# Structural Balance and Opinion Separation in Trust–Mistrust Social Networks

Weiguo Xia, Ming Cao, and Karl Henrik Johansson, *Fellow, IEEE*

*Abstract*—Structural balance theory has been developed in sociology and psychology to explain how interacting agents, for example, countries, political parties, opinionated individuals, with mixed trust and mistrust relationships evolve into polarized camps. Recent results have shown that structural balance is necessary for polarization in networks with fixed, strongly connected neighbor relationships when the opinion dynamics are described by DeGroot-type averaging rules. We develop this line of research in this paper in two steps. First, we consider fixed, *not* necessarily strongly connected, neighbor relationships. It is shown that if the network includes a strongly connected subnetwork containing mistrust, which influences the rest of the network, then no opinion clustering is possible when that subnetwork is not structurally balanced; all of the opinions become neutralized in the end. In contrast, it is shown that when that subnetwork is indeed structurally balanced, the agents of the subnetwork evolve into two polarized camps and the opinions of all other agents in the network spread between these two polarized opinions. Second, we consider *time-varying* neighbor relationships. We show that the opinion separation criteria carry over if the conditions for fixed graphs are extended to joint graphs. The results are developed for both discrete-time and continuous-time models.

*Index Terms*—Opinion separation, signed graphs, structural balance theory.

## I. INTRODUCTION

IN THEORETICAL sociology and social psychology, a strong interest has been maintained over the years in the study of the evolution of opinions of social groups [1], [2]. There is a long tradition to study how continuous interactions within an interconnected collective without isolated subgroups, might lead to the emergence of segregation, or even polarization, of communities that form homogenous opinions only internally [3], [4]. One popular theory is that the balance between trust and mistrust that dictate people's opinions to become closer or further apart, respectively, plays a major role in the dynamical process of opinion separation [5]. This theory, when explicitly expressed using signed graphs describing the trust and mistrust relationships among the interacting social agents, is called *structural balance theory* [6]–[10]. Specifically, for the graph describing the neighbor relationships between agents in a social network, positive signs are assigned to those edges corresponding to trust and negative signs to those edges corresponding to mistrust. Then, the network is *structurally balanced* if all of the vertices of its signed graph can be divided into two disjoint sets such that every edge between vertices in the same set is with a positive sign and every edge between vertices in the distinct sets has a negative sign [9].

While structural balance theory tells clearly how the trust–mistrust relationships should be distributed among the agents for the presence of stable polarized opinions, it does not specify how the agents' opinions update. Recently, there is a growing effort to introduce DeGroot type of opinion updating rules to social networks with trust and mistrust relationships [11]–[14]. The DeGroot model [15] describes how each agent repeatedly updates its opinion to the average of those of its neighbors. Since this model reflects the fundamental human cognitive capability of taking convex combinations when integrating related information [16], it has been studied extensively in the past decades [17]. But to show the process of opinion separation using the DeGroot model, more work [18]–[20] is to rely on mechanisms that lead to disconnected networks, the so-called bounded confidence Krause model, rather than to resort to trust–mistrust relationships in connected networks. Some other work has introduced an adaptive noisy updating model that characterizes individuals' diversified tendencies to explain the occurrence of clustering in human populations [21]; in this model, noise is critical in sustaining clusters of opinions in a connected network.

For DeGroot-type opinion dynamics in trust–mistrust networks, it has been proved in [11] and [12] using continuous-time models that in a strongly connected and structurally balanced network consisting of two camps, where agents only trust those within the same camp, the opinions of all the agents within the same camp become the same, which is exactly the opposite of the opinion of the other camp. It has also been shown that in a structurally unbalanced network, the opinions of all agents asymptotically converge to zero. It remains an open question about how the opinions of the agents evolve when the network is not strongly connected but structurally unbalanced. What is even more intriguing is to investigate the dynamical behavior under time-varying network topologies, since in practical situations, the relationships between agents may change over time.

In this paper, we investigate the opinion evolution of interacting agents with trust–mistrust relationships under either

W. Xia is with the School of Control Science and Engineering, Dalian University of Technology, Dalian 116024, China (e-mail: wgxiaseu@dlut.edu.cn).

M. Cao is with the Faculty of Mathematics and Natural Sciences, ENTEG, University of Groningen, Groningen 9747 AG, the Netherlands (e-mail: m.cao@rug.nl).

K. H. Johansson is with ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden (e-mail: kallej@kth.se).

fixed network topologies containing directed spanning trees or dynamically changing topologies with joint connectivity. For the fixed topology case, we show that when the network graph contains a strongly connected subgraph with negative edges, which has a directed path to every other vertex in the network, the opinions of all the agents become neutralized at zero if the strongly connected subgraph is not structurally balanced. In comparison, if the strongly connected subgraph is structurally balanced, it is shown that the opinions of the agents in this subgraph polarize at the exact opposite values, and the opinions of the rest of the agents lie in between the polarized values. For dynamically changing network topologies, similar conclusions hold when the graphical conditions are applied to the corresponding joint graphs. Our results show that in addition to getting polarized and reaching consensus, the DeGroot-type opinion dynamics can give rise to opinion clustering in a network under weaker connectivity conditions. This complements the existing mechanisms that induce clusters in social networks through introducing bounded confidence [18], updating noise [21], or delays [22].

The rest of this paper is organized as follows. In Section II, several examples are presented to motivate our study. In Section III, we introduce the opinion dynamics models and formulate the problem considered in this paper. In Sections IV and V, the behaviors of the systems with discrete-time and continuous-time dynamics are studied, respectively. We present simulation examples to verify the effectiveness of the theoretical results in Section VI. Section VII discusses the conclusions and ideas for future work.

## II. MOTIVATING EXAMPLE

In this section, to motivate introducing weaker connectivity conditions for the DeGroot-type models in trust-mistrust social networks, we present several examples showing that more complex behaviors may take place compared to what have been reported in the literature. Consider the directed graphs given in Fig. 1. Each directed edge is associated with a positive or negative sign and the weight of each edge is either 1 or $-1$. Consider the network dynamics evolved in these graphs described below. Each agent in the network is associated with a scalar $x_i \in \mathbb{R}$ that represents its opinion on a certain subject. If $(v_j, v_i)$ is an edge in the graph, then agent $i$ takes agent $j$ as a neighbor and, thus, agent $j$'s opinion is influencing agent $i$'s. Time is slotted. At each time instant, each agent updates its state to the weighted average of its neighbors' and its own, and a positive weight 1 is assigned to its own opinion. Take agent 2 in $\mathbb{G}_1$ in Fig. 1(a) as an example. Taking the weights of the edges into account, at time $t$, the state of agent 2 is updated to

$$x_2(t+1) = \frac{x_1(t) + x_2(t) + x_3(t) - x_9(t)}{4} \quad t \geq 0$$

since agents 1, 3, and 9 are the neighbors of agent 2 and $(v_9, v_2)$ with a negative weight gives the term $-x_9(t)$ in the equation. The other agents update their states in the same manner.

We are interested in the asymptotic behavior of the states of the agents and Fig. 2 shows the evolution of the agents' states under different network topologies: in (a), the graph is $\mathbb{G}_1$; in
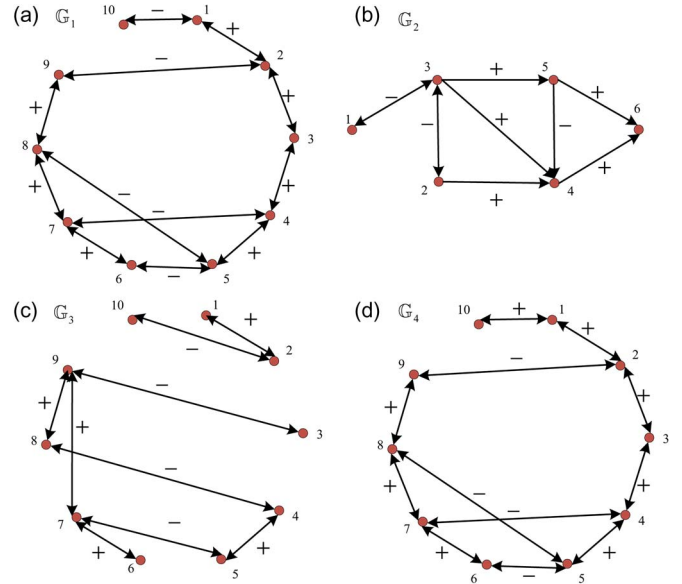


Fig. 1. Directed signed graphs $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, and $\mathbb{G}_4$.
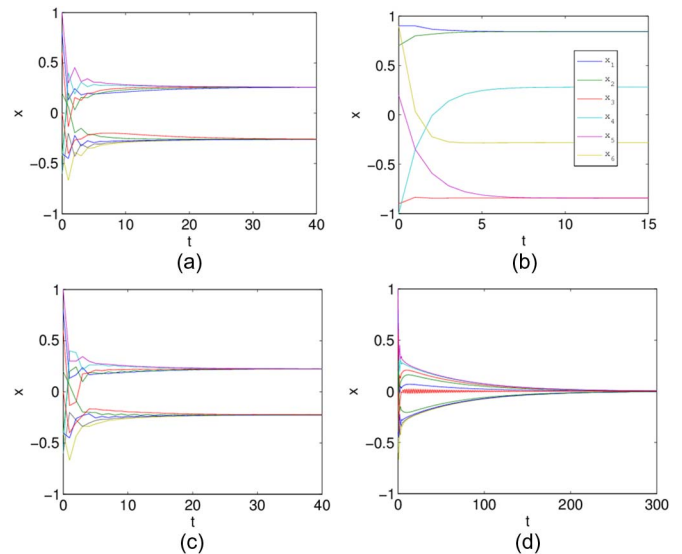


Fig. 2. Evolution of the agents' states when the graph topology (a) is $\mathbb{G}_1$; (b) is $\mathbb{G}_2$; (c) switches between $\mathbb{G}_1$ and $\mathbb{G}_3$; and (d) switches between $\mathbb{G}_1$ and $\mathbb{G}_4$.

(b), the graph is $\mathbb{G}_2$; in (c), the graph is $\mathbb{G}_1$ at even times and is $\mathbb{G}_3$ otherwise; in (d), the graph is $\mathbb{G}_1$ at even times and is $\mathbb{G}_4$ otherwise. The initial condition of each agent lies in the interval $[-1, 1]$.

