A Graph-Theoretic Approach to the $\mathcal{H}_\infty$ Performance of Leader–Follower Consensus on Directed Networks

Mohammad Pirani, Henrik Sandberg, and Karl Henrik Johansson

***Abstract*—We study a graph-theoretic approach to the $\mathcal{H}_\infty$ performance of leader following consensus dynamics in the presence of external disturbances when the underlying graph is a directed network. We first provide graph-theoretic necessary and sufficient conditions for the consensus dynamics to have the system $\mathcal{H}_\infty$ norm from external disturbances to the state of each agent to be lower than a certain value. Moreover, we discuss the tightness of the proposed conditions via examples. Then, we study the relation between the system $\mathcal{H}_\infty$ norm for directed and undirected networks for specific classes of graphs, i.e., balanced digraphs and directed trees. Moreover, we investigate the effects of adding directed edges to a directed tree on the resulting system $\mathcal{H}_\infty$ norm.**

***Index Terms*—Networked control systems, $\mathcal{H}_\infty$ performance, directed networks, leader-follower consensus.**

I. Introduction

A. Motivation

THE EXISTENCE of external communication disturbances during the plant operation is unavoidable in interconnected systems. The effect of such events can be either mitigated or propagated, depending on the structure of the underlying interaction network and can be amplified when the size of the plant (network) increases. In this direction, having knowledge about the network structure can help to address the resilience of such complex systems in a more efficient manner. This letter discusses the role of the interactions between agents on the robustness of the networked control system to communication disturbances when the underlying network is a directed graph.

Manuscript received March 1, 2019; revised May 21, 2019; accepted May 21, 2019. Date of publication May 29, 2019; date of current version June 10, 2019. This work was supported in part by the Knut and Alice Wallenberg Foundation, in part by the Swedish Foundation for Strategic Research, and in part by the Swedish Research Council. Recommended by Senior Editor C. Seatzu. *(Corresponding author: Mohammad Pirani.)*

The authors are with the Department of Automatic Control, KTH Royal Institute of Technology, 11428 Stockholm, Sweden (e-mail: pirani@kth.se; hsan@kth.se; kallej@kth.se).

Digital Object Identifier 10.1109/LCSYS.2019.2919832

B. Related Work

The notion of robustness of dynamical systems to external disturbances or parameter uncertainties has been under investigations for the past decades [1], [2]. In addition, there is a vast literature in studying the effects of network structure on both $\mathcal{H}_2$, [3]–[5], and $\mathcal{H}_\infty$ performances [6]–[8] in networked systems. Via combining the system-theoretic notions with algebraic graph theory, some works have looked at these performance metrics as network centrality measures and discussed the control node (leader) selection problems in a given large-scale network to optimize each performance metric [6], [9]. In this direction, analyzing directed (asymmetric) interactions between agents leads to some mathematical subtleties. Hence, graph-theoretical approaches to $\mathcal{H}_\infty$ performance of networked systems has received less attention [10]–[12].

C. Contributions

This letter analyzes the $\mathcal{H}_\infty$ performance of dynamical systems on directed networks. More specifically, the contributions of this letter are:

1) We present graph-theoretic measures for the $\mathcal{H}_\infty$ performance of leader-follower consensus dynamics on general directed networks (Proposition 2). Moreover, we discuss the tightness of the proposed graph-theoretic bounds on the system $\mathcal{H}_\infty$ norm via examples.
2) We discuss the relation between system $\mathcal{H}_\infty$ norms in directed and undirected networks for specific classes of networks, i.e., balanced digraphs (Theorem 1) and directed trees (Theorem 2). Moreover, we discuss the effect of the leaders' positions in directed trees on $\mathcal{H}_\infty$ performance of the system.
3) At the end, we study the effect of adding (or removing) directed edges to/from directed trees on the $\mathcal{H}_\infty$ performance of the system (Theorem 3) and show that unlike undirected networks, increasing connectivity does not necessarily increase the network robustness.

Motivating Application: One of the main motivations of the study in this letter is to quantify the effect of network topology on the performance of a specific vehicular network, called *predecessor-following* vehicle platoon, which has been investigated in the literature [13], [14]. During recent years, there has been numerous works on the effect of network topology, i.e., adding/removing certain links, in platoons on their ability to attenuate disturbances [15], [16]. In the predecessor-following structure, communications are unidirectional, in which every vehicle only uses information coming from its front vehicle (and not the vehicles from its back) for its control law. In [15], it was observed that increasing the platoon connectivity in an ad-hoc manner may deteriorate the performance of the system. This letter is an attempt to rigorously quantify the effects of the underlying network structure and connectivity as well as the position of the leader on the performance of leader-tracking consensus dynamics.

