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Convergence of Distributed Averaging and Maximizing Algorithms Part I: Time-dependent Graphs

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Abstract— **In this paper, we formulate and investigate a generalized consensus algorithm which makes an attempt to unify distributed averaging and maximizing algorithms considered in the literature. Each node iteratively updates its state as a time-varying weighted average of its own state, the minimal state, and the maximal state of its neighbors. This part of the paper focuses on time-dependent communication graphs. We prove that finite-time consensus is almost impossible for averaging under this uniform model. Then various necessary and/or sufficient conditions are presented on the consensus convergence. The results characterize some similarities and differences between distributed averaging and maximizing algorithms.**

Index Terms— **Averaging algorithms, Max-consensus, Finite-time convergence**

I. Introduction

Distributed averaging algorithms, where each node iteratively averages its neighbors' states, have been extensively studied in the literature, due to its wide applicability in engineering [11], [12], [23], computer science [8], [9], and social science [5], [6], [7]. Recently also the max-consensus algorithms have attracted attention. These algorithms compute the maximal value among the nodes, and have been used for leader election, network size estimation, and various applications in wireless networks [23], [22].

The convergence to a consensus is central in the study of averaging and maximizing algorithms but can be hard to analyze, especially when the node interactions are carried out over a switching graph. The most convenient way of modeling the switching node interactions is just to assume the communication graphs are defined by a sequence of time-dependent graphs over the node set. The connectivity of this sequence of graphs plays an important role for the network to reach consensus. Joint connectivity, i.e., connectivity of the union graph over time intervals, has been considered, and various convergence conditions have been established [11], [21], [12], [13], [15], [16], [14], [17], [16].

Few studies have discussed the fundamental similarities and differences between distributed averaging and maximizing. Averaging and maximizing consensus algorithms are both distributed information processing over graphs, where nodes communicate and exchange information with its neighbors in the aim of collective convergence. Average consensus algorithms in the literature are based on two standing assumptions: local cohesion and node self-confidence. The node states iteratively update as a weighted average of its neighbors' states, with a positive lower bound for the weight corresponding to its own state [11], [21], [12], [15], [14], [20]. Average consensus algorithms can also be viewed as the equivalent state evolution process where each node updates its state as a weighted average of its own state, and the minimum and maximum states of its neighbors. Maximizing (or minimizing) consensus algorithms are simply based on that each node updates its state to the maximal (minimal) state among its neighbors [27], [28], [29]. Asymptotic convergence is common in the study of averaging consensus algorithms [14], [15], [20], [12], while it has been shown that maximizing algorithms converge in general in finite time [28], [29]. Finite-time convergence of averaging algorithms was investigated in [23], [25], [26] for continuous-time models, and recently finite-time consensus in discrete time was discussed in [33] for a special case of gossiping [32].

In this paper, we make the simple observation that averaging and maximizing algorithms can be viewed as instances of a more general distributed processing model. Using this model the transition of the consensus convergence can be studied for the two classes of distributed algorithms in a unified way. Each node iteratively updates its state as a weighted average of its own state together with the minimum and maximum states of its neighbors. By special cases for the weight parameters, averaging and maximizing algorithms can be analyzed. This is the first part of the paper discussing time-varying communication graphs. Under this uniform model, we prove for averaging that finite-time consensus is impossible in general, and asymptotical consensus is possible only when the node self-confidence satisfies a divergence condition. Various necessary and/or sufficient conditions are presented on the consensus convergence. State-dependent graph models are studied in Part II of the paper [34], and a complete version of the paper can be found in [35].

The rest of the paper is organized as follows. In Section II we introduce the considered network model, the uniform processing algorithm, and the consensus problem. The impossibilities of finite-time or asymptotic consensus are studied in Section III. The main results are presented for time-dependent graphs in Section IV. Finally some concluding remarks are given in Section V.

II. Problem Definition

In this section, we introduce the network model, the considered algorithm, and define the problem of interest.

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A. Network

We first recall some concepts and notations in graph theory [1]. A directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of nodes and an arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An element $e = (i,j) \in \mathcal{E}$ is called an *arc* from node $i \in \mathcal{V}$ to $j \in \mathcal{V}$. If the arcs are pairwise distinct in an alternating sequence $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ of nodes $v_i \in \mathcal{V}$ and arcs $e_i = (v_{i-1}, v_i) \in \mathcal{E}$ for $i = 1, 2, \dots, k$, the sequence is called a (directed) *path* with *length* k. If there exists a path from node i to node j, then node j is said to be reachable from node i. Each node is thought to be reachable by itself. A node v from which any other node is reachable is called a *center* (or a *root*) of $\mathcal{G}$. A digraph $\mathcal{G}$ is said to be *strongly connected* if node i is reachable from j for any two nodes $i,j \in \mathcal{V}$; *quasi-strongly connected* if $\mathcal{G}$ has a center [2]. The *distance* from i to j in a digraph $\mathcal{G}$, $d(i,j)$, is the length of a shortest simple path $i \to j$ if j is reachable from i, and the *diameter* of $\mathcal{G}$ is $\mathrm{diam}(\mathcal{G}) = \max\{d(i,j) | i,j \in \mathcal{V},\ j \text{ is reachable from } i\}$. The union of two digraphs with the same node set $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$. A digraph $\mathcal{G}$ is said to be bidirectional if for every two nodes i and j, $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$. A bidirectional graph $\mathcal{G}$ is said to be *connected* if there is a path between any two nodes.