It is clear that $\mathbb{G}_1$ is structurally balanced [6] (the formal definition will be given in the next section) in the sense that it can be partitioned into $\{1, \ldots, 5\}$ and $\{6, \ldots, 10\}$ with positive edges within each set and negative edges in between. Since $\mathbb{G}_1$ is strongly connected, we know from [12, Theor. 1] that the agents' states will "polarize" in the sense that agents in $\mathcal{V}_1$ reach the same value that is opposite of the agreed value of those in $\mathcal{V}_2$. This is also called "bipartite consensus" in [12]. Although [12] only studied the switching case of strongly connected graphs at each time, one may infer that the agents still polarize since $\mathbb{G}_1$ and $\mathbb{G}_3$ are both structurally balanced and share the same bipartition.

However, when the topology switches between $\mathbb{G}_1$ and $\mathbb{G}_4$, it is unclear why the agents reach an agreement since each graph is structurally balanced though they do not share a common bipartition. What is intriguing is the phenomenon observed for Fig. 2(b) where the network topology $\mathbb{G}_2$ contains a directed spanning tree but is not strongly connected. Instead of getting polarized or reaching consensus, the agents' opinions become clustered and this clustering is a new behavior that does not take place when the network is strongly connected. More detailed theoretical analysis about such behavior is provided in Theorems 2 and 5. The new opinion clustering phenomenon has motivated us to study the system dynamics when the network topologies are not strongly connected and/or become time varying.

## III. PROBLEM FORMULATION

Consider a network of $N$ agents labeled by $1, \ldots, N$, where each agent $i$, $1 \leq i \leq N$, is associated with a scalar $x_i \in \mathbb{R}$ that represents its opinion on a certain subject. Here, $x_i$ being positive implies supportive opinions, being negative implies protesting views, and being zero implies neutral reaction. We use a directed signed graph $\mathbb{G}$ [7] with the vertex set $\mathcal{V} = \{v_1, \ldots, v_N\}$ to describe the trust–mistrust relationships between the agents. The definition of signed graphs is as follows.

*Definition 1:* A directed *signed* graph is a directed graph where each edge is associated with either a positive or negative sign.

Some notions in graph theory need to be introduced [23]. We consider only directed graphs without self-loops throughout this paper. In a directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, \ldots, v_N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, a directed *walk* is a sequence of vertices $v_{i_1}, \ldots, v_{i_k}$ such that $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{E}$ for $s = 1, \ldots, k-1$. A directed *path* is a walk with distinct vertices in the sequence. A directed *cycle* is a walk with distinct vertices $v_{i_1}, \ldots, v_{i_k}$, $k \geq 2$, and $v_{i_1} = v_{i_k}$. $\mathbb{G}$ is said to be *strongly connected* if there is a directed path from every vertex to every other vertex in $\mathbb{G}$. A directed *tree* is a graph containing a unique vertex, called *root*, which has a directed path to every other vertex. A directed *spanning tree* $\mathbb{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ of the directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ is a subgraph of $\mathbb{G}$ such that $\mathbb{G}^s$ is a directed tree and $\mathcal{V}^s = \mathcal{V}$. $\mathbb{G}$ is said to contain a directed spanning tree if a directed spanning tree is a subgraph of $\mathbb{G}$. "Directed" is omitted for the rest of the paper and we simply say $\mathbb{G}$ contains a spanning tree since we focus exclusively on directed graphs. The *root vertex set* of $\mathbb{G}$ is a set of all the roots of $\mathbb{G}$.

In the $N$-agent network, there is a directed edge $(v_j, v_i)$ from $v_j$ to $v_i$ if and only if agent $i$ takes agent $j$ as a neighbor and, thus, agent $j$'s opinion is influencing agent $i$'s. Furthermore, the directed edge $(v_j, v_i)$ is assigned with a nonzero weight $a_{ij}$, which is positive if agent $i$ trusts agent $j$ and negative otherwise; here, we assume the interagent relationship, if there is any, is either trusting or mistrusting, although the strength of the relationship may vary as reflected by the magnitude $|a_{ij}|$. We use $\mathcal{N}_i$ to denote the set of indices of agent $i$'s neighbors.

For the DeGroot-type updating rule, discrete-time and continuous-time models have been constructed in the literature. The discrete-time opinion dynamics can be described by

$$x_i(t+1) = \sum_{j \in \{\mathcal{N}_i(t), i\}} p_{ij}(t) x_j(t), \ i = 1, \ldots, N, \ t = 0, 1, \ldots \tag{1}$$

where $a_{ii}(t) > 0$ is a self-trusting weight and

$$p_{ij}(t) = \frac{a_{ij}(t)}{a_{ii}(t) + \sum_{k \in \mathcal{N}_i(t)} |a_{ik}(t)|} \tag{2}$$

which obviously satisfies

$$p_{ii}(t) + \sum_{j \in \mathcal{N}_i(t)} |p_{ij}(t)| = 1. \tag{3}$$

If we take $x(t) = [x_1(t), \ldots, x_N(t)]^T$ to be the network state, then (1) can be written in its state-space form

$$x(t+1) = P(t)x(t), \qquad t = 0, 1, \ldots \tag{4}$$

where $P(t) = (p_{ij}(t))_{N \times N}$ is an $N \times N$ matrix with positive diagonals.

Similarly, the continuous-time update equation for agent $i$ is

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i(t)} |a_{ij}(t)| \left( x_i - \text{sgn} \left( a_{ij}(t) \right) x_j \right), \ i = 1, \ldots, N \tag{5}$$

where $\text{sgn}(\cdot)$ denotes the sign function. System (5) can be written in the compact form

$$\dot{x} = -L(t)x \tag{6}$$

where $L(t)$ is the signed Laplacian matrix that is defined by

$$l_{ii}(t) = \sum_{j \in \mathcal{N}_i(t)} |a_{ij}(t)|$$

$$l_{ij}(t) = \begin{cases} -a_{ij}(t), & \text{for } j \in \mathcal{N}_i(t) \\ 0, & \text{for } j \neq i \text{ and } j \notin \mathcal{N}_i(t). \end{cases} \tag{7}$$

Since the graphs describing the interactions between agents may change with time $t$, we use $\mathbb{G}(P(t))$ and $\mathbb{G}(L(t))$ to denote the graph at time $t$ for the discrete-time system (4) and for the continuous-time system (6), respectively. Let $P_{11}(t)$ be the principal submatrix of $P(t)$ obtained from $P(t)$ by deleting the $i_1$th, $i_2$th,..., $i_m$th rows and columns, where $1 \leq i_1, \ldots, i_m \leq N$. Then, $\mathbb{G}(P_{11}(t)) = (\mathcal{V}^s, \mathcal{E}^s)$ denotes the subgraph of $\mathbb{G}(P(t)) = (\mathcal{V}, \mathcal{E})$ such that $\mathcal{V}^s = \mathcal{V} \backslash \{v_{i_1}, \ldots, v_{i_m}\}$ and $\mathcal{E}^s = \{(v_i, v_j) : (v_i, v_j) \in \mathcal{E} \text{ and } v_i, v_j \in \mathcal{V}^s\}$. When the graph is fixed, we omit $t$ and write $\mathbb{G}(P)$ and $\mathbb{G}(L)$.

Equations (4) and (6) both come from the DeGroot model and share the same intuition: the opinions of those agents that agent $i$ trusts influence its opinion positively and, thus, the averaging rule tries to bring them closer; at the same time, the opinions of those agents that agent $i$ does not trust influence its opinion negatively and, thus, the averaging rule pushes them apart. It is natural then that the distribution of positive and negative edges of a graph affects the evolution of opinions and, for this reason, the notion of "structural balance" becomes instrumental.

*Definition 2:* A directed signed graph $\mathbb{G}$ with vertex set $\mathcal{V}$ is *structurally balanced* if $\mathcal{V}$ can be partitioned into two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that all the edges $(v_i, v_j)$ with $v_i, v_j$ taken in the same set $\mathcal{V}_k$, $k = 1, 2$, are of positive signs and all of the edges $(v_i, v_j)$ with $v_i, v_j$ taken in different sets $\mathcal{V}_k$ are of negative signs.

Note that in the definition of structural balance, a network is still said to be structurally balanced if *all* the edges of the network graph are assigned with the positive sign and, thus, one of $\mathcal{V}_1$ and $\mathcal{V}_2$ in Definition 2 becomes empty.

A social network that is structurally balanced where the agents' opinions update according to the DeGroot-type averaging rules (1) or (5) may evolve into two polarized camps. We now make our notion of polarization precise.

*Definition 3:* System (4) or (6) *polarizes* if for almost all initial conditions, $\lim_{t\to\infty} |x_i(t)| = \lim_{t\to\infty} |x_j(t)| > 0$ for all $i$, $j = 1, \ldots, N$, and $\lim_{t\to\infty} x_i(t) = -\lim_{t\to\infty} x_j(t)$ for some $i \neq j$.

It has been shown in [12] that system (6) with a fixed, strongly connected network graph $\mathbb{G}(L)$ polarizes if $\mathbb{G}(L)$ is structurally balanced. It is the goal of this paper to study for both systems (4) and (6), what the relationship is between structural balance and opinion separation, for which opinion polarization is an extreme case, when the network topology is either fixed and contains a spanning tree or is time varying. In what follows, we separately study the discrete-time model (4) and the continuous-time model (6).

