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Convergence Rate of the Modified DeGroot-Friedkin Model with Doubly Stochastic Relative Interaction Matrices

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***Abstract*— In a recent paper [1], a modified DeGroot-Friedkin model was proposed to study the evolution of the social-confidence levels of individuals in a reflected appraisal mechanism in which a network of n individuals consecutively discuss a sequence of issues. The individuals update their self-confidence levels on one issue in finite time steps, via communicating with their neighbors, instead of waiting until the discussion on the previous issue reaches a consensus, while the neighbor relationships are described by a static relative interaction matrix. This paper studies the same modified DeGroot-Friedkin model, but with time-varying interactions which are characterized by a sequence of doubly stochastic matrices. It is shown that, under appropriate assumptions, the n individuals' self-confidence levels will all converge to $\frac{1}{n}$ exponentially fast. An explicit expression of the convergence rate is provided.**

I. Introduction

Opinion dynamics have a long history in social sciences, dating back to the classical DeGroot model [2], which is probably the most well-known model for opinion dynamics and closely related to consensus processes [3], [4]. Recently, there has been considerable attention in understanding how an individual's opinion evolves over time, and various new models have been proposed for opinion dynamics. Notable among them are the Friedkin-Johnsen model [5], [6], the Hegselmann-Krause model [7]–[9], the DeGroot-Friedkin model [1], [10]–[12], and the Altafini model [13]–[17].

The DeGroot-Friedkin model [10] considers the situations when a group of individuals discusses a sequence of issues, and studies the evolution of the self-confidence levels of individuals, i.e., how confident an individual is for its opinions on the sequence of issues. The evolution consists of two stages, where in the first stage, individuals update their opinions for a particular issue according to the classical DeGroot model, and in the second stage, the self-confidence levels for the next issue are governed by the reflected appraisal mechanism proposed in [18]. The DeGroot-Friedkin model provides a nice interpretation for the evolution of self-confidence levels, though the model is somewhat centralized in the sense that each individual needs to compute the normalized left eigenvector of the dominant eigenvalue of a network-wide update matrix, and requires sufficiently long time discussion for each issue in the sequence.

Recently, a modified DeGroot-Friedkin model was proposed in [1] which provides a time-efficient, distributed implementation of the original DeGroot-Friedkin model. The modified model allows individuals to update their self-confidence levels by only interacting with their neighbors, and the update of self-confidence levels to take place in finite time without waiting for the opinion process to reach a consensus on any particular issue. It has been shown in [1] that for the case when the relative interaction matrix is static and doubly stochastic, all individuals' self-confidence levels converge to $\frac{1}{n}$, where n is the number of individuals in the network, which implies that a democratic state is reached. It is worth noting that the analysis of the modified DeGroot-Friedkin model with a fixed stochastic relative interaction matrix is still an open problem.

In a realistic social network, the interaction among the individuals often changes over time. With this in mind, this paper aims to study the modified DeGroot-Friedkin model with time-varying interactions which are characterized by a sequence of doubly stochastic relative interaction matrices. We show that, under a certain joint connectivity condition and a uniform lower boundedness condition on the nonzero elements of the interaction matrices, the convergence of the system to a democratic state occurs even if the relative interaction matrix is not fixed. More importantly, we prove that the convergence is exponentially fast, and provide an explicit expression of the convergence rate.

The remainder of this paper is organized as follows. Some notations are introduced in Section I-A. In Section II, the modified DeGroot-Friedkin model is introduced. The main results of the paper are presented in Section III, whose analysis and proofs are given in Section IV. The paper ends with some concluding remarks in Section V.

A. Notations

For a fixed positive integer n, we use $\mathcal{V}$ to denote the set $\{1, \dots, n\}$. We use Δ_n to denote the simplex $\{x \in \mathbb{R}^n : x_i \geq 0, i \in \mathcal{V}, \sum_{i=1}^n x_i = 1\}$. For each $i \in \mathcal{V}$, we use e_i to denote the vector in $\mathbb{R}^n$ whose ith element equals 1 and all the other elements equal 0. We use I to denote the identity matrix with an appropriate dimension and use $\mathbf{1}$ to denote the all-one vector with an appropriate dimension. A row-stochastic matrix is a nonnegative matrix with each row sum equal to 1. A matrix is column-stochastic if its transpose is a row-stochastic matrix. A matrix is called doubly stochastic

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if it is both row-stochastic and column-stochastic. For any two real vectors $x, y \in \mathbb{R}^n$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in \mathcal{V}$ and $x > y$ if $x_i > y_i$ for all $i \in \mathcal{V}$. We use $\operatorname{diag}(x)$ to denote the diagonal matrix with the ith entry being x_i. For a scalar $a \in \mathbb{R}$, we use $\lfloor a \rfloor$ to denote the largest integer that is no larger than a. For a directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, if $(i, j) \in \mathcal{E}$, the vertex j is called an out-neighbor of vertex i and i is an in-neighbor of j. The out-neighbor set of i is denoted by $\mathcal{N}_i^-(i) = \{j | (i, j) \in \mathcal{E}\}$ and the in-neighbor set of i is denoted by $\mathcal{N}_i^+(i) = \{j | (j, i) \in \mathcal{E}\}$.