Consider a network with node set $\mathcal{V} = \{1, 2, \dots, n\}$, $n \geq 3$. Time is slotted. Denote the state of node i at time $k \geq 0$ as $x_i(k) \in \mathbb{R}$. Then $x(k) = \big(x_1(k) \dots x_n(k)\big)^T$ represents the network state. For time-varying graphs, we use the following definition.

Definition 2.1 (Time-dependent Graph): The interactions among the nodes are determined by a given sequence of digraphs with node set $\mathcal{V}$, denoted as $\mathcal{G}_k = (\mathcal{V}, \mathcal{E}_k)$, $k = 0, 1, \dots$.

Throughout this paper, we call node j a *neighbor* of node i if there is an arc from j to i in the graph. Each node is supposed to always be a neighbor of itself. Let $\mathcal{N}_i(k)$ represent the neighbor set of node i at time k.

B. Algorithm

The classical average consensus algorithm in the literature is given by

$$x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) x_j(k), \quad i = 1 \dots, n. \tag{1}$$

Two standing assumptions are fundamental in determining the nature of its dynamics:

A1 *(Local Cohesion)* $\sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) = 1$ for all i and k;

A2 *(Self-confidence)* There exists a constant $\eta > 0$ such that $a_{ii}(k) \geq \eta$ for all i and k.

These assumptions are widely imposed in the existing works, e.g., [12], [11], [19], [20], [14], [15], [21]. With A1 and A2, we have

$$\begin{aligned}\sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) x_j(k) &= \eta x_i(k) + \big(a_{ii}(k) - \eta\big) x_i(k) \\ &\quad + \sum_{j \in \mathcal{N}_i(k), j \neq i} a_{ij}(k) x_j(k)\end{aligned} \tag{2}$$

and

$$\begin{aligned}\big(1-\eta\big) \min_{j \in \mathcal{N}_i(k)} x_j(k) &\leq \big(a_{ii}(k) - \eta\big) x_i(k) \\ &\quad + \sum_{j \in \mathcal{N}_i(k), j \neq i} a_{ij}(k) x_j(k) \\ &\leq \big(1-\eta\big) \max_{j \in \mathcal{N}_i(k)} x_j(k).\end{aligned} \tag{3}$$

Noting the fact that for any $c \in [a,b]$ there exists a unique $\lambda \in [0,1]$ satisfying $c = \lambda a + (1-\lambda) b$, we see from (3) that for every i and k, there exists $\beta_k^{\langle i \rangle} \in [0,1]$ such that

$$\begin{aligned}&\big(a_{ii}(k) - \eta\big) x_i(k) + \sum_{j \in \mathcal{N}_i(k), j \neq i} a_{ij}(k) x_j(k) \\ &= \beta_k^{\langle i \rangle} \big(1-\eta\big) \min_{j \in \mathcal{N}_i(k)} x_j(k) \\ &+ \big(1 - \beta_k^{\langle i \rangle}\big)\big(1-\eta\big) \max_{j \in \mathcal{N}_i(k)} x_j(k) \\ &= \alpha_k^{\langle i \rangle} \min_{j \in \mathcal{N}_i(k)} x_j(k) + \big(1 - \eta - \alpha_k^{\langle i \rangle}\big) \max_{j \in \mathcal{N}_i(k)} x_j(k),\end{aligned} \tag{4}$$

where $\alpha_k^{\langle i \rangle} = \beta_k^{\langle i \rangle}(1-\eta) \in [0, 1-\eta]$.

Therefore, in light of (2) and (4), based on assumptions A1 and A2, we can always write the average consensus algorithm (1) into the following equivalent form:

$$\begin{aligned}x_i(k+1) &= \eta x_i(k) + \alpha_k^{\langle i \rangle} \min_{j \in \mathcal{N}_i(k)} x_j(k) \\ &\quad + \big(1 - \eta - \alpha_k^{\langle i \rangle}\big) \max_{j \in \mathcal{N}_i(k)} x_j(k),\end{aligned} \tag{5}$$

where $\alpha_k^{\langle i \rangle} \in [0, 1-\eta]$ for all i and k. Thus, the information processing principle behind distributed averaging is that each node iteratively takes a weighted average of its current state and the minimum and maximum states of its neighbor set.

The standard maximizing algorithm [27], [28], [29] is defined by

$$x_i(k+1) = \max_{j \in \mathcal{N}_i(k)} x_j(k), \tag{6}$$

so distributed maximizing is each node interacting with its neighbors and simply taking the maximal state within its neighbor set.