## IV. DISCRETE-TIME MODEL

We introduce a $2N$-dimensional system[1] [24] based on the $N$-dimensional system (4). For a matrix $P(t)$, define two non-negative matrices $P^+(t)$ and $P^-(t)$ according to

$$\left(P^+(t)\right)_{ij} = \begin{cases} p_{ij}(t), & p_{ij}(t) > 0 \\ 0, & p_{ij}(t) \le 0 \end{cases}$$
$$\left(P^-(t)\right)_{ij} = \begin{cases} -p_{ij}(t), & p_{ij}(t) < 0 \\ 0, & p_{ij}(t) \ge 0 \end{cases} \quad (8)$$

where $(P^+(t))_{ij}$ and $(P^-(t))_{ij}$ are the $ij$-th elements of $P^+(t)$ and $P^-(t)$, respectively. It is obvious that $P(t) = P^+(t) - P^-(t)$. Define $x_i^+(t) = x_i(t)$, $x_i^-(t) = -x_i(t)$. One knows that $x_i^+(t) + x_i^-(t) = 0$ for all $t \ge 0$. From system (4), we obtain the following update equations for $x_i^+(t)$ and $x_i^-(t)$:

$$x_i^+(t+1) = \sum_{j,\, p_{ij}(t) > 0} p_{ij}(t) x_j^+(t) + \sum_{j,\, p_{ij}(t) < 0} |p_{ij}(t)|\, x_j^j(t),$$
$$x_i^-(t+1) = \sum_{j,\, p_{ij}(t) > 0} p_{ij}(t) x_j^j(t) + \sum_{j,\, p_{ij}(t) < 0} |p_{ij}(t)|\, x_j^+(t). \quad (9)$$

Let $y(t) = [x_1^+(t), \ldots, x_N^+(t), x_1^-(t), \ldots, x_N^-(t)]^T$. Then system (9) can be written as

$$y(t+1) = \begin{bmatrix} P^+(t) & P^-(t) \\ P^-(t) & P^+(t) \end{bmatrix} y(t) = Q(t) y(t) \quad (10)$$

where $Q(t) = \begin{bmatrix} P^+(t) & P^-(t) \\ P^-(t) & P^+(t) \end{bmatrix}$ is a stochastic matrix.

---

[1] We are indebted to Julien Hendrickx for pointing out this reformulation of the update equations.
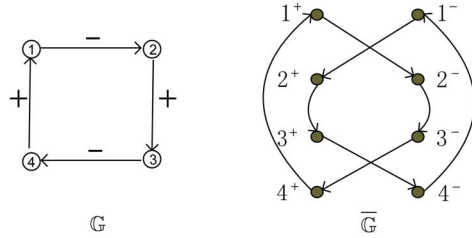


Fig. 3. A structurally balanced graph $\mathbb{G}$ and the corresponding graph $\bar{\mathbb{G}}$.
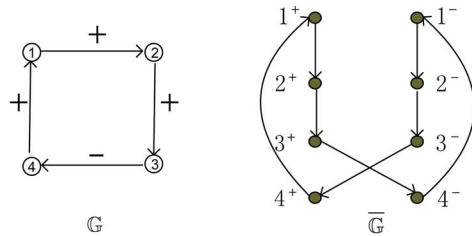


Fig. 4. A structurally unbalanced graph $\mathbb{G}$ and the corresponding graph $\bar{\mathbb{G}}$.

To study the properties of system (4), we will explore the properties of system (10) which is a classical consensus system and existing convergence results can be utilized [25]–[27]. We make the connection between the graph $\mathbb{G}(P(t))$ associated with $P(t)$ with the vertex set $\{v_1, \ldots, v_N\}$ and the graph $\mathbb{G}(Q(t))$ associated with $Q(t)$ with the vertex set $\{v_1', v_2', \ldots, v_{2N}'\}$, so that we can transform the graphical conditions on $\mathbb{G}(P(t))$ to conditions on $\mathbb{G}(Q(t))$.

Given a directed signed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, we define an enlarged directed graph $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ based on $\mathbb{G}$ as follows. $\bar{\mathbb{G}}$ has $2N$ vertices and all of its edges are positive. Denote the vertices in $\bar{\mathbb{G}}$ as $v_1^+, v_1^-, v_2^+, v_2^-, \ldots, v_N^+, v_N^-$. If there is a positive edge $(v_i, v_j) \in \mathcal{E}$, then there are two directed edges $(v_i^+, v_j^+), (v_i^-, v_j^-) \in \bar{\mathcal{E}}$; if there is a negative edge $(v_i, v_j) \in \mathcal{E}$, then there are two directed edges $(v_i^+, v_j^-), (v_i^-, v_j^+) \in \bar{\mathcal{E}}$. In this manner, it is easy to see that if there is a positive path[2] from $v_i$ to $v_j$ in $\mathbb{G}$, then there is a path from $v_i^+$ to $v_j^+$ and a path from $v_i^-$ to $v_j^-$ in $\bar{\mathbb{G}}$; if there is a negative path from $v_i$ to $v_j$ in $\mathbb{G}$, then there is a path from $v_i^+$ to $v_j^-$ and a path from $v_i^-$ to $v_j^+$ in $\bar{\mathbb{G}}$. Two examples are illustrated in Figs. 3 and 4.

*Lemma 1:* Let $\mathbb{G}$ be a strongly connected signed graph and let $\bar{\mathbb{G}}$ be the enlarged graph based on $\mathbb{G}$. Then $\mathbb{G}$ is structurally balanced if and only if $\bar{\mathbb{G}}$ is disconnected and composed of two strongly connected components.

*Proof (Necessity):* If $\mathbb{G}$ is structurally balanced, there is a bipartition $\mathcal{V}_1 = \{v_{i_1}, \ldots, v_{i_m}\}$, $\mathcal{V}_2 = \{v_{i_{m+1}}, \ldots, v_{i_N}\}$, $1 \le i_1, \ldots, i_N \le N$ of $\mathcal{V}$ such that the edges between $\mathcal{V}_1$ and $\mathcal{V}_2$ are all negative and edges within each set $\mathcal{V}_i$, $i = 1, 2$, are all positive. We claim that in $\bar{\mathbb{G}}$ there is no edge between $\bar{\mathcal{V}}_1 = \{v_{i_1}^+, \ldots, v_{i_m}^+, v_{i_{m+1}}^-, \ldots, v_{i_N}^-\}$ and $\bar{\mathcal{V}}_2 = \{v_{i_1}^-, \ldots, v_{i_m}^-, v_{i_{m+1}}^+, \ldots, v_{i_N}^+\}$, and, thus, $\bar{\mathbb{G}}$ is disconnected.

---

[2] In a directed signed graph, a path is said to be positive if it contains an even number of edges with negative weights and to be negative otherwise. A positive or negative cycle is defined similarly.

If the contrary is true and there is an edge between $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$, then there is a positive edge between $\mathcal{V}_1$ and $\mathcal{V}_2$ or a negative edge within $\mathcal{V}_1$ or $\mathcal{V}_2$, which contradicts the fact that $\mathbb{G}$ is structurally balanced. Since $\mathbb{G}$ is strongly connected, each component with the vertex set $\bar{\mathcal{V}}_i$, $i = 1, 2$, is strongly connected.

(Sufficiency) Assume that the vertex sets of the two components are $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$ and $v_{i_1}^+, \ldots, v_{i_m}^+$ are in $\bar{\mathcal{V}}_1$ and $v_{i_{m+1}}^+, \ldots, v_{i_N}^+$ are in $\bar{\mathcal{V}}_2$. We claim that $v_{i_j}^-$, $j = 1, \ldots, m$ are in $\bar{\mathcal{V}}_2$ and $v_{i_j}^-$, $j = m+1, \ldots, N$ are in $\bar{\mathcal{V}}_1$. If this is not true, then without loss of generality, assume $v_{i_1}^+$ and $v_{i_1}^-$ are both in $\bar{\mathcal{V}}_1$. Since each component is strongly connected, there is a path from $v_{i_1}^+$ to $v_{i_j}^+$ and a path from $v_{i_j}^+$ to $v_{i_1}^-$ for $j = 2, \ldots, m$. Then from the definition of $\bar{\mathbb{G}}$, there is a path from $v_{i_1}^-$ to $v_{i_j}^-$ and a path from $v_{i_j}^-$ to $v_{i_1}^+$. It follows that $v_{i_j}^-$, $j = 2, \ldots, m$ are in $\bar{\mathcal{V}}_1$ and, thus, $\bar{\mathcal{V}}_1 = \{v_{i_j}^+, v_{i_j}^-, \ j = 1, \ldots, m\}$. Similarly $\bar{\mathcal{V}}_2 = \{v_{i_j}^+, v_{i_j}^-, j = m+1, \ldots, N\}$. Since there is no edge between $\bar{\mathcal{V}}_1$ and $\bar{\mathcal{V}}_2$ in $\bar{\mathbb{G}}$, it follows that $\mathbb{G}$ is disconnected, which contradicts the assumption. One can conclude that $\bar{\mathcal{V}}_1 = \{v_{i_1}^+, \ldots, v_{i_m}^+, v_{i_{m+1}}^-, \ldots, v_{i_N}^-\}$ and $\bar{\mathcal{V}}_2 = \{v_{i_1}^-, \ldots, v_{i_m}^-, v_{i_{m+1}}^+, \ldots, v_{i_N}^+\}$. Then it is easy to see that $\mathbb{G}$ is structurally balanced if we define $\mathcal{V}_1 = \{v_{i_1}, \ldots, v_{i_m}\}$, $\mathcal{V}_2 = \{v_{i_{m+1}}, \ldots, v_{i_N}\}$. $\qquad \square$

*Lemma 2:* Let $\mathbb{G}$ be a strongly connected signed graph and let $\bar{\mathbb{G}}$ the enlarged graph based on $\mathbb{G}$. Then $\mathbb{G}$ is structurally unbalanced if and only if $\bar{\mathbb{G}}$ is strongly connected.

*Proof:* Sufficiency is obvious in view of Lemma 1 and we only prove the necessity.

Since $\mathbb{G}$ is structurally unbalanced, without loss of generality, assume that there is a negative cycle from $v_k$ to $v_k$ in $\mathbb{G}$ [7]. For any $i \neq j$, $1 \leq i, j \leq N$, since $\mathbb{G}$ is strongly connected, there is a path from $v_i$ to $v_j$. Without loss of generality, assume this path is positive. Then there is a directed path from $v_i^+$ to $v_j^+$ and a directed path from $v_i^-$ to $v_j^-$ in $\bar{\mathbb{G}}$. Since $\mathbb{G}$ is strongly connected and there is a negative cycle starting from $v_k$, we are able to find a directed negative walk from $v_j$ to $v_i$. Accordingly, there is a walk from $v_j^+$ to $v_i^-$ in $\bar{\mathbb{G}}$. Thus, in $\bar{\mathbb{G}}$ there is a directed walk from $v_i^+$ to $v_j^+$, to $v_i^-$ and to $v_i^-$. Thus, there is a directed path from $v_i^+$ to $v_j^+$, to $v_j^-$ and to $v_i^-$ and it follows that $\bar{\mathbb{G}}$ is strongly connected. $\qquad \square$

Two examples are given in Figs. 3 and 4 to illustrate these two lemmas.

