2022 IEEE 61st Conference on Decision and Control (CDC)
December 6-9, 2022. Cancún, Mexico

# Robust Risk-Aware Model Predictive Control of Linear Systems with Bounded Disturbances

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2022 IEEE 61st Conference on Decision and Control (CDC) | 978-1-6654-6761-2/22/$31.00 ©2022 IEEE | DOI: 10.1109/CDC51059.2022.9993149

***Abstract*—We propose a robust risk-aware model predictive control (MPC) algorithm for linear discrete-time systems with bounded disturbances. The MPC problem is formulated to maximize the size of the predicted disturbance sets (which are subsets of the maximal disturbance sets) at each time step, under state and control constraints and a reachability specification. It is shown that the proposed scheme has the following properties: *i*) its feasible state set (i.e., the set of states starting from which the MPC problem is feasible) is as large as that of the MPC problem for the corresponding undisturbed system; *ii*) it maintains recursive feasibility if the conventional robust MPC problem is feasible. The proposed controller enlarges the feasible state set, at the expense of the risk of possible constraint violation that is quantified by the optimal solution of the problem. When the sets are represented as zonotopes, we further provide a computationally tractable reformulation and design an online implementation algorithm with adaptive prediction horizon. We illustrate the effectiveness of the proposed methods using a simulated example.**

## I. INTRODUCTION

Model predictive control (MPC) has attracted great attention in recent decades since it enables state and input constraint-handling in controller synthesis [1], [2]. The basic operations of MPC include: solving a finite-horizon optimization problem at each time step, obtaining a sequence of optimal control inputs, only implementing the first control input in this sequence, and repeating this procedure. MPC has been successfully applied to disparate systems, e.g., process control systems [3] and robotic systems [4], [5]; for an overview of MPC the readers are referred to [6], [7], [8], [9].

Both robust and stochastic MPC have been developed for efficiently handling the uncertainties. Robust MPC [10] is concerned with hard constraint satisfaction against all the possible realizations of uncertainties, while stochastic MPC [11] allows the constraint violation with prescribed probability. Many research efforts have been made to develop computationally efficient ways to solve robust or stochastic MPC problems. Tube-based method is one of the most popular approaches in the literature; see [12], [13] and [14], [15]. Despite these efforts, existing methods still suffer from some degree of conservatism in handling uncertainties, i.e., a shrinked feasible state set in comparison with that of MPC for deterministic systems.

In this paper, we study the robust risk-aware MPC for linear discrete-time systems with bounded disturbances. Motivated by the fact that the worst-case realizations of the disturbances may not frequently occur in practice, we formulate the MPC problem to maximize the size of the predicted disturbance sets (that are subsets of the maximal disturbance sets) at each time step, under state and control constraints and a reachability specification. That is, instead of finding a controller that can tolerate all possible realizations of the disturbances as in robust MPC, our formulation aims at finding a controller that can robustly handle as large a disturbance set as possible under the system constraints. The risk in this paper is the possibility of constraint violation when the encountered disturbances come beyond the predicted disturbance sets and can be quantified by using the optimal solution of the problem. By taking this risk (i.e., allowing constraint violation), our formulation is able to provide a larger feasible set, i.e., the set of states starting from which the MPC problem is feasible, than robust MPC and thereby reduce the inherent conservativeness of robust MPC. The main contributions are summarized as follows.

1) We propose a robust risk-aware MPC algorithm for linear system with additive disturbances. Different from standard robust MPC, where worst-case disturbance realizations are assumed in prediction, the proposed approach maximizes the size of predicted disturbance sets under the system constraints. That is, the proposed algorithm gives priority to feasibility while taking the risk that future disturbance realization may exceed the predicted disturbance set.
2) We prove that the proposed scheme enjoys the following properties: *i*) its feasible state set is as large as that of the MPC problem for undisturbed systems; *ii*) it remains feasible recursively if the corresponding robust MPC problem is feasible. That is, it provides a natural bridge between MPC problem for deterministic systems and robust MPC problem for uncertain systems.
3) When the sets are represented as zonotopes, we show that the original robust risk-aware MPC problem can be reformulated as a linear program using the properties of zonotopes, which enables efficient implementation. Furthermore, we design an online implementation algorithm with adaptive prediction horizon.

**Related works** Risk usually refers to the possibility that something unexpected happens. The risk in our paper is different from the common risk notions in the existing work, which are usually defined in the stochastic setting. There is a large body of literature that incorporates risk into control problems, e.g., risk-sensitive stochastic optimal control [16], [17] and more recent risk-aware motion planning [18]. In these works, conditional Value at Risk (CVaR) is a popular measure used in motion planning [19]. In [20], the risk-sensitive MPC has been studied by considering time-consistent, dynamic risk evaluation of the cumulative cost as the objective function, which is different from the formulation of our paper. A recent work on automated car overtaking [21] introduces a new notion of risk that quantifies the probability of collision between cars under a supermartingale assumption. In the scenario-based optimization, the risk has a different meaning and refers to the probability that the optimal solution obtained from a given set of samples is not feasible for a new sample [22]. The risk in our paper (i.e., the possibility of constraint violation when the encountered disturbances come beyond the predicted disturbance sets) has a close meaning to the risk in the scenario-based optimization [22]. Another related work is [23], where a robust optimal control strategy is designed with adjustable uncertainty sets. The differences between the proposed algorithm and [23] include: *i*) we study the relation between robust MPC and deterministic MPC in terms of the feasible state set, and *ii*) we further provide a computationally tractable reformulation based on zonotopes. Some recent extensions of [23] can be found in [24], [25], [26].

The remainder of the paper is organized as follows. In Section II, we recall the robust MPC problem. In Section III, we formulate the robust risk-aware MPC problem and discuss its connection to a few existing MPC schemes. In Section IV, we provide a linear program reformulation using the properties of zonotopes and design an online implementation algorithm. In Section V, we illustrate the effectiveness of our approaches with a numerical example. In Section VI, we conclude the paper with a discussion about our current and future works.

