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Distributed event-triggered control for global consensus of multi-agent systems with input saturation ☆

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ABSTRACT

The global consensus problem for first-order continuous-time multi-agent systems with input saturation is considered. In order to reduce the overall need of communication and system updates, we propose an event-triggered consensus protocol and a triggering law, which do not require any a priori knowledge of global network parameters. It is shown that Zeno behavior is excluded for these systems and that the underlying directed graph having a directed spanning tree is a necessary and sufficient condition for global consensus. We use a new Lyapunov function to show the sufficient condition and it inspires the triggering law. Numerical simulations are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

In the past decades, consensus in multi-agent systems has been extensively investigated. In this setup, each agent updates its state based on its own state and the states of its neighbors in such a way that the final states of all agents converge to a common value. In particular, for continuous-time consensus, each agent updates its state continuously (e.g., Liu, Lu, & Chen, 2011; Olfati-Saber & Murray, 2004; Ren, Beard, & Atkins, 2007; You & Xie, 2011; Yuan, Stan, Shi, Barahona, & Goncalves, 2013). It is known that consensus is achieved if and only if the underlying fixed directed graph has a directed spanning tree (Cao, Morse, & Anderson, 2008; Ren & Beard, 2005).

However, generally physical systems are subject to physical constraints, such as input, output, communication, and sensor constraints. These constraints normally lead to nonlinearities in the closed-loop dynamics. Thus the behavior of each agent is affected and special attention to the constraints needs to be taken in order to understand their influence on the consensus convergence. Some recent investigations on this problem include, for example, Li, Xiang, and Wei (2011) who consider the global consensus problem for multi-agent systems with input saturation; Meng, Zhao, and Lin (2013) consider the leader-following consensus problem for multi-agent systems subject to input saturation; Yang, Meng, Dimarogonas, and Johansson (2014) study global consensus for discrete-time multi-agent systems with input saturation constraint; and Lim and Ahn (2016) and Wang and Sun (2016) investigate initial conditions for achieving consensus in the presence of output saturation.

The classical continuous-time consensus protocol requires continuous information exchange among the agents. It may be impractical, however, to require continuous communication in physical applications. Event-triggered control is introduced partially to tackle this problem (Åström & Bernhardsson, 1999; Heemels, Johansson, & Tabuada, 2012; Tabuada, 2007; Wang & Lemmon, 2011). Event-triggered control is often piecewise constant between triggering times. The triggering times are determined implicitly by the event conditions. Event-triggered control for multi-agent systems has been studied by many researchers recently (e.g., Dimarogonas, Frazzoli, & Johansson, 2012; Meng, Xie, & Soh, 2018; Seyboth, Dimarogonas, & Johansson, 2013; Yi, Lu, & Chen, 2016, 2017; Yi, Wei, Dimarogonas, & Johansson, 2017). Key challenges are how to design the control law, to determine the event threshold, and to avoid Zeno behavior. Zeno behavior means that there are infinite number of triggers in a finite time interval (Johansson, Egerstedt, Lygeros, & Sastry, 1999). In other words, the non-existence of Zeno behavior is equivalent to that in every finite time interval there are only finite number of triggers. Thus, if Zeno behavior does not happen, it is guaranteed that during every finite time interval, the inter-event times are greater

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than a positive constant. Under the condition that some global network parameters are known, Seyboth et al. (2013) prove the non-existence of Zeno behavior by showing that there exists a positive constant such that all inter-event times are greater than such a positive constant. It should be highlighted that this is a sufficient condition for the non-existence of Zeno behavior.

In almost all physical applications, actuators have bounded input and output. However, none of the aforementioned event-triggered papers take saturation into consideration. In fact, even for a single agent system with input saturation and event-triggered controller, the stability problem is challenging. Kiener, Lehmann, and Johansson (2014) address the influence of actuator saturation on event-triggered control. Xie and Lin (2017) study the global stabilization problem of the multiple integrator system using event-triggered bounded controls. The consensus problem with input saturation and event-triggered controllers is challenging since the constraints lead to nonlinearities in the closed-loop dynamics. Wu and Yang (2016) propose a distributed event-triggered control strategy to achieve consensus for multi-agent systems subject to input saturation through output feedback. Different from the model we will consider in this paper, the underlying graph is undirected and the analysis does not exclude Zeno behavior. Yin, Yue, and Hu (2016) use LMI techniques to study local leader-following consensus for multi-agent systems subject to input saturation. LMI techniques are often implicit and conservative (Boyd, Ghaoui, Feron, & Balakrishnan, 1994), however we will give explicit necessary and sufficient condition to guarantee global consensus. Wang, Su, Wang, and Chen (2017) investigate the event-triggered semi-global consensus problem for general linear multi-agent systems subject to input saturation. The underlying graph they consider is undirected and in order to determine the triggering times, each agent needs to continuously measure its neighbors' states, i.e., continuous communication is needed.

In this paper, we solve the global consensus problem for multi-agent systems with input saturation over directed graphs (digraphs). More specifically, we propose an event-triggered consensus protocol and a triggering law, which lead to global consensus if and only if the underlying digraph has a directed spanning tree. The triggering law is inspired by a new Lyapunov function. The Lyapunov function is different from the one in Li et al. (2011). It should be noted that the event-triggered consensus protocol together with the triggering law do not require any a priori knowledge of global network parameters and is guaranteed to be free from Zeno behavior. Noting that the classic continuous-time consensus protocol is a special case of event-triggered consensus protocol, we then conclude that the multi-agent system with input saturation and continuous-time consensus protocol achieves consensus under the same necessary and sufficient directed spanning tree. In other words, we prove the result in Li et al. (2011), but we use a different method. In fact, it is noticed that the existence of a directed spanning tree is a necessary and sufficient condition for global consensus for both multi-agent systems with and without input saturation, despite that the saturation gives rise to a more complex nonlinear dynamic behavior.

The remainder of this paper is organized as follows. Section 2 introduces the preliminaries. The main results are stated in Section 3. Simulations are given in Section 4. The paper is concluded in Section 5. Most proofs are given in the Appendix.

Notations: $\|\cdot\|$ represents the Euclidean norm for vectors or the induced 2-norm for matrices. $\mathbf{1}_n$ denotes the column one vector with dimension n. I_n is the n dimensional identity matrix. $\rho(\cdot)$ stands for the spectral radius for matrices and $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues. Given two symmetric matrices M and N, $M > N$ ($M \geq N$) means that $M - N$ is positive definite (positive semi-definite). For a matrix A, $A^\top$ denotes its transpose and $\text{rank}(A)$ is its rank. Given a vector $[x_1, \ldots, x_n]^\top \in \mathbb{R}^n$, $\text{diag}([x_1, \ldots, x_n])$ is a diagonal matrix with the ith diagonal element being x_i. The notation $A \otimes B$ denotes the Kronecker product of matrices A and B. Given a vector $s = [s_1, \ldots, s_n]^\top \in \mathbb{R}^n$, define the component operator $c_l(s) = s_l, l = 1, \ldots, n$.