II. The Modified DeGroot-Friedkin Model

Consider a network consisting of n individuals labeled 1 through n. The individuals in this network discuss a sequence of issues denoted by $s \in \{0, 1, 2, \dots\}$. The interactions among the n individuals are characterized by a directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$. A directed edge (i, j) is in $\mathbb{G}$ if individual i can communicate with j and will take j's opinion into account when updating her opinion. For each issue s, each individual updates her opinion according to the DeGroot model [2] as

$$y_i(s, t+1) = w_{ii}(s) y_i(s, t) + \sum_{j=1, j \neq i}^{n} w_{ij}(s) y_j(s, t), \quad (1)$$

or in a matrix form

$$y(s, t+1) = W(s) y(s, t), \quad (2)$$

where $y_i(s, t) \in \mathbb{R}$ is the opinion of individual i at time t about issue s, $y(s, t) = [y_1(s, t), \dots, y_n(s, t)]^T$, and $W(s) = (w_{ij}(s))_{n \times n}$ is called the influence matrix and is row-stochastic. The diagonal element $w_{ii}(s)$ of $W(s)$ is the self-weight of individual i assigned to her own opinion at the previous step t on issue s and $w_{ij}(s)$ is the weight individual i accords to the opinion of individual j. $w_{ij}(s) = 0$ if (i, j) is not an edge of the graph $\mathbb{G}$.

In [10], the self-weight $w_{ii}(s)$ is simply denoted by $x_i(s)$ and the off-diagonal elements $w_{ij}(s)$ are decomposed as $w_{ij}(s) = (1 - x_i(s)) c_{ij}$. The matrix $C = (c_{ij})_{n \times n}$ is called the relative interaction matrix, in which $c_{ii} = 0$ for all $i \in \mathcal{V}$ and c_{ij} is the relative weight that individual i assigns to her out-neighbor j. It should be clear that C is a row-stochastic matrix since the total weights i assigns to her out-neighbors are $1 - x_i(s)$ on issue s. Then, the influence matrix can be written as

$$W(x(s)) = \operatorname{diag}(x(s)) + (I - \operatorname{diag}(x(s)) C, \quad (3)$$

and the opinion dynamics (1) becomes

$$y_i(s, t+1) = x_i(s) y_i(s, t) + \sum_{j=1, j \neq i}^{n} (1 - x_i(s)) c_{ij} y_j(s, t), \quad (4)$$

or in a matrix form

$$y(s, t+1) = W(x(s)) y(s, t). \quad (5)$$

Suppose that the relative interaction matrix C is irreducible, i.e., the interaction graph $\mathbb{G}$ is strongly connected. Then, the matrix $W(x(s))$ is also irreducible from (3). By the Perron-Frobenius Theorem, there exists a unique positive normalized vector, denoted by $u(x(s))$, such that $\lim_{t \to \infty} W(x(s))^t = \mathbf{1} u(x(s))^T$. The vector $u(x(s))$ is the normalized left eigenvector of $W(x(s))$ corresponding to the eigenvalue 1 and is called the dominant left eigenvector. Then, the limit of the opinion in (5) is given by

$$\lim_{t \to \infty} y(s, t) = \lim_{t \to \infty} W(x(s))^t y(s, 0) = u(x(s))^T y(s, 0) \mathbf{1}. \quad (6)$$

The consensus opinion $u(x(s))^T y(s, 0)$ of the network on issue s is a convex combination of the initial opinions of the individuals. The coefficient $u_i(x(s))$ is regarded as the relative control of this individual over the final outcome on issue s and is referred to as the social power of the ith individual [10].

In [10], Jia et al. proposed the DeGroot-Friedkin model to study the social power evolution in the reflected appraisal mechanism. In this mechanism, each individual's self-weight is updated as $x_i(s+1) = u_i(x(s))$ after the discussion on issue s has reached a consensus, i.e., it is set to her relative control over the final outcome on the previous issue. The DeGroot-Friedkin model can be described by

$$x(s+1) = u(x(s)), \quad (7)$$

In the above model (7), the convergence of the opinion $y(s, t)$ on issue s is asymptotic and may take finitely many or infinite steps for the individuals to reach a consensus. However, individual self-weights may update before the opinion consensus has been achieved on the previous issue. It is therefore desirable to know the self-weight for the next issue without waiting for the convergence of the opinions. A distributed way to estimate the social power of each individual has been proposed in [10] and based on this a modified DeGroot-Friedkin model has been proposed in [1].