**Notations** Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{R}$ the set of real numbers. For some $q, s \in \mathbb{N}$ and $q < s$, let $\mathbb{N}_{[q,s]} = \{r \in \mathbb{N} \mid q \leq r \leq s\}$. For two sets $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{X} \oplus \mathbb{Y} = \{x + y \mid x \in \mathbb{X}, y \in \mathbb{Y}\}$. When $\leq$, $\geq$, $<$, and $>$ are applied to vectors, they are interpreted element-wise.

## II. Preliminaries and Problem Statement

Consider a discrete-time linear system with additive disturbance in the form of

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{U} \subset \mathbb{R}^{n_u}$ the control input, and $w_k \in \mathbb{W} \subseteq \mathbb{R}^{n_x}$ the disturbance. $A$, $B$ are matrices with appropriate dimensions. The disturbance set $\mathbb{W}$ is assumed to contain the origin.

Consider a reachability specification for the system (1). Let $\mathbb{X}_f \subset \mathbb{X}$ be a target set (which may be not robustly invariant). Given an initial state $x_0 \in \mathbb{X}$, the control objective is to find a sequence of feedback control inputs $u_k(x_k) \in \mathbb{U}$ such that the state $x_k$ stays in $\mathbb{X}$ and eventually reaches $\mathbb{X}_f$, for all possible realizations of $w_k \in \mathbb{W}$.

To address such reachability specification, many robust MPC algorithms have been developed to suppress the worst-case realization of the disturbance. In the following, we review the basic formulation of robust MPC.

### A. Robust MPC

Given the state $x_k$ at time step $k$, the robust MPC is by and large formulated as an optimization problem:

$$\begin{aligned}
\min_{u_{\cdot|k}} \quad & J(x_k, u_{k|k}, \cdots, u_{k+N_k-1|k}) \\
\text{s.t}: \ & x_{k|k} = x_k, \\
& \forall i \in \mathbb{N}_{[0,N_k-1]}: \\
& \qquad x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k} + w_{k+i|k} \\
& \qquad \forall w_{k+i|k} \in \mathbb{W}: \begin{cases} u_{k+i|k} \in \mathbb{U}, \\ x_{k+i|k} \in \mathbb{X}, \\ x_{k+N_k|k} \in \mathbb{X}_f, \end{cases}
\end{aligned}$$

where $J$ denotes the cost function, $u_{k+i|k}$ and $x_{k+i|k}$ represent the predicted control input and state at time $k+i$, respectively, and $N_k$ stands for the prediction horizon. Without loss of generality, the horizon $N_k$ is time-dependent, which facilitates the algorithm design with adaptive horizon. Upon using the set-based formulation, the optimization problem, denoted by $\mathcal{P}_{\text{Robust}}(x_k, N_k)$, can be equivalently expressed by:

$$\begin{aligned}
\min_{u_{\cdot|k}} \quad & J(x_k, u_{k|k}, \cdots, u_{k+N_k-1|k}) \\
\text{s.t}: \ & \mathbb{X}_{k|k} = \{x_k\}, \\
& \forall i \in \mathbb{N}_{[0,N_k-1]}: \\
& \qquad \mathbb{X}_{k+i+1|k} = A\mathbb{X}_{k+i|k} \oplus B\{u_{k+i|k}\} \oplus \mathbb{W}, \\
& \qquad \begin{cases} u_{k+i|k} \in \mathbb{U}, \\ \mathbb{X}_{k+i|k} \subseteq \mathbb{X}, \\ \mathbb{X}_{k+N_k|k} \subseteq \mathbb{X}_f, \end{cases}
\end{aligned} \tag{2}$$

where $\mathbb{X}_{k+i|k}$ denotes the set of predicted states at time $k+i$ that can be reached from the initial state $x_k$ under the control sequence $\{u_{k+j|k}\}_{j=0}^{i-1}$ and any possible disturbance realizations from $\mathbb{W}$. The set $\mathbb{X}_{k+i|k}$ can also be rewritten as

$$\mathbb{X}_{k+i|k} = \left\{ A^i x_k + \sum_{j=0}^{i-1} A^{i-1-j} B u_j \right\} \bigoplus \left\{ \bigoplus_{j=0}^{i-1} A^j \mathbb{W} \right\}.$$

One observes that the size (or volume) of the set $\mathbb{X}_{k+i|k}$ is monotonically increasing with the predict step $i$. If the horizon $N_k$ is too long or the disturbance set $\mathbb{W}$ is too large, the terminal constraint $\mathbb{X}_{k+N_k|k} \subseteq \mathbb{X}_f$ may not be satisfied. On the other hand, if $N_k$ is too short, the set of states $x_k$ for which the above optimization problem is feasible is restricted.

Thus, considering the worst-case realization of the disturbance to achieve robust constraint satisfaction leads to a conservative design. Nevertheless, it may be rarely the case in practice that the disturbance frequently takes its worst-case realization. Motivated by this, we develop a robust risk-aware MPC framework in the following, where the controller optimizes its view on disturbance realizations in the future while bearing a certain degree of risk.

## III. Robust Risk-Aware MPC

In this section, we detail the formulation of robust risk-aware MPC and study its main properties.