In this paper, we aim to present a model under which we can discuss fundamental differences of some distributed information processing mechanisms. We consider the following algorithm for the node updates:

$$\begin{aligned}x_i(k+1) &= \eta_k x_i(k) + \alpha_k \min_{j \in \mathcal{N}_i(k)} x_j(k) \\ &\quad + \big(1 - \eta_k - \alpha_k\big) \max_{j \in \mathcal{N}_i(k)} x_j(k),\end{aligned} \tag{7}$$

where $\alpha_k, \eta_k \geq 0$ and $\alpha_k + \eta_k \leq 1$. We denote the set of all algorithms of the form (7) by $\mathcal{A}$, when the parameter (α_k, η_k) takes value as $\eta_k \in [0,1], \alpha_k \in [0, 1-\eta_k]$. This model is a special case of (5) as the parameter α_k is not depending on the node index i in (7).

Note that $\mathcal{A}$ represents a uniform model for distributed averaging and maximizing algorithms. Obeying the cohesion and self-confidence assumptions, the set of (weighted) averaging algorithms, $\mathcal{A}_{\mathrm{ave}}$, consists of algorithms in the form of (7) with parameters $\eta_k \in (0,1], \alpha_k \in [0, 1-\eta_k]$. The set of maximizing algorithms, $\mathcal{A}_{\max}$, is given by the parameter set $\eta_k \equiv 0$ and $\alpha_k \equiv 0$.

Remark 2.1: Algorithm (7) is more restrictive than (5) in the sense that the averaging weight $\alpha_k^{\langle i \rangle}$ in (5) might vary for different nodes. Hence, (7) cannot in general capture the averaging algorithm (1). Except for this difference, the standing assumptions A1 and A2 of average consensus algorithms are still fulfilled for algorithm (7).

Remark 2.2: In Algorithm (7) each node's update only depends on the states of the minimum and maximum neighbor states at every time step. In other words, not all links are active explicitly in the iterations. Therefore, the existing convergence results on averaging algorithms cannot be applied directly, since these results rely on the connectivity of the communication graph.

Remark 2.3: Following Algorithm (7), it is straightforward to see that the convergence limit is a convex combination of the initial values if consensus is reached. But due to the state-dependent node update in (7), the coefficients in the convex combination of the consensus limit indeed depend on the initial condition (even with fixed communication graph).

C. Problem

Let $\big\{x(k;x^0) = \big(x_1(k;x^0)\dots x_n(k;x^0)\big)^T\big\}_0^\infty$ be the sequence generated by (7) for initial time k_0 and initial value $x^0 = x(k_0) = \big(x_1(k_0)\dots x_n(k_0)\big)^T \in \mathbb{R}^n$. We will identify $x(k;x^0)$ as $x(k)$ in the following discussions. We introduce the following definition on the convergence of the considered algorithm.

Definition 2.2: (i) Asymptotic consensus is achieved for Algorithm (7) for initial condition $x(k_0) = x^0 \in \mathbb{R}^n$ if there exists $z_*(x^0) \in \mathbb{R}$ such that

$$\lim_{k\to\infty} x_i(k) = z_*, \quad i = 1, \dots, n.$$

Global asymptotic consensus is achieved if asymptotic consensus is achieved for all $k_0 \geq 0$ and $x^0 \in \mathbb{R}^n$.

(ii) Finite-time consensus is achieved for Algorithm (7) for initial condition $x(k_0) = x^0 \in \mathbb{R}^n$ if there exist $z_*(x^0) \in \mathbb{R}$ and an integer $T_*(x^0) > 0$ such that

$$x_i(T_*) = z_*, \quad i = 1, \dots, n.$$

Global finite-time consensus is achieved if finite-time consensus is achieved for all $k_0 \geq 0$ and $x^0 \in \mathbb{R}^n$.

The algorithm reaching consensus is equivalent with that $x(k)$ converges to the manifold

$$\mathrm{C} = \Big\{x = (x_1 \dots x_n)^T : \ x_1 = \dots = x_n\Big\}.$$

We call C the consensus manifold. Its dimension is one.

In the following, we focus on the impossibilities and possibilities of asymptotic or finite-time consensus. We will show that the convergence properties drastically change when Algorithm (7) transits from averaging to maximizing.

III. Convergence Impossibilities

In this section, we discuss the impossibilities of asymptotic or finite-time convergence for the averaging algorithms in $\mathcal{A}_{\rm ave}$. One theorem for each case is proven.

Theorem 3.1: For every averaging algorithm in $\mathcal{A}_{\rm ave}$, finite-time consensus fails for all initial values in $\mathbb{R}^n$ except for initial values on the consensus manifold.

Proof. We define

$$h(k) = \min_{i\in\mathcal{V}} x_i(k); \quad H(k) = \max_{i\in\mathcal{V}} x_i(k).$$

Introduce $\Phi(k) = H(k) - h(k)$. Then clearly asymptotic consensus is achieved if and only if $\lim_{k\to\infty}\Phi(k) = 0$.