Let $p_i(s, t)$ be the perceived social power of individual i for issue s at time t. Assume that each individual knows the interpersonal weight her neighbors accord to her. Then, the perceived social power $p_i(s, t)$ can be updated in a distributive way as

$$p_i(s, t+1) = w_{ii}(s) p_i(s, t) + \sum_{j=1, j \neq i}^{n} w_{ji}(s) p_j(s, t), \quad (8)$$

or equivalently,

$$p_i(s, t+1) = x_i(s) p_i(s, t) + \sum_{j=1, j \neq i}^{n} (1 - x_j(s)) c_{ji} p_j(s, t). \quad (9)$$

Suppose that the self-weights of individuals on issue s are updated at time $t = T$. Then,

$$x_i(s+1) = p_i(s, T).$$

In [1], the case when $T = 1$ was considered and the update equation of $x_i(s)$ was then given by

$$x_i(s+1) = x_i^2(s) + \sum_{j=1, j \neq i}^{n} (1 - x_j(s)) x_j(s) c_{ji}. \quad (10)$$

In this paper, we consider system (10) and in addition we consider the case when the interaction graph $\mathbb{G}$ changes over issues. This results in a time-varying relative interaction matrix.

Note that since $T = 1$, individuals discuss a new issue at each time step. As a result, the index s, which was standing for an issue, now can also be regarded as an index of time steps.

We use $\mathbb{G}(s)$ and $C(s)$ to denote the dependence on time of the interaction graph and the relative interaction matrix, respectively. The model we consider in this paper is written as

$$x_i(s+1) = x_i^2(s) + \sum_{j=1, j\neq i}^{n} (1 - x_j(s))x_j(s)c_{ji}(s), \quad (11)$$

or in matrix form

$$x(s+1) = X(s)x(s) + C(s)^T(I - X(s))x(s), \quad (12)$$

where $x(s) = [x_1(s), \ldots, x_n(s)]^T$ and $X(s) = \text{diag}(x(s))$ is a diagonal matrix. Let $A(s) = X(s) + C(s)^T(I - X(s))$. The system (12) can be written as

$$x(s+1) = A(s)x(s). \quad (13)$$

We will focus on the case when $C(s)$ is a doubly stochastic matrix for each s, and study the limiting behavior of the system. The results and analysis will reveal how the self-confidence levels of the individuals evolve in a network as the discussion on issues proceeds, and how much social power each individual in the network will ultimately gain.

III. Main Results

To state the main results of this paper, we need the following assumptions and definitions.

Assumption 1: For each s, $C(s)$ is a doubly stochastic matrix with diagonal elements equal to zero, and there exists a constant $\gamma > 0$ such that for any $i, j \in \mathcal{V}$, if $c_{ij}(s) > 0$, then $c_{ij}(s) \geq \gamma$.

Assumption 2: There exists an integer $B \geq 1$ such that the union graph $\cup_{s=kB}^{(k+1)B-1}\mathbb{G}(s)$ is strongly connected for all nonnegative integers k.

Let $h(s) = \min_{i\in\mathcal{V}} x_i(s)$, $H(s) = \max_{i\in\mathcal{V}} x_i(s)$, and $V(s) = H(s) - h(s)$. Then, $V(s)$ is a measure of the difference between the extreme values of the self-confidence levels in the network. The sequences $h(s)$ and $H(s)$ have the following property.

Lemma 1: [1] Suppose that $C(s)$, $s \geq 0$, are doubly stochastic matrices with diagonal elements equal to zero. Then, $h(s)$ is a nondecreasing sequence and $H(s)$ is a nonincreasing sequence.

From the above lemma, $V(s)$ is a nonincreasing function of s, and $V(s) = 0$ if and only if $H(s) = h(s)$. Since $\mathbf{1}^T x(s) = \mathbf{1}^T x(0)$ for all s, if $x(0) \in \Delta_n$, then $V(s) = 0$ implies that $x(s) = \frac{1}{n}\mathbf{1}$. The following theorem describes the limiting behavior of the system (12).

Theorem 1: Suppose that $n \geq 3$ and Assumptions 1 and 2 hold. If $x(0) \in \Delta_n \backslash \{e_1, \ldots, e_n\}$ and $x(0)$ has m nonzero entries, then $\lim_{s\to\infty} x_i(s) = \frac{1}{n}$ for all $i \in \mathcal{V}$, and the convergence is exponentially fast:

$$V(s) \leq \left(1 - (\gamma\alpha(2\alpha)^{B-1})^{n-1}\right)^{\left\lfloor \frac{s-(n-m)B}{(n-1)B} \right\rfloor} V(0), \quad (14)$$

for all integers $s \geq 0$, where $\alpha = h((n-m)B) > 0$.

