15th Triennial World Congress, Barcelona, Spain

HYBRID LIMIT CYCLES AND HYBRID POINCARÉ-BENDIXSON [⋆]

Slobodan N. Simić [∗]

[∗] *Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720-1774, U.S.A. Email:* simic@eecs.berkeley.edu

Karl Henrik Johansson

Department of Signals, Sensors & Systems, Royal Institute of Technology, 100 44 Stockholm, Sweden. Email: kallej@s3.kth.se

John Lygeros

Department of Engineering, University of Cambridge, Cambridge CB2 1PZ, U.K. Email: jl290@eng.cam.ac.uk

Shankar Sastry [∗]

Abstract: We present two results about regular hybrid systems with no branching (Simić *et al.*, 2000*a*). The first one provides a condition for asymptotic stability of hybrid closed orbits in terms of contraction-expansion rates of resets and flows in a hybrid system. The second one is a generalization of the Poincaré-Bendixson theorem to planar hybrid systems.

Keywords: Regular hybrid system; hybrifold; hybrid flow; hybrid closed orbit.

1. INTRODUCTION

Research in the area of hybrid systems has been motivated by a variety of applications to air traffic management, automotive control, embedded software, process control, highway systems, manufacturing, and other areas. Numerous methods for modeling, analyzing, and controlling hybrid systems have been proposed. However, many fundamental questions in the field still remain open. The main reason is that in addition to being nonlinear, hybrid systems are not smooth. When analytic methods for analysis fail – as they generally do, which is the principal reason for the development of the modern theory of dynamical systems – the only resort we have is qualitative analysis.

This is why in (Simić *et al.*, 2000*a*) and (Simić *et al.*, 2000*b*), we proposed a framework for a geometric (i.e., qualitative) theory of hybrid systems. We restricted ourselves to a class of systems, called *regular* and *without branching*, which behave like

[⋆] This work was supported by the NASA grant NAG-2-1039, EPRI grant EPRI-35352-6089, ARO under the MURI grant DAAH04-96-1-0341, ONR under grant N00014-97-1-0946, and DARPA under contract F33615-98-C-3614.

piecewise smooth dynamical systems (without sliding) on piecewise smooth manifolds. We introduced a single state space called the *hybrifold* for the hybrid dynamics captured by the *hybrid flow*, explained the geometric reason for the Zeno phenomenon (when the system makes infinitely many switches in finite time), and locally classified it in dimension two. In (Lygeros *et al.*, 2000), we studied along similar lines, among other problems, stability of equilibria and invariant sets of hybrid systems via linearization and LaSalle's principle.

In this article we continue with our program and deal with the question of stability of hybrid closed orbits (Section 3). We propose a stability criterion in terms of expansion-contraction rates of reset maps and flows involved in creating the closed orbit in question. We then address the problem of recurrence in planar hybrid systems (Section 4) and show that the classical Poincaré-Bendixson theorem can be generalized to regular systems without branching. We close (Section 5) by outlining the conclusions and directions for future work.

Due to a space limit, we omit all proofs. They will appear in the forthcoming full version of the paper.

2. PRELIMINARIES

In this section we define the basic notions, fix the notation, and state the standing assumptions. We closely follow (Simić *et al.*, 2000*a*) (and its preliminary version (Simić *et al.*, 2000*b*)) to which the reader is referred for details.

Definition 2.1. An n-dimensional *hybrid system* is a 6-tuple $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$, where:

- $Q = \{1, \ldots, k\}$ is the collection of *(discrete) states* of $\mathbf{H}$, where $k \geq 1$ is an integer;
- $E \subset Q \times Q$ is the collection of *edges*;
- $\mathcal{D} = \{D_i : i \in Q\}$ is the collection of *domains* of $\mathbf{H}$, where $D_i \subset \{i\} \times \mathbf{R}^n$ for all $i \in Q$;
- $\mathcal{X} = \{X_i : i \in Q\}$ is the collection of vector fields such that X_i is Lipschitz on D_i for all $i \in Q$; we denote the local flow of X_i by $\{\phi_t^i\}$.
- $\mathcal{G} = \{G(e) : e \in E\}$ is the collection of *guards*, where for each $e = (i, j) \in E$, $G(e) \subset D_i$;
- $\mathcal{R} = \{R_e : e \in E\}$ is the collection of *resets*, where for each $e = (i, j) \in E$, R_e is a relation between elements of $G(e)$ and elements of D_j, i.e. $R_e \subset G(e) \times D_j$.