2. Preliminaries

In this section, we present some definitions from algebraic graph theory and the considered multi-agent system.

2.1. Algebraic graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ denote a weighted digraph with the set of agents (vertices) $\mathcal{V} = \{v_1, \ldots, v_n\}$, the set of links (edges) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $A = (a_{ij})$ with nonnegative elements a_{ij}. A link of $\mathcal{G}$ is denoted by $(v_i, v_j) \in \mathcal{E}$ if there is a directed link from agent v_i to agent v_j with weight $a_{ji} > 0$, i.e., agent v_i can send information to agent v_j. The adjacency elements associated with the links of the graph are positive, i.e., $(v_i, v_j) \in \mathcal{E} \iff a_{ji} > 0$. It is assumed that $a_{ii} = 0$ for all $i \in \mathcal{I}$, where $\mathcal{I} = \{1, \ldots, n\}$. The in-degree of agent v_i is defined as $\text{deg}_i^{\text{in}} = \sum_{j=1}^n a_{ij}$. The degree matrix of $\mathcal{G}$ is defined as $\text{Deg} = \text{diag}([\text{deg}_1^{\text{in}}, \ldots, \text{deg}_n^{\text{in}}])$. The (weighted) Laplacian matrix associated with $\mathcal{G}$ is defined as $L = \text{Deg} - A$. A directed path from agent v_i to agent v_j is a directed subgraph of $\mathcal{G}$ with distinct agents $v_i, v_{i_1}, \ldots, v_{i_k}, v_j$ and links $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{k-1}}, v_{i_k}), (v_{i_k}, v_j)$.

Definition 1. A digraph $\mathcal{G}$ is strongly connected if for any two distinct agents v_i and v_j, there exists a directed path from v_i to v_j.

$\mathcal{G}$ is strongly connected is equivalent to L is irreducible. Strong connectivity requires that any agent is accessible to all other agents, while the following weaker connectivity condition only requires that one agent can access all other agents.

Definition 2. A digraph $\mathcal{G}$ has a directed spanning tree if there exists one agent such that there exists a directed path from this agent to any other agent.

By Perron–Frobenius theorem (Horn & Johnson, 2012), we have the following result, see Lu and Chen (2006, 2007) for a proof.

Lemma 1. *If L is the Laplacian matrix associated with a digraph $\mathcal{G}$ that has a directed spanning tree, then $\text{rank}(L) = n - 1$, and there is a nonnegative vector $\xi^\top = [\xi_1, \ldots, \xi_n]$ such that $\xi^\top L = 0$ and $\sum_{i=1}^n \xi_i = 1$. Moreover, if $\mathcal{G}$ is strongly connected, then $\xi_i > 0, i \in \mathcal{I}$.*

The following result from Yi, Lu et al. (2017) is also useful for our analysis.

Lemma 2. *Suppose that L is the Laplacian matrix associated with a digraph $\mathcal{G}$ that is strongly connected and ξ is the vector defined in Lemma 1. Let $\Xi = \text{diag}(\xi)$, $U = \Xi - \xi\xi^\top$, and $R = \frac{1}{2}(\Xi L + L^\top \Xi)$. Then $R = \frac{1}{2}(UL + L^\top U)$ and*

$$U \geq \frac{\rho_2(U)}{\rho(L^\top L)} L^\top L \geq 0 \text{ and } R \geq \frac{\rho_2(R)}{\rho(U)} U \geq 0.$$

By proper row and column permutations, we can rewrite any Laplacian matrix L in the Perron–Frobenius form (see Definition 2.3 in Wu (2007)) as

$$L = \begin{bmatrix} L^{1,1} & L^{1,2} & \cdots & L^{1,M} \\ 0 & L^{2,2} & \cdots & L^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L^{M,M} \end{bmatrix}, \quad (1)$$

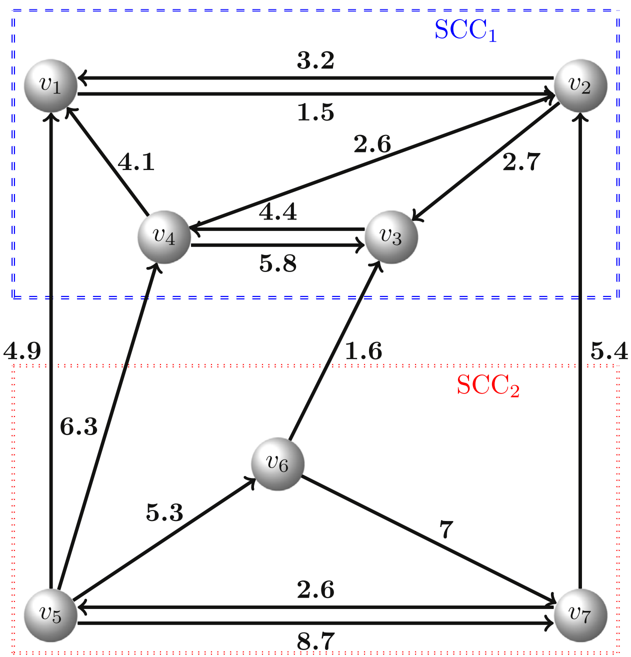

Fig. 1. An example of a digraph which contains directed spanning trees. The subgraph in the dashed lines is the first strongly connected component and the subgraph in the dotted lines is the second strongly connected component.

where $L^{m,m}$ has dimension n_m and is associated with the mth strongly connected component (SCC) of $\mathcal{G}$, denoted SCC$_m$, $m = 1, \ldots, M$. In the following, without loss of generality, we assume that L has the form (1).

For SCC$_m$, $L^{m,q} = 0$ for all $q > m$, if and only if there are no (directed) links from agents outside SCC$_m$ to agents inside SCC$_m$. In this case we say SCC$_m$ is closed. The following result, which follows from Lin, Francis, and Maggiore (2005, Lemma 1), gives an equivalent description of a digraph that has a directed spanning tree.

Lemma 3. *The digraph $\mathcal{G}$ contains a directed spanning tree if and only if for each $m = 1, \ldots, M-1$, SCC$_m$ is not closed.*

Let us illustrate this construction with an example.

Example 1. Fig. 1 shows a directed graph of 7 agents having multiple directed spanning trees. For example, one of the directed spanning tree is described by links $(v_7, v_5), (v_5, v_6), (v_6, v_3), (v_3, v_4), (v_4, v_2), (v_2, v_1)$. The graph can be divided into two strongly connected components, as indicated in the figure. The corresponding Laplacian matrix

$$L = \left[\begin{array}{cccc|ccc} 12.2 & -3.2 & 0 & -4.1 & -4.9 & 0 & 0 \\ -1.5 & 9.5 & 0 & -2.6 & 0 & 0 & -5.4 \\ 0 & -2.7 & 10.1 & -5.8 & 0 & -1.6 & 0 \\ 0 & 0 & -4.4 & 10.7 & -6.3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2.6 & 0 & -2.6 \\ 0 & 0 & 0 & 0 & -5.3 & 5.3 & 0 \\ 0 & 0 & 0 & 0 & -8.7 & -7 & 15.7 \end{array}\right],$$

has the form (1).

