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Critical Sampling Rate for Sampled-Data Consensus over Random Networks

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***Abstract*— In this paper, we consider the consensus problem for a network of nodes with random interactions and sampled-data control actions. Each node independently samples its neighbors in a random manner over a directed graph underlying the information exchange of different nodes. The relationship between the sampling rate and the achievement of consensus is studied. We first establish a sufficient condition, in terms of the inter-sampling interval, such that consensus in expectation, in mean square, and in almost sure sense are simultaneously achieved provided a mild connectivity assumption for the underlying graph. Necessary and sufficient conditions for mean-square consensus are derived in terms of the spectral radius of the corresponding state transition matrix. These conditions are then interpreted as the existence of a critical value on the inter-sampling interval, below which global mean-square consensus is achieved and above which the system diverges in mean-square sense for some initial states. Finally, we establish an upper bound of the inter-sampling interval, below which almost sure consensus is reached, and a lower bound, above which almost sure divergence is reached. An numerical example is given to validate the theoretical results.**

I. Introduction

In the traditional consensus algorithm, each node exchanges information with a few neighbors, typically given by their relative states, and then updates its own state according to a weighted average. In the sampled-data consensus algorithms [1]–[4], the agent dynamics are continuous and the control input is piecewise continuous. The closed-loop system is transformed into discrete-time dynamics. In this paper, we study the sampled-data consensus over random networks. We consider a network of N nodes indexed in the set $\mathrm{V} = \{1, 2, \ldots, \mathsf{N}\}$. Each node i holds a value $x_i(t) \in \mathbb{R}$ for $t \in [0, \infty)$. The evolution of $x_i(t)$ is described by

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $u_i \in \mathbb{R}$ is the control input. Each node samples data from a few neighbors at each sample instant, and then construct the control input according to a weighted average of its own state and those of its neighbors. We analyze the convergence of the consensus algorithm with a sampled-data controller over independent random networks.

Consensus over random networks has drawn much attention since communication networks are naturally random. The mean-square convergence of consensus algorithm with independent, identically distributed (i.i.d.) random graphs was studied in [5], [6]. Probabilistic consensus has also been investigated in the literature. Almost sure convergence of consensus algorithms was considered in [7]–[10]. Divergence in random consensus networks has also been considered. Almost sure divergence of consensus algorithms was studied in [11], [12]. Compared to the literature, the main contributions of this paper are summarized as follows. Necessary and sufficient conditions for mean-square consensus are derived in terms of the spectral radius of the corresponding state transition matrix. These conditions are then interpreted as critical thresholds on the inter-sampling interval and we show that they can be computed by a generalized eigenvalue problem, which can be stated as a quasi-convex optimization problem. We obtain an upper bound on the inter-sampling interval below which almost sure convergence is reached, and a lower bound on the inter-sampling interval above which almost sure divergence is reached. An extended version of the results of this paper is available as [13].

The remainder of the paper is organized as follows. Section II provides the system model and introduces various consensus notions. Their relations are also discussed. In Section III, we present necessary and/or sufficient conditions for expectation consensus, mean-square consensus, almost sure consensus/divergence. In Section IV, we give a numerical example. Some conclusions are drawn in Section V.

Notations: $\mathbb{N}$, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}_+$ are the sets of nonnegative integers, complex numbers, real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, $x \vee y$ and $x \wedge y$ stand for the maximum and minimum of x and y respectively. The set of n by n positive semi-definite (positive definite) matrices over the field $\mathbb{C}$ is denoted as $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$). For a matrix $X = [x_1\ x_2\ \cdots\ x_n] \in \mathbb{R}^{m\times n}$, $\|X\|$ represents the spectral norm of X. $\mathrm{vec}(X)$ is the vectorization of X, i.e., $\mathrm{vec}(X) := [x_1', x_2', \ldots, x_n']' \in \mathbb{R}^{mn}$. $\otimes$ denotes a Kronecker product of two matrices. If $m = n$, $\rho(X)$ denotes the spectral radius of X. For vectorization and Kronecker product, the following properties are frequently used in this work: *i*) $\mathrm{vec}(ABC) = (C'\otimes A)\mathrm{vec}(B)$; *ii*) $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$, where A, B, C and D are matrices of compatible dimensions. The notation $\sigma(\cdot)$ represents the σ-algebra generated by random variables.

II. System Model and Preliminaries

A. Random Networks and Sampling

The directed interaction graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ describes underlying information exchange. Here $\mathrm{E} \subseteq \mathrm{V} \times \mathrm{V}$ is an arc set and $(j, i) \in \mathrm{E}$ means there is a (possibly unreliable)

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communication link from node j to node i. The set of neighbors of node i in the underlying graph G is denoted as $\mathscr{N}_i := \{j : (j,i) \in \mathrm{E}\}$. The Laplacian matrix $L := [l_{ij}] \in \mathbb{R}^{\mathsf{N}\times\mathsf{N}}$ associated with G is defined as

$$l_{ij} = \begin{cases} -1, & \text{if } i \neq j \text{ and } (j,i) \in \mathrm{E} \\ \sum_{m\neq i} 1_{\{(m,i)\in\mathrm{E}\}}, & \text{if } i = j. \end{cases}$$

A directed *path* from node i_1 to node i_l is a sequence of nodes $\{i_1, \ldots, i_l\}$ such that $(i_j, i_{j+1}) \in \mathrm{E}$ for $j = 1, \ldots, l-1$. A directed tree is a directed subgraph of $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ such that every node has exactly one parent, except a single root node with no parents. Therefore, there must exist a directed path from the root to every other node. A directed spanning tree is a directed tree that contains all the nodes of G.