Take a node i satisfying $x_i(k) = h(k)$. We have

$$\begin{aligned} x_i(k+1) &= \eta_k x_i(k) + \alpha_k \min_{j\in\mathcal{N}_i(k)} x_j(k) \\ &\quad + \big(1-\eta_k-\alpha_k\big) \max_{j\in\mathcal{N}_i(k)} x_j(k) \\ &\leq (\alpha_k+\eta_k)h(k) + (1-\eta_k-\alpha_k)H(k). \end{aligned} \tag{8}$$

Similarly, taking another node m satisfying $x_m(k) = H(k)$, we obtain

$$\begin{aligned} x_m(k+1) &= \eta_k x_j(k) + \alpha_k \min_{m\in\mathcal{N}_i(k)} x_j(k) \\ &\quad + \big(1-\eta_k-\alpha_k\big) \max_{m\in\mathcal{N}_i(k)} x_j(k) \\ &\geq \alpha_k h(k) + (1-\alpha_k)H(k). \end{aligned} \tag{9}$$

With (8) and (9), we conclude that

$$\begin{aligned} \Phi(k+1) &= \max_{i\in\mathcal{V}} x_i(k) - \min_{i\in\mathcal{V}} x_i(k) \\ &\geq x_m(k+1) - x_i(k+1) \\ &\geq \eta_k\Phi(k). \end{aligned} \tag{10}$$

Therefore, since (10) holds for all k, we immediately obtain that for every algorithm in the averaging set $\mathcal{A}_{\rm ave}$,

$$\Phi(K) \geq \Phi(k_0)\prod_{k=k_0}^{K-1}\eta_k > 0 \tag{11}$$

for all $K \geq k_0$ as long as $\Phi(k_0) > 0$. Noticing that the initial values satisfying $\Phi(k_0) = 0$ are on the consensus manifold, the desired conclusion follows. □

Since the consensus manifold is a one-dimensional manifold in $\mathbb{R}^n$, Theorem 3.1 indicates that finite-time convergence is almost impossible for average consensus algorithms. This partially explains why finite-time convergence results are rare for averaging algorithms in the literature.

Next, we discuss the impossibility of asymptotic consensus. The following lemma is well-known.

Lemma 3.1: Let $\{b_k\}_0^\infty$ be a sequence of real numbers with $b_k \in [0,1)$ for all k. Then $\sum_{k=0}^\infty b_k = \infty$ if and only if $\prod_{k=0}^\infty (1-b_k) = 0$.

The following theorem on asymptotic convergence holds.

Theorem 3.2: For every averaging algorithm in $\mathcal{A}_{\rm ave}$, asymptotic consensus fails for all initial values in $\mathbb{R}^n$ except for initial values on the consensus manifold, if $\sum_{k=0}^\infty \big(1 - \eta_k\big) < \infty$.

Proof. In light of Lemma 3.1 and (11), we see that for every algorithm in the averaging set $\mathcal{A}_{\rm ave}$,

$$\lim_{K\to\infty}\Phi(K)\geq\Phi(k_0)\prod_{k=k_0}^{\infty}\eta_k>0 \tag{12}$$

if $\sum_{k=0}^{\infty}\big(1-\eta_k\big)<\infty$ for all initial values satisfying $\Phi(k_0)>0$. The desired conclusion thus follows. □

Theorem 3.2 indicates that $\sum_{k=0}^{\infty}\big(1-\eta_k\big)=\infty$ is a necessary condition for average algorithms to reach asymptotic consensus. Note that η_k measures node self-confidence. Thus, the condition $\sum_{k=0}^{\infty}\big(1-\eta_k\big)=\infty$ characterizes the maximal self-confidence that nodes can hold and still reach consensus.

It is worth pointing out that Theorems 3.1 and 3.2 hold for any communication graph.

IV. Main Results

In this section, we focus on time-dependent graphs. We first discuss a special case when the network topology is fixed, and then time-varying communications will be discussed.

A. Fixed Graph

For fixed communication graphs, we present the following result.

Theorem 4.1: Suppose $\mathcal{G}_k\equiv\mathcal{G}_*$ is a fixed digraph for all k.

(i) For every algorithm in $\mathcal{A}_{\rm ave}$, global asymptotic consensus can be achieved only if $\mathcal{G}_*$ is quasi-strongly connected.

(ii) For every algorithm in $\mathcal{A}_{\rm max}$, global finite-time consensus is achieved if and only if $\mathcal{G}_*$ is strongly connected.

Proof. (i) If $\mathcal{G}_*$ is not quasi-strongly connected, there exist two distinct nodes i and j such that $\mathcal{V}_1\cap\mathcal{V}_2=\emptyset$, where $\mathcal{V}_1=\{\text{nodes from which } i \text{ is reachable in } \mathcal{G}_*\}$ and $\mathcal{V}_2=\{\text{nodes from which } j \text{ is reachable in } \mathcal{G}_*\}$. Consequently, nodes in $\mathcal{V}_1$ never receive information from nodes in $\mathcal{V}_2$. Take $x_i(k_0)=0$ for $i\in\mathcal{V}_1$ and $x_i(k_0)=1$ for $i\in\mathcal{V}_2$. Obviously, consensus cannot be achieved under this initial condition. The conclusion holds.