The proof of the theorem is given in the next section.

Remark 1: Theorem 1 shows that when the relative interaction matrix $C(s)$ is doubly stochastic, then under some connectivity condition and uniform lower boundedness condition, the self-confidence levels of the individuals in the network reach the same value at $\frac{1}{n}$ exponentially fast, which corresponds to a democratic state [10]. Therefore, the individuals have equal weights on deciding the final outcome on the issue eventually.

Remark 2: In [1], the asymptotic convergence to $\frac{1}{n}\mathbf{1}$ has been established for the case of a static relative interaction matrix. The inequality (14) in Theorem 1 explicitly establishes the convergence rate of the system (12) to $\frac{1}{n}\mathbf{1}$ via quantitatively analyzing the evolution of $V(s)$.

IV. Analysis

In this section, we provide a complete analysis of the system (12) and the proofs of the main results. We begin with several fundamental properties of the system (12) that have been established in [1]. Some of them are still valid for the case of a time-varying relative interaction matrix.

Lemma 2: If $C(s)$ are row-stochastic matrices for all $s \geq 0$, then the average of self-confidence levels is preserved, i.e., $\mathbf{1}^T x(s) = \mathbf{1}^T x(s+1)$.

Lemma 3: If $C(s)$ are row-stochastic matrices for all $s \geq 0$, then e_i, $i \in \mathcal{V}$, are equilibria of system (12). In addition, if $C(s)$ are doubly stochastic, the vector $\frac{1}{n}\mathbf{1}$ is a nontrivial equilibrium of system (12).

Proofs of Lemmas 2 and 3 proceed using arguments similar to those as in proofs of Lemma 1, Lemma 2, and Theorem 2 in [1], and thus are omitted due to space limitations, and will be included in an expanded version of the paper.

Lemma 4: If $C(s)$ are row-stochastic matrices for all $s \geq 0$, then the following statements of system (12) hold:

1) If $x_i(s) > 0$, then $a_{ii}(s) > 0$ and $x_i(s+1) > 0$; if $x_i(s) = 0$, then $a_{ii}(s) = 0$.
2) If $x(s) > 0$, then $x(s+1) > 0$.
3) If $C(s)$ is irreducible, then $A(s)$ is irreducible.

Item 2) of Lemma 4 proves that when the initial state $x(0)$ is in the interior of the simplex Δ_n, the state $x(s)$ will remain in the interior for all s. The next lemma says that whenever the initial state $x(0) \neq e_i$, $i \in \mathcal{V}$, the state will enter the interior of the simplex Δ_n in finite steps.

Lemma 5: Assume that Assumptions 1 and 2 hold. If $x(0) \in \Delta_n \backslash \{e_1, \ldots, e_n\}$ and has m nonzero entries, then $x((n-m)B) > 0$.

We now present a key lemma regarding the property of the function $f(x) = x - x^2$, which will be very useful in our later discussion.

Lemma 6: Suppose that $\alpha \in \mathbb{R}^n$, $\alpha \geq 0$, $\sum_{k=1}^n \alpha_k = 1$, and $x \in \mathbb{R}^n$, $x \geq 0$, $\sum_{k=1}^n x_k = 1$. Then, there exists a constant

$$v \in \mathcal{I} = \left[\min_{\substack{k\in\mathcal{V}\\ \alpha_k\neq 0}} \{x_k\}, \max_{\substack{k\in\mathcal{V}\\ \alpha_k\neq 0}} \{x_k\} \right]$$

such that $v \leq \sum_{k=1}^{n} \alpha_k x_k$ and

$$v - v^2 = \sum_{k=1}^{n} \alpha_k (x_k - x_k^2). \tag{15}$$

The proofs of Lemmas 4, 5 and 6 are omitted due to space limitations and will be included in an expanded version of the paper.

In [1], the stability of the equilibrium $\frac{1}{n}\mathbf{1}$ has been proven via checking the evolution of $H(s)$ qualitatively for a fixed relative interaction matrix C. In what follows, we will look at the evolution of $H(s)$ quantitatively and provide an explicit upper bound for $H(s)$, and thus the convergence speed can be derived in terms of the decrease of $V(s)$.