Let $\mathscr{G}$ be the set containing all subgraphs of G and $\{\mathrm{G}_k = (\mathrm{V}, \mathrm{E}_k)\}_{k\in\mathbb{N}}$ be a sequence of random graphs, in which by definition each G_k is a random variable taking values in $\mathscr{G}$. The Laplacian matrix $L(k) := [l_{ij}(k)] \in \mathbb{R}^{\mathsf{N}\times\mathsf{N}}$ associated with G_k is defined as

$$l_{ij}(k) = \begin{cases} -1, & \text{if } i \neq j \text{ and } (j,i) \in \mathrm{E}_k \\ \sum_{m\neq i} 1_{\{(m,i)\in\mathrm{E}_k\}}, & \text{if } i = j. \end{cases}$$

The set of neighbors of node i in $\mathrm{G_k}$ is denoted as $\mathscr{N}_i(k) := \{j : (j,i) \in \mathrm{E}_k\}$. Let the triple $(\mathscr{G}^{\mathbb{N}}, \mathcal{F}, \mathbb{P})$ denote the probability space capturing the randomness contained in the random graph sequence, where $\mathcal{F}$ is the set of all subsets of $\mathscr{G}^{\mathbb{N}}$. Define a filtration $\mathcal{F}_k = \sigma(\mathrm{G}_0, \ldots, \mathrm{G}_k)$ for $k \in \mathbb{N}$.

We define a sequence of node sampling instants as $0 = t_0 < \cdots < t_k < t_{k+1} < \ldots$ with $\tau_k = t_{k+1} - t_k$ representing the inter-sampling interval. The sampled-data consensus scheme associated with the random graph sequence $\{\mathrm{G}_k\}_{k\in\mathbb{N}}$ is given by

$$u_i(t) = \sum_{j\in\mathscr{N}_i(k)} \big[x_j(t_k) - x_i(t_k)\big], \quad t \in [t_k, t_{k+1}). \tag{2}$$

The closed-loop system can then be written in the compact form

$$x(t_{k+1}) = \big[I - \tau_k L(k)\big]x(t_k) := W(k)x(t_k) \tag{3}$$

with $W(k) := [w_{ij}(k)]$.

B. Consensus Metrics

Define $x_{\max}(t_k) := \max_{i\in\mathrm{V}} x_i(t_k)$ and $x_{\min}(t_k) := \min_{i\in\mathrm{V}} x_i(t_k)$ and the agreement measure $\mathfrak{X}(k) := x_{\max}(t_k) - x_{\min}(t_k)$. We have the following consensus metrics.

Definition 1: (i) Algorithm (3) achieves (global) consensus in expectation if for any initial state $x(t_0) \in \mathbb{R}^{\mathsf{N}}$ there holds $\lim_{k\to\infty} \mathbb{E}[\mathfrak{X}(k)] = 0$.

(ii) Algorithm (3) achieves (global) consensus in mean square if for any initial state $x(t_0) \in \mathbb{R}^{\mathsf{N}}$ there holds $\lim_{k\to\infty} \mathbb{E}[\mathfrak{X}^2(k)] = 0$.

(iii) Algorithm (3) achieves (global) consensus almost surely if for any initial state $x(t_0) \in \mathbb{R}^{\mathsf{N}}$ there holds $\mathbb{P}\left(\lim_{k\to\infty} \mathfrak{X}(k) = 0\right) = 1$.

(iv) Algorithm (3) diverges almost surely if there holds $\mathbb{P}\Big(\limsup_{k\to\infty} \mathfrak{X}(k) = \infty\Big) = 1$ for any initial state $x(t_0) \in \mathbb{R}^{\mathsf{N}}$ except for $x(t_0) \perp \mathbf{1}$.

Define the difference between the state $x(t_k)$ and its average as $d(k) := x(t_k) - \frac{1}{\mathsf{N}}\mathbf{1}\mathbf{1}'x(t_k)$. Since

$$\begin{aligned} \mathfrak{X}(k) &= x_{\max}(k) - \frac{1}{\mathsf{N}}\mathbf{1}'x(t_k) - \left[x_{\min}(t_k) - \frac{1}{\mathsf{N}}\mathbf{1}'x(t_k)\right] \\ &\leq \left|x_{\max}(t_k) - \frac{1}{\mathsf{N}}\mathbf{1}'x(t_k)\right| + \left|x_{\min}(t_k) - \frac{1}{\mathsf{N}}\mathbf{1}'x(t_k)\right| \\ &\leq \sqrt{2\sum_{i=1}^{\mathsf{N}}\left[x_i(t_k) - \frac{1}{\mathsf{N}}\mathbf{1}'x(t_k)\right]^2} = \sqrt{2}\,\|d(k)\| \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathfrak{X}(k) &= \mathsf{N}^{-1/2}\sqrt{\mathsf{N}(x_{\max}(t_k) - x_{\min}(t_k))^2} \\ &\geq \mathsf{N}^{-1/2}\sqrt{\sum_{i=1}^{\mathsf{N}}\left[x_i(t_k) - \frac{1}{\mathsf{N}}\mathbf{1}'x(t_k)\right]^2} \\ &= \mathsf{N}^{-1/2}\,\|d(k)\|, \end{aligned} \tag{5}$$

$\lim_{k\to\infty} \mathbb{E}[\mathfrak{X}^2(k)] = 0$ is equivalent to $\lim_{k\to\infty} \mathbb{E}\|d(k)\|^2 = 0$.

The following lemma suggests that if the inter-sampling interval is small enough, the consensus notations in Definition 1 are equivalent. The proof can be found in [13].

Lemma 1: Suppose $\tau_k \in \big(0, (\mathsf{N}-1)^{-1}\big]$ for all $k \in \mathbb{N}$. Then expectation consensus, mean-square consensus, and almost sure consensus are all equivalent under Algorithm (3).

Remark 1: In [14], the equivalence of L^p consensus, consensus in probability, and almost sure consensus was obtained over a random network generated by i.i.d. stochastic matrices. In Lemma 1, we show that this equivalence holds regardless of the type of random process, by which the row stochastic matrices are generated.

III. Sampling Rate and Consensus Metrics over Independent Random Networks

In this section, we investigate sampled-data consensus when the random graph G_k is obtained by each node independently sampling its neighbors in a random manner over G. Regarding the connectivity of the underlying graph G, we adopt the following assumption:

(A1) *The underlying graph* G *has a directed spanning tree.*

We also impose the following assumption.