Let $\overline{\mathbb{G}(P(t))}$ be the enlarged graph based on $\mathbb{G}(P(t))$. If we denote the graph associated with $Q(t)$ by $\mathbb{G}(Q(t))$, then it is easy to see from the structure of $Q(t)$ that $\overline{\mathbb{G}(P(t))}$ and $\mathbb{G}(Q(t))$ are isomorphic [23], that is, $\overline{\mathbb{G}(P(t))} \simeq \mathbb{G}(Q(t))$. The bijection $\phi$ that maps the vertex set $\{v_1^+, v_1^-, \ldots, v_N^+, v_N^-\}$ of $\overline{\mathbb{G}(P(t))}$ to the vertex set $\{v_1', v_2', \ldots, v_{2N}'\}$ of $\mathbb{G}(Q(t))$ is given by: $v_i^+ \to v_i'$, $v_i^- \to v_{N+i}'$ for $i = 1, \ldots, N$. We simply use $\{v_1^+, v_1^-, \ldots, v_N^+, v_N^-\}$ to denote the vertex set for $\mathbb{G}(Q(t))$ in the following for the clear correspondence. We have the following result from Lemmas 1 and 2.

*Proposition 1:* Assume that $\mathbb{G}(P(t))$ is strongly connected. $\mathbb{G}(P(t))$ is structurally balanced if and only if $\mathbb{G}(Q(t))$ is disconnected and composed of two strongly connected components; $\mathbb{G}(P(t))$ is structurally unbalanced if and only if $\mathbb{G}(Q(t))$ is strongly connected.

### A. $\mathbb{G}(P(t))$ Is Fixed

When $\mathbb{G}(P(t))$ is fixed, the matrix $P(t)$ in (4) and $Q(t)$ in (10) are also fixed, that is, $P(t) \equiv P$, $Q(t) \equiv Q$, $t \geq 0$. In view of (3), we know that[3] $|P|$ is a stochastic matrix.

*Theorem 1:* Consider an irreducible $P$ with $|P|$ being stochastic. Assume that the graph $\mathbb{G}(P)$ has at least one negative edge. System (4) polarizes if and only if $\mathbb{G}(P)$ is structurally balanced. If $\mathbb{G}(P)$ is structurally unbalanced, then $\lim_{t \to \infty} x(t) = 0$ for every initial value.

*Proof (Sufficiency):* When $\mathbb{G}(P)$ is structurally balanced with at least one negative edge, from the proof of Lemma 1 and Proposition 1 we know that $\mathbb{G}(Q)$ contains two disconnected components each of which is strongly connected and $\{v_{i_1}^+, \ldots, v_{i_m}^+, v_{i_{m+1}}^-, \ldots, v_{i_N}^-\}$ and $\{v_{i_1}^-, \ldots, v_{i_m}^-, v_{i_{m+1}}^+, \ldots, v_{i_N}^+\}$ are the vertex sets of the two components, for some $m$, $1 \leq m < N$ and $1 \leq i_1, \ldots, i_N \leq N$. Thus, the $y$-system (10) is decomposed into two disconnected subsystems. It follows from the classical consensus result [25], [26] that each subsystem converges to some constant value. In system (4) the agents in $\{v_{i_1}, \ldots, v_{i_m}\}$ have the same value and the other agents in $\{v_{i_{m+1}}, \ldots, v_{i_N}\}$ have the opposite value. Since the initial conditions that renders the agreed value of each component to be zero form a set which has zero Lebesgue measure, system (4) polarizes.

(Necessity) Assume that system (4) polarizes. If $\mathbb{G}(P)$ is structurally unbalanced, then $\mathbb{G}(Q)$ is strongly connected based on Proposition 1. It follows that $y_i(t)$ converges to some constant $\alpha$ for all $i = 1, \ldots, 2N$ as $t \to \infty$. Since the $y$-system contains $x_i^+(t)$ and $x_i^-(t)$ as subsystems, $\alpha$ should always be 0 which contradicts the assumption that system (4) polarizes. One can conclude that $\mathbb{G}(P)$ is structurally balanced. $\qquad \square$

The above discussion has assumed that $P$ is irreducible or equivalently $\mathbb{G}(P)$ is strongly connected. Next, we discuss the more general case when $\mathbb{G}(P)$ is not necessarily strongly connected but contains a spanning tree. Using some permutation of rows and columns of $P$, $P$ can be transformed into

$$P = \begin{bmatrix} P_{11} & \mathbf{0} \\ P_{21} & P_{22} \end{bmatrix} \qquad (11)$$

where $P_{11} \in \mathbb{R}^{r \times r}$, $P_{22} \in \mathbb{R}^{(N-r) \times (N-r)}$, $P_{21} \in \mathbb{R}^{(N-r) \times r}$ and $\mathbf{0}$ is a zero matrix of compatible dimension. The submatrix $P_{11}$ is irreducible and the subgraph $\mathbb{G}(P_{11})$ is strongly connected, and there is a directed path from every vertex in $\mathbb{G}(P_{11})$ to every other vertex in $\mathbb{G}(P)$. Note that the vertex set of $\mathbb{G}(P_{11})$ is the root vertex set of $\mathbb{G}(P)$. If $P$ is irreducible, then $r = N$.

Without loss of generality, assume that $P$ is in the form of (11). We discuss two scenarios when $\mathbb{G}(P)$ contains edges of mixed signs: the first is that $\mathbb{G}(P_{11})$ is structurally unbalanced and the other is that $\mathbb{G}(P_{11})$ is structurally balanced with at least one negative edge.

*Case 1. $\mathbb{G}(P_{11})$ Is Structurally Unbalanced:* Since $\mathbb{G}(P_{11})$ is structurally unbalanced, the subgraph with the vertex set

---

[3]We take the absolute value of a matrix elementwise.

$\{v_1^+, \ldots, v_r^+, v_1^-, \ldots, v_r^-\}$ in the graph $\mathbb{G}(Q)$ is strongly connected. In addition, since $\mathbb{G}(P)$ contains a spanning tree, similar to the proof of Lemmas 1 and 2, one can show that $\mathbb{G}(Q)$ contains a spanning tree as well. $y(t)$ in system (10) converges to $\alpha\mathbf{1}$ for some constant $\alpha$ as $t \to \infty$, where $\mathbf{1}$ is the all-one vector of compatible dimension; in addition $\alpha$ must be 0 since $\lim_{t\to\infty}(x_i^+(t) + x_i^-(t)) = 2\alpha = 0$. Thus, $x(t) \to 0$ as $t \to \infty$.

*Case 2.* $\mathbb{G}(P_{11})$ *Is Structurally Balanced With at Least One Negative Edge:* If $\mathbb{G}(P)$ is structurally balanced, then similar to Lemma 1 one can show that $\mathbb{G}(Q)$ contains two disconnected components with $v_i^+$ in one component and $v_i^-$ in the other. In addition, each component contains a spanning tree. Thus, the agents in each component reaches the same value which is the opposite of the other component. It immediately implies that the agents in (4) polarize.

Next, we consider the case when $\mathbb{G}(P)$ is structurally unbalanced. Since $\mathbb{G}(P_{11})$ is structurally balanced, we know from Lemma 1 that in graph $\mathbb{G}(Q)$ the subgraph with the vertex set $\bar{\mathcal{V}}^s = \{v_1^+, \ldots, v_r^+, v_1^-, \ldots, v_r^-\}$ is composed of two disconnected components and each one is strongly connected. The vertex sets of the two components can be denoted as $\bar{\mathcal{V}}_1^s = \{v_{i_1}^+, \ldots, v_{i_m}^+, v_{i_{m+1}}^-, \ldots, v_{i_r}^-\}$ and $\bar{\mathcal{V}}_2^s = \{v_{i_1}^-, \ldots, v_{i_m}^-, v_{i_{m+1}}^+, \ldots, v_{i_r}^+\}$, $1 \le i_1, \ldots, i_r \le r$. Since $\mathbb{G}(P)$ contains a spanning tree, one know that for every vertex in $\bar{\mathcal{V}}\backslash\bar{\mathcal{V}}^s$ in $\mathbb{G}(Q)$, there is a directed path from some vertex in $\bar{\mathcal{V}}^s$ to it.

We check the spectral property of $Q$ and determine the limit of $Q^k$ as $k \to \infty$. Using some permutation of rows and columns of $Q$, $Q$ can be transformed into

$$Q = \begin{bmatrix} Q_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_2 & \mathbf{0} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \qquad (12)$$

with $Q_1 = Q_2$ and $Q_1$ being irreducible. The vertices in the subgraph $\mathbb{G}(Q_1)$ are $\bar{\mathcal{V}}_1^s$ and those in $\mathbb{G}(Q_2)$ are $\bar{\mathcal{V}}_2^s$, and for any vertex in $\mathbb{G}(Q_{33})$ there is a directed path from some vertex either in $\mathbb{G}(Q_1)$ or in $\mathbb{G}(Q_2)$ to it. It can be shown that the spectral radius of $Q_{33}$ is less than 1, that is, $\rho(Q_{33}) < 1$ [28]. Since $Q_1$ has positive diagonals and is irreducible, 1 is a simple eigenvalue of $Q_1$ and the magnitudes of all the other eigenvalues of $Q_1$ are less than 1. Hence $Q$ has exactly two eigenvalues equal to 1 and the magnitudes of all the other eigenvalues are less than 1. The following lemma, the proof of which is provided in Appendix A, is useful for determining the asymptotic state of system (10).

