Dynamical Properties of Hybrid Automata

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***Abstract*—Hybrid automata provide a language for modeling and analyzing digital and analogue computations in real-time systems. Hybrid automata are studied here from a dynamical systems perspective. Necessary and sufficient conditions for existence and uniqueness of solutions are derived and a class of hybrid automata whose solutions depend continuously on the initial state is characterized. The results on existence, uniqueness, and continuity serve as a starting point for stability analysis. Lyapunov's theorem on stability via linearization and LaSalle's invariance principle are generalized to hybrid automata.**

***Index Terms*—Continuity of solutions, dynamical systems; existence, LaSalle's principle, Lyapunov's indirect method, hybrid systems, uniqueness.**

I. Introduction

HYBRID systems are dynamical systems that involve the interaction of continuous and discrete dynamics. Systems of this type arise naturally in a number of engineering applications. For example, the hybrid paradigm has been used successfully to address problems in air traffic control [1], automotive control [2], bioengineering [3], process control [4], [5], highway systems [6], and manufacturing [7]. The needs of these applications have fuelled the development of theoretical and computational tools for modeling, simulation, analysis, verification, and controller synthesis for hybrid systems.

Fundamental properties of hybrid systems, such as existence and uniqueness of solutions, continuity with respect to initial conditions, etc., naturally attracted the attention of researchers fairly early on. The majority of the work in this area concentrated on developing conditions for well posedness (existence and uniqueness of solutions) for special classes of hybrid systems: variable structure systems [8], piecewise linear systems [9], [10], complementarity systems [11], [12], mixed logic dynamical systems [13], etc. Continuity of the solutions with respect to initial conditions and parameters has been somewhat less extensively studied. Motivated by questions of simulation, Tavernini [14] established a class of hybrid systems that have the property of continuous dependence of solutions for almost every initial condition. More recently, an approach to the study of continuous dependence on initial conditions based on the Skorohod topology was proposed [15]. The Skorohod topology, used in stochastic processes for the space of cadlag functions [16], is mathematically appealing, but tends to be very cumbersome to work with in practice. This fact severely limits the applicability of the results.

The first contribution of the present paper, presented in Section III, is a set of new results on existence, uniqueness and continuous dependence of executions on initial conditions. The results are very intuitive, natural and applicable to a wide class of hybrid systems, but require the computation of bounds on the set of reachable states and the set of states from which continuous evolution is impossible. We demonstrate how the computation of these quantities can be carried out on an example and refer to our earlier work [17] and [18] for a more general treatment.

Questions of stability of equilibria and invariant sets of hybrid systems have also attracted considerable attention. Most of the work in this area has concentrated on extensions of Lyapunov's direct method to the hybrid domain [19], [20]. The work of [21] provided effective computational tools, based on linear matrix inequalities (LMIs), for applying these results to a class of piecewise linear systems. For an overview of the literature in this area the reader is referred to [22].

Despite all this progress on extensions of Lyapunov's direct method, relatively little work has been done on hybrid versions of other stability analysis results. The second contribution of this paper, presented in Section IV, is to provide extensions of two more standard stability analysis tools to the hybrid domain: LaSalle's invariance theorem and Lyapunov's indirect method. The latter results can be viewed as a generalization of the approach of [23] and [24], where a direct study of the stability of piecewise linear systems is developed. The results in Section IV build on the existence, uniqueness, and continuity concepts presented in Section III and demonstrate the usefulness of the results developed there.

The development in Sections III and IV is based on a fairly standard class of autonomous hybrid systems, which we refer to as *hybrid automata*. This class has been studied extensively in the literature in a number of variations, for a number of purposes, and by a number of authors. Special cases of the class of systems considered here include switched systems [25], complementarity systems [11], mixed logic dynamic systems [13], and piecewise linear systems [26] (the autonomous versions of

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these, to be more precise). The hybrid automata considered here are a special case of the hybrid automata of [27] and the impulse differential inclusions of [28], both of which allow differential inclusions to model the continuous dynamics. They are also a special case of hybrid input–output automata of [29], which in addition allow infinite-dimensional continuous state. Each one of the references uses slightly different notation, concepts and terminology. Our development will roughly follow the conventions introduced in [18]. To avoid any ambiguity, the notation and some basic concepts will be reviewed in Section II.

II. Framework

A. Notation

For a finite collection V of variables, let $\mathbf{V}$ denote the set of valuations (possible value assignments) of these variables. We use a lower case letter to denote both a variable and its valuation; the interpretation should be clear from the context. We refer to variables whose set of valuations is finite or countable as *discrete*, and to variables whose set of valuations is a subset of a Euclidean space as *continuous*. For a set of continuous variables X with $\mathbf{X} = \mathbb{R}^n$ for some $n \geq 0$, we assume that $\mathbf{X}$ is given the Euclidean metric topology, and use $\|\cdot\|$ to denote the Euclidean norm. For a set of discrete variables Q, we assume that $\mathbf{Q}$ is given the discrete topology (every set is open), generated by the metric $d_D(q,q') = 0$ if $q = q'$ and $d_D(q,q') = 1$ if $q \neq q'$. We denote the valuations of the union $Q \cup X$ by $\mathbf{Q} \times \mathbf{X}$, with the product topology generated by the metric $d((q,x),(q',x')) = d_D(q,q') + \|x - x'\|$. The metric notation is extended to sets $M, M' \subseteq \mathbf{Q} \times \mathbf{X}$ by defining $d(M,M') = \inf\{d((q,x),(q',x'))\colon (q,x) \in M, (q',x') \in M'\}$. We assume that a subset U of a topological space is given the induced subset topology, and we use $\overline{U}$ to denote its closure, U^o its interior, $\partial U = \overline{U} \setminus U^o$ its boundary, U^c its complement, and $P(U)$ its power set (the set of all subsets of U). In logic formulas, we use $\wedge$ and $\vee$ to denote "and" and "or," respectively.