### A. Formulation

Recall the assumption that the set $\mathbb{W}$ contains the origin, which implies that $\alpha\mathbb{W} \subseteq \mathbb{W}$, $\forall\alpha \in [0,1]$. Let us introduce additional variables $\{\alpha_{k+i|k} \in [0,1]\}_{i=0}^{N_k-1}$ to tighten the original disturbance set, i.e., $\alpha_{k+i|k}\mathbb{W}$, and use the tightened set in prediction, thereby enlarging the region of states from which there exists a feasible controller. Moreover, being optimistic about the disturbance in prediction introduces risk, which is captured by $\sum_{i=0}^{N_k-1}(1-\alpha_{k+i|k})$. The corresponding optimization problem of robust risk-aware MPC, denoted by $\mathcal{P}_{\mathrm{Risk}}(x_k, N_k)$, is formulated as

$$\begin{aligned}
&\max_{u_{\cdot|k},\alpha_{\cdot|k}} \quad \sum_{i=0}^{N_k-1} \alpha_{k+i|k} \\
&\text{s.t}: \ \mathbb{X}_{k|k} = \{x_k\}, \\
&\quad \forall i \in \mathbb{N}_{[0,N_k-1]}: \\
&\qquad \mathbb{X}_{k+i+1|k} = A\mathbb{X}_{k+i|k} \oplus B\{u_{k+i|k}\} \oplus \alpha_{k+i|k}\mathbb{W}, \\
&\qquad \begin{cases} u_{k+i|k} \in \mathbb{U}, \\ \mathbb{X}_{k+i|k} \subseteq \mathbb{X}, \\ \alpha_{k+i|k} \in [0,1], \\ \mathbb{X}_{k+N_k|k} \subseteq \mathbb{X}_f, \end{cases} \\
&\quad \forall i \in \mathbb{N}_{[0,N_k-2]}: \\
&\qquad \alpha_{k+i|k} \geq \alpha_{k+i+1|k}.
\end{aligned} \tag{3}$$

Note that the prediction horizon $N_k$ is time-varying in problem (3); how it evolves with respect to $k$ will be explained in Algorithm 1. The optimal solution to problem (3) is denoted as $u^*_{k+i|k}$ and $\alpha^*_{k+i|k}$ for $i \in \mathbb{N}_{[0,N_k-1]}$. The underlying idea of problem (3) is to maximize the robustness of its solution with the guarantee of its feasibility. The maximization of the robustness can be observed from the objective function, which is to maximize the sum of shrinking coefficients $\alpha_{k+i|k}$, i.e., to maximize the size of the tolerated disturbance sets over the prediction horizon. The last constraint $\alpha_{k+i|k} \geq \alpha_{k+i+1|k}$ promotes a smaller risk in the nearer future. By doing so, a larger degree of robustness can be obtained for the nearer future, driving the state to the target set as close as possible.

The risk of problem (3) refers to that when $\alpha_{k+i|k} < 1$, the solution to (3) has the limitation in dealing with the disturbance $w \notin \alpha_{k+i|k}\mathbb{W}$, in comparison with the robust MPC problem (2). More specifically, as the time moves one step forward, the risk of the solution to the problem (3) is the possibility of $x_{k+1} \notin \mathbb{X}$, when implementing the control inputs $u^*_{k|k}$, due to $w_k \in \mathbb{W} \setminus \alpha_{k|k}\mathbb{W}$. Note that this risk can be quantified by using the optimal solution of $\mathcal{P}_{\mathrm{Risk}}(x_k, N_k)$ if we have knowledge about the disturbance realizations. For example, if the disturbance set $\mathbb{W}$ is affiliated with probability measure $\mu$, the risk of constraint violation, i.e., $x_{k+1} \notin \mathbb{X}$, can be upper bounded by $1 - \int_{\alpha^*_{k|k}\mathbb{W}} d\mu$, where $\alpha^*_{k|k}$ is obtained from the optimal solution. A special case is that when the encountered disturbance $w_k$ is uniformly sampled from $\mathbb{W}$, the risk is then upper bounded by $1 - (\alpha^*_{k|k})^{n_x}$.

**Remark III.1.** *The risk in the current paper is close to that in the scenario-based optimization, which refers to the probability that the optimal solution obtained from a given set of samples is not feasible for a new sample [22]. How to integrate the scenario optimization with our robust risk-aware formulation and analytically quantify the risk is an interesting problem, which we leave for future research.*

**Remark III.2.** *Another strategy to suppress the risk in the nearer future is to introduce a fixed discount factor $\beta \in (0,1)$ and a fixed shrinking coefficient $\alpha$ such that the dynamics in $\mathcal{P}_{\mathrm{Risk}}(x_k, N_k)$ becomes $\mathbb{X}_{k+i+1|k} = A\mathbb{X}_{k+i|k} \oplus B\{u_{k+i|k}\} \oplus \beta^{i-1}\alpha\mathbb{W}$. This will reduce the number of decision variables in $\mathcal{P}_{\mathrm{Risk}}(x_k, N_k)$ and thereby mitigate its computational complexity.*

**Remark III.3.** *Note that the cost function $J$ can be incorporated into the problem (3) to achieve a tradeoff between control performance and robustness. For example, the objective function in (3) can be written as minimizing $J(x_k, u_{k|k}, \cdots, u_{k+N_k-1|k}) - \gamma\sum_{i=0}^{N_k-1}\alpha_{k+i|k}$, where $\gamma$ is a positive scalar. Under such a formulation, the properties in the next section still hold.*

### B. Feasibility

Let us first consider the feasible state set of the robust risk-aware MPC problem in (3), which is denoted by

$$\mathbb{F}_{\mathrm{Risk}}(N_k) = \{x_k \in \mathbb{X} \mid \mathcal{P}_{\mathrm{Risk}}(x_k, N_k) \text{ is feasible}\}.$$

Similarly, denote the feasible state set of the robust MPC problem in (2) by

$$\mathbb{F}_{\mathrm{Robust}}(N_k) = \{x_k \in \mathbb{X} \mid \mathcal{P}_{\mathrm{Robust}}(x_k, N_k) \text{ is feasible}\}.$$