*Lemma 3:* Let $Q = (q_{ij})_{s\times s} = \begin{bmatrix} Q_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Q_1 & \mathbf{0} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$ be a stochastic matrix, where $Q_1 \in \mathbb{R}^{r\times r}$ is a square matrix. Assume that $Q$ has exactly two eigenvalues equal to 1 and the magnitudes of all the other eigenvalues are less than 1. Then

$$\lim_{k\to\infty} Q^k = \begin{bmatrix} \mathbf{1}\xi^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\xi^T & \mathbf{0} \\ \eta_1\xi^T & \eta_2\xi^T & \mathbf{0} \end{bmatrix} \qquad (13)$$

where $\xi \ge 0$, $\xi^T Q_1 = \xi^T$, $\xi^T\mathbf{1} = 1$, and $\eta_1 = (I - Q_{33})^{-1}Q_{31}\mathbf{1}$, $\eta_2 = (I - Q_{33})^{-1}Q_{32}\mathbf{1}$ are some non-negative column vectors. In addition, $\|\eta_1 - \eta_2\|_\infty \le 1$.

The matrix $Q$ in (12) satisfies the condition in Lemma 3. Hence, the states of all the agents in (10) converge. The agents in $\bar{\mathcal{V}}^s$ have the same final absolute value given by $|\xi^T[x_{i_1}(0), \ldots, x_{i_m}(0), -x_{i_{m+1}}(0), \ldots, -x_{i_r}(0)]^T|$, where $\xi$ is the left eigenvector of $Q_1$ corresponding to 1 defined as in the above lemma. For every agent in $\mathcal{V}\backslash\bar{\mathcal{V}}^s$, we have the following bound:

$$\lim_{t\to\infty} |y_i(t)|$$
$$\le \|\eta_1 - \eta_2\|_\infty$$
$$\times \left|\xi^T\left[x_{i_1}(0), \ldots, x_{i_m}(0), -x_{i_{m+1}}(0), \ldots, -x_{i_r}(0)\right]^T\right|$$
$$\le \left|\xi^T\left[x_{i_1}(0), \ldots, x_{i_m}(0), -x_{i_{m+1}}(0), \ldots, -x_{i_r}(0)\right]^T\right|$$

for $i = 2r + 1, \ldots, 2N$.

We summarize the above discussion into the following theorem.

*Theorem 2:* Consider $P$ in the form of (11), with $P_{11}$ being irreducible, $|P|$ being stochastic, and $\mathbb{G}(P)$ containing a spanning tree. If $\mathbb{G}(P_{11})$ is structurally unbalanced, the state of system (4) converges to zero for every initial value. If $\mathbb{G}(P_{11})$ is structurally balanced with at least one negative edge, then the agents of the subgraph $\mathbb{G}(P_{11})$ polarize, and the states of the other agents converge and lie in the interval $[-|C|, |C|]$, where $|C|$ is the absolute value of the polarized value of the agents in $\mathbb{G}(P_{11})$. Furthermore, when $\mathbb{G}(P)$ is structurally balanced with at least one negative edge, system (4) polarizes.

*Remark 1:* In [29], the authors pointed out that when the fixed graph is structurally unbalanced and contains a spanning tree, the states of the agents may converge to zero or become fragmented. Here by looking into the eigenvalues and eigenvectors of system matrix $Q$ of (10), we are able to give a complete characterization of the final state of system (4).

### B. $\mathbb{G}(P(t))$ Is Time-Varying

In this subsection, we consider the case when $\mathbb{G}(P(t))$ changes with time. Assume that there exists a constant $\gamma$, $0 < \gamma < 1$, such that the nonzero elements of $P(t)$ satisfy

$$\left|(P(t))_{ij}\right| \ge \gamma \text{ for } (P(t))_{ij} \ne 0, \ t = 0, 1, 2, \ldots. \qquad (14)$$

We have the following polarization result.

*Theorem 3:* Assume that $P(t)$, $t = 0, 1, 2, \ldots$, satisfy (3) and (14) and there exists a bipartition of $\mathcal{V}$ into two nonempty subsets, such that for each graph $\mathbb{G}(P(t))$, $t \ge 0$, the edges between the two subsets are negative and the edges within each subset are positive. Assume that there exists an infinite sequence of nonempty, uniformly bounded time intervals $[t_i, t_{i+1})$, $i \ge 0$, starting at $t_0 = 0$ with the property that across each time interval $[t_i, t_{i+1})$, the union of the graphs $\mathbb{G}(P(t))$ is strongly connected. Then system (4) polarizes.

*Proof:* Assume that the bipartition of $\mathcal{V}$ is $\mathcal{V}_1 = \{v_{i_1}, \ldots, v_{i_m}\}$, $\mathcal{V}_2 = \{v_{i_{m+1}}, \ldots, v_{i_N}\}$ for some $1 \le m < N$ and $1 \le i_1, \ldots, i_N \le N$. Consider system (10). From the assumption, one can show that across each time interval $[t_i, t_{i+1})$, the union of the graphs $\mathbb{G}(Q(t))$ contains two disconnected components with the vertex sets $\{v_{i_1}^+, \ldots, v_{i_m}^+, v_{i_{m+1}}^-, \ldots, v_{i_N}^-\}$ and $\{v_{i_1}^-, \ldots, v_{i_m}^-, v_{i_{m+1}}^+, \ldots, v_{i_N}^+\}$, and in addition, each

component is strongly connected. We conclude from [25], [26] that the states of the agents in each of the two subsystems converge to the same values, respectively, which are the opposite of each other. Since the initial conditions that render the agreed value of each component to be 0 come from a set with zero measure, we know that system (4) polarizes.  □

*Remark 2:* If $P(t)$, $t = 0, 1, 2, \ldots$, are all irreducible and structurally balanced and, furthermore, the unique bipartitions of $\mathcal{V}$ satisfying Definition 2 are the same for $\mathbb{G}(P(t))$, $t = 0, 1, 2, \ldots$, then the assumptions in Theorem 3 are satisfied and the states of the agents converge to two opposite values. Theorem 3 is a generalization of the previous results for distributed averaging algorithms in [25] and [26], where the weights are non-negative and obviously $P(t)$ are structurally balanced.

*Theorem 4:* Let $P(t)$, $t = 0, 1, 2, \ldots$, satisfy (3) and (14). Assume that $[t_i, t_{i+1})$, $i \geq 0$, $t_0 = 0$ is an infinite sequence of nonempty, uniformly bounded time intervals. Suppose that across each time interval $[t_i, t_{i+1})$, the union of the graphs is strongly connected and there does not exist a bipartition of $\mathcal{V}$ into two subsets, such that for each graph $\mathbb{G}(P(s))$, $s \in [t_i, t_{i+1})$, the edges between the two subsets are negative and the edges within each subset are positive. Then, $x(t)$ of system (4) converges to zero asymptotically.

Note that in Theorem 4, one of the two subsets may be empty.

*Remark 3:* For each time interval $[t_i, t_{i+1})$, if there always exists some $t \in [t_i, t_{i+1})$, such that $P(t)$ is strongly connected and structurally unbalanced, then the conditions in Theorem 4 are satisfied and, thus, the state of the system converges to zero. Stated differently, if structural unbalance arises in the network frequently enough, then polarization of the states of the agents will not occur and instead the opinions of the agents in the network become neutralized in the end.

*Proof of Theorem 4:* It suffices to prove that $y(t)$ of system (10) converges to $\alpha \mathbf{1}$ for some constant $\alpha$ as $t$ goes to infinity. For each time interval $[t_i, t_{i+1})$, we will prove that the union of the graphs over $[t_i, t_{i+1})$, $\cup_{s \in [t_i, t_{i+1})} \mathbb{G}(Q(s))$ is strongly connected. Then, from [25, Theor. 3.10], it follows that $y(t)$ converges to $\alpha \mathbf{1}$ as $t$ goes to infinity.

For each time interval $[t_i, t_{i+1})$, define a directed graph $\mathbb{G}^m = (\mathcal{V}^m, \mathcal{E}^m)$ with $\mathcal{V}^m = \mathcal{V}$ as follows. For two vertices $v_j$ and $v_k$, there exists a positive edge $(v_j, v_k) \in \mathcal{E}^m$ if $(v_j, v_k)$ is a positive edge in graph $\mathbb{G}(P(s))$ for some $s \in [t_i, t_{i+1})$; there is a negative edge $(v_j, v_k) \in \mathcal{E}^m$ if $(v_j, v_k)$ is a negative edge in graph $\mathbb{G}(P(s))$ for some $s \in [t_i, t_{i+1})$. Note that for an ordered pair of vertices $v_j$ and $v_k$, there may exist two directed edges $(v_j, v_k)$ in $\mathbb{G}^m$ with one being positive and the other being negative. Let the enlarged graph based on $\mathbb{G}^m$ be $\overline{\mathbb{G}^m}$. Since the union of the graphs $\mathbb{G}(P(s))$ over the interval $[t_i, t_{i+1})$ is strongly connected, $\mathbb{G}^m$ is strongly connected. In addition, from the condition that there does not exist a bipartition of $\mathcal{V}$ into two subsets, such that for each graph $\mathbb{G}(P(s))$, $s \in [t_i, t_{i+1})$, the edges between the two subsets are negative and the edges within each subset are positive, there is a negative cycle in the graph $\mathbb{G}^m$. Mimicking the proof in the necessity part of Lemma 2, it can be proved that the enlarged graph $\overline{\mathbb{G}^m}$ is strongly connected. Based on the way we define $\overline{\mathbb{G}^m}$, it can be seen that $\overline{\mathbb{G}^m}$ and $\cup_{s \in [t_i, t_{i+1})} \mathbb{G}(Q(s))$ are isomorphic and, thus, $\cup_{s \in [t_i, t_{i+1})} \mathbb{G}(Q(s))$ is strongly connected. Hence, $y(t)$

converges to $\alpha \mathbf{1}$ asymptotically, which implies the state $x(t)$ converges to zero asymptotically.  □

If the union of the graphs over $[t_i, t_{i+1})$, $\cup_{s \in [t_i, t_{i+1})} \mathbb{G}(P(s))$ is not strongly connected, but only contains a spanning tree, system (4) can give rise to some new behavior as discussed next.

*Theorem 5:* Let $P(t)$, $t = 0, 1, 2, \ldots$, satisfy (3) and (14). Let the root vertex set of the union of the graphs over $[0, \infty)$ be $\mathcal{V}^s$. Assume that there exists a bipartition of $\mathcal{V}^s$ into two nonempty subsets, such that for each graph $\mathbb{G}(P(t))$, $t \geq 0$, the edges between the two subsets are negative and the edges within each subset are positive. Assume that there exists an infinite sequence of nonempty, uniformly bounded time intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$ with the property that across each time interval $[t_i, t_{i+1})$, the union of the graphs contains a spanning tree. Then the agents in the root vertex set polarize, and the other agents' states will finally lie in between the polarized values.