II. Notations and Definitions

We use $\mathcal{G}_d = \{\mathcal{V}, \mathcal{E}\}$ to denote an unweighted directed graph (digraph) where $\mathcal{V}$ is the set of vertices (or nodes) and $\mathcal{E}$ is the set of directed edges, i.e., $(v_i, v_j) \in \mathcal{E}$ if an only if there exists a directed edge from v_i to v_j. Moreover, an unweighted undirected graph $\mathcal{G}_u = \{\mathcal{V}, \mathcal{E}\}$ is a graph such that $(v_i, v_j) \in \mathcal{E}$ if an only if there exists an undirected edge between v_i and v_j. For directed graphs in this letter, we only consider unidirectional edges, i.e., if there exists a direct edge v_i to v_j, then there is no direct edge from v_j to v_i. Let $|\mathcal{V}| = n$ and define the adjacency matrix for $\mathcal{G}_d$, denoted by $A_{n\times n}$, to be a binary matrix where $A_{ij} = 1$ if and only if there is a directed edge from v_j to v_i in $\mathcal{G}_d$. The *neighbors* of vertex $v_i \in \mathcal{V}$ in graph $\mathcal{G}_d$ are given by the set $\aleph_i = \{v_j \in \mathcal{V} \mid (v_j, v_i) \in \mathcal{E}\}$. We define the in-degree for node v_i as $\Delta_i = \sum_{v_j\in\aleph_i} A_{ij}$ and the out-degree as $\delta_i = \sum_{v_j\in\aleph_i} A_{ji}$. An (unweighted) balanced digraph is a digraph that for each node v_i we have $\delta_i = \Delta_i$, i.e., it has the same in and out neighbors. For a symmetric matrix M, the eigenvalues are ordered as $\lambda_1(M) \le \lambda_2(M) \le \ldots \le \lambda_n(M)$ and the singular values of a matrix $\mathcal{M}$ are ordered as $\sigma_1(\mathcal{M}) \le \sigma_2(\mathcal{M}) \le \cdots \le \sigma_n(\mathcal{M})$. The Laplacian matrix of the graph is $\mathcal{L} = D - A$, where $D = \mathrm{diag}(\Delta_1, \Delta_2, \ldots, \Delta_n)$. We will be considering a nonempty subset of vertices $\mathfrak{S} \subset \mathcal{V}$ to be *leaders*, whose in-degree is zero, and assume without loss of generality that the leaders are placed last in an ordering of the agents. Vertices in $\mathcal{V} \setminus \mathfrak{S}$ are called *followers*. The *grounded Laplacian* induced by the leader set $\mathfrak{S}$ is denoted by $\mathcal{L}_g(\mathfrak{S})$ or simply $\mathcal{L}_g$, and is obtained by removing the rows and columns of $\mathcal{L}$ corresponding to the nodes in $\mathfrak{S}$ [17]. If the underlying graph is directed, the grounded Laplacian is denoted by $\mathcal{L}_{g,d}$ and if the graph is undirected, it is $\mathcal{L}_{g,u}$. The state-space representation of a linear time-invariant system with n states, m inputs and k outputs is denoted by the triple $(A_{n\times n}, B_{n\times m}, C_{k\times n})$, where A is the state matrix, B is the input matrix and C is the output matrix. We use $\mathbf{e}_i$ to indicate the i-th vector of the canonical basis. We denote $\lfloor x\rfloor$ the largest integer less than x and $\lceil x\rceil$ the smallest integer larger than x.

III. Problem Statement

Consider a connected network consisting of n agents $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$. The set of agents is partitioned into a set of followers $\mathfrak{F}$, and a set of leaders[1] $\mathfrak{S}$. We assume that there exists at least one leader in the network. The number of leaders which are directly connected to follower v_i is denoted by Γ_i, i.e., $\Gamma_i \triangleq |\aleph_i \cap \mathfrak{S}|$. Examples of such directed graphs are shown in Fig. 1, where black nodes are leaders and white nodes are the followers. Each agent v_i has a scalar and real valued state $\psi_i(t)$, where t is the time index. The state of each follower agent $v_j \in \mathfrak{F}$ evolves based on the interactions with its neighbors as

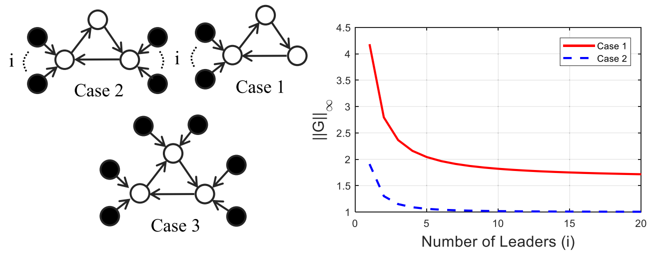


Fig. 1. Examples showing the applicability of sufficient condition (5).

$$\dot{\psi}_j(t) = \sum_{v_i\in\aleph_j} (\psi_i(t) - \psi_j(t)). \tag{1}$$

Since the leaders state is not influenced by the followers, it is assumed to be constant and thus

$$\dot{\psi}_j(t) = 0, \quad \forall v_j \in \mathfrak{S}. \tag{2}$$

Aggregating the states of all followers into a vector $\boldsymbol{\psi}_{\mathfrak{F}}(t) \in \mathbb{R}^{n-|\mathfrak{S}|}$, and the states of all leaders into a vector $\boldsymbol{\psi}_{\mathfrak{S}}(t) \in \mathbb{R}^{|\mathfrak{S}|}$ (note that $\boldsymbol{\psi}_{\mathfrak{S}}(t) = \boldsymbol{\psi}_{\mathfrak{S}}(0)$ for all $t \ge 0$), equations (1) and (2) yield the following dynamics for followers

$$\dot{\boldsymbol{\psi}}_{\mathfrak{F}}(t) = -\mathcal{L}_{g,d}\boldsymbol{\psi}_{\mathfrak{F}}(t) + \mathcal{L}_{12}\boldsymbol{\psi}_{\mathfrak{S}}(0). \tag{3}$$

Here, $\mathcal{L}_{g,d}$ is the grounded Laplacian matrix, representing the interaction between the followers. When the graph is undirected, the grounded Laplacian is denoted by $\mathcal{L}_{g,u}$. The submatrix $\mathcal{L}_{12}$ of the graph Laplacian captures the influence of the leaders on the followers. We make the following assumption in this letter.