Let us further consider the MPC problem of the deterministic system, denoted by $\mathcal{P}_{\mathrm{Deter}}(x_k, N_k)$, as follows:

$$\begin{aligned}
&\max_{u_{\cdot|k}} \quad J(x_k, u_{k|k}, \cdots, u_{k+N_k-1|k}) \\
&\text{s.t}: \ \mathbb{X}_{k|k} = \{x_k\}, \\
&\quad \forall i \in \mathbb{N}_{[0,N_k-1]}: \\
&\qquad \mathbb{X}_{k+i+1|k} = A\mathbb{X}_{k+i|k} \oplus B\{u_{k+i|k}\}, \\
&\qquad \begin{cases} u_{k+i|k} \in \mathbb{U}, \\ \mathbb{X}_{k+i|k} \subseteq \mathbb{X}, \\ \mathbb{X}_{k+N_k|k} \subseteq \mathbb{X}_f. \end{cases}
\end{aligned} \tag{4}$$

The feasible state set of $\mathcal{P}_{\mathrm{Deter}}(x_k, N_k)$ is denoted by

$$\mathbb{F}_{\mathrm{Deter}}(N_k) = \{x_k \in \mathbb{X} \mid \mathcal{P}_{\mathrm{Deter}}(x_k, N_k) \text{ is feasible}\}.$$

**Remark III.4.** *The feasible state set is consistent with the notion of region of attraction. In this paper, we use the former instead of the later, since there may exist some trajectories with initial states belonging to the feasible state set that are not admissible to the robust risk-aware MPC at succeeding time instants.*

The following proposition shows that the feasible state set of the problem (3) is the same as that of the problem (4). This implies that the new formulation admits a lager feasible state set than the robust MPC problem.

**Proposition III.1.** *For any $N_k \in \mathbb{N}_{\geq 1}$,*

$$\mathbb{F}_{\text{Robust}}(N_k) \subseteq \mathbb{F}_{\text{Risk}}(N_k) = \mathbb{F}_{\text{Deter}}(N_k).$$

*Proof:* The proof is straightforward and omitted for brevity. ■

**Remark III.5.** *The feasible state set $\mathbb{F}_{\text{Robust}}(N_k)$ in our work is defined in an open-loop control manner. Instead, a larger feasible state set can be obtained by using a close-loop controller, e.g., the tube MPC. In general, the feasible state set under a closed-loop controller leverages the computation of backward reachable sets [27].*

The following proposition suggests that the feasibility of the robust MPC problem $\mathcal{P}_{\text{Robust}}(x_k, N_k)$ ensures the recursive feasibility of robust risk-aware MPC $\mathcal{P}_{\text{Risk}}(x_k, N_k)$.

**Proposition III.2.** *Consider the state $x_k$ at the time step $k$ and a horizon $N_k \in \mathbb{N}$. Suppose that the problem $\mathcal{P}_{\text{Robust}}(x_k, N_k)$ is feasible. Then, there exists a sequence $\{N_k - j\}_{j=0}^{N_k-1}$ such that $\mathcal{P}_{\text{Risk}}(x_{k+j}, N_k - j)$ is feasible for all $j \in \mathbb{N}_{[0,N_k-1]}$.*

*Proof:* Note that problem $\mathcal{P}_{\text{Robust}}(x_k, N_k)$ considers the worst-case realization of disturbance. Thus, if it is feasible at time $k$ then $\mathcal{P}_{\text{Robust}}(x_{k+j}, N_k - j)$ is feasible for all $j \in \mathbb{N}_{[0,N_k-1]}$. Upon using Proposition III.1, we have that $\mathcal{P}_{\text{Risk}}(x_{k+j}, N_k - j)$ is also feasible for all $j \in \mathbb{N}_{[0,N_k-1]}$. ■

**Remark III.6.** *From Proposition III.2, one observes that whenever $\mathcal{P}_{\text{Robust}}(x_k, N_k)$ is feasible, the optimization problem* (3) *is recursively feasible with a decreasing prediction horizon. For initial conditions with which $\mathcal{P}_{\text{Robust}}(x_k, N_k)$ is infeasible, the problem* (3) *may admit a feasible solution by optimizing its view on future disturbance realizations. This essentially enlarges the feasible state set. Indeed, in the most optimistic case, $\mathcal{P}_{\text{Risk}}(x_k, N_k)$ has the same feasible state set with the deterministic setup in* (4)*, as illustrated by Proposition III.1.*

## IV. Implementation Algorithm

In this section, we address the computational tractability of the above optimizations using zonotopes, design the receding-horizon algorithm for implementation, and explore its properties.

### A. Zonotope

Let us begin with some preliminaries on zonotopes.

**Definition IV.1.** *A zonotope is a polytope that can be written as an affine transformation of the unit box. That is,*

$$\mathbb{Y} = \langle y, Y \rangle = \{z \in R^n \mid z = y + \sum_{i=1}^{p} b_i Y^{(i)}, -1 \leq b_i \leq 1\}.$$

Zonotopes have the following properties.

**Lemma IV.1.** *[28], [29] Consider two zonotopes $\mathbb{Y}_1 = \langle y_1, Y_1 \rangle$ and $\mathbb{Y}_2 = \langle y_2, Y_2 \rangle$, where $Y_1 \in \mathbb{R}^{n \times p_1}$ and $Y_2 \in \mathbb{R}^{n \times p_2}$. The following statements hold:*

*i)* $\mathbb{Y}_1 \oplus \mathbb{Y}_2 = \langle y_1 + y_2, [Y_1 \ Y_2] \rangle$*;*

*ii)* $L\mathbb{Y}_1 = \langle Ly_1, LY_1 \rangle$ *for any* $L \in \mathbb{R}^{l \times n}$*;*

*iii)* $\mathbb{Y}_1 \subseteq \mathbb{Y}_2$ *if there exist* $\Gamma \in \mathbb{R}^{p_1 \times p_1}$ *and* $v \in \mathbb{R}^{p_2}$ *such that*

$$\begin{cases} Y_1 = Y_2 \Gamma, \\ y_2 - y_1 = Y_2 v, \\ \|[\Gamma \ v]\|_\infty \leq 1. \end{cases}$$

### B. Tractable Reformulation using Zonotopes

Denote by $\mathbb{W} = \langle \underline{w}, \underline{W} \rangle$, $\mathbb{X} = \langle \underline{x}, \underline{X} \rangle$, $\mathbb{X}_f = \langle \underline{x}_f, \underline{X}_f \rangle$, and $\mathbb{X}_{k+i|k} = \langle \underline{x}_{k+i|k}, \underline{X}_{k+i|k} \rangle$, where $\underline{w}, \underline{x}, \underline{x}_f$, and $\underline{x}_{k+i|k}$ are vectors in $\mathbb{R}^n$ and $\underline{W}, \underline{X}, \underline{X}_f$, and $\underline{X}_{k+i|k}$ are matrices with appropriate dimensions. Assume that $\mathbb{U} = \{z \in \mathbb{R}^m \mid Qz \leq q\}$, where $Q$ and $q$ are a given matrix and vector with appropriate dimensions, respectively.