2.2. Multi-agent systems with input saturation

We consider a network of n agents that are modeled as single integrators with input saturation:

$$\dot{x}_i(t) = \mathrm{sat}_h(u_i(t)),\ i \in \mathcal{I},\ t \geq 0, \tag{2}$$

where $x_i(t) \in \mathbb{R}^p$ is the state and $u_i(t) \in \mathbb{R}^p$ is the control input of agent v_i, respectively, $p > 0$ is the state dimension, and $\mathrm{sat}_h(\cdot)$ is the saturation function defined as

$$\mathrm{sat}_h(s) = [\mathrm{sat}_h(s_1), \ldots, \mathrm{sat}_h(s_l)]^\top,$$

where $s = [s_1, \ldots, s_l]^\top \in \mathbb{R}^l$ with $l > 0$ and

$$\mathrm{sat}_h(s_i) = \begin{cases} h, & \text{if } s_i \geq h \\ s_i, & \text{if } |s_i| < h \\ -h, & \text{if } s_i \leq -h, \end{cases}$$

where h is a positive constant referred to as the saturation level.

Remark 1. For the ease of presentation, we focus on the case where all the agents have the same saturation level. The analysis can be readily extended to the case where the agents have different saturation levels.

The following properties about the saturation function are useful for our analysis.

Lemma 4. *For any real constants a and b,*

$$\int_0^a \mathrm{sat}_h(s)ds = \begin{cases} \frac{1}{2}a^2, & \text{if } |a| \leq h, \\ h(a-h) + \frac{1}{2}h^2, & \text{if } a \geq h, \\ h(-a-h) + \frac{1}{2}h^2, & \text{if } a \leq -h, \end{cases}$$

$$\frac{1}{2}a^2 \geq \int_0^a \mathrm{sat}_h(s)ds \geq \frac{1}{2}(\mathrm{sat}_h(a))^2,$$

$$(a-b)^2 \geq (\mathrm{sat}_h(a) - \mathrm{sat}_h(b))^2.$$

Lemma 5. *Suppose that L is the Laplacian matrix associated with a digraph $\mathcal{G}$ that has a directed spanning tree. For $x_1, \ldots, x_n \in \mathbb{R}^p$, define $y_i = \mathrm{sat}_h(-\sum_{j=1}^n L_{ij}x_j)$. Then $y_1 = \cdots = y_n$ if and only if $x_1 = \cdots = x_n$.*

Proof. The sufficiency is straightforward. Let us show the necessity. Let $z_i = -\sum_{j=1}^n L_{ij}x_j$. From $y_1 = \cdots = y_n$, we know that for any $l = 1, \ldots, p$, $c_l(z_i) > 0, \forall i \in \mathcal{I}$, or $c_l(z_i) < 0, \forall i \in \mathcal{I}$, or $c_l(z_i) = 0, \forall i \in \mathcal{I}$.

From Li et al. (2011, Lemma 2), we know that neither $c_l(z_i) > 0, \forall i \in \mathcal{I}$ nor $c_l(z_i) < 0, \forall i \in \mathcal{I}$ holds. Thus $-\sum_{j=1}^n L_{ij}c_l(x_j) = c_l(z_i) = 0, \forall i \in \mathcal{I}$. From Lemma 1, we know rank$(L) = n-1$. Thus, we have $c_l(x_i) = c_l(x_j), \forall i, j \in \mathcal{I}$. Hence $x_1 = \cdots = x_n$. □

3. Event-triggered control for multi-agent systems with input saturation

In this section, we consider the multi-agent system (2) defined over a digraph $\mathcal{G}$. In the literature, the following distributed continuous-time consensus protocol is often considered, e.g., Li et al. (2011)

$$u_i(t) = -\sum_{j=1}^n L_{ij}x_j(t). \tag{3}$$

To implement consensus protocol (3), continuous states from neighbors are needed. However, continuous communication is impractical in physical applications. To avoid continuously sending information among agents and updating actuators, we equip the consensus protocol (3) with an event-triggered communication scheme. The control signal is only updated when the triggering condition is satisfied. It results in the following multi-agent system with input saturation and event-triggered consensus protocol

$$\dot{x}_i(t) = \text{sat}_h(\hat{u}_i(t)),\ i \in \mathcal{I},\ t \geq 0, \tag{4}$$

$$\hat{u}_i(t) = -\sum_{j=1}^{n} L_{ij} x_j(t^j_{k_j(t)}), \tag{5}$$

where $t^j_{k_j(t)} = \max\{t^j_k : t^j_k \leq t\}$. The increasing time sequence $\{t^j_k\}_{k=1}^{\infty}$, $j \in \mathcal{I}$, named *triggering time sequence* of agent v_j, will be determined later. Note that the consensus protocol (5) only updates at the triggering times and is constant between two consecutive triggering times. For simplicity, let $\hat{x}_i(t) = x_i(t^i_{k_i(t)})$, and $e_i(t) = \hat{x}_i(t) - x_i(t)$.

In the following, we show that global consensus is achieved for the multi-agent system (4) with event-triggered consensus protocol (5) under a properly designed triggering time sequence.

Theorem 1. *Consider the multi-agent system (4)– (5). Given $\alpha_i > 0$, $\beta_i > 0$ and the first triggering time t^i_1, agent v_i determines the triggering times $\{t^i_k\}_{k=2}^{\infty}$ by*

$$t^i_{k+1} = \max_{r \geq t^i_k} \left\{ r : \|e_i(t)\|^2 \leq \alpha_i e^{-\beta_i t}, \forall t \in [t^i_k, r] \right\}. \tag{6}$$

Then, (i) there is no Zeno behavior; and (ii) global consensus is achieved if and only if the underlying digraph $\mathcal{G}$ has a directed spanning tree.

The proof of Theorem 1 is given in the Appendix. Zeno behavior is excluded by contradiction. The necessity in the second result is a direct result of Lemma 3. We illustrate the main idea of the proof of sufficiency in the second result here, while the detailed proof is given in the Appendix. We first consider the case where $\mathcal{G}$ is strongly connected, i.e., $M = 1$ in (1), and show that global consensus is achieved. We next consider the case where $\mathcal{G}$ has a directed spanning tree but it is not strongly connected, i.e., $M \geq 2$. From the first case ($M = 1$), it follows that all agents in SCC_M achieve consensus since SCC_M is either strongly connected or of dimension one. Then, we consider SCC_{M-1} and note that all agents in SCC_{M-1}, which is either strongly connected or of dimension one, achieve the same consensus value as those in SCC_M, since the agents in SCC_M and SCC_{M-1} are not influenced by $\text{SCC}_1, \dots, \text{SCC}_{M-2}$ and the consensus problem of this subsystem can be treated as a leader–follower problem where agents in SCC_M are leaders and agents in SCC_{M-1} are followers. Notice that $\text{SCC}_1, \dots, \text{SCC}_{M-2}$, are either strongly connected or of dimension one. By applying a similar analysis, the consensus of $\text{SCC}_m, \text{SCC}_{m+1}, \dots, \text{SCC}_M$ can be treated as a leader–follower consensus problem with agents in $\text{SCC}_M, \text{SCC}_{M-1}, \dots, \text{SCC}_{m+1}$ being leaders and agents in SCC_m being followers. Therefore, the result follows.