Given $\mathbf{H}$, the basic idea is that starting from a point in some domain D_i we flow according to X_i until (and if) we reach some guard $G(i, j)$, then switch via the reset $R_{(i,j)}$, continue flowing in D_j according to X_j and so on. This is formalized in the following two definitions.

Definition 2.2. A *(forward) hybrid time trajectory* is a sequence (finite or infinite) $\tau = \{I_j\}_{j=0}^N$ of intervals such that $I_j = [\tau_j, \tau_j']$ for all $j \geq 0$ if the sequence is infinite; if N is finite, then $I_j = [\tau_j, \tau_j']$ for all $0 \leq j \leq N-1$ and I_N is either of the form $[\tau_N, \tau_N']$ or $[\tau_N, \tau_N')$. The sequences τ_j and τ_j' satisfy: $\tau_j \leq \tau_j' = \tau_{j+1}$, for all j.

One thinks of τ_j's as time instants when discrete transitions (or switches) from one domain to another take place. If τ is a hybrid time trajectory, we will call N its *size* and denote it by $N(\tau)$. Also, we use $\langle \tau \rangle$ to denote the set $\{0, \ldots, N(\tau)\}$ if $N(\tau)$ is finite, and $\{0, 1, 2, \ldots\}$ if $N(\tau)$ is infinite.

Definition 2.3. An *execution* (or *forward execution*) of a hybrid system $\mathbf{H}$ is a triple $\chi = (\tau, q, x)$, where τ is a hybrid time trajectory, $q : \langle \tau \rangle \to Q$ is a map, and $x = \{x_j : j \in \langle \tau \rangle\}$ is a collection of C^1 maps such that $x_j : I_j \to D_{q(j)}$ and for all $t \in I_j$,

$$\dot{x}_j(t) = X_{q(j)}(x_j(t)).$$

Furthermore, for all $j \in \langle \tau \rangle$ such that $j < N(\tau)$, we have $(q(j), q(j+1)) \in E$, and

$$(x_j(\tau_j'), x_{j+1}(\tau_{j+1})) \in R_{(q(j), q(j+1))}.$$

For an execution $\chi = (\tau, q, x)$, denote by $\tau_\infty(\chi)$ its *(forward) execution time* $\tau_\infty(\chi) = \sum_{j=0}^{N(\tau)} (\tau_j' - \tau_j)$. We distinguish several types of executions (see (Simić *et al.*, 2000*a*) or (Lygeros *et al.*, 2000) for details). *Infinite* executions make infinitely many switches or have infinite execution time, while *maximal* ones are maximal with respect to a natural ordering on executions. An execution χ is called a *Zeno execution* if $N(\tau) = \infty$ and $\tau_\infty(\chi) < \infty$. That is, it makes infinitely many switches in finite time. A Zeno execution is called *dynamic* if for every $l > 0$ there exists $j \geq l$ such that $\tau_j' > \tau_j$, i.e., it doesn't cease to make time progress.

In (Simić *et al.*, 2000*a*) we studied a class of hybrid systems which behave like piecewise smooth dynamical systems on piecewise smooth manifolds. We called such systems *regular hybrid systems without branching*. Roughly speaking, $\mathbf{H}$ is regular and without branching if each guard can be glued to the image of the corresponding reset in such a way to obtain a topological manifold on which the corresponding projected dynamics look like that of a piecewise smooth flow. So for instance, for every initial condition there is a unique infinite execution, the domains are piecewise smooth manifolds, the guards are smooth submanifolds of the boundary of the domains, resets are diffeomorphisms, executions *cross* (i.e., are not tangent to) the guards except possibly along the boundary, and guards and images of reset maps can meet only along their boundaries. For the complete list of assumptions, please see the above reference.