**Theorem IV.1.** *The following LP, denoted by $\hat{\mathcal{P}}_{\text{Risk}}(x_k, N_k)$, provides a suboptimal solution to the problem $\mathcal{P}_{\text{Risk}}(x_k, N_k)$:*

$$\begin{aligned} &\max_{u_{\cdot|k}, \alpha_{\cdot|k}, \Gamma_{\cdot|k}, v_{\cdot|k}} \quad \sum_{i=0}^{N_k-1} \alpha_{k+i|k} \\ &\text{s.t}: \ \underline{x}_{k|k} = x_k, \ \underline{X}_{k|k} = \mathbf{0}, \\ &\quad \forall i \in \mathbb{N}_{[0,N_k-1]}: \\ &\quad\quad \underline{X}_{k+i+1|k} = [A\underline{X}_{k+i|k} \ \alpha_{k+i|k}\underline{W}] \\ &\quad \underline{x}_{k+i+1|k} = A\underline{x}_{k+i|k} + Bu_{k+i|k} + \alpha_{k+i|k}\underline{w}, \\ &\quad \begin{cases} Qu_{k+i|k} \leq q, \\ \underline{X}_{k+i|k} = \underline{X}\Gamma_{k+i|k}, \\ \underline{x} - \underline{x}_{k+i|k} = \underline{X}v_{k+i|k}, \\ \|[\Gamma_{k+i|k} \ v_{k+i|k}]\|_\infty \leq 1, \\ \alpha_{k+i|k} \in [0,1], \\ \underline{X}_{k+N_k|k} = \underline{X}_f \Gamma_{k+N_k|k}, \\ \underline{x}_f - \underline{x}_{k+N_k|k} = \underline{X}_f v_{k+N_k|k}, \\ \|[\Gamma_{k+N_k|k} \ v_{k+N_k|k}]\|_\infty \leq 1, \end{cases} \\ &\quad \forall i \in \mathbb{N}_{[0,N_k-2]}: \\ &\quad\quad \alpha_{k+i|k} \geq \alpha_{k+i+1|k}. \end{aligned} \tag{5}$$

*Proof:* Given the zonotope representation of the sets $\mathbb{W}$, $\mathbb{X}$, and $\mathbb{X}_f$, it follows from the first two properties in Lemma IV.1 that the set-valued dynamics in problem (3), i.e., $\mathbb{X}_{k+i+1|k} = A\mathbb{X}_{k+i|k} \oplus B\{u_{k+i|k}\} \oplus \alpha_{k+i|k}\mathbb{W}$, can be written as $\underline{X}_{k+i+1|k} = [A\underline{X}_{k+i|k} \ \alpha_{k+i|k}\underline{W}]$ and $\underline{x}_{k+i+1|k} = A\underline{x}_{k+i|k} + Bu_{k+i|k} + \alpha_{k+i|k}\underline{w}$. According to the third property, we have that the constraint $\mathbb{X}_{k+i|k} \subseteq \mathbb{X}$ holds if there exist $\Gamma_{k+i|k}$ and $v_{k+i|k}$ such that

$$\begin{cases} \underline{X}_{k+i|k} = \underline{X}\Gamma_{k+i|k}, \\ \underline{x}_{k+i|k} - \underline{x} = \underline{X}v_{k+i|k}, \\ \|[\Gamma_{k+i|k} \ v_{k+i|k}]\|_\infty \leq 1. \end{cases}$$

Similarly, we can obtain that the last three constraints in (5) are sufficient for $\mathbb{X}_{k+N_k|k} \subseteq \mathbb{X}_f$. Thus, we conclude that the optimal solution to the problem $\hat{\mathcal{P}}_{\text{Risk}}(x_k, N_k)$ in (5) is a suboptimal solution to the problem $\mathcal{P}_{\text{Risk}}(x_k, N_k)$ in (3). ■

**Remark IV.1.** *If the convex uncertainty set is represented in other forms, e.g., ball, ellipsoid, and rectangle, the optimization problem $\mathcal{P}_{\text{Risk}}(x_k, N_k)$ in this paper can also be reformulated as a tractable convex optimization problem by following the idea of [23]. It is also of interest to consider the disturbance-dependent affine control structure [23] in the future work.*

Before proceeding to an online implementation algorithm in the next section, we turn to consider the deterministic version, denoted by $\hat{\mathcal{P}}_{\text{Deter}}(x_k, N_k)$, of the problem $\hat{\mathcal{P}}_{\text{Risk}}(x_k, N_k)$, where the decision variables $\alpha_{k+i|k}$ are set to be 0. Then, it reads:

$$
\begin{aligned}
&\max_{u_{\cdot|k}, v_{\cdot|k}} \quad 0 \\
&\text{s.t}: \ \underline{x}_{k|k} = x_k, \\
&\quad \forall i \in \mathbb{N}_{[0,N_k-1]}: \\
&\underline{x}_{k+i+1|k} = A\underline{x}_{k+i|k} + Bu_{k+i|k}, \\
&\qquad\qquad \begin{cases} Qu_{k+i|k} \leq q, \\ \underline{x}_{k+i|k} - \underline{x} = \underline{X} v_{k+i|k}, \\ \|v_{k+i|k}\|_\infty \leq 1, \\ \underline{X}_{k+N_k|k} = \underline{X}_f \Gamma_{k+N_k|k}, \\ \underline{x}_{k+N_k|k} - \underline{x}_f = \underline{X}_f v_{k+N_k|k}, \\ \|v_{k+N_k|k}\|_\infty \leq 1. \end{cases}
\end{aligned} \tag{6}
$$

