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Synchronization of Coupled Nonlinear Dynamical Systems: Interplay Between Times of Connectivity and Integral of Lipschitz Gain

Tao Yang, *Member, IEEE,* Ziyang Meng, *Member, IEEE,* Wei Ren *Senior Member, IEEE,* and Karl H. Johansson, *Fellow, IEEE*

***Abstract*—This paper considers the synchronization problem of coupled nonlinear dynamical systems over time-varying interaction graphs. We first show that infinite joint connectivity is necessary for achieving globally asymptotic synchronization. We then show that the commonly used Lipschitz condition on the nonlinear self dynamics is not sufficient to ensure synchronization even for an arbitrarily large coupling strength. A sufficient synchronization condition is established in terms of the times of connectivity, the integral of the Lipschitz gain, and the network parameters.**

***Index Terms*—Multi-agent systems, nonlinear dynamical systems, time-varying interaction, synchronization.**

I. Introduction

In recent years, the study on synchronization of coupled nonlinear dynamical systems has attracted considerable attention, partly due to that an increasing number of circuits and systems can be described in such a framework. Examples include arrays of Chua circuits [1], Lorenz systems [2], chaotic systems [3]–[5], and other physical systems surveyed in [6].

The interaction among the dynamical systems is often modeled by a graph. Most works in the literature studied the case where the graph is fixed, e.g., [7]–[11]. It has been shown that if the nonlinear self dynamics is globally Lipschitz, then synchronization is achieved for a connected graph provided that the coupling strength is sufficiently large.

For the case where the graph is time-varying, most attention has been devoted to a few special cases, where the self dynamics is a single integrator [12]–[18] or a neutrally stable system [19]–[22]. For such systems, it has been shown that synchronization is achieved if the interaction graph is uniformly jointly quasi-strongly connected or infinitely jointly connected. When the self dynamics is nonlinear, the problem becomes much more challenging and existing works mainly focused on the special case where the interaction graph has some particular structure. In particular, the authors of [23] assumed that the graph is weakly connected and balanced at all time and the self dynamics is globally Lipschitz. They showed that synchronization is achieved if the coupling strength is sufficiently large. A similar result was obtained in [24] for a more general case where the interaction graph frequently has a directed spanning tree. These special time-varying graphs are rather restrictive compared to joint connectivity where the interaction can be lost at any particular time.

The goal of this paper is to study synchronization of coupled nonlinear dynamical systems over jointly connected graphs. We begin to show that a weak form of graph connectivity, infinite joint connectivity, is necessary for achieving global asymptotic synchronization. For infinitely jointly connected graphs, we show through an example that the commonly used global Lipschitz condition on the nonlinear self dynamics alone is not sufficient to ensure synchronization even for an arbitrarily large coupling strength. We then establish a sufficient condition for reaching synchronization in terms of the times of connectivity, the integral of the Lipschitz gain, and the network parameters.

The rest of the paper is organized as follows: In Section II, we provide some background on graph theory and Dini derivative. Section III formulates the three synchronization problems considered in this paper. Our main results are presented in Section IV followed by concluding remarks.

II. Preliminaries

Let us first recall some basic concepts from graph theory [25]. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a nonempty finite set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, where an edge $(j, i) \in \mathcal{E}$ denotes that nodes i and j can obtain each other's information mutually. All neighbors of node i are denoted $\mathcal{N}_i := \{j : (j, i) \in \mathcal{E}\}$. A path is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots$. The graph $\mathcal{G}$ is connected if each node has a path to any other node. For the graph $\mathcal{G}$, the weighted adjacent matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is defined such that $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The weighted adjacency matrix A associated with the undirected graph is not necessarily symmetric since $a_{ij} \neq a_{ji}$ in general.

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T. Yang and Z. Meng contributed equally to this work.

T. Yang and K. H. Johansson are with with the ACCESS Linnaeus Centre, School of Electrical Engineering, Royal Institute of Technology, Stockholm 10044, Sweden (e-mail: taoyang@kth.se, kallej@kth.se).

Z. Meng is with the Department of Precision Instrument and the State Key Laboratory of Precision Measurement Technology and Instruments, Tsinghua University, Beijing 100084, China (e-mail: ziyangmeng@mail.tsinghua.edu.cn).

W. Ren is with the Department of Electrical and Computer Engineering, University of California, Riverside, CA 92521 USA (e-mail: ren@ee.ucr.edu).