In the same paper, we introduced the notion of the *hybrifold* $M_{\mathbf{H}}$ of a hybrid system $\mathbf{H}$ and its *hybrid flow* $\Psi^{\mathbf{H}}$. The hybrifold is the single state space for the hybrid dynamics and is obtained by "gluing" the domains along guards via reset maps. Just like the flow of a smooth system, the hybrid flow satisfies $\Psi^{\mathbf{H}}_t(\Psi^{\mathbf{H}}_s(x)) = \Psi^{\mathbf{H}}_{t+s}(x)$, for all $x \in M_{\mathbf{H}}$ and $t, s \in \mathbf{R}$ for which both sides are defined. So instead of studying executions of $\mathbf{H}$ in several different locations, we study orbits (see below for a definition) of the corresponding hybrid flow on a single hybrifold. The advantage of this is that it allows the use of techniques from the theory classical continuous-time dynamical systems. Furthermore, it provides a convenient setting for global analysis of hybrid systems.

Standing assumption: *Every hybrid system* $\mathbf{H}$ *henceforth is regular and without branching. Its hybrifold is denoted by* $M_{\mathbf{H}}$ *and its hybrid flow by* $\Psi^{\mathbf{H}}$.

Recall that, for $x \in M_{\mathbf{H}}$, $t \mapsto \Psi^{\mathbf{H}}_t(x)$ denotes the unique execution (viewed in $M_{\mathbf{H}}$) starting at x at time 0. Let $J(x)$ be the set of all real numbers t for which $\Psi^{\mathbf{H}}_t(x)$ is defined and let $\tau_\infty(x) = \sup J(x)$. For each x, we call the collection of points $\Psi^{\mathbf{H}}_t(x)$, $t \in J(x)$, the *orbit* of x. Also, denote by π the projection map $\bigcup_i D_i \to M_{\mathbf{H}}$, which assigns to each p the set of points p is identified with in the hybrifold construction (Simić *et al.*, 2000*a*).

Since our goal is to study asymptotic behavior of orbits, analogously to the classical case (Palis Jr. and de Melo, 1982) and following (Simić *et al.*, 2000*a*), we introduce the notion of the ω-limit set.

Definition 2.4. A point $y \in M_{\mathbf{H}}$ is called an *ω-limit point of* $x \in M_{\mathbf{H}}$ if

$$y = \lim_{m \to \infty} \Psi^{\mathbf{H}}_{t_m}(x),$$

for some increasing sequence (t_m) in $J(x)$ such that $t_m \to \tau_\infty(x)$, as $m \to \infty$. The set of all ω-limit points of x is called the *ω-limit set of* x and is denoted by $\omega(x)$.

One final note: smooth will mean of class C^∞. If $f : M \to N$ is a smooth map between smooth manifolds and $p \in M$, $T_p f$ will denote the tangent map (or derivative in the case when $M = \mathbf{R}^n$) of f at p; it maps the tangent space of M at p denoted by $T_p M$ to $T_{f(p)} N$.

3. HYBRID CLOSED ORBITS

The simplest types of recurrence in any dynamical system are exhibited by equilibria and closed orbits. The notion of an equilibrium of a hybrid system we use was defined in (Simić *et al.*, 2000*b*) (as well as (Lygeros *et al.*, 2000) for more general hybrid systems). Namely, a point $x \in M_{\mathbf{H}}$ is an *equilibrium* for $\Psi^{\mathbf{H}}$ if $\Psi^{\mathbf{H}}_t(x) = x$, for all $t \in J(x)$. Note that Zeno executions which make no time progress (i.e., $J(x)$ is a singleton) also give rise to equilibria.

Definition 3.1. An orbit γ of a hybrid flow $\Psi^{\mathbf{H}}$ on $M_{\mathbf{H}}$ is *closed* if it is not an equilibrium and there exists a positive number T such that for some (and therefore all) $x \in \gamma$, $\Psi^{\mathbf{H}}_T(x) = x$. The smallest such T is called the *period* of γ. If γ is not contained in a single domain $\pi(D_i)$, it is called a *hybrid closed orbit*.

We also speak of closed orbits of the hybrid system $\mathbf{H}$ itself (as opposed to its hybrid flow $\Psi^{\mathbf{H}}$). Those are the executions of $\mathbf{H}$ which project to closed orbits of $\Psi^{\mathbf{H}}$ via π.