Define the feasible state set of the problem $\hat{\mathcal{P}}_{\text{Deter}}(x_k, N_k)$ as

$$\hat{\mathbb{F}}_{\text{Deter}}(N_k) = \{x_k \in \mathbb{X} \mid \hat{\mathcal{P}}_{\text{Deter}}(x_k, N_k) \text{ is feasible}\}.$$

### C. Algorithm

Algorithm 1 details the MPC implementation with adaptive horizon. Let $N_{\max}$ be the maximal horizon that can be specified in the optimization problems $\hat{\mathcal{P}}_{\text{Risk}}(x_k, N_k)$ and $\hat{\mathcal{P}}_{\text{Deter}}(x_k, N_k)$. At the initialization phase (lines 2-10), we first check if the initial state $x_0 \in \hat{\mathbb{F}}_{\text{Deter}}(N_{\max})$. If so, we solve the problem $\hat{\mathcal{P}}_{\text{Risk}}(x_0, N_0)$ with $N_0 = N_{\max}$ and implement the first optimal control input $u^*_{0|0}$; otherwise, we stop running with the output Infeasible. In the online implementation phase (lines 12-31), we begin with checking if the current state reaches the target set $\mathbb{X}_f$ (lines 15-17). If not, we need to check if the risk-aware solution can tolerate the injected disturbance, i.e., if $w_{k-1} \in \alpha^*_{k-1|k-1}\mathbb{W}$. If yes, we can also gradually decrease the horizon, i.e., $N_k = N_{k-1} - 1$. Otherwise, we need to check if $x_k \in \hat{\mathbb{F}}_{\text{Deter}}(L)$ for some $L \in \mathbb{N}_{[1,N_{\max}]}$. If so, it implies that the current state still belongs to the feasible state set for some allowed horizon, for which we reset $N_k = N_k = \min\{L \in \mathbb{N}_{[1,N_{\max}]} \mid x_k \in \hat{\mathbb{F}}_{\text{Deter}}(L)\}$. Otherwise, we stop running with the output Infeasible.

**Algorithm 1** Robust Risk-Aware MPC Algorithm

1: **Initialization**:
2: Set $k = 0$ and TerInd $= 1$;
3: Let the maximal horizon be $N_{\max}$;
4: **if** $x_0 \in \hat{\mathbb{F}}_{\text{Deter}}(N_{\max})$ **then**
5:     Let $N_0 = N_{\max}$ and solve the problem $\hat{\mathcal{P}}_{\text{Risk}}(x_0, N_0)$;
6:     Implement $u^*_{0|0}$; ▷ $u^*_{0|0}$ is obtained from the optimal solution to $\hat{\mathcal{P}}_{\text{Risk}}(x_0, N_0)$;
7: **else**
8:     TerInd $= 0$;
9:     **Output**: Infeasible;
10: **end if**
11: **Implementation**:
12: **while** TerInd **do**
13:     Set $k = k + 1$;
14:     Measure $x_k$;
15:     **if** $x_k \in \mathbb{X}_f$ **then**
16:         TerInd $= 0$;
17:         **Output**: Successful;
18:     **else**
19:         Let $w_{k-1} = x_k - Ax_{k-1} - Bu^*_{k-1|k-1}$
20:         **if** $w_{k-1} \in \alpha^*_{k-1|k-1}\mathbb{W}$ **then**
21:             Let $N_k = N_{k-1} - 1$ and go to line 31;
22:         **else**
23:             **if** $x_k \notin \hat{\mathbb{F}}_{\text{Deter}}(L)$, $\forall L \in \mathbb{N}_{[1,N_{\max}]}$ **then**
24:                 TerInd $= 0$;
25:                 **Output**: Infeasible;
26:             **else**
27:                 Let $N_k = \min\{L \in \mathbb{N}_{[1,N_{\max}]} \mid x_k \in \hat{\mathbb{F}}_{\text{Deter}}(L)\}$ and go to line 31;
28:             **end if**
29:         **end if**
30:     **end if**
31:     Solve the problem $\hat{\mathcal{P}}_{\text{Risk}}(x_k, N_k)$ and implement $u^*_{k|k}$; ▷ $u^*_{k|k}$ is obtained from the optimal solution to $\hat{\mathcal{P}}_{\text{Risk}}(x_k, N_k)$;
32: **end while**

## V. Examples

In this section, we demonstrate our method and perform comparisons with tube MPC on one simulated example. The numerical experiments are run in Matlab R2021b with YALMIP toolbox [30] and MOSEK toolbox [31] on a MacBook Pro laptop with Apple M1 chip and 8.0 GB Memory.

Consider the following system

$$x_{k+1} = Ax_k + Bu_k + w_k$$

where $A = [1, 0.9; 0, 1.1]$, $B = [0; 1]$, $\mathbb{X} = \{x \in \mathbb{R}^2 | [-10, -10]^T \leq x \leq [10, 10]^T\}$, and $\mathbb{U} = \{u \in \mathbb{R} | -1 \leq u \leq 1\}$. We consider the following five different disturbance sets in the form of $\mathbb{W} = \{w \in \mathbb{R}^2 | [-0.2\gamma, -0.2\gamma]^T \leq w \leq [0.2\gamma, 0.2\gamma]^T\}$, where $\gamma$ takes value in $\{1, 2, 4, 6, 8\}$. The realization of $w_k$ is drawn from $\mathbb{W}$ uniformly at random. The target set is set as $\mathbb{X}_f = \{x \in \mathbb{R}^2 | [-0.9, -0.9]^T \leq x \leq [0.9, 0.9]^T\}$. Set the prediction horizon $N_{\max} = N_0 = 10$ and the maximum time step is $T_{\max} = 15$. We stop running either when the state trajectory enters $\mathbb{X}_f$ or when the time step exceeds $T_{\max}$. The run in which the actual state trajectory does not enter $\mathbb{X}_f$ before termination is deemed infeasible.