(A2) *The random variables* $1_{\{(j,i)\in\mathrm{E}_k\}}$, $(j,i) \in \mathrm{E}$, $k \in \mathbb{N}$, *are i.i.d. Bernoulli with mean* $q > 0$.

In order to simplify the analysis, we also make the following assumption.

(A3) Let $\tau_k = \tau_*$ for all $k \in \mathbb{N}$ with $\tau_* > 0$.

When each node samples its neighbors as Assumption (A2) describes, $\{L(k)\}_{k\in\mathbb{N}}$ are i.i.d. random variables, whose randomness originates from the primitive random variables $1_{\{(j,i)\in\mathrm{E}_k\}}$'s. We denote the sample space of $L(k)$ by $\mathscr{L} := \{L^{(1)}, L^{(2)}, \ldots, L^{(\mathsf{M})}\}$ where $\mathsf{M} = |\mathscr{G}|$ and $L^{(l)} := \big[l_{ij}^{(l)}\big] \in \mathbb{R}^{\mathsf{N}\times\mathsf{N}}$ is the Laplacian matrix associated with a subgraph $\mathrm{G}^{(l)} \in \mathscr{G}$. By counting how many edges are

present in G_k and how many are absent from G_k respectively, the distribution of $L(k)$ is computed by

$$\mathbb{P}(L(k) = L^{(i)}) = q^{\mathrm{Tr}(L^{(i)})}(1-q)^{\mathrm{Tr}(L-L^{(i)})} := \pi_i \quad (6)$$

for $i = 1, \ldots, \mathsf{M}$. When $\tau_k = \tau_*$, $W(k)$ inherits the same distribution as $L(k)$ from G_k. Then, we denote $W^{(l)} =: I - \tau_* L^{(l)}$.

A. Conjunction of Various Consensus Metrics

When the inter-sampling interval is small enough (to be precise $\tau_* < (\mathsf{N}-1)^{-1}$), each node recursively updates its state as a convex combination of the previous states of its own and its neighbors. Under Assumptions (A1), (A2), and (A3), we show in the following theorem that, as long as G has a directed spanning tree, Algorithm (3) achieves consensus, simultaneously in expectation, in mean square, and in almost sure sense.

Theorem 1: Let Assumptions (A1), (A2), and (A3) hold. Then expectation consensus, mean-square consensus, and almost sure consensus are achieved under Algorithm (3) if $\tau_* \in \big(0, (\mathsf{N}-1)^{-1}\big)$.

Proof: By Lemma 1, it suffices to show that Algorithm (3) achieves consensus in expectation.

Fix a directed spanning tree G_T of graph G and a sampling time t_k. Let the root of G_T be $i_1 \in \mathrm{V}$, and define a set of nodes $\mathscr{M}_1 := \{i_1\}$. Denote $\eta := (\tau_*) \wedge (1-(\mathsf{N}-1)\tau_*)$. Then, there holds $\eta > 0$ when $\tau_* \in \big(0, (\mathsf{N}-1)^{-1}\big)$. We assume $x_{i_1}(t_k) \le 1/2(x_{\max}(t_k) + x_{\min}(t_k))$ while the other case for $x_{i_1}(t_k) > 1/2(x_{\max}(t_k) + x_{\min}(t_k))$ will be discussed later.

Choose a node $i_2 \in \mathrm{V}$ such that $i_2 \notin \mathscr{M}_1$ and $(i_1, i_2) \in \mathrm{G}_T$. Define $\mathscr{M}_2 := \mathscr{M}_1 \cup \{i_2\}$. Consider the event $\mathscr{E}_2 := \{(i_1, i_2) \in \mathrm{E}_{k+1}\}$. When $\mathscr{E}_2$ happens, $x_{i_2}(t_{k+1})$ evolves as follows:

$$\begin{aligned} x_{i_2}(t_{k+1}) &= w_{i_2 i_1}(k) x_{i_1}(t_k) + \sum_{j \neq i_1} w_{i_2 j}(k) x_j(t_k) \\ &\le \frac{1}{2} w_{i_2 i_1}(k)(x_{\min}(t_k) + x_{\max}(t_k)) \\ &\quad + (1 - w_{i_2 i_1}(k)) x_{\max}(t_k) \\ &\le \frac{1}{2}\eta x_{\min}(t_k) + (1 - \frac{1}{2}\eta) x_{\max}(t_k), \end{aligned}$$

where the last inequality holds because $\eta \le w_{i_2 i_1}(k)$. Since $\eta \le w_{i_1 i_1}(k)$, we show that $x_{i_1}(t_{k+1})$ is bounded by

$$x_{i_1}(t_{k+1}) \le \frac{1}{2}\eta x_{\min}(t_k) + (1 - \frac{1}{2}\eta) x_{\max}(t_k).$$

At time t_{k+2},

$$\begin{aligned} x_{i_2}(t_{k+2}) &= w_{i_2 i_2}(k+1) x_{i_2}(t_{k+1}) + \sum_{j \neq i_1} w_{i_2 j}(k+1) x_j(t_{k+1}) \\ &\le w_{i_2 i_2}(k+1) \left[\frac{1}{2}\eta x_{\min}(t_k) + (1 - \frac{1}{2}\eta) x_{\max}(t_k)\right] \\ &\quad + \big(1 - w_{i_2 i_2}(k+1)\big) x_{\max}(t_{k+1}) \\ &\le \frac{1}{2}\eta^2 x_{\min}(t_k) + (1 - \frac{1}{2}\eta^2) x_{\max}(t_k), \end{aligned}$$

where the last inequality is due to $x_{\max}(t_{k+1}) \le x_{\max}(t_k)$ and $\eta \le w_{i_2 i_2}(k+1)$. The same is true of node i_1, i.e., $x_{i_1}(t_{k+2}) \le \frac{1}{2}\eta^2 x_{\min}(t_k) + (1-\eta^2) x_{\max}(t_k)$. Recursively, we see that $x_{i_1}(t_{k+n}) \le \frac{1}{2}\eta^n x_{\min}(t_k) + (1 - \frac{1}{2}\eta^n) x_{\max}(t_k)$ and $x_{i_2}(t_{k+n}) \le \frac{1}{2}\eta^n x_{\min}(t_k) + (1 - \frac{1}{2}\eta^n) x_{\max}(t_k)$.