In general, it is hard to find closed orbits even of smooth dynamical systems. In the plane, we have Bendixson's criterion in terms of divergence which tells us when there are *no* closed orbits (Sastry, 1999), and the Poincaré-Bendixson theorem (see below) on the 2-sphere or the 2-disk which is only an existence result. In higher dimensions, however, looking for closed orbits is a matter of hard hands-on analysis and simulation. Similarly, for non-smooth or hybrid systems, little is known about existence and stability of closed orbits (see, for instance, (Guckenheimer and Johnson, 1994; Johansson *et al.*, 1997; Matveev and Savkin, 2000)). The recent book *Qualitative Theory of Hybrid Dynamical Systems* (Matveev and Savkin, 2000) deals with similar questions (among many others) as we do in this paper but in a different setting; for instance, Zeno executions are not allowed and in the study of limit cycles, only constant vector fields are permitted. It should also be mentioned that planar switching systems were investigated by A. A. Andronov and his group in the Soviet Union before 1950 (for a historical account and references, please see (Bissell, 2001)). Another good reference for stability of closed orbits of smooth and discontinuous systems is (Leonov *et al.*, 1996).

Closed orbits which attract other orbits are of special significance: a closed orbit γ is called a *limit cycle* if there exists a point $x \notin \gamma$ such that $\omega(x) = \gamma$. Some limit cycles have an additional property of attracting a whole neighborhood of orbits around them.

Definition 3.2. A closed orbit γ of a hybrid flow $\Psi^{\mathbf{H}}$ is called *asymptotically stable* if for every neighborhood U of γ in $M_{\mathbf{H}}$ there is a neighborhood $V \subset U$ of γ such that $\Psi^{\mathbf{H}}_t(V) \subset U$, for all $t > 0$, and for every $x \in V$,

$$\lim_{t \to \infty} d(\Psi^{\mathbf{H}}_t(x), \gamma) = 0. \qquad (1)$$

Here $d(x, \gamma)$ denotes the minimum distance from x to γ measured by the metric on $M_{\mathbf{H}}$ (Simić *et al.*, 2000*a*) defined in a standard way as the infimum of the length of curves between points.

Remark. We briefly remind the reader of the basic result on stability of closed orbits for smooth systems (cf. (Hirsch and Smale, 1974)). Suppose X is a smooth vector field on (for simplicity) $\mathbf{R}^n$ which has a closed orbit γ with period τ. Denote the flow of X by ϕ_t. Take a point $p \in \gamma$ and let H be a hyperplane through p transverse to γ (i.e., $X(p)$ and H span $\mathbf{R}^n$) which is *invariant* under $T_p\phi_\tau$. Recall that $T_p\phi_\tau$ has the eigenvalue 1 corresponding to the eigenvector $X(p)$, so all other eigenvalues correspond to directions in H. Consider the first-return map g from some neighborhood U of p in H into H. Then $g(p) = p$ and p is asymptotically stable for g if and only if γ is asymptotically stable. Further, it can be shown that

$$T_p g = T_p\phi_\tau|_H.$$

Therefore, *if $n-1$ eigenvalues of $T_p\phi_\tau$ are less than 1 in absolute value, then γ is asymptotically stable.*

As far as it is known to the authors, there exists no similar result which does not require integrating the vector field. Consequently, it is not reasonable to expect it for hybrid systems.

The main result of this section which we now state is in the spirit of the above remark.

Theorem 3.1. Let γ be a hybrid closed orbit of $\Psi^{\mathbf{H}}$. Denote by $\Gamma = \pi^{-1}(\gamma)$ the execution of $\mathbf{H}$ which gives rise to γ, and assume that Γ cyclically visits an ordered collection of distinct domains which we, without loss of generality, denote by $D_1, \ldots, D_l$. Assume that $\Gamma_j = \Gamma \cap \overline{D_j}$ is a single smooth arc (as opposed to a collection of them) starting at a_j and ending at b_j.

Let $e_j = (j, j+1)$, for $1 \le j \le l-1$, and $e_l = (l, 1)$ be the edges of E corresponding to transitions between domains D_j. Let $A_j = \text{image } R_{e_{j-1}}$ and $B_j = G(e_j)$ so that $a_j \in A_j$ and $b_j \in B_j$. Suppose that a_j, b_j lie in the interior (relative to the boundary of D_j) of A_j, B_j respectively, and that A_j, B_j are smooth at a_j, b_j (see Fig. 1).