For comparison, we introduce the following parameters

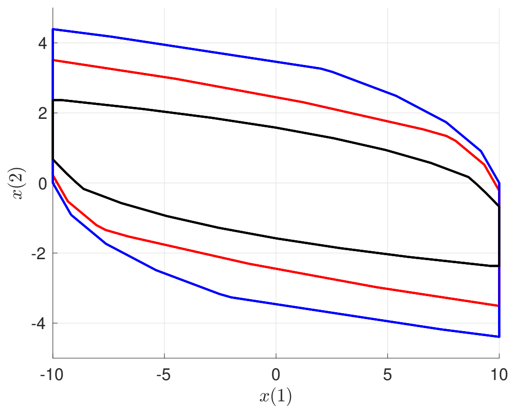


Fig. 1: Feasible state sets of robust risk-aware MPC and robust MPC for $\gamma = 1$. The region with blue boundary represents the feasible state set of risk-aware MPC, i.e., $\mathbb{F}_{\text{Risk}}$. The region circled by red line is the robust backward reachable set from $\mathbb{X}_f$ within $N = 10$ steps; The region circled black line is the feasible state set of tube MPC $\mathbb{F}_{\text{Tube}}$.

for tube MPC [10]. We use weight matrices $Q = I$ and $R = 0.1$, under which the LQR gain is chosen as the feedback gain, denoted by $K$, to construct tubes. Set the solution to the corresponding algebraic Riccati equation as the weight matrix, denoted by $P$, for terminal cost. Denote by $\mathbb{Z}_f$ the corresponding minimal robust controlled invariant set. At time step $k$, given the state $x_k$, the adapted tube MPC problem is formulated as follows:

$$\begin{aligned} \min_{v_{\cdot|k}, z_{\cdot|k}} \quad & \sum_{i=0}^{N_k-1} (\|z_{k+i|k}\|_Q^2 + \|v_{k+i|k}\|_R^2) + \|z_{k+N_k|k}\|_P^2 \\ \text{s.t :} \quad & z_{k|k} \in \{x_k\} \oplus \mathbb{Z}_f, \\ & \forall i \in \mathbb{N}_{[0,N_k-1]} : \\ & \quad z_{k+i+1|k} = Az_{k+i|k} \oplus Bv_{k+i|k}, \\ & \quad \begin{cases} v_{k+i|k} \in \mathbb{U} \ominus K\mathbb{Z}_f, \\ z_{k+i|k} \subseteq \mathbb{X} \ominus \mathbb{Z}_f, \\ z_{k+N_k|k} \subseteq \mathbb{X}_f \ominus \mathbb{Z}_f, \end{cases} \end{aligned} \tag{7}$$

where $z_{k+i|k}$ and $v_{k+i|k}$ are the predicted nominal state and control input of the nominal system, respectively. Given the optimal solution, denoted by $v^*_{k+i|k}$ and $z^*_{k+i|k}$, to the problem (7), the implemented controller is $u_k = v^*_{k|k} + K(x_k - z^*_{k|k})$. The feasible state set of the problem (7) is denoted by $\mathbb{F}_{\text{Tube}}(N_k)$. The online implementation of tube MPC is similar to Algorithm 1 with adaptive horizon.

### A. *Comparison with Tube MPC*

We begin with the comparison of feasible state sets for risk-aware MPC, robust MPC, and tube MPC. Here the feasible state set of robust MPC is computed by performing the backward reachability. The feasible state sets for $\gamma = 1$ are shown in Fig. 1. The region circled by blue line is $\mathbb{F}_{\text{Risk}}(N_{\max})$ and the region with black line represents $\mathbb{F}_{\text{Tube}}(N_{\max})$. The set $\mathbb{F}_{\text{Robust}}(N_{\max})$ defined by the open-loop control is empty. As a comparison, we compute the robust backward reachable set from $\mathbb{X}_f$ within $N = 10$ steps [27], which is the largest feasible set for closed-loop robust control, as shown by the region with red black line. We can observe that the risk-ware MPC admits a larger feasible state set than its robust counterparts. In this example, the feasible state set of the reformulated robust risk-aware MPC, i.e., $\hat{\mathbb{F}}_{\text{Deter}}(N_{\max})$, is the same as $\mathbb{F}_{\text{Risk}}(N_{\max})$. We note that for larger disturbance sets, i.e., when $\gamma = 2, 4, 6, 8$, the feasible state set of risk-aware MPC is invariant with respect to $\gamma$ while those of the other two methods are empty.

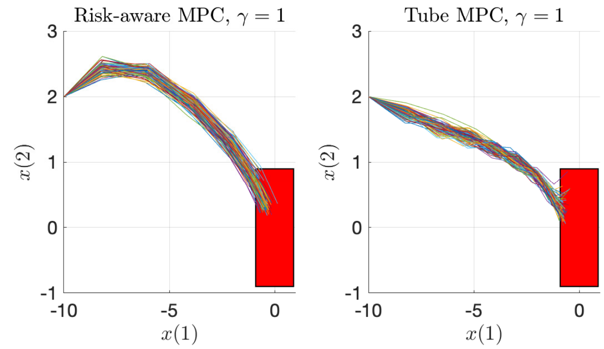


Fig. 2: Feasible state trajectories for 100 realizations of disturbances with $\gamma = 1$ by robust risk-aware MPC and tube MPC. The rectangle shaded in red denotes the target set.