We choose nodes $i_1, \ldots, i_{\mathsf{N}}$ in sequel and accordingly define $\mathscr{M}_1, \ldots, \mathscr{M}_{\mathsf{N}}$ and $\mathscr{E}_2, \ldots, \mathscr{E}_{\mathsf{N}}$. Consider $\mathscr{E}_2, \ldots, \mathscr{E}_{\mathsf{N}}$ sequentially happen, then $x_{i_m}(t_{k+n}) \le \frac{1}{2}\eta^n x_{\min}(t_k) + (1 - \frac{1}{2}\eta^n) x_{\max}(t_k)$ holds for all $1 \le m \le \mathsf{N}$ and $n \ge \mathsf{N}-1$, which entails

$$\begin{aligned} x_{\max}(t_{k+\mathsf{N}-1}) &= \max_i x_i(t_{k+\mathsf{N}-1}) \\ &\le \frac{1}{2}\eta^{\mathsf{N}-1} x_{\min}(t_k) + (1 - \frac{1}{2}\eta^{\mathsf{N}-1}) x_{\max}(t_k). \end{aligned}$$

In this case, the relationship between $\mathfrak{X}(t_{k+\mathsf{N}-1})$ and $\mathfrak{X}(k)$ is given by

$$\begin{aligned} &\mathfrak{X}(k+\mathsf{N}-1) \\ &= x_{\max}(t_{k+\mathsf{N}-1}) - x_{\min}(t_{k+\mathsf{N}-1}) \\ &\le \frac{1}{2}\eta^{\mathsf{N}-1} x_{\min}(t_k) + (1 - \frac{1}{2}\eta^{\mathsf{N}-1}) x_{\max}(t_k) - x_{\min}(t_k) \\ &= \left(1 - \frac{1}{2}\eta^{\mathsf{N}-1}\right) \mathfrak{X}(k). \end{aligned} \quad (7)$$

If $x_{i_1}(t_k) > 1/2\,(x_{\max}(t_k) + x_{\min}(t_k))$ is assumed, a symmetric analysis leads to that, when $\mathscr{E}_2, \ldots, \mathscr{E}_{\mathsf{N}}$ sequentially occur, $x_{\min}(t_{k+\mathsf{N}-1}) \ge \frac{1}{2}\eta^{\mathsf{N}-1} x_{\max}(t_k) + (1 - \frac{1}{2}\eta^{\mathsf{N}-1}) x_{\min}(t_k)$. Then we obtain exactly the same result as (7). Therefore, the inequality (7) holds irrespective of the state of $x_{i_1}(t_k)$.

In addition, we know that probability that the events $\mathscr{E}_2, \ldots, \mathscr{E}_{\mathsf{N}}$ sequentially occur is

$$\mathbb{P}\left(1_{\cap_{i=2}^{\mathsf{N}} \mathscr{E}_i} = 1\right) = \prod_{i=2}^{\mathsf{N}} \mathbb{P}(1_{\mathscr{E}_i} = 1) \ge q^{\mathsf{N}-1}.$$

Combining all the above analysis,

$$\begin{aligned} &\mathbb{E}[\mathfrak{X}(k+\mathsf{N}-1)] \\ &\le q^{\mathsf{N}-1}\left(1 - \frac{1}{2}\eta^{\mathsf{N}-1}\right)\mathbb{E}[\mathfrak{X}(k)] + (1 - q^{\mathsf{N}-1})\mathbb{E}[\mathfrak{X}(k)] \\ &= \left(1 - \frac{1}{2}(q\eta)^{\mathsf{N}-1}\right)\mathbb{E}[\mathfrak{X}(k)]. \end{aligned} \quad (8)$$

Since $0 < q\eta < 1$, then $\lim_{k\to\infty} \mathbb{E}[\mathfrak{X}(k)] = 0$, which completes the proof. ∎

B. The Mean-square Consensus Threshold

In this part, we focus on the relationship between sampling rate and mean-square consensus. The key step to our main result is the following proposition. Due to limited space, readers can refer to [13] for the proof.

Proposition 1: Let Assumptions (A1), (A2), and (A3) hold. Then the following statements are equivalent:

(i) Algorithm (3) achieves mean-square consensus;

(ii) There holds $\rho\Big(\mathbb{E}[W(0) \otimes W(0)](J \otimes J)\Big) < 1$, where

$$J := I - \frac{1}{\mathsf{N}}\mathbf{1}\mathbf{1}'; \quad (9)$$

(iii) There exists a matrix $S > 0$ such that

$$\phi(S) := \sum_{i=1}^{\mathsf{M}} \pi_i J W^{(i)} J S J (W^{(i)})' J < S, \tag{10}$$

where π_i is defined in (6).

Theorem 2: Let Assumptions (A1), (A2), and (A3) hold. Then Algorithm (3) achieves mean-square consensus if and only if $\tau_* \leq \tau_\dagger$, where $\tau_\dagger$ is given by the following quasi-convex optimization problem:

$$\begin{aligned} &\arg\min_{\tau} \ -\tau \\ &\text{subject to } Y, Z, \Psi > 0 \qquad (11a) \\ &\qquad Y - \tau Z \geq 0, \qquad (11b) \end{aligned}$$

where Ψ is defined in (12).