Lemma 7: Suppose $n \geq 3$. Assume that Assumptions 1 and 2 hold. If $x(T) > 0$ and $x(T) \in \Delta_n$, then the following inequality holds:

$$V(T + (n-1)B) \leq \Big(1 - (\gamma h(T)(2h(T))^{B-1})^{n-1}\Big) V(T). \tag{16}$$

Proof: Note that $h(s)$ is a nondecreasing sequence and $H(s)$ is a nonincreasing sequence from Lemma 1. We bound $H(T+(n-1)B)$ from above so that the inequality (16) can be established. We divide the proof into four steps.

Step 1. Let $\mathcal{V}_0 = \{i \in \mathcal{V} | x_i(T) = h(T)\}$. For any $i_0 \in \mathcal{V}_0$,

$$x_{i_0}(T+1) = x_{i_0}^2(T) + \sum_{j=1, j\neq i}^{n} c_{ji_0}(T)(1 - x_j(T))x_j(T).$$

Since $C(T)$ is a doubly stochastic matrix, $\sum_{j=1}^{n} c_{ji}(T) = 1$ for all i. The vector $x(T)$ satisfies that $x(T) > 0$ and $\sum_{j=1}^{n} x_j(T) = 1$. Applying Lemma 6, one has that there exists a constant

$$v_{i_0}(T) \in \left[\min_{j\in\mathcal{N}_{i_0}^{+}(T)} \{x_j(T)\}, \max_{j\in\mathcal{N}_{i_0}^{+}(T)} \{x_j(T)\} \right] \tag{17}$$

such that

$$\sum_{j=1}^{n} c_{ji_0}(T)(1 - x_j(T))x_j(T) = v_{i_0}(T) - v_{i_0}^2(T).$$

Hence,

$$\begin{aligned} x_{i_0}(T+1) &= x_{i_0}^2(T) + v_{i_0}(T) - v_{i_0}^2(T) \\ &= \Big(x_{i_0}(T) + v_{i_0}(T)\Big)x_{i_0}(T) \\ &\quad + \Big(1 - x_{i_0}(T) - v_{i_0}(T)\Big)v_{i_0}(T). \end{aligned} \tag{18}$$

From (17) and $\sum_{i=1}^{n} x_i(T) = 1$, one has that $h(T) \leq v_{i_0}(T) \leq H(T)$ and

$$0 \leq x_{i_0}(T) + v_{i_0}(T) \leq x_{i_0}(T) + \max_{j\in\mathcal{N}_{i_0}^{+}(T)} \{x_j(T)\} \leq 1.$$

Therefore, $x_{i_0}(T+1)$ is a convex combination of $x_{i_0}(T)$ and $v_{i_0}(T)$. In view of the fact that $x_{i_0}(T) = h(T)$, one has

$$x_{i_0}(T) + v_{i_0}(T) \geq 2h(T). \tag{19}$$

Combining with (18) yields

$$\begin{aligned} x_{i_0}(T+1) &\leq \Big(x_{i_0}(T) + v_{i_0}(T)\Big)h(T) \\ &\quad + \Big(1 - x_{i_0}(T) - v_{i_0}(T)\Big)H(T) \\ &\leq 2h(T)h(T) + \Big(1 - 2h(T)\Big)H(T). \end{aligned}$$

One further calculates $x_{i_0}(T+2)$ as

$$\begin{aligned} x_{i_0}(T+2) &= x_{i_0}^2(T+1) \\ &\quad + \sum_{j=1}^{n} c_{ji_0}(T+1)(1 - x_j(T+1))x_j(T+1) \\ &= \Big(x_{i_0}(T+1) + v_{i_0}(T+1)\Big)x_{i_0}(T+1) \\ &\quad + \Big(1 - x_{i_0}(T+1) - v_{i_0}(T+1)\Big)v_{i_0}(T+1), \end{aligned}$$

for some

$$v_{i_0}(T+1) \in \left[\min_{j\in\mathcal{N}_{i_0}^{+}(T+1)} \{x_j(T+1)\}, \max_{j\in\mathcal{N}_{i_0}^{+}(T+1)} \{x_j(T+1)\} \right].$$

Note that $h(s)$ is a nondecreasing sequence and $H(s)$ is a nonincreasing sequence from Lemma 1. Similar to (19), one has that

$$2h(T) \leq x_{i_0}(T+1) + v_{i_0}(T+1) \leq h(T) + H(T).$$

It follows that

$$\begin{aligned} x_{i_0}(T+2) &\leq 2h(T)\Big(2h(T)h(T) + \Big(1 - 2h(T)\Big)H(T)\Big) \\ &\quad + \Big(1 - 2h(T)\Big)H(T) \\ &= \Big(2h(T)\Big)^2 h(T) + \Big(1 - (2h(T))^2\Big)H(T). \end{aligned}$$

Recursively, we obtain that

$$x_{i_0}(T+k) \leq \Big(2h(T)\Big)^k h(T) + \Big(1 - (2h(T))^k\Big)H(T) \tag{20}$$

for all nonnegative integers k.