Next let us compare the performance under our approach and the tube MPC. We choose an initial state $x_0 = [-10, 2]^T$, which is in $\mathbb{F}_{\text{Tube}}(N_{\max})$. For each approach, we perform 100 runs. Their state trajectories are shown in Fig. 2. To roughly compare the performance, we identify the average cost per step for each strategy, defined by

$$J_{\text{Strategy}} = \frac{1}{100} \sum_{k=1}^{100} \frac{1}{T_k} \sum_{i=1}^{T_k} x_i^T Q x_i + u_i^T R u_i,$$

where $T_k$ represents the number of time steps in the $k$-th run. The results are $J_{\text{Risk}} = 36.8588$ and $J_{\text{Tube}} = 40.7146$, suggesting that the proposed strategy has a comparable performance with tube MPC while significantly enlarging the feasible state set. The main reason that tube MPC leads to inferior performance may be twofold: *i*) nominal but not actual trajectories are optimized within tube MPC; *ii*) the presence of terminal cost deteriorates the optimality of system trajectories. The average computation time of each run under our approach is 1.9917 seconds, a slightly longer than 1.7308 seconds for tube MPC. The reason is that the reformulated robust risk-aware MPC problem (5) may encounter more constraints along the horizon due to the augmented matrices.

### B. *Robust Risk-Aware MPC for Large Disturbance Sets*

To illustrate the strength of the proposed method, we further report the performance of the proposed method for large disturbance sets, i.e., when $\gamma = 2, 4, 6, 8$. We choose the initial state to be $x_0 = [-10, 4]^T$, which is in $\mathbb{F}_{\text{Risk}} \setminus \mathbb{F}_{\text{Tube}}$.

| | $\gamma = 2$ | $\gamma = 4$ | $\gamma = 6$ | $\gamma = 8$ |
|---|---|---|---|---|
| $\alpha^*_{0\|0}$ | 0.4152 | 0.2129 | 0.1470 | 0.1126 |
| Probability of success | 100% | 100% | 94% | 78% |
| Ave. computation time per run | 2.5400 | 4.2922 | 4.9052 | 5.8236 |

TABLE I: The shrinking coefficients $\alpha^*_{0|0}$, the probability of successful runs, and the average computation time (in seconds) per run under Algorithm 1 for 100 realizations of disturbances for $\gamma = 2, 4, 6$, and 8.

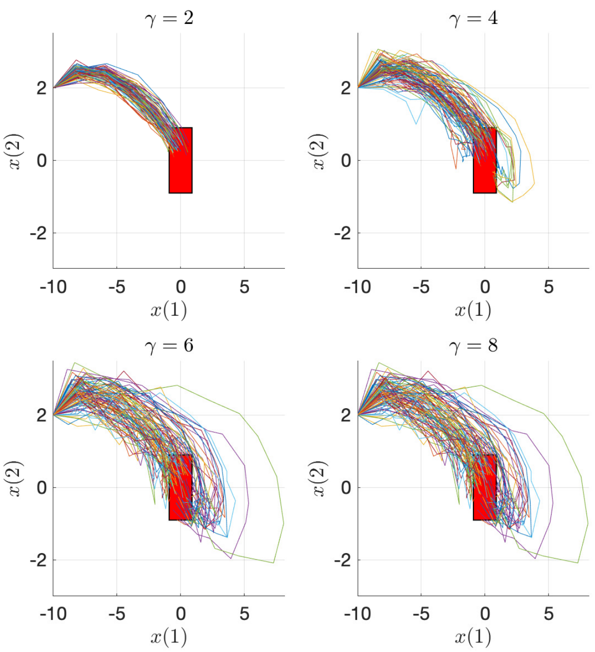


Fig. 3: Feasible state trajectories for 100 realizations of disturbances under different $\gamma$. The rectangle shaded in red denotes the target set.

In the experiments, we perform 100 runs of Algorithm 1 for each $\gamma \in \{2, 4, 6, 8\}$. The risk at time $t = 0$, probability of success, and computation time are summarized in Table I. Note that we only report $\alpha^*_{0|0}$ at $t = 0$, because typically it is the most critical in the sense that the value of $\alpha^*_{k|k}$ increases as $x_k$ approaches the target set. From Table I, we can see that Algorithm 1 can effectively (with high probability) drive the state to the target set even when the initial state is infeasible for the robust MPC. When the disturbance bound grows, the controller has relatively low robustness (captured by the size of $\alpha^*_{0|0}$) to achieve feasibility. The average computation of each run increases with respect to $\gamma$, since more time steps are needed for driving the state to the target set against larger disturbance realizations.

The feasible state trajectories by Algorithm 1 under different $\gamma$ are shown in Fig. 3. The results show that in most runs the state is successfully driven to the target set. They also demonstrate the trend that, as we increase $\gamma$, the trajectories become less concentrated because larger noises impose a stronger effect on the state trajectories. Note that the disturbance bounds for $\gamma = 6$ or 8 (which are 1.2 or 1.6, respectively) are greater than the control input bound (which is 1). To the best of our knowledge, these two cases cannot be handled by most existing control methods. The probabilities of success shown in Table I justify that our method has greater robustness in practice than the theoretical robustness quantified using the shrinking coefficients $\alpha$.

## VI. Conclusion

In this work, we developed a robust risk-aware MPC algorithm for linear discrete-time systems subject to bounded disturbances. The proposed algorithm takes a certain degree of risk on future disturbance realizations when necessary, therefore enlarging the feasible state set of conventional robust MPC. A computationally efficient reformulation was introduced for the case where the sets can be expressed as zonotopes, and an online implementation algorithm was designed. Finally, we demonstrated the effectiveness of the proposed method via one numerical example.

Future directions of great interest include the extension to the closed-loop control design, the integration of data-driven uncertainty quantification into our method, theoretical analysis of the inherent robustness, and the experimental evaluation on hardware.

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