In this paper, we model the time-varying interaction among the coupled dynamical systems by a time-varying graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, where $\sigma : [0, +\infty) \to \mathcal{P}$ is a piecewise constant function and $\mathcal{P}$ is a finite set of all possible graphs. $\mathcal{G}_{\sigma(t)}$ remains constant for $t \in [t_\ell, t_{\ell+1})$, $\ell = 0, 1, \ldots$ and switches at $t = t_\ell$, $\ell = 1, \ldots$. Throughout the paper, we assume that $\inf_\ell(t_{\ell+1} - t_\ell) \geq \tau_d > 0$, $\ell = 1, \ldots$ with $\lim_{\ell\to\infty} t_\ell = \infty$, where τ_d is a constant denoting the dwell time [26]. The joint graph of $\mathcal{G}_{\sigma(t)}$ during time interval $[t_a, t_b)$ with $t_a < t_b \leq \infty$ is defined by $\mathcal{G}([t_a, t_b)) = \bigcup_{t\in[t_a,t_b)} \mathcal{G}(t) = (\mathcal{V}, \bigcup_{t\in[t_a,t_b)} \mathcal{E}_{\sigma(t)})$. Moreover, j is a neighbor of i at time t when $(j, i) \in \mathcal{E}_{\sigma(t)}$, and $\mathcal{N}_i(\sigma(t))$ represents the set of node neighbors of i at time t. We denote $\{A_p\}_{p\in\mathcal{P}}$ as the set of adjacency matrices associated with the graph $\{\mathcal{G}_p\}_{p\in\mathcal{P}}$. The upper Dini derivative of $V(t, x(t))$ at t is defined as [27, pp.659]

$$D^+V(t, x(t)) = \limsup_{\delta\to0^+} \frac{V(t + \delta, x(t + \delta)) - V(t, x(t))}{\delta}.$$

The following lemma holds [28].

Lemma 1: Let $V_i(t, x) : \mathbb{R}\times\mathbb{R}^p \to \mathbb{R}$ $(i = 1, \ldots, n)$ be continuously differentiable and $V(t, x) = \max_{i=1,\ldots,n} V_i(t, x)$. If $\mathcal{I}(t) = \{i \in \{1, 2, \ldots, n\} : V(t, x(t)) = V_i(t, x(t))\}$ is the set of indices where the maximum is reached at t, then $D^+V(t, x(t)) = \max_{i\in\mathcal{I}(t)} \dot{V}_i(t, x(t))$.

III. Problem Formulation

Consider a network with n coupled nonlinear dynamical systems. The dynamics of the systems are described by the following equations:

$$\dot{x}_i = f(t, x_i) + \gamma \sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))(x_j - x_i),\ i \in \mathcal{V}, \quad (1)$$

where $x_i \in \mathbb{R}^p$ is the state of node i, $\gamma > 0$ is a coupling gain, $a_{ij}(p) > 0$ is the (i, j)-th entry of the adjacency matrix A_p associated with the graph $\mathcal{G}_p$ for all $p \in \mathcal{P}$, and $f(t, x_i) : [0,\infty)\times\mathbb{R}^p \to \mathbb{R}^p$ is piecewise continuous in t and continuous in x_i representing the nonlinear self dynamics of system i.

It is not hard to show that $a_* \leq a_{ij}(p) \leq a^*$, for all $a_{ij}(p) \neq 0$, all $i, j \in \mathcal{V}$, and all $p \in \mathcal{P}$, where $a^* = \max_{p\in\mathcal{P},i,j\in\mathcal{V}} a_{ij}(p)$ and $a_* = \min_{p\in\mathcal{P},i,j\in\mathcal{V}}\{a|a \in \{a_{ij}(p)\}$ and $a \neq 0\}$. We denote $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{pn}$ and assume that the initial time is $t_0 \geq 0$, and the initial state $x(t_0) = (x_1^{\mathrm{T}}(t_0), \ldots, x_n^{\mathrm{T}}(t_0))^{\mathrm{T}} \in \mathbb{R}^{pn}$.

For single integrators (i.e., $f(t, x_i) = 0$ in (1)) over an undirected graph, the following assumption is a necessary and sufficient condition for achieving global asymptotic synchronization [16].

Assumption 1: The time-varying graph $\mathcal{G}_{\sigma(t)}$ is infinitely jointly connected, i.e., $\mathcal{G}([t,\infty))$ is connected for all $t \geq t_0$.

Throughout the paper, we assume that Assumption 1 is satisfied. We are interested in the following synchronization problems.