(ii) The result was proved in [27], and here we provide a simple graphical proof.

(Sufficiency.) Let v_0 be a node with the maximal value initially. Then after one step all the nodes for which v_0 is a neighbor will reach the maximal value. Proceeding the analysis we see that the whole network will converge to the initial maximum in finite time.

(Necessity.) Assume that $\mathcal{G}_*$ is not strongly connected. There will be two different nodes i_* and j_* such that j_* is not reachable from i_*. Introduce $\mathcal{V}_*=\{j: j \text{ is reachable from } i_*\}$. Then $\mathcal{V}_*\neq\mathcal{V}$ because $j_*\notin\mathcal{V}_*$. Moreover, the definition of $\mathcal{V}_*$ guarantees that all the nodes in $\mathcal{V}\setminus\mathcal{V}_*$ will never be influenced by the nodes in $\mathcal{V}_*$. Therefore, letting the initial maximum be unique and reached by some node in $\mathcal{V}_*$, consensus will not be reached.

The proof is complete. □

As will be shown in the following discussions, quasi-strong connectivity is not only necessary, but also sufficient to guarantee global asymptotic consensus for the algorithms in the averaging set $\mathcal{A}_{\rm ave}$ under some mild conditions on the parameters (α_k,η_k). Therefore, Theorem 4.1 clearly states that quasi-strong connectivity is critical for averaging, as is strong connectivity for maximizing.

B. Time-varying Graph

We now turn to time-varying graphs. Denote the joint graph of $\mathcal{G}_k$ over time interval $[k_1,k_2]$ as $\mathcal{G}\big([k_1,k_2]\big)=(\mathcal{V},\cup_{k\in[k_1,k_2]}\mathcal{E}(k))$, where $0\leq k_1\leq k_2\leq+\infty$. We introduce the following definitions on the joint connectivity of time-varying graphs.

Definition 4.1: (i) $\mathcal{G}_k$ is *uniformly jointly quasi-strongly connected* (strongly connected) if there exists an integer $B\geq 1$ such that $\mathcal{G}\big([k,k+B-1]\big)$ is quasi-strongly connected (strongly connected) for all $k\geq 0$.

(ii) $\mathcal{G}_k$ is *infinitely jointly strongly connected* if $\mathcal{G}\big([k,\infty)\big)$ is strongly connected for all $k\geq 0$.

(iii) Suppose $\mathcal{G}_k$ is bidirectional for all k. Then $\mathcal{G}_k$ is *infinitely jointly connected* if $\mathcal{G}\big([k,\infty)\big)$ is connected for all $k\geq 0$.

Remark 4.1: The uniformly joint connectivity, which requires the union graph to be connected over each bounded interval, has been extensively studied in the literature, e.g., [11], [12], [14], [15]. The infinitely joint connectivity is a more general case since it does not impose an upper bound for the length of the interval where connectivity is guaranteed for the union graph. Convergence results for consensus algorithms based on infinitely joint connectivity are given in [16], [17], [18].

The following conclusion holds for uniformly jointly quasi-strongly connected graphs.

Theorem 4.2: Suppose $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected. Algorithms in the averaging set $\mathcal{A}_{\rm ave}$ achieve global asymptotic consensus if either $\sum_{s=0}^{\infty}\big[\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1}\alpha_k\big]=\infty$ or $\sum_{s=0}^{\infty}\big[\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1}\big(1-\alpha_k-\eta_k\big)\big]=\infty$.

Theorem 4.2 hence states that divergence of certain products of the algorithm parameters guarantees global asymptotic consensus.

It is straightforward to see that for a non-negative sequence $\{b_k\}$ with $b_k\geq b_{k+1}$ for all k, $\sum_{s=0}^{\infty}\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2-1}b_k=\infty$ if and only if $\sum_{k=0}^{\infty}b_k^{(n-1)^2B}=\infty$. Thus, the following corollary follows from Theorem 4.2.

Corollary 4.1: Suppose $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected.

(i) Assume that $\alpha_k\geq\alpha_{k+1}$ for all k. Algorithms in the averaging set $\mathcal{A}_{\rm ave}$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty}\alpha_k^{(n-1)^2B}=\infty$.

(ii) Assume that $\alpha_k+\eta_k\leq\alpha_{k+1}+\eta_{k+1}$ for all k. Algorithms in the averaging set $\mathcal{A}_{\rm ave}$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty}\big(1-\alpha_k-\eta_k\big)^{(n-1)^2B}=\infty$.

For uniformly jointly strongly connected graphs, it turns out that consensus can be achieved under weaker conditions on (α_k,η_k).