Step 2. Define

$$\begin{aligned} t_1 &= \min\{s \geq T | c_{ji}(s) > 0, \text{for some } j \in \mathcal{V}_0,\ i \in \mathcal{V}\backslash\mathcal{V}_0\}, \\ \mathcal{V}_1 &= \{i \in \mathcal{V}\backslash\mathcal{V}_0 | c_{ji}(t_1) > 0, \text{for some } j \in \mathcal{V}_0\}. \end{aligned}$$

From Assumption 2, t_1 is well defined and satisfies that $T \leq t_1 \leq T + B - 1$. For any $i \in \mathcal{V}_0$, the relation (20) implies that

$$x_i(t_1) \leq \Big(2h(T)\Big)^{t_1 - T} h(T) + \Big(1 - (2h(T))^{t_1 - T}\Big)H(T). \tag{21}$$

For any $i_1 \in \mathcal{V}_1$, it follows from Lemma 6 that

$$\begin{aligned} x_{i_1}(t_1+1) &= x_{i_1}^2(t_1) + \sum_{j=1, j\neq i}^{n} c_{ji_1}(t_1)(1-x_j(t_1))x_j(t_1) \\ &= \Big(x_{i_1}(t_1)+v_{i_1}(t_1)\Big)x_{i_1}(t_1) \\ &\quad + \Big(1-x_{i_1}(t_1)-v_{i_1}(t_1)\Big)v_{i_1}(t_1) \\ &\leq \Big(x_{i_1}(t_1)+v_{i_1}(t_1)\Big)H(T) \\ &\quad + \Big(1-x_{i_1}(t_1)-v_{i_1}(t_1)\Big)v_{i_1}(t_1), \end{aligned} \tag{22}$$

where

$$v_{i_1}(t_1) \in \left[\min_{j\in\mathcal{N}_{i_1}^+(t_1)}\{x_j(t_1)\}, \max_{j\in\mathcal{N}_{i_1}^+(t_1)}\{x_j(t_1)\}\right]$$

and satisfies that

$$v_{i_1}(t_1) \leq \sum_{k\in\mathcal{N}_{i_1}^+(t_1)} c_{ki_1}(t_1)x_k(t_1). \tag{23}$$

From the definition of $\mathcal{V}_1$, there exists some $i_0 \in \mathcal{V}_0$ such that $c_{i_0 i_1}(t_1) > 0$. It then follows from (23) that

$$\begin{aligned} v_{i_1}(t_1) &\leq \sum_{k\in\mathcal{N}_{i_1}^+(t_1)} c_{ki_1}(t_1)x_k(t_1) \\ &\leq c_{i_0i_1}(t_1)x_{i_0}(t_1) + (1-c_{i_0i_1}(t_1))H(t_1) \\ &\leq \gamma\Big(\Big(2h(T)\Big)^{t_1-T}h(T) \\ &\quad + \Big(1-(2h(T))^{t_1-T}\Big)H(T)\Big) + (1-\gamma)H(T) \\ &= \gamma\Big(2h(T)\Big)^{t_1-T}h(T) + \Big(1-\gamma(2h(T))^{t_1-T}\Big)H(T), \end{aligned}$$

where the third inequality makes use of Assumption 1 that $c_{i_0i_1}(t_1) \geq \gamma$, the inequality (21) and the fact that $H(t_1) \leq H(T)$.

Since

$$v_{i_1}(t_1) \in \left[\min_{j\in\mathcal{N}_{i_1}^+(t_1)}\{x_j(t_1)\}, \max_{j\in\mathcal{N}_{i_1}^+(t_1)}\{x_j(t_1)\}\right],$$

$v_{i_1}(t_1) \geq h(T)$. By Lemma 2, $\sum_{k=1}^{n} x_k(t_1) = 1$. Since $n \geq 3$ and $x(s) > 0$ for all integers $s \geq T$, we have

$$\begin{aligned} x_{i_1}(t_1)+v_{i_1}(t_1) &\leq x_{i_1}(t_1) + \max_{k\in\mathcal{N}_{i_1}^+} x_k(t_1) \\ &\leq 1-h(t_1) \leq 1-h(T). \end{aligned}$$

It follows from (22) that

$$\begin{aligned} x_{i_1}(t_1+1) &\leq h(T)v_{i_1}(t_1) + \Big(1-h(T)\Big)H(T) \\ &\leq \gamma h(T)\Big(2h(T)\Big)^{t_1-T}h(T) \\ &\quad + \Big(1-\gamma h(T)(2h(T))^{t_1-T}\Big)H(T). \end{aligned}$$