Definition 1: System (1) achieves global asymptotic synchronization if $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$, $\forall i, j \in \mathcal{V}$, $\forall x_i(t_0) \in \mathbb{R}^p$.

Definition 2: System (1) achieves global exponential synchronization if there exist constants $\xi_1 > 0$ and $\lambda_1 > 0$ such that $\|x_i(t) - x_j(t)\| \leq \xi_1 e^{-\lambda_1(t-t_0)}\|x_i(t_0) - x_j(t_0)\|$, $\forall i, j \in \mathcal{V}$, $\forall x_i(t_0) \in \mathbb{R}^p$, $\forall t \geq t_0$.

Definition 3: System (1) achieves global polynomial synchronization if there exist constants $\xi_2 > 0$ and $\lambda_2 > 0$ such that $\|x_i(t) - x_j(t)\| \leq \frac{\xi_2}{(t-t_0)^{\lambda_2}}\|x_i(t_0) - x_j(t_0)\|$, $\forall i, j \in \mathcal{V}$, $\forall x_i(t_0) \in \mathbb{R}^p$, $\forall t \geq t_0$.

IV. Main Results

In this section, we present our main results.

A. Necessity of Infinite Joint Connectivity

We begin to show that infinite joint connectivity given in Assumption 1, is necessary for achieving global asymptotic synchronization of (1).

Theorem 1: Assume that the equilibrium point $x = x^*$ of $\dot{x} = f(t, x)$ is not asymptotically stable. If global asymptotic synchronization is achieved for (1), then $\mathcal{G}_{\sigma(t)}$ is infinitely jointly connected.

Proof: We prove Theorem 1 by contraposition. Suppose that $\mathcal{G}_{\sigma(t)}$ is not infinitely jointly connected. Then there exists $t^* \geq t_0$ such that the union graph $\mathcal{G}([t^*,\infty))$ is not connected. This implies that there exist two nonempty, disjoint subsets $\mathcal{V}_a \subset \mathcal{V}$ and $\mathcal{V}_b \subset \mathcal{V}$ such that there is no link between sets $\mathcal{V}_a$ and $\mathcal{V}_b$ for all $t \geq t^*$. Let us choose $x_i(t^*) = x^*$ for all $i \in \mathcal{V}_a$ and $x_i \neq x^*$ for all $i \in \mathcal{V}_b$, where $x = x^*$ is the equilibrium point of $\dot{x} = f(t, x)$. Then $x_i(t) = x^*$ for all $i \in \mathcal{V}_a$ and for all $t \geq t^*$. In addition, $\dot{x}_i(t) = f(t, x_i)$, for all $i \in \mathcal{V}_b$ and for all $t \geq t^*$. Based on the fact that the equilibrium point $x = x^*$ of $\dot{x} = f(t, x)$ is not asymptotically stable, we know that $\lim_{t\to\infty}(x_i(t) - x^*) \neq 0$ for all $i \in \mathcal{V}_b$. This shows that global asymptotic synchronization cannot be achieved for (1). Hence, the result follows. ■

B. Globally Lipschitz Self Dynamics

In the literature, it has been established that for fixed connected graphs [2], [7]–[9], [11] and for some special switching graphs [23], [24], synchronization is achieved for a sufficiently large coupling γ if the self dynamics satisfies the following global Lipschitz assumption.

Assumption 2: The self dynamics $f(t, x)$ is globally Lipschitz continuous in x with the Lipschitz constant $L > 0$, i.e., $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$, $\forall x, y \in \mathbb{R}^p$, $\forall t \geq t_0$.

The following example shows that for the general time-varying graph satisfying Assumption 1, Assumption 2 alone is not sufficient for achieving synchronization even if the coupling strength γ is arbitrarily large.

Example 1: Consider a group of two agents switching between two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with adjacency matrices $A_1 = \left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, respectively. The self dynamics is $f(t, x_i) = Lx_i$, where x_i is scalar. The dynamics of each system are described by

$$\dot{x}_1 = Lx_1 + \gamma a_{12}(\sigma(t))(x_2 - x_1),$$
$$\dot{x}_2 = Lx_2 + \gamma a_{21}(\sigma(t))(x_1 - x_2).$$