*Proof:* From Theorem 3 we know that the agents in the root vertex set $\mathcal{V}^s$ polarize. Assume that the polarized values are $C$ and $-C$, where $C$ is a non-negative constant. We prove that the states of the other agents will asymptotically be bounded by $C$.

Let

$$C(t) = \max_{v_i \in \mathcal{V}^s} |x_i(t)|, \; M(t) = \max_{v_i \in \mathcal{V} \backslash \mathcal{V}^s} |x_i(t)|$$

for $t \geq 0$. From the result in the previous paragraph, one knows that $\lim_{t \to \infty} C(t) = C$. If $M(t) > C(t)$ holds only for a finite time $t$, then there exists a $t'$ such that $M(t) \leq C(t)$ for $t \geq t'$ and hence

$$\limsup_{t \to \infty} M(t) \leq \lim_{t \to \infty} C(t) = C.$$

For this case, the desired conclusion follows.

Next, we assume that $M(t) > C(t)$ holds for an infinite time sequence $t = t_1^*, t_2^*, \ldots$. We pick the specific time $t_1^*$ to carry out the discussion. It is easy to see from (3) and (4) that

$$C(t_1^* + l) \leq C(t_1^*) < M(t_1^*), \; M(t_1^* + l) \leq M(t_1^*) \quad (15)$$

for all $l \geq 0$. Pick an integer $r$ such that $t_{r-1} \leq t_1^* < t_r$ and consider the time interval $[t_r, t_{r+1})$. Since the union of the graphs across $[t_r, t_{r+1})$ contains a spanning tree, there exists some time $s_1 \in [t_r, t_{r+1})$ such that $(v_{i_0}, v_{i_1})$ is an edge of the graph $\mathbb{G}(P(s_1))$ with $v_{i_0} \in \mathcal{V}^s$ and $v_{i_1} \in \mathcal{V} \backslash \mathcal{V}^s$. One has

$$
\begin{aligned}
|x_{i_1}(s_1 + 1)| &= \left| \sum_{j=1}^N p_{i_1 j}(s_1) x_j(s_1) \right| \\
&\leq |p_{i_1 i_0}(s_1) x_{i_0}(s_1)| + \sum_{j \neq i_0} |p_{i_1 j}(s_1) x_j(s_1)| \\
&\leq \gamma C(t_1^*) + (1 - \gamma) M(t_1^*) \\
&= C(t_1^*) + (1 - \gamma)(M(t_1^*) - C(t_1^*))
\end{aligned}
$$

where $\gamma$ is the constant in (14). Since $P(t)$ has positive diagonals, further calculation shows that

$$
\begin{aligned}
|x_{i_1}(s_1 + 2)| & \\
&\leq \gamma \left( C(t_1^*) + (1 - \gamma)(M(t_1^*) - C(t_1^*)) \right) + (1 - \gamma) M(t_1^*) \\
&= C(t_1^*) + \left( 1 - \gamma^2 \right)(M(t_1^*) - C(t_1^*)).
\end{aligned}
$$

Recursively, for $l \geq 0$

$$|x_{i_1}(s_1 + l)| \leq C(t_1^*) + (1 - \gamma^l)(M(t_1^*) - C(t_1^*)).$$

Specifically, the following inequality is true for $l \geq 0$

$$
\begin{aligned}
|x_{i_1}(t_{r+1} + l)| &\leq C(t_1^*) + (1 - \gamma^{t_{r+1} - s_1 + l})(M(t_1^*) - C(t_1^*)) \\
&\leq C(t_1^*) + (1 - \gamma^{t_{r+1} - t_r + l})(M(t_1^*) - C(t_1^*)).
\end{aligned}
\tag{16}
$$

Define $\mathcal{V}_1 = \{v_j | (v_i, v_j)$ as an edge in the union of the graphs across the time interval $[t_r, t_{r+1})$ for some $v_i \in \mathcal{V}^s$ and $v_j \in \mathcal{V} \backslash \mathcal{V}^s\}$. Then the above inequality (16) holds for any $v_{i_1} \in \mathcal{V}_1$.

Consider the time interval $[t_{r+1}, t_{r+2})$. Define $\mathcal{V}_2 = \{v_j | (v_i, v_j)$ as an edge in the union of the graphs across the time interval $[t_{r+1}, t_{r+2})$ for some $v_i \in \mathcal{V}^s \cup \mathcal{V}_1$ and $v_j \in \mathcal{V} \backslash (\mathcal{V}^s \cup \mathcal{V}_1)\}$. If $v_{i_2} \in \mathcal{V}_2$, $v_{i_0} \in \mathcal{V}^s \cup \mathcal{V}_1$ and $(v_{i_0}, v_{i_2})$ is an edge of the graph $\mathbb{G}(P(s_2))$ for some $s_2 \in [t_{r+1}, t_{r+2})$, one has

$$
\begin{aligned}
&|x_{i_2}(s_2 + 1)| \\
&\leq |p_{i_2 i_0}(s_2) x_{i_0}(s_2)| + \sum_{j \neq i_0} |p_{i_2 j}(s_2) x_j(s_2)| \\
&\leq \gamma \left( C(t_1^*) + (1 - \gamma^{s_2 - t_r})(M(t_1^*) - C(t_1^*)) \right) + (1 - \gamma) M(t_1^*) \\
&= C(t_1^*) + (1 - \gamma^{s_2 - t_r + 1})(M(t_1^*) - C(t_1^*)).
\end{aligned}
$$

Thus, for $l \geq 0$, the following inequality holds:

$$|x_{i_2}(s_2 + l)| \leq C(t_1^*) + (1 - \gamma^{s_2 - t_r + l})(M(t_1^*) - C(t_1^*)).$$

For all $v_i \in \mathcal{V}_1 \cup \mathcal{V}_2$, it holds that

$$|x_i(t_{r+2} + l)| \leq C(t_1^*) + (1 - \gamma^{t_{r+2} - t_r + l})(M(t_1^*) - C(t_1^*)).$$

Continuing this process, one derives that for all $v_i \in \mathcal{V} \backslash \mathcal{V}_r$

$$
\begin{aligned}
|x_i(t_{r+N-1})| &\leq C(t_1^*) + (1 - \gamma^{t_{r+N-1} - t_r})(M(t_1^*) - C(t_1^*)) \\
&\leq C(t_1^*) + \left(1 - \gamma^{(N-1)T}\right)(M(t_1^*) - C(t_1^*))
\end{aligned}
$$

where $T$ is a uniform upper bound for $t_{r+1} - t_r$. Repeating the above calculation, we have that for all $v_i \in \mathcal{V} \backslash \mathcal{V}_r$

$$\left| x_i \left( t_{r+(N-1)l} \right) \right| \leq C(t_1^*) + \left(1 - \gamma^{(N-1)T}\right)^l (M(t_1^*) - C(t_1^*)).$$

Combining with (15), one has

$$
\begin{aligned}
M\left(t_{r+(N-1)l} + s\right) &\leq \max\left\{ M\left(t_{r+(N-1)l}\right), C\left(t_{r+(N-1)l}\right) \right\} \\
&\leq C(t_1^*) + \left(1 - \gamma^{(N-1)T}\right)^l (M(t_1^*) - C(t_1^*))
\end{aligned}
$$

for all $0 \leq s < t_{r+(N-1)(l+1)} - t_{r+(N-1)l}$. From the above inequality, we can conclude that $\limsup_{t \to \infty} M(t) \leq C(t_1^*)$. Since the above discussion applies to all $t_r^*$, it holds that $\limsup_{t \to \infty} M(t) \leq C(t_r^*)$ for all $r = 1, 2, \ldots$. In view of the fact that $\lim_{t \to \infty} C(t) = C$, one has $\limsup_{t \to \infty} M(t) \leq C$. This completes the proof. $\quad\square$

*Remark 4:* In the fixed topology case in the previous section, when the graph contains a spanning tree, it is shown in Theorem 2 that the agents in the root vertex set polarize and the states of the other agents converge and lie in between the polarized values. However, when the network topologies are time-varying, the states of the other agents may not converge but they will finally lie in between the polarized values, which will be illustrated though an example in Section VI.

*Theorem 6:* Let $P(t)$, $t = 0, 1, 2, \ldots$, satisfy (3) and (14). Assume that $[t_i, t_{i+1})$, $i \geq 0$, $t_0 = 0$, is an infinite sequence of nonempty, uniformly bounded time intervals. Suppose that across each time interval $[t_i, t_{i+1})$, the union of the graphs contains a spanning tree and for the root vertex set of the union graph, there does not exist a bipartition of this set into two subsets, such that for each graph $\mathbb{G}(P(s))$, $s \in [t_i, t_{i+1})$, the edges between the two subsets are negative and the edges within each subset are positive. Then, the state of system (4) converges to zero asymptotically.

*Proof:* Using similar arguments to the proof of Theorem 4, we can show that the union of the graphs $\mathbb{G}(Q(t))$ across each time interval $[t_i, t_{i+1})$, $i \geq 0$, contains a spanning tree. It immediately follows that the $y$-system (10) converges to $\alpha \mathbf{1}$ for some constant $\alpha$ from [25]. Thus, system (4) converges to zero asymptotically. $\quad\square$

*Remark 5:* In [24], Hendrickx formally introduced the transformation (9) and studied discrete-time and continuous-time systems with reciprocal interactions between agents and nonreciprocal interactions under joint strong connectivity conditions. The convergence of the system to polarized values or to zero were derived based on studies on consensus systems with "type-symmetric" interactions and by looking into the persistent interactions between agents [27]. Here, we have considered nonreciprocal interactions between agents with joint graphs containing spanning trees, where opinion separation of the agents may appear.