Proof: Necessity: Without loss of generality, choose for $(v_1, \ldots, v_{\mathsf{N}})$ an orthonormal basis of $\mathbb{R}^{\mathsf{N}}$ with $v_1 = \frac{1}{\mathsf{N}}\mathbf{1}$. Then, any vector $0 \neq x \in \mathbb{R}^n$ can be expressed as $x = \sum_{i=1}^{\mathsf{N}} a_i v_i$ with coefficients $a_1, \ldots, a_{\mathsf{N}}$ not all 0. We have $x'\phi(S)x = \Big(\sum_{i=2}^{\mathsf{N}} a_i v_i\Big)' \phi(S) \Big(\sum_{i=2}^{\mathsf{N}} a_i v_i\Big)$ and $x'(JSJ + \mathbf{1}\mathbf{1}')x = \Big(\sum_{i=2}^{\mathsf{N}} a_i v_i\Big)' S \Big(\sum_{i=2}^{\mathsf{N}} a_i v_i\Big) + a_1^2$. Since $a_1, \ldots, a_{\mathsf{N}}$ are not all 0 and $\phi(S) < S$, there holds $\sum_{i=1}^{\mathsf{M}} \pi_i J W^{(i)} J S J (W^{(i)})' J < JSJ + \mathbf{1}\mathbf{1}'$. Finally, let $Z = S$ and $Y = \tau_* S$. By Schur complement lemma, we see that (12) and (11b) hold. In addition, the optimization is a generalized eigenvalue problem, which is quasiconvex [15].

Sufficiency: For any given $\tau_* \leq \tau_\dagger$, there always exist Y and Z such that (12) and (11b) hold. According to Schur complement lemma, (12) is equivalent to

$$JZJ + \mathbf{1}\mathbf{1}' - \sum_{i=1}^{\mathsf{M}} \pi_i (JZ - JL^{(i)}JY) Z^{-1} (JZ - JL^{(i)}JY)^* > 0,$$

which gives

$$\begin{aligned} &JZJ + \mathbf{1}\mathbf{1}' \\ &> \sum_{i=1}^{\mathsf{M}} \pi_i (JZ - JL^{(i)}JY) Z^{-1} (JZ - JL^{(i)}JY)^* \\ &\geq \sum_{i=1}^{\mathsf{M}} \pi_i \Big[\tau_* JL^{(i)}JYJ(L^{(i)})'J - JYJ(L^{(i)})'J - JL^{(i)}JYJ\Big] \\ &\quad + JZJ \\ &\geq JZJ - {\tau_*}^{-1} JYJ + {\tau_*}^{-1}\phi(Y), \end{aligned} \tag{13}$$

where the second inequality holds by substituting Z^{-1} with $\tau_* Y^{-1}$ in accordance with (11b). Therefore, it leads to $JYJ + \tau_* \mathbf{1}\mathbf{1}' > \phi(Y)$. Letting $S = JYJ + \tau_* \mathbf{1}\mathbf{1}'$, we have

$$\phi(Y) = \sum_{i=1}^{\mathsf{M}} \pi_i J W^{(i)} J (JYJ + \tau_* \mathbf{1}\mathbf{1}') J (W^{(i)})' J = \phi(S)$$

and $S > \phi(S)$. In addition, the positive definiteness of S can be seen from the following lemma.

Lemma 2: There holds $JMJ + \epsilon \mathbf{1}\mathbf{1}' > 0$ for all $M > 0$ and $\epsilon > 0$, where J is defined in (9).

Proof: Choose for $(v_1, \ldots, v_{\mathsf{N}})$ an orthonormal basis with $v_1 = \frac{1}{\mathsf{N}}\mathbf{1}$. For any nonzero vector $x = \sum_{i=1}^{\mathsf{N}} a_i v_i$, $x'(JMJ + \epsilon\mathbf{1}\mathbf{1}')x = \Big(\sum_{i=2}^{\mathsf{N}} a_i v_i\Big)' M \Big(\sum_{i=2}^{\mathsf{N}} a_i v_i\Big) + \epsilon a_1^2$. Since $a_1, \ldots, a_{\mathsf{N}}$ are not all 0 and $M > 0$, we have $x'(JMJ + \epsilon\mathbf{1}\mathbf{1}')x > 0$. ■

By Proposition 1, Algorithm (3) achieves mean-square consensus, which completes the proof. ■

C. Almost Sure Consensus/Divergence

In this part, we focus on the impact of sampling intervals on almost sure consensus and almost sure divergence of Algorithm (3).

Theorem 3: Let Assumptions (A1), (A2), and (A3) hold.

(i) If $\tau_* \leq \tau_\dagger$ with $\tau_\dagger$ given in Theorem 2, Algorithm (3) achieves almost sure consensus.

(ii) If $\tau_* > \tau_\natural$, where $\tau_\natural \in \mathbb{R}_+$ is given by

$$\tau_\natural := \inf\left\{\tau : \log\frac{2\mathsf{N}(\tau-1)}{\mathsf{N}-1} > \frac{(1-q)\log(2\mathsf{N})}{q_* q}, s(\tau) \geq 0\right\}$$

with $q_* := \min\{(1-q)^{|\mathcal{N}_i|+|\mathcal{N}_j|} : (j,i) \in \mathrm{E}\}$ and $s(\tau) := \min\Big\{\lambda_{\min}\big(\tau(L^{(i)})'JL^{(i)} - JL^{(i)} - (L^{(i)})'J\big) : L^{(i)} \in \mathscr{L}\Big\}$, Algorithm (3) diverges almost surely for any initial state $x(t_0) \in \mathbb{R}^{\mathsf{N}}$ except $x(t_0) \perp \mathbf{1}$.

Proof: We start by presenting a supporting lemma.

Lemma 3 ([16, Lemma (5.6.10)]): Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$ be given. There is a matrix norm $\|\cdot\|_\dagger$ such that $\rho(A) \leq \|A\|_\dagger \leq \rho(A) + \epsilon$.