Note that $a_{12}(\sigma(t)) = a_{21}(\sigma(t)) = 1$ or 0 for all $t \geq t_0$. Then, the relative dynamics can be written as

$$\dot{\overline{x}} = (L - 2\gamma a_{12}(\sigma(t)))\overline{x}, \quad (3)$$

where $\overline{x} = x_1 - x_2$. Let the switching signal $\sigma(t)$ be equal to 2 when $t \in [t_0 + \varrho^2 - 1, t_0 + \varrho^2)$ and equal to 1 when $t \in [t_0 + \varrho^2, t_0 + (\varrho+1)^2 - 1)$ for $\varrho = 1, 2, \ldots$. It is easy to see that Assumption 1 is satisfied. It follows that the solution of (3) is

$$\overline{x}(t) = e^{(L-2\gamma)(\rho-1)+L(\rho+1)\rho} e^{(L-2\gamma)(t-t_0-\varrho^2+1)} \overline{x}(t_0)$$

for $t \in [t_0 + \varrho^2 - 1, t_0 + \varrho^2)$, and

$$\overline{x}(t) = e^{(L-2\gamma)\rho+L(\rho-1)\rho} e^{(L-2\gamma)(t-t_0-\varrho^2)} \overline{x}(t_0)$$

for $t \in [t_0 + \varrho^2, t_0 + (\varrho+1)^2 - 1)$. Thus $\lim_{t\to\infty} \overline{x}(t) = \infty$ for any $L > 0$ and arbitrarily large γ. Hence, it is not enough that the coupling strength γ is arbitrarily large to achieve synchronization.

C. Synchronization Conditions

In this section, we shall investigate, besides the global Lipschitz condition, what additional condition is needed to guarantee synchronization of (1). We make the following global Lipschitz-like assumption regarding the self dynamics.

Assumption 3: There exists a continuous nonnegative bounded function $L(t) \geq 0$ such that $\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|$, $\forall x, y \in \mathbb{R}^p$, $\forall t \geq t_0$.

We introduce the concept of times of connectivity. To do so, for the case of switching graph $\mathcal{G}_{\sigma(t)}$, we first introduce a subsequence of the switching time sequence $\{t_\ell\}_0^\infty$ as $t_0 = T_0 < T_1 < T_2 \ldots$, where T_ℓ, $\ell = 1, 2, \ldots$ is iteratively obtained by

$$T_\ell = \inf\{t \geq T_{\ell-1} : \mathcal{G}([T_{\ell-1}, t)) \text{ is connected, } T_{\ell-1} \in \{t_l\}_0^\infty\}.$$

Let $J(t)$ denote how many jointly connected graphs can be found during $[t_0, t)$: $J(t) = \max\{\ell : t > T_\ell\}$.

We are now ready to present the sufficient synchronization condition in terms of the times of connectivity $J(t)$ and the integral of the Lipschitz gain $L(t)$.

Theorem 2: Let Assumptions 1 and 3 hold. Global asymptotic synchronization is achieved for (1) if

$$\lim_{t\to\infty} (J(t) - \frac{2}{\rho}\int_{t_0}^t L(s)\mathrm{d}s) = \infty, \tag{4}$$

where ρ is a constant depending on the network parameters, explicitly given in the proof as (21).

The proof is based on the convergence analysis of the scalar quantity

$$V(t,x) = \max_{\{i,j\}\in\mathcal{V}\times\mathcal{V}} V_{ij}(t,x), \tag{5}$$

where

$$V_{ij}(t,x) = \frac{1}{2\gamma} e^{-2\int_{t_0}^t L(s)\mathrm{d}s} \|x_i(t) - x_j(t)\|^2. \tag{6}$$

In order to prove Theorem 2, the following lemma is needed.

Lemma 2: Along solutions to (1), $D^+V(t,x) \leq 0$ for all $t \geq 0$.

Proof: Let $\mathcal{I}_1(t)\times\mathcal{I}_2(t)$ be the set containing all the node pairs that reach the maximum at time t, i.e., $\mathcal{I}_1(t)\times\mathcal{I}_2(t) = \{\{i,j\} \in \mathcal{V}\times\mathcal{V} | V_{ij}(t) = V(t)\}$. It is not hard to obtain that