Assumption 1: In the directed graph $\mathcal{G}_d$, every follower can be reached through a directed path from some leader.[2]

If Assumption 1 holds, the states of the follower agents will converge to some convex combination of the states of the leaders [21]. Moreover, under Assumption 1, it is shown that the grounded Laplacian matrix is non-singular and $\lambda_1(\mathcal{L}_{g,d})$ is real and strictly positive and $\mathcal{L}_{g,d}^{-1}$ is a non-negative matrix [11]. In addition to the nominal dynamics (3), we assume that there exists some disturbances (or perturbations) in the communications between the followers. In particular, consider the updating rule of each follower agent $v_j \in \mathfrak{F}$ is affected by a disturbance signal $w_j(t)$ which turns (3) into

$$\begin{aligned}\dot{\boldsymbol{\psi}}_{\mathfrak{F}}(t) &= -\mathcal{L}_{g,d}\boldsymbol{\psi}_{\mathfrak{F}}(t) + \mathcal{L}_{12}\boldsymbol{\psi}_{\mathfrak{S}}(0) + \boldsymbol{w}(t),\\ z(t) &= \boldsymbol{\psi}_{\mathfrak{F}}(t),\end{aligned} \tag{4}$$

[1]These agents may also be referred to as *anchors* [18] or *stubborn agents* [19] depending on the context.

[2]Digraphs which satisfy Assumption 1 and do not have directed cycle are called *rooted-out-branching* in the literature [20].

where $z(t)$ is the (full state) measurement. Here $\boldsymbol{w}(t)$ is a vector representing the disturbances. We assume that all followers are prone to be affected by the disturbances while the leaders are unaffected by the disturbances, since they do not update their state. The objective is to quantify the effect of the external disturbance signals on the state of the follower agents. Let $\bar{\boldsymbol{\psi}}_{\mathfrak{F}}(t)$ denote the value of $\boldsymbol{\psi}_{\mathfrak{F}}(t)$ when there is no disturbance signal, and define the error term $\boldsymbol{e}(t) = \bar{\boldsymbol{\psi}}_{\mathfrak{F}}(t) - \boldsymbol{\psi}_{\mathfrak{F}}(t)$. We use the system $\mathcal{H}_\infty$ norm of the transfer function $G(s) = (sI + \mathcal{L}_{g,d})^{-1}$ from $\boldsymbol{w}(t)$ to $\boldsymbol{e}(t)$ defined as $||G||_\infty \triangleq \sup_{\omega \in \mathbb{R}} \sigma_{n-|\mathfrak{S}|}(G(j\omega))$. Since $\mathcal{L}_{g,d}$ is Hurwitz, this norm is finite. We refer to $||G||_{\infty,u}$ and $||G||_{\infty,d}$ as system $\mathcal{H}_\infty$ norms when the underlying graph is undirected or directed, respectively. Before discussing this system norm, we present the following definitions.

Definition 1 (Positive Systems): A linear system is called (internally) positive if and only if its state and output are non-negative for every non-negative input and every non-negative initial state.

Proposition 1 [22]: A continuous linear system (A, B, C) is positive if A is a Metzler-matrix and B and C are non-negative element-wise. Moreover, for such a positive system with transfer function $G(s) = C(sI - A)^{-1}B$, the system $\mathcal{H}_\infty$ norm is obtained from the DC gain of the system, i.e., $||G||_\infty = \sigma_n(G(0))$, where σ_n is the maximum singular value of matrix $G(0)$.

It is clear that the evolution of follower agents (4) together with full state measurements form a positive system. According to Proposition 1, the system $\mathcal{H}_{\infty,d}$ norm from external disturbances to state error of followers is $||G||_\infty = \sigma_{n-|\mathfrak{S}|}(\mathcal{L}_{g,d}^{-1}) = \frac{1}{\sigma_1(\mathcal{L}_{g,d})}$. Hence, characterizing the system $\mathcal{H}_\infty$ norm of (4) is equivalent to determining the smallest singular value (or the eigenvalue for undirected networks) of the grounded Laplacian matrix, $\sigma_1(\mathcal{L}_{g,d})$.

IV. Graph-Theoretic Conditions for $||G||_{\infty,d} \leq \gamma$ in Directed Networks

In this section, we present graph-theoretic conditions for the error dynamics of (4) to have sufficiently small $\mathcal{H}_\infty$ norm. The proof of the following proposition is omitted due to space limitation and it is presented [23].

Proposition 2: Let Γ_i be the number of leaders in follower v_i's neighborhood. A *sufficient* condition for the error dynamics of (4) to have $||G||_{\infty,d} \leq \gamma$ is

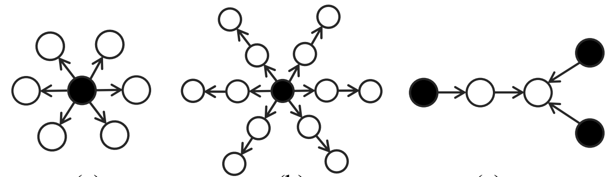
$$\min_{v_i \in \mathcal{V} \setminus \mathfrak{S}} \{\frac{\Delta_i - \delta_i + \Gamma_i}{2}\} \geq \lceil \frac{1}{\gamma} \rceil. \quad (5)$$

Moreover, in order to have $||G||_{\infty,d} \leq \gamma$ it is *necessary* to satisfy both conditions

$$\sum_{i=1}^{n-|\mathfrak{S}|} \Gamma_i^2 \geq \lfloor \frac{n - |\mathfrak{S}|}{\gamma^2} \rfloor, \quad \min_{v_i \in \mathcal{V} \setminus \mathfrak{S}} \{\Delta_i^2 + \delta_i\} \geq \lfloor \frac{1}{\gamma^2} \rfloor. \quad (6)$$

A. Discussion on Proposition 2

The conditions mentioned in Proposition 2 can give clues in designing networks with desired robustness. The following example shows the applicability of the sufficient condition (5).



(a) (b) (c)

Fig. 2. Examples of graphs which show the tightness of conditions (5) and (6).