Next we provide a bound on the state $x_{i_1}(t_1+2)$. One has that

$$\begin{aligned} x_{i_1}(t_1+2) &= \Big(x_{i_1}(t_1+1)+v_{i_0}(t_1+1)\Big)x_{i_1}(t_1+1) \\ &\quad + \Big(1-x_{i_1}(t_1+1)-v_{i_1}(t_1+1)\Big)v_{i_1}(t_1+1), \end{aligned}$$

for some

$$v_{i_1}(t_1+1) \in \left[\min_{j\in\mathcal{N}_{i_1}^+(t_1+1)}\{x_j(t_1+1)\}, \max_{j\in\mathcal{N}_{i_1}^+(t_1+1)}\{x_j(t_1+1)\}\right].$$

The following inequality holds:

$$x_{i_0}(t_1+1) + v_{i_0}(t_1+1) \geq 2h(T).$$

Applying this inequality, one has that

$$\begin{aligned} x_{i_1}(t_1+2) &\leq \gamma h(T)\Big(2h(T)\Big)^{t_1-T+1}h(T) \\ &\quad + \Big(1-\gamma h(T)(2h(T))^{t_1-T+1}\Big)H(T). \end{aligned}$$

Recursively, one can show that for any nonnegative integers k, the following inequality holds

$$\begin{aligned} x_{i_1}(t_1+1+k) &\leq \gamma h(T)\Big(2h(T)\Big)^{t_1-T+k}h(T) \\ &\quad + \Big(1-\gamma h(T)(2h(T))^{t_1-T+k}\Big)H(T). \end{aligned} \tag{24}$$

This inequality holds for all $i_1 \in \mathcal{V}_1$. Furthermore, since $\gamma \leq 1$, the relation (20) implies that the above inequality also holds for $i_1 \in \mathcal{V}_0$. Hence, (24) is true for all $i_1 \in \mathcal{V}_0 \cup \mathcal{V}_1$.

Step 3. Define

$$\begin{aligned} t_2 = \min\{s \geq t_1+1 | c_{ji}(s) > 0, &\text{for some} \\ &j \in \mathcal{V}_0\cup\mathcal{V}_1,\ i \in \mathcal{V}\backslash(\mathcal{V}_0\cup\mathcal{V}_1)\}, \end{aligned}$$

$$\mathcal{V}_2 = \{i \in \mathcal{V}\backslash(\mathcal{V}_0\cup\mathcal{V}_1) | c_{ji}(t_2) > 0, \text{for some } j \in \mathcal{V}_0\cup\mathcal{V}_1\}.$$

t_2 is well defined and satisfies $t_1+1 \leq t_2 \leq t_1+B$ from Assumption 2. For any $i_2 \in \mathcal{V}_3$ and any nonnegative integers k, similar to the calculations in step 2, one can establish an upper bound on $x_{i_2}(t_2+1+k)$ as

$$\begin{aligned} x_{i_2}(t_2+1+k) &\leq \Big(\gamma h(T)\Big)^2\Big(2h(T)\Big)^{t_2-T-1+k}h(T) \\ &\quad + \Big(1-(\gamma h(T))^2(2h(T))^{t_2-T-1+k}\Big)H(T). \end{aligned} \tag{25}$$

The above inequality also holds for $i_2 \in \mathcal{V}_0\cup\mathcal{V}_1$ in view of (24).

Step 4. Continuing this process, we can define a time sequence $t_0, t_1, \ldots, t_p$, with $t_0 = T-1$ and a sequence of sets $\mathcal{V}_0, \ldots, \mathcal{V}_p$ as

$$\begin{aligned} t_{k+1} = \min\{s \geq t_k+1 | c_{ji}(s) > 0, &\text{for some} \\ &j \in \cup_{l=0}^{k}\mathcal{V}_l,\ i \in \mathcal{V}\backslash\cup_{l=0}^{k}\mathcal{V}_l\}, \end{aligned}$$

$$\mathcal{V}_{k+1} = \{i \in \mathcal{V}\backslash\cup_{l=0}^{k}\mathcal{V}_l | c_{ji}(t_{k+1}) > 0, \text{for some } j \in \cup_{l=0}^{k}\mathcal{V}_l\},$$

for $0 \leq k \leq p-1$, such that $\mathcal{V} = \cup_{i=0}^{p}\mathcal{V}_i$. t_i satisfies $t_i+1 \leq t_{i+1} \leq t_i+B$ for $i = 0, \ldots, p-1$ from Assumption 2. For all $i \in \mathcal{V}$ and any $k \geq 0$, we have the following inequality

$$\begin{aligned} x_i(t_p+1+k) &\leq \Big(\gamma h(T)\Big)^p\Big(2h(T)\Big)^{t_p-T-(p-1)+k}h(T) \\ &\quad + \Big(1-(\gamma h(T))^p(2h(T))^{t_p-T-(p-1)+k}\Big)H(T). \end{aligned} \tag{26}$$