Remark 2. The event-triggered consensus protocol (5) together with the triggering law (6) is fully distributed. That is, each agent only requires its own state information and its neighbors' state information, without any a priori knowledge of global parameters, such as the eigenvalue of the Laplacian matrix.

If we let $\alpha_i = 0$, $\forall i \in \mathcal{I}$, in the triggering law (6), then $t^i_{k_i(t)} = t$, i.e., the event-triggered consensus protocol (5) becomes the consensus protocol (3). In this case, from the proof of Theorem 1, we know that the second statement of Theorem 1 still holds. Thus, we have the following corollary.

Corollary 1. *Consider the multi-agent system (2)– (3). Global consensus is achieved if and only if the digraph $\mathcal{G}$ has a directed spanning tree.*

Remark 3. This result appears also in Li et al. (2011). Our proof of Corollary 1 (following Theorem 1) is however based on the Lyapunov function

$$V(x) = \sum_{i=1}^{n} \xi_i \sum_{l=1}^{p} \int_0^{-\sum_{j=1}^{n} L_{ij} c_l(x_j)} \text{sat}_h(s) ds, \tag{7}$$

where $x = [x_1^\top, \dots, x_n^\top]^\top$ and $\xi^\top = [\xi_1, \dots, \xi_n]$ was defined in Lemma 1, which is different from the proof in Li et al. (2011). In addition, our Lyapunov function facilitates the design of event-triggered consensus protocol as shown in Theorem 1.

Remark 4. When $h \to \infty$, i.e., the multi-agent system is free from saturation, Theorem 1 (Corollary 1) corresponds to the well known result for the consensus problem of multi-agent systems without saturation which has been shown by Cao et al. (2008) Dimarogonas et al. (2012) and Ren and Beard (2005). The main differences between the cases with and without saturation are the convergence speed and the consensus value. For the saturation case, the convergence speed is slower and the consensus value is not fully determined by the Laplacian matrix L and the initial states of the agents. From the proof of Theorem 1, we know that the saturation is no longer active after a finite time $T \geq 0$ which depends on the initial value of each agent, the saturation level, and the network topology. Thus after T the convergence speed is exponential and the consensus value is determined by the state of each agent at T.

4. Simulations

In this section, we demonstrate the theoretical results by simulations. Consider again the digraph in Fig. 1 and the corresponding multi-agent system. Let the saturation level be $h = 10$. We choose an arbitrary initial state $x(0) = [6.2945, 8.1158, -7.4603, 8.2675, 2.6472, -8.0492, -4.4300]^\top$.

Fig. 2(a) shows the state evolutions of the multi-agent system (4)–(5) under the triggering law (6) with $\alpha_i = 10$ and $\beta_i = 1$. Fig. 2(b) shows the saturated input of each agent. Fig. 2(c) shows the corresponding triggering times for each agent. We see that global consensus is achieved also in this case. Moreover, from Fig. 2(c), we see also that each agent only needs to broadcast its state to its neighbors at its triggering times. Thus continuous broadcasting is avoided. Note however that the event-triggered control gives rise to a less smooth state evolutions because of the large variability in the control action.

5. Conclusions

In this paper, we studied the global consensus problem for multi-agent systems with input saturation constraints. We considered the event-triggered control and presented a distributed triggering law to reduce the overall need of communication and system updates. We showed that global consensus is achieved if and only if the underlying directed communication topology has a directed spanning tree. Furthermore, the triggering law was shown to be free of Zeno behavior. Future research directions include considering more general systems such as double integrator systems, time delays, and self-triggered control.

Appendix. Proof of Theorem 1

(i) Noting that the triggering time sequence is monotonically increasing, from the property of limits and the monotone convergence theorem, we conclude that non-existence of Zeno behavior is equivalent to that the triggering time sequence tends to infinity.

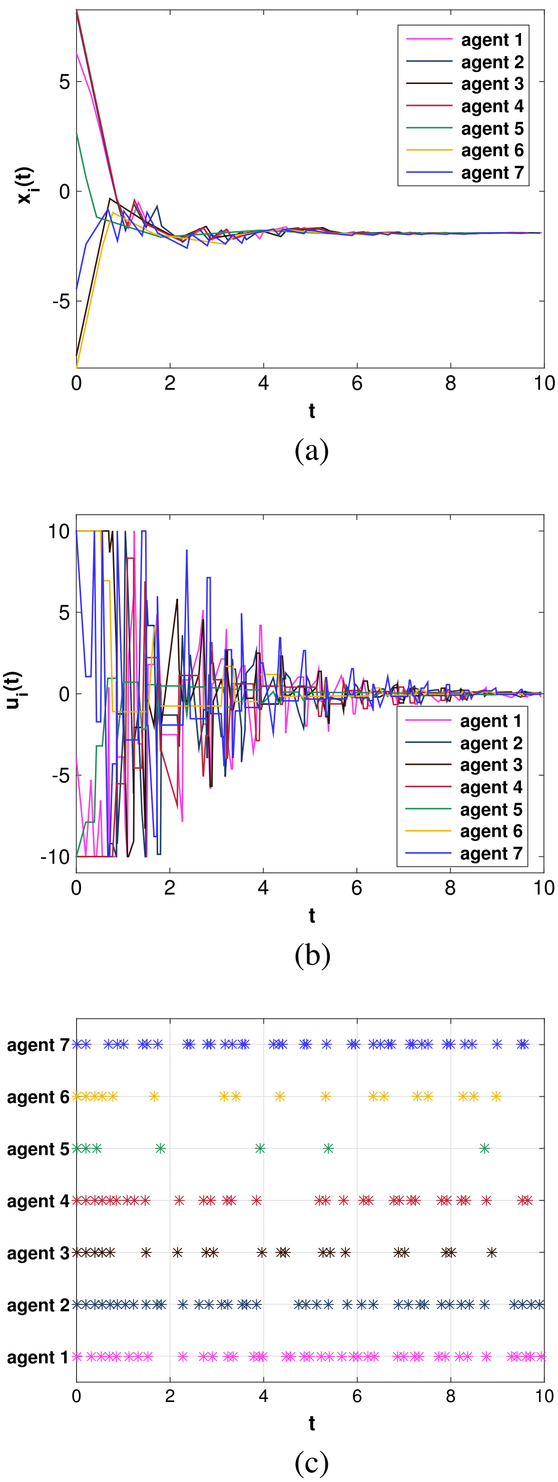

Fig. 2. (a) The state evolutions of the multi-agent system (4)–(5) under the triggering law (6). (b) The saturated input of each agent. (c) The triggering times for each agent.