Example 1 (Evaluating Sufficient Condition (5)): Suppose that a network with fixed topology (directed cycles of size 3 as shown in Fig. 1) is given and the problem is to add leaders to one or two nodes in order to ensure that the system is non-expansive, i.e., $||G||_{\infty,d} \leq 1$.[3] For both Case 1 and Case 2 in this figure, necessary conditions (6) hold when we add two leaders or more. However, based on sufficient condition (5) we should have $\min_{v_i \in \mathcal{V} \setminus \mathfrak{S}} \{\Delta_i - \delta_i + \Gamma_i\} \geq 2$ to ensure that $||G||_{\infty,d} \leq 1$. The graphs shown in Fig. 1 do not satisfy this sufficient condition. Here, even if we add many leaders to one or two of the nodes (Cases 1 and 2) we can not get $||G||_{\infty,d} \leq 1$ (as shown in the plot of Fig. 1, right, for actual values of $||G||_{\infty,d}$). However, for sufficient condition (5) to satisfy we need to add two leaders to each node to get $||G||_{\infty,d} \leq 1$.

The following example discusses the tightness of necessary conditions mentioned in Proposition 2.

Example 2 (Evaluating Necessary Conditions (6)): Consider again that the objective is to have $||G||_{\infty,d} \leq 1$, i.e., $\gamma = 1$. Two networks are shown in Fig. 2. For graph (a) both the sufficient and necessary conditions are tight, i.e., inequalities (5) and (6) become equality, and we get $||G||_{\infty,d} = 1$. For graph (b), the necessary condition (6), right, is satisfied, however, (6), left, does not hold. By calculating the actual value of the norm, we see that $||G||_{\infty,d} = 1.67 > 1$ which shows the applicability of necessary conditions. Moreover, in some cases, condition (6), right, is tighter. An example of this is graph (c). Here, for the left condition in (6) we have $\frac{\sum_{i=1}^{n-|\mathfrak{S}|} \Gamma_i^2}{n-|\mathfrak{S}|} = \frac{5}{2}$ and for the right condition we have $\min_{v_i \in \mathcal{V} \setminus \mathfrak{S}} \{\Delta_i^2 + \delta_i\} = 2$. Thus, the right condition provides a tighter upper bound for $\lfloor \frac{1}{\gamma^2} \rfloor$.

Proposition 2 provides graph-theoretic conditions for $\mathcal{H}_\infty$ performance of (4) on digraphs. However, it does not reflect the improvement of the robustness on digraphs compared to that on undirected graphs. The following section discusses this problem.

V. Relations Between $\mathcal{H}_\infty$ Norm of Directed and Undirected Networks

In this section, we compare the system $\mathcal{H}_\infty$ norms in some directed graphs and their undirected counterparts. Since there is no specific relationship between the two cases in general graphs, we focus on particular classes of networks, namely *balanced digraphs* and *directed trees*, for which we can derive explicit expressions for the relation between system $\mathcal{H}_\infty$ norms in directed and undirected networks.

[3] Adding leaders to node v physically means to increase the weight of the edge between v and the grounded node (leader).

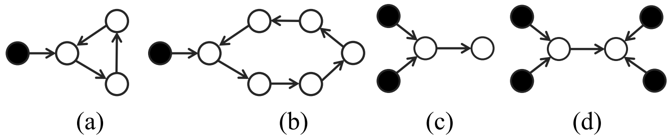


Fig. 3. **Balanced Digraphs**: (a) example showing $||G||_{\infty,d} > ||G||_{\infty,u}$, (b) example showing $||G||_{\infty,d} < ||G||_{\infty,u}$. **Directed Trees**: (c) example showing $||G||_{\infty,d} < ||G||_{\infty,u}$, (d) example showing $||G||_{\infty,d} > ||G||_{\infty,u}$.

A. Balanced Digraphs

Balanced Digraphs are directed graphs for which the in-degree and out-degree of each node are equal. For these graphs, we will show that the system $\mathcal{H}_\infty$ norm of a directed network is no worst than twice of that of the undirected network.

Theorem 1: Consider a directed graph $\mathcal{G}_d$ with leader set $\mathfrak{S}$ which satisfies Assumption 1. If the subgraph of the follower agents is balanced, then the system $\mathcal{H}_\infty$ norm of (4) satisfies $\|G\|_{\infty,d} \le 2||G||_{\infty,u}$.

Proof: From [24] we know that the $\mathcal{H}_\infty$ norm of a positive system with asymmetric interactions is upper bounded by the $\mathcal{H}_\infty$ norm of the symmetric parts of the dynamic matrix, i.e.,

$$\|G\|_{\infty,d} \le \frac{1}{|\lambda_1(\frac{\mathcal{L}_{g,d}+\mathcal{L}_{g,d}^T}{2})|}. \tag{7}$$

Moreover, for balanced graphs we have $\mathcal{L}_{g,d}+\mathcal{L}_{g,d}^T = \bar{\mathcal{L}}_g + E$ for some diagonal and positive semidefinite matrix E, where $\bar{\mathcal{L}}_g$ is the grounded Laplacian matrix corresponding to the undirected network. Thus based on Weyl's inequality we have $\lambda_1(\bar{\mathcal{L}}_g) \le \lambda_1(\mathcal{L}_{g,d}+\mathcal{L}_{g,d}^T)$, and together with (7) the result is obtained. ■

The following example shows that one can not modify Theorem 1 to get $\|G\|_{\infty,d} \le ||G||_{\infty,u}$ in balanced digraphs.

Example 3: As shown in Fig. 3 (a), which is a balanced graph of followers, we have $||G||_{\infty,u} = 3.73$ and $||G||_{\infty,d} = 4.18$. Moreover, if we increase the length of the loop from 3 to 6, Fig. 3 (b), we have $||G||_{\infty,u} = 9.19 > 8.85 = ||G||_{\infty,d}$. This shows that the bound proposed in Theorem 1 is tight and $||G||_{\infty,d} < ||G||_{\infty,u}$ does not always hold.