Theorem 4.3: Suppose $\mathcal{G}_k$ is uniformly jointly strongly connected. Algorithms in the averaging set $\mathcal{A}_{\rm ave}$ achieve global asymptotic consensus if either $\sum_{s=0}^{\infty}\big[\prod_{k=s(n-1)B}^{(s+1)(n-1)B-1}\alpha_k\big]=\infty$ or $\sum_{s=0}^{\infty}\big[\prod_{k=s(n-1)B}^{(s+1)(n-1)B-1}\big(1-\alpha_k-\eta_k\big)\big]=\infty$.

Similarly, Theorem 4.3 leads to the following corollary.

Corollary 4.2: Suppose $\mathcal{G}_k$ is uniformly jointly strongly connected.

(i) Assume that $\alpha_k\ge\alpha_{k+1}$ for all k. Averaging algorithms in the set $\mathcal{A}_{\rm ave}$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty}\alpha_k^{(n-1)B}=\infty$.

(ii) Assume that $\alpha_k+\eta_k\le\alpha_{k+1}+\eta_{k+1}$ for all k. Averaging algorithms in the set $\mathcal{A}_{\rm ave}$ achieve global asymptotic consensus if $\sum_{k=0}^{\infty}\big(1-\alpha_k-\eta_k\big)^{(n-1)B}=\infty$.

For bidirectional graphs, the conditions are much simpler to state. We present the following result.

Theorem 4.4: Suppose $\mathcal{G}_k$ is bidirectional for all k and $\mathcal{G}_k$ is infinitely jointly connected. Averaging algorithms in the set $\mathcal{A}_{\rm ave}$ achieves achieve global asymptotic consensus if there exists a constant $\alpha_*\in(0,1)$ such that either $\alpha_k\ge\alpha_*$ or $1-\alpha_k-\eta_k\ge\alpha_*$ for all k.

The convergence of algorithms in the maximizing set $\mathcal{A}_{\max}$ is stated as follows.

Theorem 4.5: Maximizing algorithms in the set $\mathcal{A}_{\max}$ achieve global finite-time consensus if $\mathcal{G}_k$ is infinitely jointly strongly connected.

Theorems 4.2–4.5 together provide a comprehensive understanding of the convergence conditions for the considered model (7) under time-varying graphs. Infinitely jointly strong connectivity is sufficient for global finite-time consensus for algorithms in $\mathcal{A}_{\max}$ according to Theorem 4.5, while infinitely joint connectivity cannot ensure global asymptotic consensus for algorithms in $\mathcal{A}_{\rm ave}$ in general. Thus, in this sense algorithms in $\mathcal{A}_{\rm ave}$ and $\mathcal{A}_{\max}$ are fundamentally different under infinitely jointly connected graphs.

The rest of this section contains the proofs of Theorems 4.2–4.5.

1) Proof of Theorem 4.2: Following any solution of (7), it is obvious to see that $h(k)$ is non-decreasing and $H(k)$ is non-increasing. Due to the symmetry of the algorithm we just need to show that $\sum_{s=0}^{\infty}\big[\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1}\alpha_k\big]=\infty$ is a sufficient condition for asymptotic consensus.

Take $k_*\ge 0$ as any moment in the iterative algorithm. Take $(n-1)^2$ intervals $[k_*,k_*+B-1]$, $[k_*+B,k_*+2B-1],\dots,[k_*+(n^2-2n)B,k_*+(n-1)^2B-1]$. Since $\mathcal{G}_k$ is uniformly jointly quasi-strongly connected, the union graph on each of these intervals has at least one center node. Consequently, there must be a node v_0 and $n-1$ integers $0\le b_1<b_2<\cdots<b_{n-1}\le n^2-2n$ such that v_0 is a center of $\mathcal{G}\big([k_*+b_iB,k_*+(b_i+1)B-1]\big), i=1,\dots,n-1$. Assume that

$$x_{v_0}(k_*)\le\frac{1}{2}h(k_*)+\frac{1}{2}H(k_*).\qquad(13)$$

Then through recursive estimation we can obtain (details can be found in [35])

$$\Phi\big(k_*+(n-1)^2B\big)\le\Big(1-\frac{\prod_{k=k_*}^{k_*+(n-1)^2B-1}\alpha_k}{2}\Big)\Phi(k_*).\qquad(14)$$

From a symmetric analysis by bounding $h(k_*+(n-1)^2B)$ from below, we know that (14) also holds for the other condition with $x_{v_0}(k_*)\ge\frac{1}{2}h(k_*)+\frac{1}{2}H(k_*)$. Therefore, since k_* is selected arbitrarily, we can assume the initial time is $k_0=0$, without loss of generality, and then conclude that

$$\Phi\big(K(n-1)^2B\big)\le\Phi(0)\prod_{s=0}^{K-1}\left(1-\frac{1}{2}\prod_{k=s(n-1)^2B}^{(s+1)(n-1)^2B-1}\alpha_k\right).$$

The desired conclusion follows immediately from Lemma 3.1.

2) Proof of Theorem 4.3: Notice that in a strongly connected graph, every node is a center node. Therefore, when $\mathcal{G}_k$ is uniformly jointly strongly connected, taking $k_*\ge 0$ as any moment in the iteration and $n-1$ intervals $[k_*,k_*+B-1]$, $[k_*+B,k_*+2B-1],\dots,[k_*+(n-2)B,k_*+(n-1)B-1]$, any node $i\in\mathcal{V}$ is a center node for the union graph over each of these intervals. As a result, the desired conclusion follows repeating the analysis used in the proof of Theorem 4.2.