We prove that there is no Zeno behavior by contradiction. Suppose that there exists Zeno behavior. Then there exists an agent v_i, such that $\lim_{k\to\infty} t_k^i = T_0$ for some constant T_0. Let $\varepsilon_0 = \frac{\sqrt{\alpha_i}}{2\sqrt{p}h} e^{-\frac{1}{2}\beta_i T_0} > 0$. Then from the property of limits, there exists a positive integer $N(\varepsilon_0)$ such that

$$t_k^i \in [T_0 - \varepsilon_0, T_0],\ \forall k \geq N(\varepsilon_0). \tag{A.1}$$

Noting that $\|\text{sat}_h(s)\| \leq h\sqrt{p}$ for any $s \in \mathbb{R}^p$, we have $\|\text{sat}_h(\hat{u}_i(t))\| \leq h\sqrt{p}$.

Also noting that

$$\left|\frac{d\|e_i(t)\|}{dt}\right| \leq \|\dot{x}_i(t)\| = \|\text{sat}_h(\hat{u}_i(t))\| \leq h\sqrt{p},$$

and $\|\hat{x}_i(t_k^i) - x_i(t_k^i)\| = 0$ for any triggering time t_k^i, we conclude that one sufficient condition to guarantee $\|e_i(t)\|^2 \leq \alpha_i e^{-\beta_i t}$, $t \geq t_k^i$ is

$$(t - t_k^i)h\sqrt{p} \leq \sqrt{\alpha_i} e^{-\frac{1}{2}\beta_i t},\ t \geq t_k^i. \tag{A.2}$$

Now suppose that the $N(\varepsilon_0)$th triggering time of v_i, $t^i_{N(\varepsilon_0)}$, has been determined. Let $t^i_{N(\varepsilon_0)+1}$ and $\tilde{t}^i_{N(\varepsilon_0)+1}$ denote the next triggering time determined by (6) and (A.2), respectively. Then $t^i_{N(\varepsilon_0)+1} \geq \tilde{t}^i_{N(\varepsilon_0)+1}$, and

$$t^i_{N(\varepsilon_0)+1} - t^i_{N(\varepsilon_0)} \geq \tilde{t}^i_{N(\varepsilon_0)+1} - t^i_{N(\varepsilon_0)} = \frac{\sqrt{\alpha_i}}{\sqrt{p}h} e^{-\frac{1}{2}\beta_i \tilde{t}^i_{N(\varepsilon_0)+1}}$$
$$\geq \frac{\sqrt{\alpha_i}}{\sqrt{p}h} e^{-\frac{1}{2}\beta_i t^i_{N(\varepsilon_0)+1}} \geq \frac{\sqrt{\alpha_i}}{\sqrt{p}h} e^{-\frac{1}{2}\beta_i T_0} = 2\varepsilon_0,$$

which contradicts (A.1). Therefore, there is no Zeno behavior.

(ii) (Necessity) Necessity follows from Lemma 3.

(Sufficiency) The proof of sufficiency follows from the structure outlined after Theorem 1 stated in Section 3. More specifically, we first show global consensus for the case where $M = 1$ which corresponds to only one SCC. Then, we consider the case where $M = 2$, and show that the agents in SCC_1 and SCC_2 reach consensus. We finally argue that the general case where $M > 2$ follows in a similar way.

(ii.a) In this part, we consider the situation where $\mathcal{G}$ is strongly connected, i.e., $M = 1$ in (1).

We first show that global consensus is achieved. Let $f_i(t) = \text{sat}_h(\hat{u}_i(t)) - \text{sat}_h(u_i(t))$. The derivative of $V(x)$ defined in (7), but along the trajectories of (4)–(5), satisfies

$$\begin{aligned}
&\dot{V}(x(t)) \\
&= \sum_{i=1}^n \xi_i \sum_{l=1}^p [\text{sat}_h(-\sum_{j=1}^n L_{ij} c_l(x_j(t)))][-\sum_{j=1}^n L_{ij} c_l(\dot{x}_j(t))] \\
&= \sum_{i=1}^n \xi_i \sum_{l=1}^p [\text{sat}_h(c_l(u_i(t)))][-\sum_{j=1}^n L_{ij} \text{sat}_h(c_l(\hat{u}_j(t)))] \\
&= -\sum_{i=1}^n \xi_i [\text{sat}_h(u_i(t))]^\top \sum_{j=1}^n L_{ij} \text{sat}_h(\hat{u}_j(t)) \\
&= -\sum_{i=1}^n \xi_i [\text{sat}_h(u_i(t))]^\top \sum_{j=1}^n L_{ij} [\text{sat}_h(u_j(t)) - f_j(t)] \\
&= -\sum_{i=1}^n \sum_{j=1}^n \xi_i L_{ij} [\text{sat}_h(u_i(t))]^\top \text{sat}_h(u_j(t)) \\
&\quad - \sum_{i=1}^n \sum_{j=1}^n \xi_i L_{ij} [f_j(t)]^\top \text{sat}_h(u_i(t)) \\
&\overset{*}{=} -\sum_{i=1}^n \xi_i q_i(t) \\
&\quad - \sum_{i=1}^n \sum_{j=1, j\neq i}^n \xi_i L_{ij} [f_i(t)]^\top [\text{sat}_h(u_j(t)) - \text{sat}_h(u_i(t))] \\
&\leq -\sum_{i=1}^n \xi_i q_i(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\Big\{-\xi_i L_{ij}\frac{1}{4}\|\mathrm{sat}_h(u_j(t))-\mathrm{sat}_h(u_i(t))\|^2\\
&-\xi_i L_{ij}\|f_i(t)\|^2\Big\}\\
=&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\sum_{i=1}^{n}\xi_i L_{ii}\|f_i(t)\|^2\\
=&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\sum_{i=1}^{n}\xi_i L_{ii}\|\mathrm{sat}_h(\hat{u}_i(t))-\mathrm{sat}_h(u_i(t))\|^2\\
\overset{**}{\le}&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\sum_{i=1}^{n}\xi_i L_{ii}\|\hat{u}_i(t)-u_i(t)\|^2\\
=&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\sum_{i=1}^{n}\xi_i L_{ii}\Big\|\sum_{j=1}^{n}L_{ij}e_j(t)\Big\|^2\\
\le&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\max_{i\in\mathcal{I}}\Big\{\xi_i L_{ii}\Big\}e^\top(t)(L^\top L\otimes I_p)e(t)\\
\le&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\max_{i\in\mathcal{I}}\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\|e_i(t)\|^2,
\end{aligned}
\tag{A.3}
$$

where

$$
q_i(t)=-\frac{1}{2}\sum_{j=1}^{n}L_{ij}\|\mathrm{sat}_h(u_j(t))-\mathrm{sat}_h(u_i(t))\|^2\ge 0.
$$