We should note that, as shown in [25], changing the direction of the edges in a balanced digraph does not change the system $\mathcal{H}_\infty$ norm.

B. Directed Trees

In this section, we focus on directed networks whose undirected counterparts are trees, i.e., connected graphs without cycles. In the following theorem, we discuss the relation between the system $\mathcal{H}_\infty$ norm of (4) in directed and undirected trees. The proof is in Appendix A.

Theorem 2: Consider a directed graph $\mathcal{G}_d$ with leader set $\mathfrak{S}$ which satisfies Assumption 1. If the subgraph of the followers is a tree, the system $\mathcal{H}_\infty$ norm of (4) satisfies

$$\frac{1}{\min_{i\in\mathcal{V}\setminus\mathfrak{S}} \Delta_i} \le ||G||_{\infty,d} \le ||G||_{\infty,u}^{\frac{1}{2}}, \tag{8}$$

with $||G||_{\infty,d} = ||G||_{\infty,u}^{\frac{1}{2}}$ if there exists a single leader, i.e., $|\mathfrak{S}| = 1$.

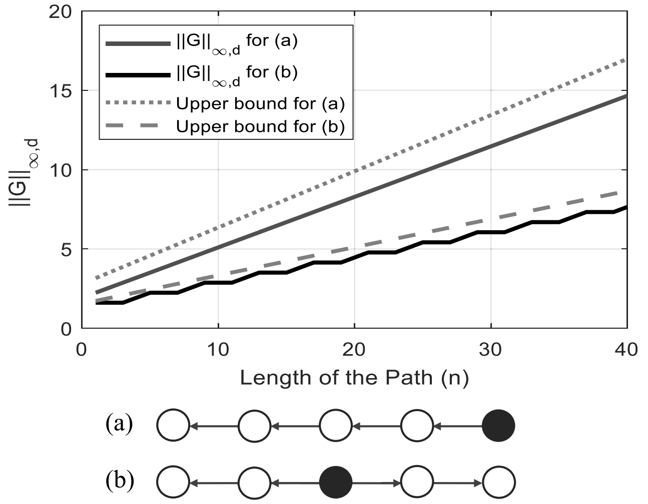


Fig. 4. The effect of the leader's location on the $\mathcal{H}_\infty$ performance on directed path graphs.

Example 4: As shown in Fig. 3 (c) and (d), the system $\mathcal{H}_\infty$ norm of the directed tree with multiple leaders can be larger or smaller than that of the undirected graph, depends on if $||G||_{\infty,d} \ge 1$ or not. In this figure, for graph (c) we have $||G||_{\infty,d} = 1.14 < 1.7 = ||G||_{\infty,u}$ and for graph (d) we have $||G||_{\infty,d} = 0.54 > 0.5 = ||G||_{\infty,u}$. The lower bound mentioned in Theorem 2 is $\frac{1}{\min_{i\in\mathcal{V}\setminus\mathfrak{S}} \Delta_i} = 1$ for graph (c) and $\frac{1}{\min_{i\in\mathcal{V}\setminus\mathfrak{S}} \Delta_i} = 0.5$ for graph (d) which shows that it is close to the actual value of $||G||_{\infty,d}$.

Theorem 2 yields the following corollary which provides an upper bound on $||G||_{\infty,d}$ based on the distances of leaders to the rest of the nodes in the network. Let $\mathcal{C}(v)$ be the sum of the distances of all (shortest) paths from leader v to the rest of the follower nodes in the network. The proof of this corollary is omitted due to space limitation and is included in [23].

Corollary 1: Consider a directed graph $\mathcal{G}_d$ with leader set $\mathfrak{S}$ which satisfies Assumption 1. If the subgraph of the followers is a tree, the system $\mathcal{H}_\infty$ norm of (4) satisfies $||G||_{\infty,d} \le \min_{v\in\mathfrak{S}} \mathcal{C}(v)^{\frac{1}{2}}$.

The following example discusses the effect of the leader's location on system $\mathcal{H}_\infty$ norm based on Corollary 1.

Example 5: Fig. 4 shows two digraphs where the leader in (a) is located in one end and in (b) the leader is located in the middle. For case (a) with n followers we have $\mathcal{C}(v) = \frac{n(n+1)}{2}$ and for (b) we have $\mathcal{C}(v) = \frac{n(n+2)}{8}$.[4] The plot shown in Fig. 4 shows the bounds predicted by Corollary 1 and the actual value of $||G||_{\infty,d}$ which shows the tightness of this upper bound as well as the effect of the leader's location in the tree network on $\mathcal{H}_\infty$ performance.

C. Effect of Adding Edges to Directed Trees

In the previous subsection, we discussed directed graphs whose undirected counterpart is a tree. In this subsection, we consider the effect of adding extra directed edges to a directed

[4] Assuming that n is even, otherwise the right and left hand side of the leader do not have the same number of followers.

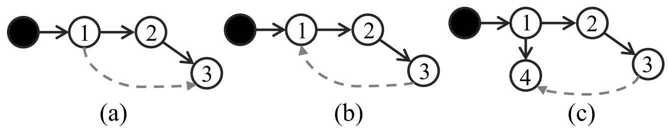


Fig. 5. (a), (b) Adding an edge to a directed path with opposite directions, (c) Adding an edge to a directed tree.