Set $\nu_j = \|T_{b_j} R_{e_j}\|$ and $\mu_j = \|T_{a_j}\phi^j_{\tau_j}|_{T_{a_j}A_j}\|$. If

$$c_\gamma = \prod_{j=1}^{l} 2\mu_j\nu_j < 1,$$

then γ is asymptotically stable. Furthermore, the convergence in (1) is exponential.

Note that it is the *interplay* of the contractive-expansive properties of both resets and flows that determines stability of γ. In fact, the result can be strengthened by assuming a weaker (but less tractable) condition as follows. For each $j = 1, \ldots, l$ there is a a diffeomorphism h_j from a neighborhood of a_j in A_j to a neighborhood of b_j in B_j, defined by:

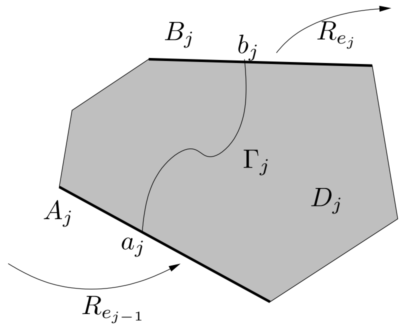

Fig. 1. Illustration for Theorem 3.1.

$h_j(x)$ is the first intersection of the forward X_j-orbit of x with B_j. Let $\kappa_j = \|T_{a_j} h_j\|$. If $k_\gamma = \prod_j \kappa_j\nu_j < 1$, then γ is asymptotically stable. However, κ_j may be even more difficult to compute than μ_j. The reason for the curious (but unfortunate) presence of the number two in the above product is explained by the following lemma which is an important step in the proof of Theorem 3.1 and says that $\kappa_j \le 2\mu_j$. Here for a linear map $L : \mathbf{R}^n \to \mathbf{R}^n$ and a subspace $E \subset \mathbf{R}^n$, $\|L|_E\| = \sup\{\|Lv\| : v \in E, \|v\| = 1\}$.

Lemma 3.1. Let X be a smooth vector field on $\mathbf{R}^n$ with local flow ϕ_t without equilibria. Let A and B be smooth disjoint hypersurfaces transverse to X such that for some $a \in A$ and $\tau > 0$, $\phi_\tau(a) = b \in B$, and for all $t \in [0, \tau)$, $\phi_t(a) \notin B$. Assume a, b lie in the interior of A, B, respectively. Then there exist a neighborhood U of a in A, a neighborhood V of b in B, and a diffeomorphism $h : U \to V$ such that if $x \in U$, then $h(x)$ is the unique first intersection of the forward X-orbit of x and B. Furthermore,

$$\|T_a h\| \le 2\|T_a\phi_\tau|_{T_aA}\|.$$

Note that a useful and often only tool available to estimate $\|T_a\phi_\tau|_{T_aA}\|$ is the well known *second variational equation* (Hirsch and Smale, 1974)

$$\frac{d}{dt}(T_p\phi_t) = T_{\phi_t(p)}X \circ T_p\phi_t.$$

Example 3.1. We force a damped pendulum into cyclic motion using impulse control. Consider the pendulum equation $\ddot{\theta} + \dot{\theta} + \sin\theta = 0$, or equivalently the system

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\sin\theta - \omega$$

near the stable equilibrium $(0, 0)$ and apply the following control strategy: *when $\theta = 0$ and $\omega < 0$ increase the angular velocity ω in modulus by a suitably chosen factor > 1. Otherwise do nothing.* This can be formalized in the following way.

Define a hybrid system $\mathbf{H}$ by setting:

- $Q = \{1, 2\}$, $E = \{(1, 2), (2, 1)\}$;
- $D_1 = \{(1, \theta, \omega) : \theta \le 0\}$, $D_2 = \{(2, \theta, \omega) : \theta \ge 0\}$;

- $X_1(1,\theta,\omega) = (\omega, -\sin\theta - \omega)$ and $X_2(2,\theta,\omega) = (\omega, -\sin\theta - \omega)$;
- $G(1,2) = \{(1,\theta,\omega) \in D_1 : \theta = 0,\ \omega \geq 0\}$, $G(2,1) = \{(2,\theta,\omega) \in D_2 : \theta = 0,\ \omega \leq 0\}$.