$$\begin{aligned}
D^+V = & \max_{\{i,j\}\in\mathcal{I}_1\times\mathcal{I}_2} \Big\{ \frac{1}{\gamma} e^{-2\int_{t_0}^t L(s)\mathrm{d}s} (x_i - x_j)^{\mathrm{T}} \\
& \times (f(t,x_i) - f(t,x_j)) - e^{-2\int_{t_0}^t L(s)\mathrm{d}s} (x_i - x_j)^{\mathrm{T}} \\
& \times \sum_{k_1\in\mathcal{N}_i(\sigma(t))} a_{ik_1}(\sigma(t))(x_i - x_{k_1}) + e^{-2\int_{t_0}^t L(s)\mathrm{d}s} \\
& \times (x_i - x_j)^{\mathrm{T}} \sum_{k_2\in\mathcal{N}_j(\sigma(t))} a_{jk_2}(\sigma(t))(x_j - x_{k_2}) \\
& -\frac{1}{\gamma} L(t) e^{-2\int_{t_0}^t L(s)\mathrm{d}s} \|x_i - x_j\|^2 \Big\} \\
\leq & -\frac{1}{2} e^{-2\int_{t_0}^t L(s)\mathrm{d}s} \max_{\{i,j\}\in\mathcal{I}_1\times\mathcal{I}_2} \Big\{ \sum_{k_1\in\mathcal{N}_i(\sigma(t))} a_{ik_1}(\sigma(t)) \\
& \times (\|x_i - x_j\|^2 - \|x_j - x_{k_1}\|^2) \\
& + \sum_{k_2\in\mathcal{N}_j(\sigma(t))} a_{jk_2}(\sigma(t))(\|x_j - x_i\|^2 - \|x_i - x_{k_2}\|^2) \Big\} \\
\leq & -\gamma \max_{\{i,j\}\in\mathcal{I}_1\times\mathcal{I}_2} \Big\{ \sum_{k_1\in\mathcal{N}_i(\sigma(t))} a_{ik_1}(\sigma(t))(V_{ij} - V_{jk_1}) \\
& + \sum_{k_2\in\mathcal{N}_j(\sigma(t))} a_{jk_2}(\sigma(t))(V_{ij} - V_{ik_2}) \Big\} \leq 0,
\end{aligned}$$

where the equality follows from Lemma 1 and (1), the first inequality follows from Assumption 3 and the fact that $\pm ab \leq \frac{a^2+b^2}{2}$ for all $a, b \in \mathbb{R}$, and the last inequality follows from (6). Therefore, $V(t, x(t)) \leq V(t_0, x(t_0)) \triangleq V_0$. ■

Remark 1: In view of Lemma 2, we see that if the initial state $x_0 \in \Omega_\beta$ at $t = t_0$, where $\Omega_\beta = \{x \in \mathbb{R}^{pn} | V(t,x) \leq \beta\}$, then every solution of (1) lies in Ω_β. Also note that Ω_β is compact. Together with the facts that $f(t,x)$ is piecewise continuous in t and globally Lipschitz in x and $a_* \leq a_{ij}(p) \leq a^*$, for all $a_{ij}(p) \neq 0$, all $i, j \in V$, and all $p \in \mathcal{P}$, it follows that (1) has a unique solution over $t \in [t_0, \infty)$ [27, Theorem 3.3].

Proof of Theorem 2: For any node $i_1 \in \mathcal{V}$, let us define a constant $\bar{t}_1 \geq t_0$ as

$$\bar{t}_1 = \inf\{t \geq t_0 : \exists i_2, \text{ such that } \{i_1, i_2\} \in \mathcal{E}_{\sigma(t)}\}.$$

Note that $\bar{t}_1 + \tau_d \leq T_1$. Then, for $t \in [\bar{t}_1, \bar{t}_1 + \tau_d)$, it follows that

$$\begin{aligned}
\dot{V}_{i_1i_2} \leq & -\gamma \sum_{k_1\in\mathcal{N}_{i_1}(\sigma(t))\setminus\{i_2\}} a_{i_1k_1}(\sigma(t))(V_{i_1i_2} - V_{i_2k_1}) \\
& -\gamma \sum_{k_2\in\mathcal{N}_{i_2}(t)\setminus\{i_1\}} a_{i_2k_2}(\sigma(t))(V_{i_1i_2} - V_{i_1k_2}) \\
& - a_{i_1i_2}(\sigma(t))\gamma V_{i_1i_2} - a_{i_2i_1}(\sigma(t))\gamma V_{i_1i_2} \\
\leq & -\alpha\gamma(V_{i_1i_2}(t) - \frac{2(n-2)a^*}{\alpha} V_0),
\end{aligned}$$

where a_* and a^* are given in Section III and $\alpha = 2(n-2)a^* + 2a_*$. Therefore, we obtain

$$V_{i_1i_2}(\bar{t}_1 + \tau_d) \leq \beta_1 V_0, \tag{7}$$

where

$$\beta_1 = 1 - \frac{2a_*}{\alpha}(1 - e^{-\alpha\gamma\tau_d}) \in (0,1). \tag{8}$$