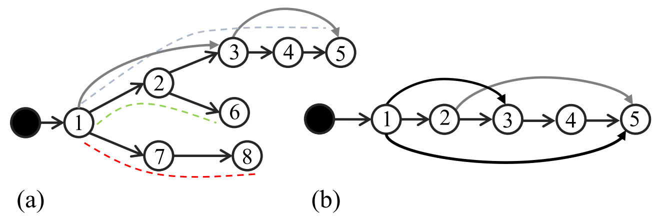


Fig. 6. (a) Example of a tree with three leader-rooted paths, (b) Example of additional interfering paths.

tree on the system $\mathcal{H}_\infty$ norm. We present an observation in Fig. 5. For graph (a), an additional directed edge (grey dashed line) is added to a directed path which does not make a directed cycle; however, for case (b) the additional edge makes a cycle. The system $\mathcal{H}_\infty$ norm of (4) for the path graph (before adding the directed edge) is $||G||_{\infty,d} = 2.25$, for case (a) is $||G||_{\infty,d} = 1.99$ and for case (b) is $||G||_{\infty,d} = 4.18$. Hence, it implies that adding a directed edge to a directed path to make a cycle deteriorates the $\mathcal{H}_\infty$ performance while adding an edge which does not make a cycle improves the performance. This observation is intuitive, since adding a cycle to the network results in the information (and uncertainties) to circulate (and thus propagate) in a part of the network. However, the opposite is not true for general trees, as shown in graph (c). In particular, for general trees, the $\mathcal{H}_\infty$ performance can be deteriorated even for an edge addition which does not make a cycle in the network. For graph (c) in Fig. 5, before adding a directed edge we have $||G||_{\infty,d} = 2.40$ and after adding that (which does not make a cycle) we have $||G||_{\infty,d} = 2.56$. Based on these observations, we present a result in Theorem 3. Before that, we need the following definition.

Definition 2 (Leader-Rooted Path and Interfering Edges): A *leader-rooted-path* in a directed tree is a directed path which starts from a leader's neighbor and ends at one of the leaf nodes, i.e., nodes with $\Delta_i = 1$ and $\delta_i = 0$. Moreover, given a leader-rooted-path $\mathcal{P}$, additional two directed edges to $\mathcal{P}$, named (v_i, v_j) and (v_k, v_h), are called interfering if the paths from v_i to v_j and from v_k to v_h in $\mathcal{P}$ do not have an edge in common.

Three different leader-rooted-paths in a directed tree are shown in Fig. 6 (a) by dashed lines. Moreover, the two additional edges (in grey) are not interfering. However, in Fig. 6 (b), the grey and the black edges are interfering. Figure 6 (b) is an example which shows that even for a path graph, if the additional edges are interfering, then they may increase the $\mathcal{H}_\infty$ norm. In this example, before adding the grey edge, we have $||G||_{\infty,d} = 2.65$ and after that we have $||G||_{\infty,d} = 2.66$. Based on the above observations and definitions, we will present Theorem 3 which is proven in Appendix B.

Theorem 3: Consider a directed graph $\mathcal{G}_d$ with leader set $\mathfrak{S}$ which satisfies Assumption 1 and suppose the subgraph of the followers is a tree. Let $||G||_\infty$ be the $\mathcal{H}_\infty$ norm of (4) on $\mathcal{G}_d$. Then for any leader rooted path $\mathcal{P}$ in $\mathcal{G}_d$ if a set of non-interfering edges $\bar{\mathcal{E}}$ is added, then the resulting system $\mathcal{H}_\infty$ norm, $||G||_{\infty,d,\bar{\mathcal{E}}}$, satisfies $||G||_{\infty,d,\bar{\mathcal{E}}} \leq ||G||_{\infty,d}$ if $\bar{\mathcal{E}}$ does not make a cycle in $\mathcal{G}_d$. Moreover, we have $||G||_{\infty,d,\bar{\mathcal{E}}} \geq ||G||_{\infty,d}$ if all edges in $\bar{\mathcal{E}}$ make cycles in $\mathcal{G}_d$.

Theorem 3 proved a general scenario which includes observations in Fig. 5 (a) and (b). Clearly, in graph shown in Fig. 5 (c) nodes 3 and 4 are not in the same leader-rooted path; hence, the sufficient condition mentioned in Theorem 3 does not hold. Observations in Fig. 5 together with Theorem 3 show that, unlike undirected networks [6], increasing connectivity in directed networks does not lead to increasing the robustness of the system to disturbances.

VI. Conclusion

In this letter, a graph-theoretic approach to the $\mathcal{H}_\infty$ performance of leader following consensus dynamics on directed graphs was studied. The relation between the system $\mathcal{H}_\infty$ norm for directed and undirected networks for specific classes of graphs, i.e., balanced digraphs and directed trees, was discussed. Moreover, the effects of adding directed edges to a directed tree on the resulting system $\mathcal{H}_\infty$ norm was investigated. A future avenue for research is to extend these results to the $\mathcal{H}_\infty$ performance of second-order consensus dynamics on directed networks and apply those results to cooperative adaptive cruise control algorithms for vehicular platooning, supported by various numerical simulations.

Appendix A
Proof of Theorem 2

For the case where there exists a single leader in the network, for each follower node there is exactly one incoming edge, as otherwise a cycle will be made in the underlying undirected graph. If we write the grounded Laplacian matrices of directed and undirected graphs with a single leader by $\hat{\mathcal{L}}_{g,d}$ and $\hat{\mathcal{L}}_{g,u}$, respectively, we know that $\hat{\mathcal{L}}_{g,d}$ is triangular with diagonal elements 1. In this case we have $\hat{\mathcal{L}}_{g,d}^T\hat{\mathcal{L}}_{g,d} = \hat{\mathcal{L}}_{g,u}$. It is due to the fact that each diagonal element of $\hat{\mathcal{L}}_{g,d}^T\hat{\mathcal{L}}_{g,d}$ is $\Delta_i^2 + \delta_i = 1 + \delta_i$ which is equal to the degree of each node in the undirected network. Moreover, every off-diagonal element $[\hat{\mathcal{L}}_{g,d}^T\hat{\mathcal{L}}_{g,d}]_{ij}$ is generated by multiplying i-th row of $\hat{\mathcal{L}}_{g,d}^T$ to j-th column of $\hat{\mathcal{L}}_{g,d}$. This is either -1, when there is a directed edge from v_i to v_j, or 0 otherwise. Hence, we have $\lambda_1(\hat{\mathcal{L}}_{g,d}^T\hat{\mathcal{L}}_{g,d}) = \lambda_1(\hat{\mathcal{L}}_{g,u})$ which shows that the $\mathcal{H}_{\infty,u}$ norm for the undirected graph is equal to the square of its directed version, i.e., $||G||_{\infty,u} = ||G||_{\infty,d}^2$. For $|\mathfrak{S}| > 1$ we can write the grounded Laplacians as $\mathcal{L}_{g,d} = \hat{\mathcal{L}}_{g,d} + E$ and $\mathcal{L}_{g,u} = \hat{\mathcal{L}}_{g,u} + E$, where E shows the effect of the rest of the leaders. Considering the fact that $E^TE \succeq E$, i.e., $E^TE - E$ is positive semidefnite, and $\hat{\mathcal{L}}_{g,d}^TE + E^T\hat{\mathcal{L}}_{g,d}$ is also positive semidefnite, we get