Before we define the reset maps corresponding to control impulses, we introduce the following notation. For each point $p = (1,0,\omega)$ [1] in D_1 with $\omega > 0$, denote by $f_-(\omega)$ the unique positive number such that the first intersection of the forward X_1-orbit of $(1,0,-f_-(\omega))$ with the boundary of D_1 is p (see Fig. 2). For each point $q = (2,0,\omega) \in D_2$ with $\omega > 0$, denote by $f_+(\omega)$ the unique positive number such that the first intersection of the forward X_2-orbit of q with the boundary of D_2 is $(2,0,-f_+(\omega))$. Clearly, f_+, f_- are smooth monotonic functions from $(0,\infty)$ to $(0,\infty)$, $f_-(\omega) > \omega$, and $f_+(\omega) < \omega$, for all $\omega > 0$.

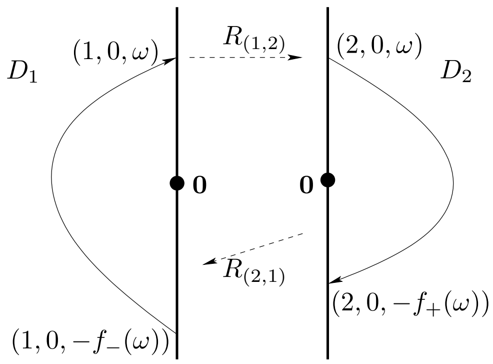

Fig. 2. Example 3.1.

Let $\rho : [0,\infty) \to [0,\infty)$ be a smooth increasing map such that $\rho(0) = 0$, $\rho(f_+(1)) = f_-(1)$, and for all $\omega \neq 1$, $\rho(f_+(\omega)) \neq f_-(\omega)$. Now define the resets

- $R_{(1,2)}(1,0,\omega) = (2,0,\omega)$ and $R_{(2,1)}(2,0,-\omega) = (1,0,-\rho(\omega))$, for all $\omega \geq 0$.

It can be verified that $\mathbf{H}$ is a regular hybrid system without branching with a unique closed orbit γ through the point $\pi(1,0,1)$.

Let t_1 and t_2 be the smallest positive numbers such that $\phi^1_{t_1}(1,0,-f_-(1)) \in G(1,2)$ and $\phi^2_{t_2}(2,0,1) \in G(2,1)$, where ϕ^j_t is the flow of X_j. Observe that $t_1 + t_2$ is the period of γ. It can be shown (in the notation of Theorem 3.1 and the comment following it) that

$$k_\gamma = \left| \rho'(f_+(1)) \frac{f'_+(1)}{f'_-(1)} \right|.$$

Furthermore, using the second variational equation and Grönwall's inequality (Sastry, 1999) we can estimate

$$c_\gamma \leq 4\,|\rho'(f_+(1))|\, e^{3(t_1+t_2)}.$$

If ρ is chosen so that $k_\gamma < 1$ (weaker condition) or $c_\gamma < 1$ (stronger condition), then Theorem 3.1 implies that γ is asymptotically stable.

[1] Recall that "1" refers to the domain to which p belongs.

4. HYBRID POINCARÉ-BENDIXSON THEOREM

In this section we show that regular hybrid systems without branching in the plane exhibit only trivial recurrence. We refer the reader to (Palis Jr. and de Melo, 1982) for the classical Poincaré-Bendixson theorem which states the same for smooth systems. Namely, if a smooth vector field on the 2-sphere S^2 (or 2-disk) has only finitely many equilibria, then for any $x \in S^2$ its ω-limit set $\omega(x)$ is either an equilibrium, a limit cycle, or a union of saddles and their connections.

