB^T P^{-1}$, is robustly synchronized for all incrementally sector bounded $\phi(\cdot, t)$.

Proof. Similar to the proof of Theorem 1, we have the same synchronization error dynamics (7) here. Choose the same Lyapunov function candidate $V_3(e) = \sum_{i=1}^N \xi_i \hat{e}_i^T P^{-1}e_i$, where $P > \mathbf{0}$ satisfies (14). The time derivative of $V_3(e)$ along the trajectories of (7) is given by (15), where $\Phi = [\phi(Cx_1, t), \dots, \phi(Cx_N, t)]^T$. In addition, we have (16), where $\bar{y} = C\bar{x}$. On the other hand, by (14) and the Schur complement lemma, we get

$$\left[\begin{array}{cc} P^{-1}A + A^T P^{-1} - 2kcP^{-1}BB^T P^{-1} & -P^{-1}E \\ -E^T P^{-1} & \mathbf{0} \\ \hline C^T(S_1 S_2 + S_2 S_1)C & -C^T(S_1 + S_2) \\ -(S_1 + S_2)C & 2I_s \end{array} \right] < \mathbf{0},$$

where $\tau = 1/\rho > 0$. Thus $\dot{V}_3(e)$ is negative definite while $\phi(\cdot, t)$ is incrementally sector bounded and the proof is completed. \square

The following theorem will address the directed spanning tree case for all incrementally sector bounded $\phi(\cdot, t)$ within sector $[S_1, S_2]$.

Theorem 4: Assume that the graph \mathcal{G} contains a directed spanning tree and $\phi(\cdot, t)$ satisfies (13) for all $y_1, y_2 \in \mathbb{R}^s$ and $t \in \mathbb{R}^+$. If there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$

and two positive real numbers k, ρ such that

$$\left[\begin{array}{c|c} P(A - \frac{1}{2}E(S_1 + S_2)C)^T & \\ \hline + (A - \frac{1}{2}E(S_1 + S_2)C)P & PC^T \\ + \frac{1}{2}\rho EE^T & \\ -2k \min_{1 \leq q \leq p} \{c_q\} BB^T & \\ \hline CP & 2\rho(S_2 - S_1)^{-2} \end{array} \right] < \mathbf{0}, \tag{17}$$

then the Lur'e network (3), where $F := -kB^T P^{-1}$, is robustly synchronized for all incrementally sector bounded $\phi(\cdot, t)$.

Proof. By Theorem 3 and (17), the agents on the first strongly connected subgraph are robustly synchronized. The rest analysis is similar to the proof of Theorem 2. Consider the same Lyapunov function candidate

$$V_4(\hat{e}) = \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1}\hat{e}_i,$$

where $P > \mathbf{0}$ satisfies (17). The time derivative of $V_4(\hat{e})$ along the trajectories of (11) is given by (18),

where $\eta_q = \left[\hat{e}_q^T, \left(\hat{\Phi}_q - \mathbf{1}_{n_q} \otimes \phi(Cs, t) \right)^T \right]^T$, $\hat{\Phi}_q = [\phi(Cx_{N_{q-1}+1}, t)^T, \dots, \phi(Cx_{N_{q-1}+n_q}, t)^T]^T$. Since (17) holds, similar to the analysis in the proof of Theorem 3, we have $Q_q M_{qq} + M_{qq}^T Q_q < \mathbf{0}$, $q = 2, \dots, p$. Then, by Lemma 1, there exist positive real numbers δ_q , $q = 2, \dots, p$, such that the sum of the first two terms after the second equality in (18) is negative definite. Following the same idea in the proof of Theorem 2, we can complete the proof. \square