Moreover, the inequality denoted by $\overset{**}{\le}$ holds due to Lemma 4 and the equality denoted by $\overset{*}{=}$ holds due to

$$
\begin{aligned}
&-\sum_{i=1}^{n}\xi_i q_i(t)\\
=&\sum_{i=1}^{n}\frac{1}{2}\sum_{j=1}^{n}\xi_i L_{ij}\|\mathrm{sat}_h(u_j(t))-\mathrm{sat}_h(u_i(t))\|^2\\
=&\sum_{i=1}^{n}\frac{1}{2}\sum_{j=1}^{n}\xi_i L_{ij}\Big[\|\mathrm{sat}_h(u_j(t))\|^2+\|\mathrm{sat}_h(u_i(t))\|^2\Big]\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i L_{ij}[\mathrm{sat}_h(u_j(t))]^\top\mathrm{sat}_h(u_i(t))\\
=&\frac{1}{2}\sum_{j=1}^{n}\|\mathrm{sat}_h(u_j(t))\|^2\sum_{i=1}^{n}\xi_i L_{ij}\\
&+\frac{1}{2}\sum_{i=1}^{n}\xi_i\|\mathrm{sat}_h(u_i(t))\|^2\sum_{j=1}^{n}L_{ij}\\
&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i L_{ij}[\mathrm{sat}_h(u_j(t))]^\top\mathrm{sat}_h(u_i(t))\\
=&-\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_i L_{ij}[\mathrm{sat}_h(u_j(t))]^\top\mathrm{sat}_h(u_i(t))\\
=&-[\mathrm{sat}_h(u(t))]^\top(R\otimes I_p)\mathrm{sat}_h(u(t))
\end{aligned}
\tag{A.4}
$$

where we have used $\xi^\top L=0$ and $L\mathbf{1}_n=0$ in (A.4)

Let us treat $z_i(t)=e^{-\beta_i t}, t\ge 0$ as an additional state to agent $v_i, i\in\mathcal{I}$, and let $z(t)=[z_1(t),\dots,z_n(t)]^\top$. Consider a Lyapunov candidate:

$$
W(x,z)=V(x)+2\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}|z_i|.
$$

From (A.3) and (6), the derivative of $W(x,z)$ along the trajectories of (4)–(5) and $\dot{z}_i(t)=-\beta_i z_i(t)$ is

$$
\begin{aligned}
&\dot{W}(x(t),z(t))\\
=&\dot{V}(x(t))-2\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\alpha_i e^{-\beta_i t}\\
\le&-\sum_{i=1}^{n}\frac{\xi_i}{2}q_i(t)+\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\|e_i(t)\|^2\\
&-2\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\alpha_i e^{-\beta_i t}\\
\le&-\frac{1}{2}[\mathrm{sat}_h(u(t))]^\top(R\otimes I_p)\mathrm{sat}_h(u(t))\\
&-\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\alpha_i z_i(t)\le 0.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
W(x,z)=&V(x)+2\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}z_i\\
=&\sum_{i=1}^{n}\xi_i\sum_{l=1}^{p}\int_0^{-\sum_{j=1}^{n}L_{ij}c_l(x_j)}\mathrm{sat}_h(s)ds\\
&+2\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}|z_i|\\
=&\sum_{i=1}^{n}\xi_i\sum_{l=1}^{p}\int_0^{-\sum_{j=1}^{n}c_l(u_j)}\mathrm{sat}_h(s)ds\\
&+2\max_i\Big\{\xi_i L_{ii}\Big\}\rho(L^\top L)\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}|z_i|\\
=:&\tilde{W}(u,z)
\end{aligned}
$$

where $u=[u_1^\top,\dots,u_n^\top]^\top$ and u_i is given in (3). From Lemma 4, we know that $\tilde{W}(u,z)$ is radially unbounded and $\tilde{W}(u,z)=0$ if and only if $u=0$ and $z=0$. From $\mathrm{rank}(R)=n-1$ as shown in Lemma 2, we know that $[\mathrm{sat}_h(u)]^\top(R\otimes I_p)\mathrm{sat}_h(u)=0$ if and only if $u_i=u_j,\ \forall i,j\in\mathcal{I}$. Then, from Lemma 5, this is equivalent to $x_i=x_j,\ \forall i,j\in\mathcal{I}$. Thus, this is equivalent to $u_i=0,\ \forall i\in\mathcal{I}$ since $\mathrm{rank}(L)=n-1$. Hence, $\dot{\tilde{W}}(u(t),z(t))<0$ for all $u\ne 0$. Thus, by Lyapunov's Second Method (Khalil, 2002), we have that $\lim_{t\to\infty}u_i(t)=0,\ \forall i\in\mathcal{I}$. Then, from Lemma 5, we have

$$
\lim_{t\to\infty}\|x_j(t)-x_i(t)\|=0,\ \forall i,j\in\mathcal{I},
\tag{A.5}
$$

i.e., global consensus is achieved.

We next show that the input of each agent enters into the saturation level in finite time.

Since $c_l(\hat{u}_i(t))=-\sum_{j=1}^{n}L_{ij}c_l(x_j(t))-\sum_{j=1}^{n}L_{ij}c_l(e_j(t))$, (6), $-\sum_{j=1}^{n}L_{ij}c_l(x_j(t)), i\in\mathcal{I}, l=1,\dots,p$ are continuous with respect to t, it then follows from (A.5) that there exists a constant $T_1\ge 0$ such that

$$
\begin{aligned}
|c_l(\hat{u}_i(t))|\le&\Big|-\sum_{j=1}^{n}L_{ij}c_l(x_j(t))\Big|+\Big|-\sum_{j=1}^{n}L_{ij}c_l(e_j(t))\Big|\\
\le&\ h,\ \forall t\ge T_1.
\end{aligned}
$$

In other words, the saturation function in (4) is no longer active after T_1. Thus, the multi-agent system (4) with the event-triggered consensus protocol (5) reduces to

$$
\dot{x}_i(t)=-\sum_{j=1}^{n}L_{ij}\hat{x}_j(t),\ t\ge T_1.
$$

Finally, we estimate the convergence speed, which will be used later. Consider the following function:

$$\tilde{V}(x) = \frac{1}{2}x^\top (U \otimes I_p)x.$$

From Lemma 2, we know that $\tilde{V}(x) \geq 0$. Similar to the proof of Yi et al. (2016, Theorem 2), we conclude that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\tilde{V}(x(t)) \leq C_1 e^{-C_2 t}, \ \forall t \geq T_1,$$

Noting that $\tilde{V}(x(t))$ is continuous with respect to t, there exists a positive constant C_3 such that

$$\tilde{V}(x(t)) \leq C_3, \ \forall t \in [0, T_1].$$

Then

$$\tilde{V}(x(t)) \leq C_4 e^{-C_2 t}, \ \forall t \geq 0,$$

where $C_4 = \max\{C_1, \ C_3 e^{C_2 T_1}\}$ is a positive constant.