## V. CONTINUOUS-TIME MODEL

In this section, we present our main results for the continuous-time model (6). For each $A(t)$, similar to (8), we can define two non-negative matrices $A^+(t)$ and $A^-(t)$ based on $A(t)$. Let $x_i^+(t) = x_i(t)$, $x_i^-(t) = -x_i(t)$ and $y(t) = [x_1^+(t), \ldots, x_N^+(t), x_1^-(t), \ldots, x_N^-(t)]^T$. From system (6), we obtain the following update equations for $y(t)$:

$$\dot{y}(t) = \left( \begin{bmatrix} A^+(t) & A^-(t) \\ A^-(t) & A^+(t) \end{bmatrix} - \begin{bmatrix} D(t) & \mathbf{0} \\ \mathbf{0} & D(t) \end{bmatrix} \right) y(t) = -W(t) y(t) \tag{17}$$

where $D(t) = \text{diag}\{d_1(t), \ldots, d_N(t)\}$ with $d_i(t) = \sum_{j=1, j \neq i}^N |a_{ij}(t)|$ and $W(t) = \begin{bmatrix} D(t) & \mathbf{0} \\ \mathbf{0} & D(t) \end{bmatrix} - \begin{bmatrix} A^+(t) & A^-(t) \\ A^-(t) & A^+(t) \end{bmatrix}$ is the Laplacian matrix with nonpositive off-diagonal elements. The dynamical behavior of system (6) can be revealed by studying system (17).

### A. $\mathbb{G}(L(t))$ Is Fixed

Consider the continuous-time system (6) under fixed topologies. Let $A(t) \equiv A \in \mathbb{R}^{N \times N}$ be the signed adjacency matrix, and let $L(t) \equiv L$ be the signed Laplacian matrix given by (7) for all $t \geq 0$. When the graph $\mathbb{G}(L)$ contains a spanning tree, by a suitable permutation of rows and columns of its associated signed Laplacian matrix $L$, $L$ can be brought into the following form:

$$L = \begin{bmatrix} L_{11} & \mathbf{0} \\ L_{21} & L_{22} \end{bmatrix} \tag{18}$$

where $L_{11} \in \mathbb{R}^{r \times r}$ is irreducible, $L_{22} \in \mathbb{R}^{(N-r) \times (N-r)}$, and $L_{21} \in \mathbb{R}^{(N-r) \times r}$. By looking into system (17) and in view of Lemmas 1 and 2, similar to Theorem 2, we have the following theorem.

*Theorem 7:* Let $\mathbb{G}(L)$ be a signed graph containing a spanning tree and let $L$ be its associated signed Laplacian matrix in the form (18). If the subgraph $\mathbb{G}(L_{11})$ is structurally unbalanced, the state of system (6) converges to zero for every initial value. If $\mathbb{G}(L_{11})$ is structurally balanced with at least one negative edge, then the agents in the subgraph $\mathbb{G}(L_{11})$ polarize and the states of the other agents converge and lie in between the polarized values; furthermore, if $\mathbb{G}(L)$ is structurally balanced with at least one negative edge, system (6) polarizes.

### B. $\mathbb{G}(L(t))$ Is Time-Varying

In this subsection, we consider the case when the interaction graph topologies are dynamically changing. Assume that $A(t)$ and $L(t)$ are piecewise constant functions and the interaction graph topologies or the weights of the edges change at time instants $t_1, t_2, \ldots$. System (6) can be rewritten as

$$\dot{x}(t) = -L(t_i)x(t), \quad t \in [t_i, t_i + \tau_i) \tag{19}$$

where $t_0 = 0$ is the initial time, and $\tau_i = t_{i+1} - t_i, i = 0, 1, \ldots$ are the dwell times. Let $\tau$ be a finite set of positive numbers and let $\mathcal{T}$ be an infinite set generated from $\tau$, which is closed under addition, and multiplication by positive integers. Assume that $\tau_i \in \mathcal{T}, i = 0, 1, 2, \ldots$. Let the nonzero elements $a_{jk}(t_i)$ of $A(t_i)$ satisfy that $a_{jk}(t_i) \in [\gamma_1, \gamma_2]$, where $\gamma_1, \gamma_2$ are positive constants.

The $y$-system (17) can be written as

$$\dot{y}(t) = -W(t_i)y(t), \quad t \in [t_i, t_i + \tau_i). \tag{20}$$

Employing similar ideas as in the previous section for the discrete-time model (4) and (10) and in view of [25, Theor. 3.12], we can prove the following two theorems.

*Theorem 8:* Let the root vertex set of the union of the graphs over $[0, \infty)$ be $\mathcal{V}^s$. Assume that there exists a bipartition of $\mathcal{V}^s$ into two nonempty subsets, such that for each graph $\mathbb{G}(L(t))$, $t \geq 0$, the edges between the two subsets are negative and the edges within each subset are positive. Assume that there exists an infinite sequence of nonempty, uniformly bounded time intervals $[t_{i_k}, t_{i_{k+1}}), k \geq 0$, starting at $t_{i_0} = 0$ with the property that across each time interval $[t_{i_k}, t_{i_{k+1}})$, the union of the graphs contains a spanning tree. Then, the agents in the root vertex set of system (6) polarize, and the states of the other agents will finally lie in between the polarized values.

*Theorem 9:* Assume that $[t_{i_k}, t_{i_{k+1}}), k \geq 0, t_{i_0} = 0$, is an infinite sequence of nonempty, uniformly bounded time intervals. Suppose that across each time interval $[t_{i_k}, t_{i_{k+1}})$, the union of the graphs contains a spanning tree and for the root vertex set of the union graph, there does not exist a bipartition of this set into two subsets, such that for each graph $\mathbb{G}(L(s)), s \in [t_{i_k}, t_{i_{k+1}})$, the edges between the two subsets are negative and the edges within each subset are positive. Then, the state of system (6) converges to zero asymptotically.

### VI. ILLUSTRATIVE EXAMPLES

In this section, we perform simulation studies on system (4) with topologies containing spanning trees. Consider the two graphs shown in Fig. 5, where the edges with negative weights are labeled by "$-$" signs and those with positive weights are
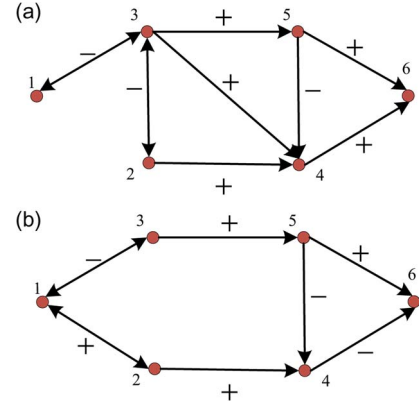


Fig. 5. Two graphs $\mathbb{G}(P_1)$ and $\mathbb{G}(P_2)$ both contain spanning trees. (a) $\mathbb{G}(P_1)$. (b) $\mathbb{G}(P_2)$.

labeled by "$+$" signs. Their corresponding matrices $P_1$ and $P_2$ are given by

$$P_1 = \begin{bmatrix} (P_1)_{11} & \mathbf{0} \\ (P_1)_{21} & (P_1)_{22} \end{bmatrix} = \left[ \begin{array}{ccc|ccc} \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right]$$

$$P_2 = \begin{bmatrix} (P_2)_{11} & \mathbf{0} \\ (P_2)_{21} & (P_2)_{22} \end{bmatrix} = \left[ \begin{array}{ccc|ccc} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array} \right].$$

One can see that $\mathbb{G}(P_1)$ is structurally unbalanced but the subgraph $\mathbb{G}((P_1)_{11})$ is structurally balanced. $\mathbb{G}(P_2)$ is structurally balanced. Let the initial state of the system be $x(0) = [0.9, 0.7, -0.9, -1, 0.2, 0.9]^T$.

The evolution of the states of the agents under the graph topology $\mathbb{G}(P_1)$ in Fig. 5(a) has been illustrated in Fig. 2(b) in Section II. As indicated in Theorem 2, agents 1, 2, 3 in the subgraph $\mathbb{G}(P_{11})$ achieve opposite values and the states of agents 4, 5, and 6 converge and lie in between the opposite values. For system (4) with

$$P(t) = \begin{cases} P_1, & t \text{ is even} \\ P_2, & t \text{ is odd} \end{cases} \tag{21}$$

the evolution of the states of the agents are shown in Fig. 6, from which we can see that the states of agents 4 and 6 do not converge but they still lie in between the opposite values of agents 1, 2, 3.

### VII. CONCLUSION

In this paper, we have studied the relationship between structural balance and opinion separation in social networks that contain both trust and mistrust relationships. When the opinion update rules are described by DeGroot-type models, we have shown that under conditions that are closely related to whether a network is structurally balanced or not, the opinions sometime get separated, for which in the extreme case the network evolves
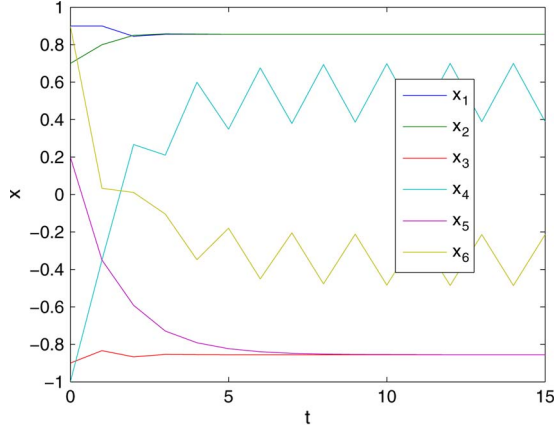
Fig. 6. Evolution of the agents' states with the graphs switching between Fig. 5(a) and (b).

into two polarized camps, and sometimes become neutralized. Our results complement the existing results in the literature.

We are interested in further developing opinion separation models that rely less on the DeGroot averaging rules. One promising direction is to look into the biased assimilation behavior in social groups. The nonlinearity inherently associated with such behavior is a main challenge that we want to attack.

## APPENDIX A

*Proof of Lemma 3:* Since $Q_1$ is a stochastic matrix, 1 is an eigenvalue of $Q_1$ with the corresponding eigenvector $\mathbf{1}$. From the assumption of the lemma, we know that 1 is a simple eigenvalue of $Q_1$ and the magnitudes of all other eigenvalues of $Q_1$ are less than 1. In addition, $\rho(Q_{33}) < 1$. Thus, from the Perron-Frobenius theorem [30],

$$\lim_{k \to \infty} Q_1^k = \mathbf{1}\xi^T$$

where $\xi \geq 0$, $\xi^T Q_1 = \xi^T$, and $\xi^T \mathbf{1} = 1$.