We next define that

$$\bar{t}_2 = \inf\{t \geq \bar{t}_1 : \exists i_3, \text{ s.t. } \{i_1, i_3\} \in \mathcal{E}_{\sigma(t)} \text{ or } \{i_2, i_3\} \in \mathcal{E}_{\sigma(t)}\}.$$

It follows from this definition that there is no edge between the set $\{i_1, i_2\}$ and the set $\mathcal{V}\backslash\{i_1, i_2\}$ for $t \in [\bar{t}_1 + \tau_d, \bar{t}_2]$. It is then not hard to see that $\dot{V}_{i_1 i_2}(t) \leq 0$ for all $t \in [\bar{t}_1 + \tau_d, \bar{t}_2]$. This together with (7) implies that

$$V_{i_1 i_2}(\bar{t}_2) \leq \beta_1 V_0. \tag{9}$$

Note that for all $t \in [\bar{t}_2, \bar{t}_2 + \tau_d]$,

$$\dot{V}_{i_1 i_2} \leq -2(n-1)a^* \gamma (V_{i_1 i_2} - V_0). \tag{10}$$

By using the above relation, (8) and (9), we obtain that for all $t \in [\bar{t}_2, \bar{t}_2 + \tau_d]$,

$$V_{i_1 i_2}(t) \leq \overline{\beta} V_0, \tag{11}$$

where

$$\overline{\beta} = 1 - (1 - e^{-\alpha\gamma\tau_d}) \frac{2a_*}{\alpha} e^{-\alpha_1 \gamma \tau_d}, \tag{12}$$

and $\alpha_1 = 2(n-1)a^*$.

We next estimate $V_{i_1 i_3}$ by considering two cases.

- Case I: $(i_1, i_3) \in \mathcal{E}_{\sigma(\bar{t}_2)}$. Following a similar analysis to that to obtain (7) for $V_{i_1 i_2}$, we obtain

$$V_{i_1 i_3}(\bar{t}_2 + \tau_d) \leq \beta_1 V_0. \tag{13}$$

- Case II: $(i_1, i_3) \notin \mathcal{E}_{\sigma(\bar{t}_2)}$. By the definition of $\bar{t}_2$, we know that $(i_2, i_3) \notin \mathcal{E}_{\sigma(\bar{t}_2)}$. It then follows that for all $t \in [\bar{t}_2, \bar{t}_2 + \tau_d)$,

$$\begin{aligned} \dot{V}_{i_1 i_3}(t) \leq & -2(n-1)a^* \gamma (V_{i_1 i_3} - V_0) \\ & - a_{i_3 i_2} \gamma (V_{i_1 i_3} - V_{i_1 i_2}). \end{aligned} \tag{14}$$

  We proceed our analysis for two subcases.

  - Case II(a): $V_{i_1 i_3}(t) > V_{i_1 i_2}(t)$ for all $t \in [\bar{t}_2, \bar{t}_2 + \tau_d)$. It then follows that

$$\dot{V}_{i_1 i_3}(t) \leq -\alpha_2 \gamma (V_{i_1 i_3} - \frac{2(n-1)a^* + a_* \overline{\beta}}{\alpha_2} V_0),$$

    where $\alpha_2 = 2(n-1)a^* + a_*$. This shows that

$$V_{i_1 i_3}(\bar{t}_2 + \tau_d) \leq \left(1 - \frac{a_*(1 - \overline{\beta})}{\alpha_2}(1 - e^{-\alpha_2 \gamma \tau_d})\right) V_0. \tag{15}$$

  - Case II(b): there exists a time $t^* \in [\bar{t}_2, \bar{t}_2 + \tau_d)$ such that

$$V_{i_1 i_3}(t^*) \leq V_{i_1 i_2}(t^*) \leq \overline{\beta} V_0. \tag{16}$$

    Applying the same analysis as we obtained (10) to (14) yields,

$$\dot{V}_{i_1 i_3}(t) \leq -2(n-1)a^* \gamma (V_{i_1 i_3} - V_0). \tag{17}$$

    By using (16), (17) and $\alpha_1 < \alpha_2$, we obtain that for all $t \in [t^*, \bar{t}_2 + \tau_d)$,

$$V_{i_1 i_3}(\bar{t}_2 + \tau_d) \leq \left(1 - e^{-\alpha_2 \gamma \tau_d}(1 - \overline{\beta})\right) V_0. \tag{18}$$