Proof of (i): Note that

$$\mathbb{E}[\|d(k)\|^2] = \mathrm{Tr}\left(\mathbb{E}[d(k)d(k)^*]\right) \leq \mathsf{N}^{1/2} \Big\| \mathrm{vec}\left(\mathbb{E}[d(k)d(k)^*]\right)\Big\|.$$

The inequality results from the fact that, for any $X := [x_{ij}] \in \mathbb{S}_+^n$, $\|\mathrm{vec}(X)\|^2 = \sum_{i=1}^n \sum_{j=1}^n x_{ij}^2 \geq \sum_{i=1}^n x_{ii}^2 \geq \frac{1}{n}(\sum_{i=1}^n x_{ii})^2 = \frac{1}{n}(\mathrm{Tr}(X))^2$. Moreover,

$$\begin{aligned} &\mathrm{vec}\big(\mathbb{E}[d(k)d(k)^*]\big) \qquad (14) \\ &= (J\otimes J)\mathbb{E}[W(0)\otimes W(0)]\,\mathrm{vec}\left(\mathbb{E}[d(k-1)d(k-1)^*]\right) \\ &= \Big((J\otimes J)\mathbb{E}[W(0)\otimes W(0)]\Big)^k \mathrm{vec}\left(d(0)d(0)^*\right) \\ &= \Big((J\otimes J)\mathbb{E}[W(0)\otimes W(0)]\Big)^k (J\otimes J)\,\mathrm{vec}\left(x(t_0)x(t_0)^*\right) \\ &= (J\otimes J)\Big(\mathbb{E}[W(0)\otimes W(0)](J\otimes J)\Big)^k \mathrm{vec}\left(x(t_0)x(t_0)^*\right), \end{aligned}$$

If $\tau_* < \tau_\dagger$ or equivalently $\rho\Big(\mathbb{E}[W(0)\otimes W(0)](J\otimes J)\Big) < 1$ by Theorem 2, there exists a matrix norm $\|\cdot\|_\dagger$ such that $\big\|\mathbb{E}[W(0)\otimes W(0)](J\otimes J)\big\|_\dagger < \lambda < 1$ by Lemma 3. Moreover, by the equivalence of norms on a finite-dimensional vector space, for the two norms $\|\cdot\|$ and $\|\cdot\|_\dagger$, there exists a real number $c \in \mathbb{R}_+$ implying $\|X\| \leq c\|X\|_\dagger$ for all $X \in \mathbb{R}^{n\times n}$. From the forgoing observations, (14) and the

$$\Psi := \begin{bmatrix} JZJ + \mathbf{1}\mathbf{1}' & \sqrt{\pi_1}\left(JZ - JL^{(1)}JY\right) & \dots & \sqrt{\pi_{\mathsf{M}}}(JZ - JL^{(\mathsf{M})}JY) \\ * & Z & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & Z \end{bmatrix} \tag{12}$$

with $*$'s standing for entries that are the Hermitian conjugates of entries in the upper triangular part.

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submultiplicativity of a matrix norm,

$$\begin{aligned}
&\mathbb{E}[\|d(k)\|^2] \\
\le&\mathsf{N}^{1/2}\left\|(J\otimes J)\Big(\mathbb{E}[W(0)\otimes W(0)](J\otimes J)\Big)^k \mathrm{vec}\,(x(t_0)x(t_0)^*)\right\| \\
\le&\mathsf{N}^{1/2}c\left\|\Big(\mathbb{E}[W(0)\otimes W(0)](J\otimes J)\Big)^k\right\|_\dagger \left\|\mathrm{vec}\,(x(t_0)x(t_0)^*)\right\| \\
<&c\lambda^k\mathsf{N}^{1/2}\left\|\mathrm{vec}\,(x(t_0)x(t_0)^*)\right\|.
\end{aligned}$$

Therefore, $\sum_{k=0}^{\infty}\mathbb{E}[\|d(k)\|^2] < c(1-\lambda)^{-1}\mathsf{N}^{1/2}\left\|\mathrm{vec}\,(x(t_0)x(t_0)^*)\right\| < \infty$, together with Markov's inequality resulting in that $\sum_{k=0}^{\infty}\mathbb{P}\big(\|d(k)\| > \delta\big) \le (1/\delta^2)\sum_{k=0}^{\infty}\mathbb{E}[\|d(k)\|^2] < \infty$ holds for any $\delta > 0$. According to the Borel-Cantelli lemma, $\lim_{k\to\infty}\|d(k)\| = 0$ almost surely for any initial state $x(t_0) \in \mathbb{R}^{\mathsf{N}}$. Then, the result follows from (4) and (5).

Proof of (ii): The rest of the proof consists of three steps.

Step 1. First of all, observe that for all $k \in \mathbb{N}$ and $\omega \in \mathscr{G}^{\mathbb{N}}$

$$\begin{aligned}
\|d(k+1,\omega)\|^2 =& d(k,\omega)^*W(k,\omega)'JJW(k,\omega)d(k,\omega) \\
\ge& \min_{\substack{\|v\|=1,\\ v\perp\mathbf{1}}}\left\|v^*W(k,\omega)'JW(k,\omega)v\right\|\|d(k,\omega)\|^2,
\end{aligned}$$

where the inequality holds because $d(k,\omega) \perp \mathbf{1}$. If $\lambda_{\min}\Big(W(k,\omega)'JW(k,\omega) + \frac{1}{\mathsf{N}}\mathbf{1}\mathbf{1}'\Big) \ge 1$ for any $k \in \mathbb{N}$ and $\omega \in \mathscr{G}^{\mathbb{N}}$, then $\min_{\substack{\|v\|=1,\\ v\perp\mathbf{1}}}\left\|v^*W(k,\omega)'JW(k,\omega)v\right\| \ge 1$, which together with (4) and (5) implies that $\mathbb{P}\left(\mathfrak{X}^2(k) \ge \frac{\mathfrak{X}^2(k-1)}{2\mathsf{N}}\right) = 1$ holds for all $k \in \mathbb{N}$. Therefore, $\mathfrak{X}(k) > 0$ for all $k \in \mathbb{N}$ provided that $\mathfrak{X}(0) > 0$. The random variables $\xi(k) := \frac{\mathfrak{X}^2(k+1)}{\mathfrak{X}^2(k)},\ k \in \mathbb{N}.$ are well defined.