Moreover, from Lemma 2 and (6), we know that

$$\begin{aligned}
\sum_{i=1}^{n} \|\hat{u}_i(t)\|^2 &= \sum_{i=1}^{n} \|u_i(t) - \sum_{j=1}^{n} L_{ij} e_j(t)\|^2 \\
&\leq 2\sum_{i=1}^{n} \|u_i(t)\|^2 + 2\rho(L^\top L)\sum_{i=1}^{n} \|e_i(t)\|^2 \\
&= x^\top(t)(L^\top L \otimes I_p)x(t) + 2\rho(L^\top L)\sum_{i=1}^{n} \|e_i(t)\|^2 \\
&\leq \frac{\rho(L^\top L)}{\rho_2(U)} x^\top(t)(U \otimes I_p)x(t) + 2\rho(L^\top L)\sum_{i=1}^{n} \|e_i(t)\|^2 \\
&= 2\frac{\rho(L^\top L)}{\rho_2(U)}\tilde{V}(x(t)) + 2\rho(L^\top L)\sum_{i=1}^{n} \|e_i(t)\|^2 \\
&\leq C_5 e^{-C_6 t}, \ \forall t \geq 0,
\end{aligned} \tag{A.6}$$

where C_5 and C_6 are two positive constants.

(ii.b) In this part, we consider the case where $M \geq 2$, but we first introduce some notations which will be used later. Define an auxiliary matrix $\tilde{L}^{m,m} = [\tilde{L}^{m,m}_{ij}]^{n_m}_{i,j=1}$ as

$$\tilde{L}^{m,m}_{ij} = \begin{cases} L^{m,m}_{ij} & i \neq j, \\ -\sum\limits_{r=1, r\neq i}^{n_m} L^{m,m}_{ir} & i = j. \end{cases}$$

Let $\xi^m = [\xi^m_1, \ldots, \xi^m_{n_m}]^\top$ be the positive left eigenvector of the irreducible $\tilde{L}^{m,m}$ corresponding to the eigenvalue zero and the sum of its components is 1. Denote $\Xi^m = \text{diag}[\xi^m]$ and $Q^m = \frac{1}{2}[\Xi^m L^{m,m} + (\Xi^m L^{m,m})^\top]$, $m = 1, \ldots, M$. Then, under the setup above, Q^m is positive definite for all $m < M$, see Wu (2005, Lemma 3.1).

Let $N_0 = 0$, $N_l = \sum_{m=1}^{l} n_m$, $l = 1, \ldots, M$, where n_m is the dimension of $L^{m,m}$. Then the ith agent in SCC_m is the $N_{m-1} + i$th agent of the whole graph. In the following, we exchangeably use v^m_i and $v_{N_{m-1}+i}$ to denote this agent. Accordingly, denote $x^m_i(t) = x_{N_{m-1}+i}(t)$, $\hat{x}^m_i(t) = \hat{x}_{N_{m-1}+i}(t)$, $u^m_i(t) = u_{N_{m-1}+i}(t)$ and define $u^m(t) = [(u^m_1)^\top(t), \ldots, (u^m_{n_m})^\top(t)]^\top$. For simplicity, let $\hat{u}^m_i(t) = \hat{u}_{N_{m-1}+i}(t)$, $e^m_i(t) = e_{N_{m-1}+i}(t)$, $f^m_i(t) = f_{N_{m-1}+i}(t)$, $\alpha^m_i = \alpha_{N_{m-1}+i}$, $\beta^m_i = \beta_{N_{m-1}+i}$, and $\hat{u}^m(t) = [(\hat{u}^m_1)^\top(t), \ldots, (\hat{u}^m_{n_m})^\top(t)]^\top$.

In the following we only consider the case where $M = 2$. The case where $M > 2$ can be treated in a similar manner, as discussed in the proof sketch in Section 3.

First, let us consider SCC_2 and note that no agent in SCC_2 is influenced by any agent in SCC_1. Thus, SCC_2 can be treated as a strongly connected digraph. Then, from the analysis in part (ii.a), we have that

$$\lim_{t\to\infty} \|x^2_i(t) - x^2_j(t)\| = 0, \ \forall i, j = 1\ldots, n_2,$$

and that there exists a constant $T_2 \geq 0$ such that

$$|c_l(\hat{u}^2_i(t))| = \Big| -\sum_{j=1}^{n_2} L^{2,2}_{ij} c_l(\hat{x}^2_j(t))\Big| \leq h, \ \forall t \geq T_2.$$

In addition, similar to (A.6), we have

$$\|\hat{u}^2(t)\|^2 = \sum_{j=1}^{n_2} \|\hat{u}^2_j(t)\|^2 \leq C_7 e^{-C_8 t}, \ \forall t \geq 0, \tag{A.7}$$

where C_7 and C_8 are two positive constants.

Second, let us consider SCC_1. Similar to $V(x)$ defined in (7), define

$$V_1(x) = \sum_{i=1}^{n_1} \xi^1_i \sum_{l=1}^{p} \int_0^{c_l(u^1_i)} \text{sat}_h(s)ds, \tag{A.8}$$

$$V_2(x) = \sum_{i=1}^{n_2} \xi^2_i \sum_{l=1}^{p} \int_0^{c_l(u^2_i)} \text{sat}_h(s)ds. \tag{A.9}$$

From the definition of the component operator $c_l(\cdot)$, we know that $c_l(u^1_i(t)) = -\sum_{j=1}^{n_1} L^{1,1}_{ij} c_l(x^1_i(t)) - \sum_{j=1}^{n_2} L^{1,2}_{ij} c_l(x^2_i(t))$ and $c_l(u^2_i(t)) = -\sum_{j=1}^{n_2} L^{2,2}_{ij} c_l(x^2_i(t))$. From Lemma 4, we have $V_1(x) \geq 0$ and $V_2(x) \geq 0$. Similar to (A.3), the derivative of $V_2(x)$ defined in (A.9) along the trajectories of system (4)–(5), satisfies

$$\dot{V}_2(x(t)) \leq -\sum_{i=1}^{n_2} \frac{\xi^2_i}{2} q^2_i(t) + d_1 \sum_{i=1}^{n_2} \|e^2_i(t)\|^2, \tag{A.10}$$

where $d_1 = \max_{i\in\mathcal{I}}\left\{\xi^2_i L^{2,2}_{ii}\right\}\rho((L^{2,2})^\top L^{2,2})$.