We shall find an upper bound for $V_{i_1 i_3}$ for the above cases. It follows from (12) and (18) that for Case II(b),

$$V_{i_1 i_3}(\bar{t}_2 + \tau_d) \leq \left(1 - e^{-(\alpha_1 + \alpha_2)\gamma\tau_d}(1 - e^{-\alpha\gamma\tau_d}) \frac{2a_*}{\alpha}\right) V_0.$$

Also note that from (8) and (13), for Case I, we have

$$V_{i_1 i_3}(\bar{t}_2 + \tau_d) \leq \left(1 - (1 - e^{-\alpha\gamma\tau_d}) \frac{2a_*}{\alpha}\right) V_0.$$

Therefore, the bound in Case II(b) is larger than that of Case I. By noting that $\frac{a_*(1-\overline{\beta})}{\alpha_2}(1 - e^{-\alpha_2\gamma\tau_d}) e^{-\alpha_2\gamma\tau_d} \leq \min\{e^{-\alpha_2\gamma\tau_d}(1 - \overline{\beta}), \frac{a_*(1-\overline{\beta})}{\alpha_2}(1 - e^{-\alpha_2\gamma\tau_d})\}$ and comparing (15) and (18), it is not hard to see that

$$\begin{aligned} V_{i_1 i_3}(\bar{t}_2 + \tau_d) &\leq \left(1 - \frac{a_*(1 - \overline{\beta})}{\alpha_2}(1 - e^{-\alpha_2\gamma\tau_d}) e^{-\alpha_2\gamma\tau_d}\right) V_0 \\ &= \beta_2 V_0, \end{aligned} \tag{19}$$

where

$$\beta_2 = 1 - \frac{2a_*^2}{\alpha\alpha_2}(1 - e^{-\alpha_2\gamma\tau_d}) e^{-\alpha_2\gamma\tau_d}(1 - e^{-\alpha\gamma\tau_d}) e^{-\alpha_1\gamma\tau_d} \in (0, 1).$$

Note that $\bar{t}_2 + \tau_d \leq T_2$. It follows from (11) and (19) that

$$V_{i_1 k}(\bar{t}_2 + \tau_d) \leq \beta_2 V_0, \; k \in \{i_2, i_3\}.$$

We then proceed the above analysis for other nodes $k \in \mathcal{V}\backslash\{i_1\}$. Eventually, we obtain that

$$V_{i_1 k}(\bar{t}_{n-1} + \tau_d) \leq \beta_{n-1} V_0, \; k \in \mathcal{V}\backslash\{i_1\},$$

where $\bar{t}_{n-1} + \tau_d \leq T_{n-1}$.

Let us now consider node i_2 and try to bound $V_{i_2 i_3}, \ldots, V_{i_2 i_n}$. By going through a similar analysis, we obtain that $V_{i_2 k}(\bar{t}_n + \tau_d) \leq \beta_{2n-3} V_0$, $k \in \mathcal{V}\backslash\{i_2\}$. Continuing, we eventually obtain that $V_{ij}(\bar{t}_{(n-1)n/2} + \tau_d) \leq \beta_{(n-1)n/2} V_0$, $\forall i, j \in \mathcal{V}$, where $\beta_{(n-1)n/2}$ is a constant depending on the network parameters, namely, τ_d, n, a^*, a_* and γ. Also note that $\bar{t}_{(n-1)n/2} + \tau_d \leq T_{(n-1)n/2}$. Therefore, it follows that $V(T_{(n-1)n/2+1}) \leq \beta_{(n-1)n/2} V_0$. We then have that

$$V(t, x(t)) \leq \beta_{(n-1)n/2}^{\lfloor \frac{J(t)}{(n-1)n/2+1} \rfloor} V_0 \leq \frac{1}{\beta_{(n-1)n/2}} e^{-\rho J(t)} V_0, \tag{20}$$

where $\lfloor c \rfloor$ denotes the largest integer not greater than $c \in \mathbb{R}$ and

$$\rho = \frac{1}{(n-1)n/2 + 1} \ln \frac{1}{\beta_{(n-1)n/2}} > 0. \tag{21}$$

Note that $\lim_{t\to\infty} J(t) = \infty$ based on Assumption 1. It thus follows from (5), (6) and (20) that

$$\begin{aligned} &\max_{\{i,j\} \in \mathcal{V}\times\mathcal{V}} \|x_i(t) - x_j(t)\|^2 \\ &\leq \frac{2\gamma}{\beta_{(n-1)n/2}} e^{2\int_{t_0}^t L(s)\mathrm{d}s - \rho J(t)} V(t_0, x(t_0)). \end{aligned}$$