One condition guaranteeing $\lambda_{\min}\Big(W(k,\omega)'JW(k,\omega) + \frac{1}{\mathsf{N}}\mathbf{1}\mathbf{1}'\Big) \ge 1$ is established as follows. For any $L^{(i)} \in \mathscr{L}$,

$$\begin{aligned}
&\lambda_{\min}\Big(W(k,\omega)'JW(k,\omega) + \frac{1}{\mathsf{N}}\mathbf{1}\mathbf{1}'\Big) \\
=&\tau\lambda_{\min}\Big(\tau(L^{(i)})'JL^{(i)} - JL^{(i)} - (L^{(i)})'J\Big) + 1.
\end{aligned}$$

Introduce

$$\tau_\sharp = \inf\Big\{\tau : \lambda_{\min}\Big(\tau(L^{(i)})'JL^{(i)} - JL^{(i)} - (L^{(i)})'J\Big) \ge 0, \ \forall L^{(i)} \in \mathscr{L}\Big\}. \tag{15}$$

A basic but vital observation is that $\tau_\sharp < \infty$, which makes $\tau_\sharp$ well defined. According to Weyl Theorem (Theorem 4.3.1 in [16]), $\lambda_{\min}\Big(\tau(L^{(i)})'JL^{(i)} - JL^{(i)} - (L^{(i)})'J\Big) \ge 0$ whenever $\tau > \tau_\sharp$ for each $L^{(i)} \in \mathscr{L}$. Recalling that $L(k,\omega) \in \mathscr{L}$, we see that $\tau > \tau_\sharp$ guarantees $\lambda_{\min}\Big(W(k,\omega)'JW(k,\omega) + \frac{1}{\mathsf{N}}\mathbf{1}\mathbf{1}'\Big) \ge 1$ for all $k \in \mathbb{N}$ and $\omega \in \mathscr{G}^{\mathbb{N}}$.

Step 2. First, we propose the following claim, which can be proved by contradiction.

Claim. There always exist two (random) nodes $i, j \in \mathrm{V}$ at each time k such that $(j,i) \in \mathrm{E}$ and $|x_i(t_k) - x_j(t_k)| \ge \frac{1}{\mathsf{N}-1}\mathfrak{X}(k)$.

In view of this claim, for each $\omega \in \mathscr{G}^{\mathbb{N}}$, we choose two nodes $i_k(\omega), j_k(\omega) \in \mathrm{V}$ at time k such that $(j_k(\omega), i_k(\omega)) \in \mathrm{E}$ and $|x_{i_k(\omega)}(t_k) - x_{j_k(\omega)}(t_k)| \ge \frac{1}{\mathsf{N}-1}\mathfrak{X}(k,\omega)$. The dependence of the node selections on a specific sample path gives rise to a challenge in the subsequent analysis. To get rid of this, we introduce auxiliary random variables. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables defined on $\big((0,1)^{\mathbb{N}}, (\mathcal{B}(0,1))^{\mathbb{N}}, \ell\big)$, where $\mathcal{B}(0,1)$ denotes the Borel algebra on $(0,1)$, with $z_k(\zeta) = \zeta_k$ for all $\zeta \in (0,1)^{\mathbb{N}}$ and each z_k uniformly distributed in $(0,1)$. Let $z_0, z_1, \dots$ and $\mathrm{G}_0, \mathrm{G}_1, \dots$ be independent. Formally, we are allowed to define a product probability space $(\mathscr{S}, \mathcal{S}, \mu)$ where $\mathscr{S} = \mathscr{G}^{\mathbb{N}} \times (0,1)^{\mathbb{N}}$, $\mathcal{S}$ is the σ-algebra generated by $\big\{\mathscr{A}\times\mathscr{B} : \mathscr{A} \in \mathcal{F}, \mathscr{B} \in (\mathcal{B}(0,1))^{\mathbb{N}}\big\}$, and μ is the probability measure satisfying $\mu(\mathscr{A}\times\mathscr{B}) = \mathbb{P}(\mathscr{A})\ell(\mathscr{B})$. Define $\mathcal{S}_k = \sigma\left((\mathrm{G}_0, z_0), \dots, (\mathrm{G}_k, z_k)\right)$. Introduce a sequence of events associated with $i_k(\omega)$, $j_k(\omega)$ and z_k:

$$\begin{aligned}
\mathscr{D}(k) = \Big\{ &\cup_{\omega\in\mathscr{G}^{\mathbb{N}}}(\omega\times\mathscr{B}_k(\omega)) : \\
&\mathscr{N}_{i_k(\omega)}(k,\omega) = \{j_k(\omega)\}, \mathscr{N}_{j_k(\omega)}(k,\omega) = \emptyset\Big\}
\end{aligned}$$

with

$$\mathscr{B}_k(\omega) = \Big\{\zeta \in (0,1)^{\mathbb{N}} : z_k(\zeta) < q_*/(1-q)^{|\mathscr{N}_{i_k(\omega)}|+|\mathscr{N}_{j_k(\omega)}|}\Big\}.$$

Since $i_k(\omega), j_k(\omega) \in \mathcal{F}_{k-1}$, one can verify $\mathscr{D}(k) \in \mathcal{S}_k$. If $\tau_* > 1$, for all $(\omega,\zeta) \in \mathscr{D}(k)$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\mathfrak{X}(k+1,\omega) \ge& |x_{i_k(\omega)}(t_{k+1}) - x_{j_k(\omega)}(t_{k+1})| \\
=&(\tau_*-1)|x_{i_k(\omega)}(t_k) - x_{j_k(\omega)}(t_k)| \\
\ge&\frac{\tau_*-1}{\mathsf{N}-1}\mathfrak{X}(k,\omega).
\end{aligned} \tag{16}$$

Direct calculation yields $\mu((\omega,\zeta) \in \mathscr{D}(k)) = \frac{q_*q}{1-q}$.