3) Proof of Theorem 4.4: Similar to the proof of Theorem 4.2, we only need to show that the existence of a constant $\alpha_*\in(0,1)$ such that $\alpha_k\ge\alpha_*$ is sufficient for asymptotic consensus.

Take $k_1^*\ge 0$ as an arbitrary moment in the iterative algorithm. Take a node $u_0\in\mathcal{V}$ satisfying $x_{u_0}(k_1^*)=h(k_1^*)$. We define $k_1=\inf\big\{k\ge k_1^*$: there exists another node connecting u_0 at time $k\big\}$, and then $\mathcal{V}_1=\big\{k\ge k_1^*$: nodes which are connected to u_0 at time $k_1\big\}$. Thus, we have

$$\begin{aligned}x_{u_0}(k_1+1)&=\eta_{k_1}x_{u_0}(k_1)+\alpha_{k_1}\min_{j\in\mathcal{N}_{u_0}(k_1)}x_j(k_1)\\&\le\alpha_*h(k_1^*)+\big(1-\alpha_*\big)H(k_1^*)\end{aligned}\qquad(15)$$

and

$$x_i(k_1+1)\le\alpha_*h(k_1^*)+\big(1-\alpha_*\big)H(k_1^*)\qquad(16)$$

for all $i\in\mathcal{V}_1$.

Note that if nodes in $\{u_0\}\cup\mathcal{V}_1$ are not connected with other nodes in $\mathcal{V}\setminus(\{u_0\}\cup\mathcal{V}_1)$ during $[k_1+1,k_1+s]$, $s\ge 1$, we have that for all $i\in\{u_0\}\cup\mathcal{V}_1$,

$$x_i(k_1+m)\le\alpha_*h(k_1^*)+\big(1-\alpha_*\big)H(k_1^*)\qquad(17)$$

for all $m=1,\dots,s+1$. Continuing the estimate, $k_2,\dots,k_d$ and $\mathcal{V}_2,\dots,\mathcal{V}_d$ can be defined correspondingly until $\mathcal{V}=\{u_0\}\cup(\cup_{i=1}^d\mathcal{V}_i)$, so eventually we have

$$x_i(k_d+1)\le\alpha_*^dh(k_1^*)+\big(1-\alpha_*^d\big)H(k_1^*),\quad i=1,\dots,n,\qquad(18)$$

which implies

$$H(k_d+1)\le\alpha_*^dh(k_1^*)+\big(1-\alpha_*^d\big)H(k_1^*).\qquad(19)$$

We denote $k_2^* = k_d + 1$. Because it holds that $d \leq n-1$, we see from (19) that

$$\Phi(k_2^*) \leq \big(1 - \alpha_*^{n-1}\big)\Phi(k_1^*). \quad (20)$$

Since $\mathcal{G}_k$ is infinitely jointly connected, this process can be carried on for an infinite sequence $k_1^* < k_2^* < \dots$. Thus, asymptotic consensus is achieved for all initial conditions. This completes the proof.

4) Proof of Theorem 4.5: Let v_0 be a node with the maximal value initially. Because $\mathcal{G}_k$ is infinitely jointly strongly connected, we can define $k_1 = \inf\big\{k \geq k_1^*$: there exists another node for which v_0 is a neighbor at time $k\big\}$ and then $\mathcal{V}_1 = \big\{k \geq k_1^*$: nodes for which v_0 is a neighbor at time $k_1\big\}$. Then at time $k_1 + 1$ all the nodes in $\mathcal{V}_1$ will reach the maximal value. Proceeding the analysis we know that the whole network will converge to the initial maximum in finite time. □

V. Conclusions

This paper focused on a uniform model for distributed averaging and maximizing. Each node iteratively updated its state as a weighted average of its own state, the minimal state, and maximal state among its neighbors. We proved that finite-time consensus is almost impossible for averaging under the uniform model. This part of the paper investigate time-dependent communication graphs. Necessary and sufficient conditions were established on the graph to ensure a global consensus. We showed that quasi-strong connectivity is critical for averaging algorithms, as is strong connectivity for maximizing algorithms. The results revealed the fundamental connection and difference between distributed averaging and maximizing.

References

[1] C. Godsil and G. Royle. *Algebraic Graph Theory.* New York: Springer-Verlag, 2001.

[2] C. Berge and A. Ghouila-Houri. *Programming, Games, and Transportation Networks*. John Wiley and Sons, New York, 1965.

[3] J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices," *Proc. Amer. Math. Soc.*, vol. 15, pp. 733-736, 1963.

[4] J. Hajnal, "Weak ergodicity in non-homogeneous Markov chains," *Proc. Cambridge Philos. Soc.*, no. 54, pp. 233-246, 1958.

[5] M. H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118-121, 1974.