$$\begin{aligned}\lambda_1(\mathcal{L}_{g,d}^T\mathcal{L}_{g,d}) &= \lambda_1(\hat{\mathcal{L}}_{g,d}^T\hat{\mathcal{L}}_{g,d} + \hat{\mathcal{L}}_{g,d}^TE + E^T\hat{\mathcal{L}}_{g,d} + E^TE) \\ &\geq \lambda_1(\hat{\mathcal{L}}_{g,u} + E) = \lambda_1(\mathcal{L}_{g,u}), \qquad (9)\end{aligned}$$

which yields $||G||_{\infty,d}^2 \leq ||G||_{\infty,u}$.

For the lower bound, via an appropriate permutation of rows, matrix $\mathcal{L}_{g,d}$ can be put into a triangular form. Then we have $\lambda_1(\mathcal{L}_{g,d}) = \min_{i\in\mathcal{V}\setminus\mathfrak{S}} \Delta_i$ (the minimum in-degree in the subgraph of followers). According to the fact that $\sigma_1(\mathcal{L}_{g,d}) \leq \lambda_1(\mathcal{L}_{g,d})$, we have $||G||_{\infty,d} \geq \frac{1}{\min_{i\in\mathcal{V}\setminus\mathfrak{S}} \Delta_i}$. ■

Appendix B
Proof of Theorem 3

Before proving the theorem, we have the following lemma.

Lemma 1 [26]: If one element in a non-negative matrix A is increased, then the largest eigenvalue is also increased. The increase is strict for irreducible matrices.

According to Lemma 1, we can conclude that if one element in a non-negative matrix A is decreased (but still be positive), then the largest eigenvalue is also decreased. Now, we prove Theorem 3.

Proof: When an edge is added from node j to node i, we can write the new Laplacian matrix as $\tilde{\mathcal{L}}_{g,d} = \mathcal{L}_{g,d} + \mathbf{e}_i\mathbf{e}_{ij}^T$, where $\mathbf{e}_i$ is a vector which is 1 in $i-th$ place and zero elsewhere and $\mathbf{e}_{ij} = \mathbf{e}_i - \mathbf{e}_j$. We know that $\tilde{\mathcal{L}}_{g,d}^{-1}$ is a non-negative matrix. Without loss of generality, we label the nodes in the leader-rooted-path containing i and j as $1, 2, \ldots, j, \ldots, i, \ldots, r$, where 1 belongs to the leader's neighbor and r is the length of the path. First, we will show that adding a directed edge from j to i decreases some elements of $\tilde{\mathcal{L}}_{g,d}^{-1}$ and adding an edge from i to j increases them. If we use Sherman-Morrison formula [27] we get

$$\tilde{\mathcal{L}}_{g,d}^{-1} = \left(\mathcal{L}_{g,d} + \mathbf{e}_i\mathbf{e}_{ij}^T\right)^{-1} = \mathcal{L}_{g,d}^{-1} - \frac{\mathcal{L}_{g,d}^{-1}\mathbf{e}_i\mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}}{1 + \mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}\mathbf{e}_i}. \quad (10)$$

As both nodes i and j are in the same leader-rooted-path, the block of matrix $\mathcal{L}_{g,d}^{-1}$ from row 1 to row r and column 1 to column r is in the form of a lower triangular matrix whose lower triangle elements are all 1 (it can be easy verified by solving the corresponding block in $\mathcal{L}_{g,d}^{-1}$ from $\mathcal{L}_{g,d}\mathcal{L}_{g,d}^{-1} = I$). Hence, $\mathcal{L}_{g,d}^{-1}\mathbf{e}_i\mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}$ is a non-negative matrix and $\mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}\mathbf{e}_i = 1$. Thus, based on (10), it implies that $\tilde{\mathcal{L}}_{g,d}^{-1}$ is non-negative and its elements are not larger than those of $\mathcal{L}_{g,d}^{-1}$. Likewise, the elements of $\tilde{\mathcal{L}}_{g,d}^{-T}\tilde{\mathcal{L}}_{g,d}^{-1}$ are not larger than the elements of $\mathcal{L}_{g,d}^{-T}\mathcal{L}_{g,d}^{-1}$ and based on Lemma 1 we conclude that $\lambda_{n-|\mathfrak{S}|}(\mathcal{L}_{g,d}^{-T}\mathcal{L}_{g,d}^{-1}) \geq \lambda_{n-|\mathfrak{S}|}(\tilde{\mathcal{L}}_{g,d}^{-T}\tilde{\mathcal{L}}_{g,d}^{-1})$ which proves the claim. For the case where the additional edge makes a cycle, i.e., from i to j, with the similar argument we can show that the elements of $\frac{\mathcal{L}_{g,d}^{-1}\mathbf{e}_i\mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}}{1+\mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}\mathbf{e}_i}$ are non positive and the rest of the proof is similar.