In particular,

$$H(t_p+1) \leq \Big(\gamma h(T)\Big)^p \Big(2h(T)\Big)^{t_p-T-(p-1)} h(T) + \Big(1-(\gamma h(T))^p (2h(T))^{t_p-T-(p-1)}\Big)H(T). \quad (27)$$

It is obvious that $p \leq n-1$. Since $t_0 \leq t_1 \leq t_0+B-1$ and $t_i+1 \leq t_{i+1} \leq t_i+B-1$ for $i=1,\dots,p-1$, $t_p \leq T+pB-1$. Since $\gamma h(T)<1$ and $2h(T)<1$, one has $(\gamma h(T))^p \geq (\gamma h(T))^{n-1}$ and

$$(2h(T))^{t_p-T-(p-1)} \geq (2h(T))^{p(B-1)} \geq (2h(T))^{(n-1)(B-1)}.$$

Combining with the inequality (27) yields

$$\begin{aligned} &H(T+(n-1)B) \\ \leq &H(t_p+1+k) \\ \leq &\Big(\gamma h(T)\Big)^{n-1}\Big(2h(T)\Big)^{(n-1)(B-1)} h(T) \\ &+ \Big(1-(\gamma h(T))^{n-1}(2h(T))^{(n-1)(B-1)}\Big)H(T). \end{aligned}$$

Therefore, the contraction of $V(T+(n-1)B)$ can be calculated as

$$\begin{aligned} &V(T+(n-1)B) \\ =&H(T+(n-1)B)-h(T+(n-1)B) \\ \leq &H(T+(n-1)B)-h(T) \\ =&\Big(1-(\gamma h(T))^{n-1}(2h(T))^{(n-1)(B-1)}\Big)V(T). \end{aligned}$$

This completes the proof. ■

We are now in a position to prove the main theorem.

Proof of Theorem 1: Applying Lemma 7, one has

$$\begin{aligned} &V((n-m)B+k) \\ \leq &\Big(1-(\gamma\alpha(2\alpha)^{(B-1)})^{n-1}\Big)^{\lfloor \frac{k}{(n-1)B} \rfloor} V((n-m)B), \end{aligned}$$

for all nonnegative integers k. The fact that $V((n-m)B) \leq V(0)$ gives us (14) for $s \geq (n-m)B$.

For $0 \leq s < (n-m)B$, (14) holds since $V(s) \leq V(0)$ and $\Big(1-(\gamma\alpha(2\alpha)^{B-1})^{n-1}\Big)^{\lfloor \frac{s-(n-m)B}{(n-1)B} \rfloor} > 1$. ■

Note that Theorem 1 also shows that $\frac{1}{n}\mathbf{1}$ is the unique nontrivial equilibrium of system (12).

For a time-invariant irreducible matrix, i.e., $C(s)=C$ for all s, and C irreducible, Assumption 2 is naturally satisfied with $B=1$. We immediately have a corollary from Lemma 5, which is item (iii) of Lemma 3 in [1].

Corollary 1: Suppose that $C(s)=C$ for all s and C is an irreducible, row-stochastic matrix with diagonal entries all equal to 0. If $x(0) \in \Delta_n \backslash \{e_1,\dots,e_n\}$ has m nonzero elements, then $x(n-m)>0$.

The convergence result and convergence rate for the time-invariant case can be obtained as follows.

Proposition 1: Suppose that $n \geq 3$ and $C(s)=C$ for all s where C is an irreducible doubly stochastic matrix with diagonal entries all equal to 0. If $x(0) \in \Delta_n \backslash \{e_1,\dots,e_n\}$ and $x(0)$ has m nonzero entries, then $\lim_{s\to\infty} x_i(s) = \frac{1}{n}$ for all $i \in \mathcal{V}$, and the convergence is exponentially fast:

$$V(s) \leq \Big(1-\gamma^{n-1}\alpha^{n-1}\Big)^{\lfloor \frac{s-(n-m)}{n-1} \rfloor} V(0), \quad (28)$$

for all integers $s \geq n-m$, where $\alpha = h(n-m) > 0$.

V. Conclusion

In this paper, we have revisited the modified DeGroot-Friedkin model introduced in [1], with a time-varying relative interaction matrix. We have shown that if the matrix is doubly stochastic at all time instances, then the individuals' self-confidence levels will reach a democratic state exponentially fast. An explicit expression of the convergence rate has also been provided. Analysis of more general cases when the relative interaction matrix is row-stochastic, is a subject of future work, which was studied earlier in a continuous-time setting [19].

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