[6] P. M. DeMarzo, D. Vayanos, J. Zwiebel, "Persuasion bias, social influence, and unidimensional opinions," *Quarterly Journal of Economics*, vol. 118, no. 3, pp. 909-968, 2003.

[7] B. Golub and M. O. Jackson, "Naïve learning in social networks and the wisdom of crowds," *American Economic Journal: Microeconomics*, vol. 2, no. 1, pp. 112-149, 2007.

[8] S. Muthukrishnan, B. Ghosh, and M. Schultz, "First and second order diffusive methods for rapid, coarse, distributed load balancing," *Theory of Computing Systems*, vol. 31, pp. 331-354, 1998.

[9] R. Diekmann, A. Frommer, and B. Monien, "Efficient schemes for nearest neighbor load balancing," *Parallel Computing*, vol. 25, pp. 789-812, 1999.

[10] N. Malpani, J. L. Welch and N. Vaidya, "Leader election algorithms for mobile ad hoc networks," *Proc. 4th international workshop on Discrete algorithms and methods for mobile computing and communications*, pp. 96-104, 2000.

[11] J. Tsitsiklis, D. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Trans. Autom. Control*, vol. 31, pp. 803-812, 1986.

[12] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988-1001, 2003.

[13] R. Olfati-Saber and R. Murray, "Consensus problems in the networks of agents with switching topology and time dealys," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520-1533, 2004.

[14] M. Cao, A. S. Morse and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: a graphical approach," *SIAM J. Control Optim.*, vol. 47, no. 2, 575-600, 2008.

[15] W. Ren and R. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655-661, 2005.

[16] L. Moreau, "Stability of multi-agent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 50, pp. 169-182, 2005.

[17] G. Shi and Y. Hong, "Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies," *Automatica*, vol. 45, no. 5, pp. 1165-1175, 2009.

[18] G. Shi, Y. Hong and K. H. Johansson, "Connectivity and set tracking of multi-agent systems guided by multiple moving leaders," *IEEE Trans. Autom. Control*, vol. 57, no. 3, pp. 663-676, 2012.

[19] A. Nedić, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis, "On distributed averaging algorithms and qantization effects," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2506-2517, 2009.

[20] A. Olshevsky and J. N. Tsitsiklis, "Convergence speed in distributed consensus and averaging," *SIAM Review*, vol. 53, no. 4, pp. 747-772, 2011.

[21] V. Blondel, J. M. Hendrickx, A. Olshevsky and J. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," *IEEE Conf. Decision and Control*, pp. 2996-3000, 2005.

[22] J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Distributed anonymous discrete function computation," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2276-2289, 2011.

[23] J. Cortés, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no.11, pp. 1993-2000, 2006.

[24] J. Cortés, "Analysis and design of distributed algorithms for χ-consensus," *IEEE Conference on Decision and Control*, pp. 3363-3368, 2006.

[25] Q. Hui, W. M. Haddad, and S. P. Bhat, "Finite-Time semistability and consensus for nonlinear dynamical networks," *IEEE Trans. Autom. Contr.*, vol.53, pp. 1887-1900, 2008.

[26] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Contr*, vol. 55, no.4, pp. 950-955, 2010.

[27] B. M. Nejad, S. A. Attia and J. Raisch, "Max-consensus in a max-plus algebraic setting: The case of fixed communication topologies," in *International Symposium on Information, Communication and Automation Technologies*, pp.1-7, 2009.

[28] B. M. Nejad, S. A. Attia and J. Raisch, "Max-consensus in a max-plus algebraic setting: The case of fixed communication topologies," in *International Symposium on Information, Communication and Automation Technologies*, pp.1-7, 2009.

[29] F. Iutzeler, P. Ciblat, and J. Jakubowicz, "Analysis of max-consensus algorithms in wireless channels," *IEEE Transactions on Signal Processing*, vol. 60, no.11, pp. 6103-6107, 2012.

[30] D. Varagnolo, G. Pillonetto, and L. Schenato, "Distributed statistical estimation of the number of nodes in sensor networks, in *IEEE Conference on Decision and Control*, 2011.

[31] R. Carli, F. Fagnani, A. Speranzon and S. Zampieri, "Communication constraints in the average consensus problem," *Automatica*, vol.44, no. 3, pp. 671-684, 2008.

[32] S. Boyd, A. Ghosh, B. Prabhakar and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Information Theory*, vol. 52, no. 6, pp. 2508-2530, 2006.

[33] G. Shi, M. Johansson and K. H. Johansson, "When do gossip algorithms converge in finite time?" avaiable at http://arxiv.org/pdf/1206.0992v1.pdf.

[34] G. Shi and K. H. Johansson, "Convergence evolution of distributed averaging and maximizing algorithms: Part II: state-dependent graphs," *America Control Conference*, 2013.

[35] G. Shi and K. H. Johansson, "Finite-time and asymptotic convergence of distributed averaging and maximizing algorithms," submitted for journal publication, available at http://arxiv.org/pdf/1205.1733v2.pdf.