Step 3. Now we define random variable $\mathfrak{M}(k) = \frac{\tau_*-1}{\mathsf{N}-1}$ if $(\omega,\zeta) \in \mathscr{D}(k)$ and $\frac{1}{2\mathsf{N}}$ otherwise, which together with (16) leads to $\mu\left(\xi_k = \frac{\mathfrak{X}^2(k+1)}{\mathfrak{X}^2(k)} \ge \mathfrak{M}^2(k)\right) = 1$. Therefore, $\mu\left(\prod_{k=0}^t \xi_k = \frac{\mathfrak{X}^2(t+1)}{\mathfrak{X}^2(0)} \ge \prod_{k=0}^t \mathfrak{M}^2(k)\right) = 1$, which gives

$$\mu\left(\log\mathfrak{X}(t+1) - \log\mathfrak{X}(0) \ge \sum_{k=0}^t \log\mathfrak{M}(k)\right) = 1. \tag{17}$$

Since each node samples the neighbors independently, where the "independence" is in both spatial and temporal sense (Assumption (A2)), therefore, for any $k \in \mathbb{N}$,

$$\mu\left((\omega,\zeta) \in \mathscr{D}(k) \mid \mathcal{S}_{k-1}\right) = \frac{p_* p}{1-p} = \mu\left((\omega,\zeta) \in \mathscr{D}(k)\right),$$

indicating that $\mathfrak{M}(k)$'s are independent random variables for $\mathscr{D}(0), \ldots, \mathscr{D}(k-1) \in \mathcal{S}_{k-1}$. By induction, we eventually have $\{\mathfrak{M}(k)\}_{k\in\mathbb{N}}$ are i.i.d. with the mean computed as

$$\mathbb{E}[\log \mathfrak{M}(k)] = \frac{q_* q}{1-q} \log \frac{\tau_* - 1}{\mathsf{N}-1} + (1 - \frac{q_* q}{1-q}) \log \frac{1}{2\mathsf{N}} := m(\tau_*).$$

Additionally, as $\mathfrak{M}(k)$'s have uniformly bounded covariances, Kolmogorov's strong law of large numbers [17] shows that $\mu\left(\lim_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t} \log \mathfrak{M}(k) = m(\tau_*)\right) = 1$,which together with (17) implies that, when $m(\tau_*) > 0$, $\mathbb{P}\left(\liminf_{k\to\infty} \mathfrak{X}(k) = \infty\right) = 1$. Notice that $m(\tau_*)$ is increasing in τ_*. Defining $\tau_\flat = \inf\{\tau : m(\tau_*) > 0\}$ and choosing $\tau_* > \tau_\sharp \vee \tau_\flat := \tau_\natural$, the conclusion follows. ■

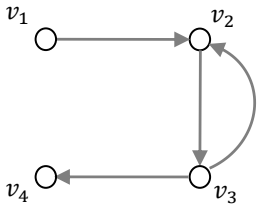

Fig. 1: The underlying graph G consisting of four nodes.

IV. Simulation Results

In this section, we use a numerical example to illustrate the existence of the threshold on τ_*, which decides the mean-square convergence or divergence (see Theorem 2). We consider a network consisting of $\mathsf{N} = 4$ nodes indexed by $\mathrm{V} = \{v_1, v_2, v_3, v_4\}$. Let $\mathrm{E} = \{(v_1,v_2),(v_2,v_3),(v_3,v_2),(v_3,v_4)\}$. The underlying graph $\mathrm{G} = (\mathrm{V},\mathrm{E})$ is illustrated in Figure 1. It has a directed spanning tree. The random variables $1_{\{(j,i)\in \mathrm{E}_k\}}$, $(j,i) \in \mathrm{E}$ and $k \in \mathbb{N}$, are i.i.d. Bernoulli ones with $\mathbb{P}\big((j,i) \in \mathrm{E}_k\big) = 0.5$. We choose a uniform inter-sampling interval, i.e., $\tau_k = \tau_*$ for all $k \in \mathbb{N}$. According to Theorem 2, we compute that Algorithm (3) achieves consensus in mean square if and only if $\tau_* \leq 1.07$. We next illustrate this conclusion using simulations. Choose $x(t_0) = [5\ 2\ 1\ 1]'$, run 10^6 Monte Carlo simulations, and then use the average as an approximation of $\mathbb{E}[\mathfrak{X}^2(k)]$. Figure 2 illustrates the convergence/divergence behaviors of $\mathbb{E}[\mathfrak{X}^2(k)]$ for different τ_*'s, validating the result of Theorem 2.

V. Conclusions

In this paper, we have investigated the relationship between sampling rate and sampled-data consensus over independent random networks. Three types of consensus were shown to be simultaneously achieved if the underlying graph contains a directed spanning tree and the inter-sampling interval is small enough. Then, necessary and sufficient conditions for mean-square consensus were derived in terms of the inter-sampling interval. Sufficient conditions for almost sure convergence/divergence were also provided, respectively.

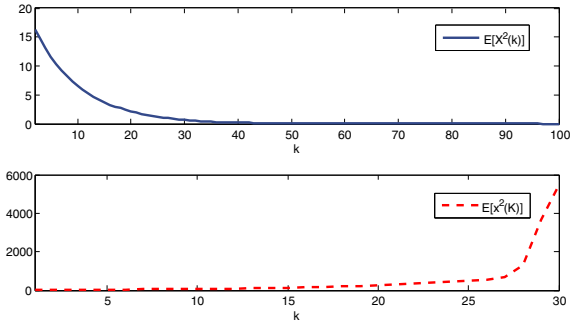

Fig. 2: The evolutions of $\mathbb{E}[\mathfrak{X}^2(k)]$ under different sampling rates over an independent random network with $q = 0.5$. In the upper figure, $\lim_{k\to\infty} \mathbb{E}[\mathfrak{X}^2(k)] = 0$ when $\tau_* = 1$. In the bottom figure, $\lim_{k\to\infty} \mathbb{E}[\mathfrak{X}^2(k)] = \infty$ when $\tau_* = 1.14$.

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