The derivative of $V_1(x)$ defined in (A.8) along the trajectories of system (4)–(5), satisfies

$$\begin{aligned}
\dot{V}_1(x(t)) &= \sum_{i=1}^{n_1} \xi^1_i \sum_{l=1}^{p} \text{sat}_h(c_l(u^1_i(t))) c_l(\dot{u}^1_i(t)) \\
&= \sum_{i=1}^{n_1} \xi^1_i \sum_{l=1}^{p} c_l(\text{sat}_h(u^1_i(t)))\Big[-\sum_{j=1}^{n_1} L^{1,1}_{ij} c_l(\text{sat}_h(\hat{u}^1_j(t))) \\
&\quad - \sum_{j=1}^{n_2} L^{1,2}_{ij} c_l(\text{sat}_h(\hat{u}^2_j(t)))\Big] \\
&= \sum_{i=1}^{n_1} \xi^1_i [\text{sat}_h(u^1_i(t))]^\top \Big[-\sum_{j=1}^{n_1} L^{1,1}_{ij} \text{sat}_h(\hat{u}^1_j(t)) \\
&\quad - \sum_{j=1}^{n_2} L^{1,2}_{ij} \text{sat}_h(\hat{u}^2_j(t))\Big] \\
&= \sum_{i=1}^{n_1} \xi^1_i [\text{sat}_h(u^1_i(t))]^\top \Big[-\sum_{j=1}^{n_1} L^{1,1}_{ij} (\text{sat}_h(u^1_j(t)) + f^1_j(t)) \\
&\quad - \sum_{j=1}^{n_2} L^{1,2}_{ij} \text{sat}_h(\hat{u}^2_j(t))\Big] \\
&= -[\text{sat}_h(u^1(t))]^\top (Q^1 \otimes I_p)\text{sat}_h(u^1(t)) \\
&\quad + \sum_{i=1}^{n_1} \xi^1_i [\text{sat}_h(u^1_i(t))]^\top \sum_{j=1}^{n_2} L^{1,2}_{ij} \text{sat}_h(\hat{u}^2_j(t)) \\
&\quad + \sum_{i=1}^{n_1} \xi^1_i [\text{sat}_h(u^1_i(t))]^\top \sum_{j=1}^{n_1} L^{1,1}_{ij} f^1_j(t)
\end{aligned}$$

$$\le -\rho_2(Q^1)\|\mathrm{sat}_h(u^1(t))\|^2 + \frac{\rho_2(Q^1)}{4}\sum_{i=1}^{n_1}\|\mathrm{sat}_h(u_i^1(t))\|^2 + \frac{1}{\rho_2(Q^1)}\sum_{i=1}^{n_1}\Big\|\xi_i^1\sum_{j=1}^{n_2}L_{ij}^{1,2}\mathrm{sat}_h(\hat{u}_j^2(t))\Big\|^2 + \frac{\rho_2(Q^1)}{4}\sum_{j=1}^{n_1}\|\mathrm{sat}_h(u_j^1(t))\|^2 + \frac{1}{\rho_2(Q^1)}\sum_{i=1}^{n_1}\Big\|\sum_{j=1}^{n_1}\xi_i^1L_{ij}^{1,1}f_j^1(t)\Big\|^2 \le -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(u^1(t))\|^2 + d_2\sum_{i=1}^{n_1}\|f_i^1(t)\|^2 + d_3\|\mathrm{sat}_h(\hat{u}^2(t))\|^2, \tag{A.11}$$

where

$$d_2 = \frac{(n_1)^2}{\rho_2(Q^1)}\max_{i\in\{1,\ldots,n_1\}}\{(\xi_i^1L_{ij}^{1,1})^2\},$$
$$d_3 = \frac{2n_1n_2}{\rho_2(Q^1)}\max_{i\in\{1,\ldots,n_1\}}\{(\xi_i^1L_{ij}^{1,2})^2\}.$$

Similar to the analysis obtaining (A.3), from (A.11), we have

$$\dot{V}_1(x(t)) \le -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(\hat{u}^1(t))\|^2 + d_4\sum_{i=1}^{n_1}\|e_i^1(t)\|^2 + d_3\|\hat{u}^2(t)\|^2, \tag{A.12}$$

where $d_4 = d_2\rho(L^\top L)$

Let us treat $\eta_i^r(t) = e^{-\beta_i^r t}$, $t \ge 0$, as an additional state of agent v_i^r, $r = 1, 2$, $i = 1, \ldots, n_2$, $\theta_i^2(t) = e^{-C_8 t}$, $t \ge 0$, as an additional state of agent v_i^2, $i = 1, \ldots, n_2$, and $\theta_i^1(t) = 0$, $t \ge 0$, as an additional state of agent v_i^1, $i = 1, \ldots, n_1$. Let $\eta(t) = [\eta_1^1(t), \ldots, \eta_{n_1}^1(t), \eta_1^2(t), \ldots, \eta_{n_2}^1(t)]^\top$ and $\theta(t) = [\theta_1^1(t), \ldots, \theta_{n_1}^1(t), \theta_1^2(t), \ldots, \theta_{n_2}^1(t)]^\top$.

Consider the following Lyapunov candidate:

$$W_r(x, \eta, \theta) = V_1(x) + V_2(x) + \frac{2C_7d_3}{C_8}\sum_{i=1}^{n_2}|\theta_i^2| + 2\sum_{i=1}^{n_2}\frac{d_1\alpha_i^2}{\beta_i^2}|\eta_i^2| + 2\sum_{i=1}^{n_1}\frac{d_4\alpha_i^1}{\beta_i^1}|\eta_i^1|.$$

The derivative of $W_r(x, \eta, \theta)$ along the trajectories of system (4)–(5) satisfies

$$\dot{W}_r(x(t), \eta(t), \theta(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) - 2C_7d_3\sum_{i=1}^{n_2}\theta_i^2(t) - 2\sum_{i=1}^{n_2}d_1\alpha_i^2\eta_i^2(t) - 2\sum_{i=1}^{n_1}d_4\alpha_i^1\eta_i^1(t).$$

Then, from (A.10), (A.12), and (A.7), for any $t \ge 0$, we have

$$\dot{W}_r(x(t), \eta(t), \theta(t)) \le -\frac{\rho_2(Q^1)}{2}\|\mathrm{sat}_h(u^1(t))\|^2 + \sum_{i=1}^{n_2}-\frac{\xi_i^2}{2}q_i^2(t) - C_7d_3\sum_{i=1}^{n_2}\theta_i^2(t) - \sum_{i=1}^{n_2}d_1\alpha_i^2\eta_i^2(t) - \sum_{i=1}^{n_1}d_4\alpha_i^1\eta_i^1(t).$$

Similar to the derivation of (A.5), by Lyapunov's Second Method, we have

$$\lim_{t\to\infty}\|x_j(t) - x_i(t)\| = 0, \ \forall i, j \in \mathcal{I}.$$

Thus, global consensus is achieved. Moreover, similar to the analysis in part (ii.a), we can show that after a finite time the saturation is no longer active. Thus concludes the proof.

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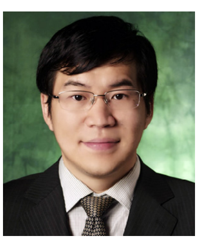

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