If another edge, called (v_x, v_y), is added to the same leader-rooted path which is not interfering with edge (v_j, v_i), since $\mathcal{L}_{g,d}^{-1}\mathbf{e}_i\mathbf{e}_{ij}^T\mathcal{L}_{g,d}^{-1}$ affects columns $j+1$ to i and we have $x \geq i$, then the block of matrix $\mathcal{L}_{g,d}^{-1}$ from row x to row y and column x to column y is not affected by the previous edge (v_j, v_i) and the rest of the proof is the same. ■

References

[1] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.

[2] G. E. Dullerud and F. Paganini, *A Course in Robust Control Theory: A Convex Approach*. New York, NY, USA: Springer-Verlag, 2010.

[3] A. Chapman and M. Mesbahi, "Semi-autonomous consensus: Network measures and adaptive trees," *IEEE Trans. Autom. Control*, vol. 58, no. 1, pp. 19–31, Jan. 2013.

[4] M. Siami and N. Motee, "Fundamental limits and tradeoffs on disturbance propagation in linear dynamical networks," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4055–4062, Dec. 2016.

[5] G. F. Young, L. Scardovi, and N. E. Leonard, "Robustness of noisy consensus dynamics with directed communication," in *Proc. Amer. Control Conf.*, 2010, pp. 6312–6317.

[6] M. Pirani, E. M. Shahrivar, B. Fidan, and S. Sundaram, "Robustness of leader–follower networked dynamical systems," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 4, pp. 1752–1763, Dec. 2018.

[7] P. Lin, Y. Jia, and L. Li, "Distributed robust H_∞ consensus control in directed networks of agents with time-delay," *Syst. Control Lett.*, vol. 57, no. 8, pp. 643–653, 2008.

[8] I. Herman, D. Martinec, Z. Hurák, and M. Sebek, "Nonzero bound on Fiedler eigenvalue causes exponential growth of H-infinity norm of vehicular platoon," *IEEE Trans. Autom. Control*, vol. 60, no. 8, pp. 2248–2253, Aug. 2015.

[9] K. E. Fitch and N. E. Leonard, "Joint centrality distinguishes optimal leaders in noisy networks," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 4, pp. 366–378, Dec. 2016.

[10] G. F. Young, L. Scardovi, and N. E. Leonard, "A new notion of effective resistance for directed graphs—Part I: Definition and properties," *IEEE Trans. Autom. Control*, vol. 61, no. 7, pp. 1727–1736, Jul. 2016.

[11] W. Xia and M. Cao, "Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks," *Automatica*, vol. 80, pp. 10–16, Jun. 2017.

[12] G. Giordano, F. Blanchini, E. Franco, V. Mardanlou, and P. L. Montessoro, "The smallest eigenvalue of the generalized Laplacian matrix, with application to network-decentralized estimation for homogeneous systems," *IEEE Trans. Netw. Sci. Eng.*, vol. 3, no. 4, pp. 312–324, Oct./Dec. 2016.

[13] H. Hao and P. Barooah, "Stability and robustness of large platoons of vehicles with double-integrator models and nearest neighbor interaction," *Int. J. Robust Nonlin. Control*, vol. 23, no. 18, pp. 2097–2122, 2013.

[14] A. A. Peters, R. H. Middleton, and O. Mason, "Leader tracking in homogeneous vehicle platoons with broadcast delays," *Automatica*, vol. 50, no. 1, pp. 64–74, 2014.

[15] J. I. Ge and G. Orosz, "Dynamics of connected vehicle systems with delayed acceleration feedback," *Transp. Res. C Emerg. Technol.*, vol. 46, pp. 46–64, Sep. 2014.

[16] L. Zhang and G. Orosz, "Motif-based design for connected vehicle systems in presence of heterogeneous connectivity structures and time delays," *IEEE Trans. Intell. Transp. Syst.*, vol. 17, no. 6, pp. 1638–1651, Jun. 2016.

[17] M. Pirani and S. Sundaram, "Spectral properties of the grounded Laplacian matrix with applications to consensus in the presence of stubborn agents," in *Proc. Amer. Control Conf.*, 2014, pp. 2160–2165.

[18] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 162–186, 2009.

[19] J. Ghaderi and R. Srikant, "Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate," *Automatica*, vol. 50, no. 12, pp. 3209–3215, 2014.

[20] S. G. Williamson, *Combinatorics for Computer Science*. Mineola, NY, USA: Dover, 1985.

[21] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, *Submodularity in Dynamics and Control of Networked Systems*. Cham, Switzerland: Springer, 2016.

[22] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. New York, NY, USA: Wiley, 2000.

[23] M. Pirani, H. Sandberg, and K. H. Johansson, "A graph-theoretic approach to the $\mathcal{H}_\infty$ performance of dynamical systems on directed and undirected networks," *arXiv preprint arXiv:1804.10483*, 2018.

[24] N. K. Dhingra, X. Wu, and M. R. Jovanovic, "Sparsity-promoting optimal control of systems with invariances and symmetries," in *Proc. 10th IFAC Symp. Nonlin. Control Syst.*, vol. 53, 2016, pp. 648–653.

[25] N. K. Dhingra, M. Colombino, and M. R. Jovanović, "Leader selection in directed networks," in *Proc. 55th Conf. Decis. Control (CDC)*, vol. 53, 2016, pp. 2715–2720.

[26] P. V. Mieghem, *Graph Spectra for Complex Networks*. Cambridge, U.K.: Cambridge Univ. Press, 2010.

[27] P. W. H. Flannery and S. A. Teukolsky, *Numerical Recipes. The Art of Scientific Computing*. Cambridge, U.K.: Cambridge Univ. Press, 1986.