Further recall (Simić *et al.*, 2000*a*) that for a hybrid system $\mathbf{H}$, a point $z \in M_{\mathbf{H}}$ is called a *Zeno state* if $z \in \omega(x)$ and the execution starting from x is a dynamic Zeno execution. Isolated Zeno states were investigated in (Simić *et al.*, 2000*a*) and locally classified in dimension two. It was shown there that near a planar Zeno state, the hybrid flow is topologically equivalent to a smooth spiral sink. Recall that two flows (hybrid or smooth) are said to be *topologically equivalent* if there exists a homeomorphism sending orbits of one to the orbits of the other preserving their time direction (though not necessarily time itself). For more details see (Palis Jr. and de Melo, 1982) for smooth and (Simić *et al.*, 2000*a*) for hybrid systems.

Let us remark that in (Matveev and Savkin, 2000) a version of the Poincaré-Bendixson theorem is stated and proved, but in a setting which does not permit Zeno executions, which is allowed in our framework.

We now state the main result of this section.

Theorem 4.1. Let $\mathbf{H}$ be 2-dimensional regular hybrid system without branching. Suppose that $M_{\mathbf{H}}$ is homeomorphic to the 2-sphere (or the unit 2-disk) and that there are only finitely many equilibria, Zeno states, and closed orbits. Then for every $x \in M_{\mathbf{H}}$, $\omega(x)$ is either

(a) an equilibrium,
(b) a Zeno state,
(c) a limit cycle,
(d) a union of saddles and their connections.

In particular, the system exhibits no nontrivial recurrence.

Corollary 4.1. Suppose that $\mathbf{H}$ is 2-dimensional and there exists a compact invariant set $K \subset M_{\mathbf{H}}$ such that in K, $\Psi^{\mathbf{H}}$ has no equilibria and Zeno states, and has only finitely many closed orbits. Then for every $x \in K$, $\omega(x)$ is a closed orbit.

5. CONCLUSION AND FUTURE WORK

We provide a relatively simple stability criterion for hybrid closed orbits which, as for smooth systems, unfortunately requires integrating the vector fields along the closed orbit. The result is applied to study the periodic motion of a pendulum under impulsive forcing. We also derive a generalization of the classical Poincaré-Bendixson theorem, which rules out nontrivial recurrence in planar regular hybrid systems without branching.

It remains to see if there can be more complicated types of recurrence in more general planar hybrid systems. Further, it would be desirable to develop an index theory (cf. (Sastry, 1999)) for equilibria (including Zeno states) of hybrid systems, especially those of greater generality than studied in this paper. These are, among others, some possible directions for future work.

6. REFERENCES

Bissell, C. (2001). A. A. Andronov and the development of Soviet control engineering. *IEEE Control Systems Magazine* **18**(1), 56–62.

Guckenheimer, J. and S. Johnson (1994). Planar hybrid systems. In: *Hybrid Systems and Autonomous Control Workshop*.

Hirsch, M. W. and S. Smale (1974). *Differential equations, dynamical systems, and linear algebra*. Academic Press.

Johansson, K. H., A. Barabanov and K. J. Åström (1997). Limit cycles with chattering in relay feedback systems. In: *Proc. 36th IEEE Conference on Decision and Control*. San Diego, CA.

Leonov, G. A., I. M. Burkin and A. I. Shepeljavyi (1996). *Frequency methods in oscillation theory*. number 357 In: *Mathematics and Its Applications*. Kluwer. Dodrecht.

Lygeros, J., K. H. Johansson, S.N. Simić, J. Zhang and S. Sastry (2000). Dynamical properties of hybrid automata. (submitted to *IEEE Transactions on Automatic Control*).

Matveev, A. S. and A. V. Savkin (2000). *Qualitative theory of hybrid dynamical systems*. Control Enegineering. Birkhäuser. Boston.

Palis Jr., J. and W. de Melo (1982). *Geometric Theory of Dynamical Systems*. Springer Verlag. New York.

Sastry, S. Shankar (1999). *Nonlinear systems: analysis, stability, and control*. Springer-Verlag.

Simić, S. N., K. H. Johansson, J. Lygeros and S. Sastry (2000*a*). Towards a geometric theory of hybrid systems. *Mathematics of Control, Signals, and Systems*. submitted.

Simić, S. N., K. H. Johansson, S. Sastry and J. Lygeros (2000*b*). Towards a geometric theory of hybrid systems. In: *Hybrid Systems: Computation and Control* (Nancy Lynch and Bruce H. Krogh, Eds.). pp. 421–436. Number 1790 In: *LNCS*. Springer Verlag.