V. CONCLUSIONS

In this paper we have discussed the roles of general algebraic connectivities of strongly connected graphs and subgraphs in robust synchronization problems for directed Lur'e networks. The results we obtain here have extended our previous work for undirected Lur'e networks [14], [16]. A possible topic for future research is to consider fully

$$\dot{V}_3(e) \leq \begin{bmatrix} e \\ \Phi - \mathbf{1}_N \otimes \phi(C\bar{x}, t) \end{bmatrix}^T \begin{bmatrix} \Xi \otimes (P^{-1}A + A^T P^{-1} - 2kcP^{-1}BB^T P^{-1}) & -\Xi \otimes P^{-1}E \\ -\Xi \otimes E^T P^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} e \\ \Phi - \mathbf{1}_N \otimes \phi(C\bar{x}, t) \end{bmatrix} \quad (15)$$

$$\begin{aligned} & \begin{bmatrix} e \\ \Phi - \mathbf{1}_N \otimes \phi(C\bar{x}, t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2}\Xi \otimes C^T(S_1 S_2 + S_2 S_1)C & -\frac{1}{2}\Xi \otimes C^T(S_1 + S_2) \\ -\frac{1}{2}\Xi \otimes (S_1 + S_2)C & \Xi \otimes I_s \end{bmatrix} \begin{bmatrix} e \\ \Phi - \mathbf{1}_N \otimes \phi(C\bar{x}, t) \end{bmatrix} \\ &= \sum_{i=1}^N \xi_i (\phi(y_i, t) - \phi(\bar{y}, t) - S_1(y_i - \bar{y}))^T (\phi(y_i, t) - \phi(\bar{y}, t) - S_2(y_i - \bar{y})) \leq 0 \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{V}_4(\hat{e}) &= 2 \sum_{q=2}^p \delta_q \left(\eta_q^T \begin{bmatrix} \Xi_q \otimes P^{-1}A - k\Xi_q L_{qq} \otimes P^{-1}BB^T P^{-1} & \Xi_q \otimes P^{-1}E \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \eta_q \right. \\ &\quad \left. - \sum_{j=2}^{q-1} \eta_j^T \begin{bmatrix} k\Xi_q L_{qj} \otimes P^{-1}BB^T P^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \eta_j \right) - 2 \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1} f \\ &= \sum_{q=2}^p \delta_q \eta_q^T \left(\underbrace{\begin{bmatrix} \Xi_q \otimes I_n & \mathbf{0} \\ \mathbf{0} & I_{n_q n} \end{bmatrix}}_{Q_q} \underbrace{\begin{bmatrix} I_{n_q} \otimes P^{-1}A - kL_{qq} \otimes P^{-1}BB^T P^{-1} & I_{n_q} \otimes P^{-1}E \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{M_{qq}} \right) \\ &\quad + \begin{bmatrix} I_{n_q} \otimes P^{-1}A - kL_{qq} \otimes P^{-1}BB^T P^{-1} & I_{n_q} \otimes P^{-1}E \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \Xi_q \otimes I_n & \mathbf{0} \\ \mathbf{0} & I_{n_q n} \end{bmatrix} \eta_q \\ &\quad - \sum_{q=2}^p \delta_q \sum_{j=2}^{q-1} \eta_j^T \left(\begin{bmatrix} \Xi_q \otimes I_n & \mathbf{0} \\ \mathbf{0} & I_{n_q n} \end{bmatrix} \underbrace{\begin{bmatrix} kL_{qj} \otimes P^{-1}BB^T P^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{-M_{qj}} \right) \\ &\quad + \begin{bmatrix} kL_{qj} \otimes P^{-1}BB^T P^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^T \begin{bmatrix} \Xi_q \otimes I_n & \mathbf{0} \\ \mathbf{0} & I_{n_q n} \end{bmatrix} \eta_j - 2 \sum_{q=2}^p \delta_q \sum_{i=N_{q-1}+1}^{N_{q-1}+n_q} \xi_{q,i-N_{q-1}} \hat{e}_i^T P^{-1} f \end{aligned} \quad (18)$$

distributed synchronization of directed Lur'e networks as we achieved for undirected Lur'e networks in [16]. Another one is to study dynamic feedback synchronization of directed Lur'e networks as in [15], [17].

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