Hence, global asymptotic synchronization is achieved provided that $\lim_{t\to\infty}(J(t) - \frac{2}{\rho}\int_{t_0}^t L(s)\mathrm{d}s) = \infty$. ■

Remark 2: A necessary condition to ensure (4) is $\lim_{t\to\infty} J(t) = \infty$. Since the dwell time assumption is imposed, the maximum of $J(t)$ is bounded because each time interval between two consecutive switching instants is connected and at least τ_d long. In such a case, $\frac{1}{\tau_d}(t - t_0) - 1 \leq J(t) \leq \frac{1}{\tau_d}(t - t_0)$, for all $t \geq t_0 + \tau_d$. This together with the synchronization condition (4) implies that $L(t)$ needs to be bounded for all $t \geq t_0$. Therefore, in view of Theorem 1, the global Lipschitz condition is necessary to achieve global asymptotic synchronization but not sufficient.

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The following two corollaries provide sufficient conditions for achieving global exponential synchronization and global polynomial synchronization, respectively, provided that the times of connectivity $J(t)$ satisfies certain conditions.

Corollary 1: Let Assumptions 1 and 3 hold. If there exist positive constants $\kappa > 0$ and $\xi \geq 0$ such that $J(t) \geq \kappa t - \xi$, for all $t \geq t_0$, then global exponential synchronization of (1) is achieved when $L(t) < \frac{\rho\kappa}{2}$.

Corollary 2: Let Assumptions 1 and 3 hold. If there exists positive constants $\kappa > 0$ and $\xi \geq 0$ such that $J(t) \geq \kappa \ln t - \xi$, for all $t \geq t_0$, then global polynomial synchronization of (1) is achieved when $L(t) < \frac{\rho\kappa}{2t}$.

Based on Corollaries 1 and 2, we can slightly revise Example 1 so that global asymptotic synchronization is achieved.

Example 2: Let the switching signal $\sigma(t)$ be equal to 2 when $t \in [t_0+\varrho-1, t_0+\varrho)$ and equal to 1 when $t \in [t_0+\varrho, t_0+\varrho+1)$, for $\varrho = 1, 3, \ldots$. Then, the solution of (3) is

$$\overline{x}(t) = e^{(L-2\gamma)(\rho-1)+L\rho} e^{(L-2\gamma)(t-t_0-\varrho+1)} \overline{x}(t_0)$$

for $t \in [t_0 + \varrho - 1, t_0 + \varrho)$, and

$$\overline{x}(t) = e^{L(\rho-1)+(L-2\gamma)\rho} e^{L(t-t_0-\varrho)} \overline{x}(t_0)$$

for $t \in [t_0 + \varrho, t_0 + \varrho + 1)$. Therefore, $\lim_{t\to\infty} \overline{x}(t) = \infty$ if $L < \frac{\gamma}{2}$. This can be easily checked by the sufficient condition in Corollary 1 by noting that $J(t) \geq \frac{t}{2}$, for all $t \geq t_0$.

Example 3: Let the switching signal $\sigma(t)$ be equal to 2 when $t \in [t_0 + \varrho^2 - 1, t_0 + \varrho^2)$ and equal to 1 when $t \in [t_0 + \varrho^2, t_0 + (\varrho + 1)^2 - 1)$ for $\varrho = 1, 2, \ldots$, but the self dynamics now be $f(t, x_i) = \frac{Lx_i}{t}$, where x_i is scalar. Then, global asymptotic synchronization is achieved if $L < 1$. This can be easily checked by the sufficient condition in Corollary 2 by noting that $J(t) \geq \sqrt{t} - 2 \geq \ln t - 2$, for all $t \geq t_0$.

V. Concluding Remarks

This paper studied synchronization of coupled nonlinear dynamical systems over time-varying graphs. We first showed that infinite joint connectivity is necessary for achieving global asymptotic synchronization. We then constructed a simple example to show that the commonly used Lipschitz condition on the self dynamics is not sufficient for achieving synchronization in the case where the graph is infinitely jointly connected. Finally, we established sufficient synchronization conditions in terms of the times of connectivity, the integral of the Lipschitz gain, and the network parameters.

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