It is easy to see that $v_1^T = [\xi^T \ \mathbf{0}^T \ \mathbf{0}^T]$ and $v_2^T = [\mathbf{0}^T \ \xi^T \ \mathbf{0}^T]$ are two independent left eigenvectors of $Q$ corresponding to 1. One can verify that $u_1 = [\mathbf{1}^T \ \mathbf{0}^T \ \eta_1^T]^T$ and $u_2 = [\mathbf{0}^T \ \mathbf{1}^T \ \eta_2^T]^T$ are two independent right eigenvectors of $Q$ corresponding to 1, where $\eta_1$ and $\eta_2$ are given by

$$\eta_1 = (I - Q_{33})^{-1} Q_{31} \mathbf{1}, \ \eta_2 = (I - Q_{33})^{-1} Q_{32} \mathbf{1}.$$

$I - Q_{33}$ is invertible because $\rho(Q_{33}) < 1$. In addition, the following equalities hold $v_1^T u_1 = 1$, $v_2^T u_2 = 1$, $v_1^T u_2 = 0$, and $v_2^T u_1 = 0$. By using the Jordan canonical form, we can show that $Q^k$ converges as $k$ goes to infinity and

$$\lim_{k \to \infty} Q^k = u_1 v_1^T + u_2 v_2^T = \begin{bmatrix} \mathbf{1}\xi^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\xi^T & \mathbf{0} \\ \eta_1\xi^T & \eta_2\xi^T & \mathbf{0} \end{bmatrix}.$$

Since $\mathbf{1} = (Q_{31} + Q_{32} + Q_{33})\mathbf{1}$, it follows that $\eta_1 + \eta_2 = (I - Q_{33})^{-1}(Q_{31} + Q_{32})\mathbf{1} = \mathbf{1}$. From the non-negativity of the vectors $\eta_1$ and $\eta_2$, one has that

$$\|\eta_1 - \eta_2\|_\infty \leq \|\eta_1 + \eta_2\|_\infty = 1.$$

□

## REFERENCES

[1] I. Ajzen, "Nature and operation of attitudes," *Annu. Rev. Psychol.*, vol. 52, pp. 27–58, 2001.
[2] D. Easley and J. Kleinberg, *Networks, Crowds, Markets: Reasoning About a Highly Connected World*. Cambridge, U.K.: Cambridge University Press, 2010.
[3] O. D. Duncan and S. Lieberson, "Ethnic segregation and assimilation," *Amer. J. Sociol.*, vol. 64, p. 364, 1959.
[4] S. B. Sarason, *The Psychological Sense of Community: Prospects for a Community Psychology*. San Francisco, CA, USA: Jossey-Bass, 1974.
[5] N. E. Friedkin, *A Structural Theory of Social Influence*. Cambridge, U.K.: Cambridge University Press, 2006.
[6] F. Harary, "On the notion of balance of a signed graph," *Mich. Math. J.*, vol. 2, pp. 143–146, 1953–1954.
[7] D. Cartwright and F. Harary, "Structural balance: A generalization of Heider's theory," *Psychol. Rev.*, vol. 63, pp. 277–292, 1956.
[8] N. T. Feather, "A structural balance approach to the analysis of communication effects," *Adv. Exper. Soc. Psychol.*, vol. 3, pp. 99–165, 1967.
[9] S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications*. Cambridge, U.K.: Cambridge University Press, 1994.
[10] P. Abell and M. Ludwig, "Structural balance: A dynamic perspective," *J. Math. Sociol.*, vol. 33, pp. 129–155, 2009.
[11] C. Altafini, "Dynamics of opinion forming in structurally balanced social networks," *PLoS ONE*, vol. 7, no. 6, p. e38135, 2012.
[12] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.
[13] A. V. Proskurnikov, A. Matveev, and M. Cao, "Opinion dynamics in social networks with hostile camps: Consensus vs. polarization," *IEEE Trans. Autom. Control*, to be published.
[14] Z. Meng, G. Shi, K. H. Johansson, M. Cao, and Y. Hong, "Modulus consensus over networks with antagonistic interactions and switching topologies," arXiv:1402.2766.
[15] M. H. DeGroot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.
[16] N. H. Anderson, *Foundations of Information Integration Theory*. London, U.K.: Academic Press, 1981.
[17] C. Chamley, A. Scaglione, and L. Li, "Models for the diffusion of beliefs in social networks: An overview," *IEEE Signal Process. Mag.*, vol. 30, no. 3, pp. 16–29, May 2013.
[18] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence: Models, analysis and simulation," *J. Artif. Soc. Soc. Simul.*, vol. 5, pp. 1–24, 2002.
[19] J. Lorenz, "Continuous opinion dynamics under bounded confidence: A survey," *Int. J. Modern Phys. C*, vol. 18, pp. 1819–1838, 2007.
[20] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis, "On Krause's multi-agent consensus model with state-dependent connectivity," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2586–2597, Nov. 2009.
[21] M. Mas, A. Flache, and D. Helbing, "Individualization as driving force of clustering phenomena in humans," *PLoS Comput. Biol.*, vol. 6, p. e1000959, 2010.
[22] W. Xia and M. Cao, "Clustering in diffusively coupled networks," *Automatica*, vol. 47, pp. 2395–2405, 2011.
[23] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*. Berlin, Germany: Springer-Verlag, 2000.
[24] J. M. Hendrickx, "A lifting approach to models of opinion dynamics with antagonisms," in *Proc. 53th IEEE Conf. Dec. Control*, 2014, pp. 2118–2123.
[25] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.
[26] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: A graphical approach," *SIAM J. Control Optimiz.*, vol. 47, no. 2, pp. 575–600, 2008.
[27] J. M. Hendrickx and J. Tsitsiklis, "Convergence of type-symmetric and cut-balanced consensus seeking systems," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 214–218, Jan. 2013.
[28] F. Xiao and L. Wang, "State consensus for multi-agent systems with switching topologies and time-varying delays," *Int. J. Control*, vol. 79, no. 10, pp. 1277–1284, 2006.
[29] J. Hu and W. Zheng, "Bipartite consensus for multi-agent systems on directed signed networks," in *Proc. 52nd IEEE Conf. Dec. Control Eur. Control Conf.*, 2013, pp. 3451–3456.
[30] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge University Press, 1985.

**Weiguo Xia** received the B.Sc. and M.Sc. degrees in applied mathematics from Southeast University, Nanjing, China, in 2006 and 2009, respectively, and the Ph.D. degree in mathematics and natural sciences from ENTEG, University of Groningen, Groningen, The Netherlands, in 2013.

Currently, he is an associate professor with the School of Control Science and Engineering, Dalian University of Technology, Dalian, China. From 2013 to 2015, he was a Postdoctoral Researcher at ACCESS Linnaeus Centre, Royal Institute of Technology, Stockholm, Sweden. His research interests include complex networks, social networks, and multiagent systems.

**Ming Cao** received the B.Sc. and M.Sc. degrees in electrical engineering from Tsinghua University, Beijing, China, in 1999 and 2002, respectively, and the Ph.D. degree in electrical engineering from Yale University, New Haven, CT, USA, in 2007.

Currently, he is an Associate Professor with tenure and responsible for the research direction of network analysis and control with the Faculty of Mathematics and Natural Sciences at the University of Groningen, Groningen, the Netherlands, where he started as a tenure-track Assistant Professor in 2008. From 2007 to 2008, he was a Postdoctoral Research Associate with the Department of Mechanical and Aerospace Engineering at Princeton University, Princeton, NJ, USA. He was a Research Intern during the summer of 2006 with the Mathematical Sciences Department at the IBM T. J. Watson Research Center, Yorktown Heights, NY, USA. His main research interest is in autonomous agents and multiagent systems, mobile-sensor networks, and complex networks.

Prof. Cao is an Associate Editor for *Systems and Control Letters*, and for the Conference Editorial Board of the IEEE Control Systems Society. He is also a member of the IFAC Technical Committee on Networked Systems.

**Karl Henrik Johansson** (F'13) received the M.Sc. and Ph.D. degrees in electrical engineering from Lund University, Lund, Sweden, in 1992 and 1997, respectively.

Currently, he is Director of the ACCESS Linnaeus Centre and Professor at the School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, Sweden. He is a Wallenberg Scholar and has held a six-year Senior Researcher Position with the Swedish Research Council. He is also heading the Stockholm Strategic Research Area ICT The Next Generation. He held visiting positions at the University of California, Berkeley, CA, USA, in 1998–2000 and California Institute of Technology, Pasadena, CA, USA, from 2006 to 2007. He is on the editorial board of IEEE TRANSACTIONS ON AUTOMATIC CONTROL and IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS, and has been Guest Editor for IEEE TRANSACTIONS ON AUTOMATIC CONTROL and *IEEE Control Systems Magazine*. His research interests are in networked control systems, cyber-physical systems, and applications in transportation, energy, and automation systems.

Dr. Johansson has been a member of the IEEE Control Systems Society Board of Governors and the Chair of the IFAC Technical Committee on Networked Systems. He has been on the editorial boards of several journals, including *Automatica* and *IET Control Theory and Applications*. He is currently on the Editorial Board of the *European Journal of Control*. He was the General Chair of the ACM/IEEE Cyber-Physical Systems Week 2010 in Stockholm and IPC Chair of many conferences. He has served on the executive committees of several European research projects in the area of networked embedded systems. He received the Best Paper Award of the IEEE International Conference on Mobile Ad-hoc and Sensor Systems in 2009 and the Best Theory Paper Award of the World Congress on Intelligent Control and Automation in 2014. In 2009, he was awarded the Wallenberg Scholar, as one of the first ten scholars from all sciences, by the Knut and Alice Wallenberg Foundation. He was awarded Future Research Leader from the Swedish Foundation for Strategic Research in 2005. He received the triennial Young Author Prize from IFAC in 1996 and the Peccei Award from the International Institute of System Analysis, Austria, in 1993. He received Young Researcher Awards from Scania in 1996 and from Ericsson in 1998 and 1999.