If A is a piecewise smooth submanifold of $\mathbb{R}^n$, we define distance $d_A(p,q)$ between two points $p, q \in A$ as the infimum arc length $\ell(c)$ of all piecewise smooth curves c in A that connect p and q. $d_A(p,q)$ makes A into a metric space. For a map $h\colon A \to B$ between two metric spaces (A, d_A) and (B, d_B), the Lipschitz constant of h at a point $p \in A$ is the real number

$$\mathrm{Lip}_p(h) = \sup_{a \in A\setminus\{p\}} \frac{d_B(h(a),h(p))}{d_A(a,p)}.$$

We say h is globally Lipschitz continuous if $\mathrm{Lip}_p(h)$ is a bounded function of p.

We assume that the reader is familiar with the standard definitions of vector fields and flows for smooth manifolds. Here, we consider vector fields parameterised by discrete variables $f\colon \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$, where Q is a collection of discrete variables, and X is a collection of continuous variables, with $\mathbf{X}$ a smooth manifold. As usual, $T\mathbf{X}$ denotes the tangent bundle of $\mathbf{X}$ and $T_x\mathbf{X}$ the tangent space of $\mathbf{X}$ at $x \in \mathbf{X}$. For each $q \in \mathbf{Q}$, we use $\psi(t,q,x)$ to denote the flow of the vector field $f(q,\cdot)$. For a function $\sigma\colon \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ we use $L_f\sigma\colon \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ to denote the Lie derivative of σ along f defined by

$$L_f\sigma(q,x) = \left.\frac{d}{dt}\right|_{t=0} \sigma(q,\psi(t,q,x)) = \frac{\partial\sigma}{\partial x} f(q,x)$$

(assuming all derivatives are defined). We use $Df(q,x)$ to denote the linearization of f with respect to x.

B. Hybrid Automata and Executions

A hybrid automaton is a dynamical system that describes the evolution in time of the valuations of a set of discrete and continuous variables.

Definition II.1 (Hybrid Automaton): A hybrid automaton H is a collection $H = (Q, X, f, \mathrm{Init}, D, E, G, R)$, where

Q	finite set of discrete variables;
X	finite set of continuous variables;
$f\colon \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$	vector field;
$\mathrm{Init} \subseteq \mathbf{Q} \times \mathbf{X}$	set of initial states;
$D\colon \mathbf{Q} \to P(\mathbf{X})$	a domain[1];
$E \subseteq \mathbf{Q} \times \mathbf{Q}$	set of edges;
$G\colon E \to P(\mathbf{X})$	guard condition;
$R\colon E \times \mathbf{X} \to P(\mathbf{X})$	reset map.

We refer to $(q,x) \in \mathbf{Q} \times \mathbf{X}$ as the *state* of H. We impose the following standing assumption.

Assumption II.1: The number of discrete states is finite. $\mathbf{X} = \mathbb{R}^n$, for some $n \geq 0$. For all $q \in \mathbf{Q}$, the vector field $f(q,\cdot)$ is globally Lipschitz continuous. For all $e \in E$, $G(e) \neq \emptyset$, and for all $x \in G(e)$, $R(e,x) \neq \emptyset$.

Most of the results presented in this paper trivially extend to hybrid automata where the discrete state is countably infinite and the continuous state takes values in a smooth manifold. It can be shown that the last part of the assumption can effectively be imposed without loss of generality [17].

Definition II.2 (Hybrid Time Trajectory): A hybrid time trajectory is a finite or infinite sequence of intervals $\tau = \{I_i\}_{i=0}^N$, such that

- $I_i = [\tau_i, \tau_i']$, for all $i < N$;
- if $N < \infty$, then either $I_N = [\tau_N, \tau_N']$, or $I_N = [\tau_N, \tau_N')$;
- $\tau_i \leq \tau_i' = \tau_{i+1}$ for all i.

The interpretation is that the τ_i are the times at which discrete transitions take place. Since all hybrid automata discussed here are time invariant we assume that $\tau_0 = 0$, without loss of generality. Each hybrid time trajectory τ is linearly ordered by the relation $\prec$, defined by $t_1 \prec t_2$ for $t_1 \in [\tau_i, \tau_i']$ and $t_2 \in [\tau_j, \tau_j']$ if $t_1 < t_2$ or $i < j$. We say that $\tau = \{I_i\}_{i=0}^N$ is a *prefix* of $\tau' = \{J_i\}_{i=0}^M$ and write $\tau \sqsubseteq \tau'$ if either they are identical, or τ is finite, $N \leq M$, $I_i = J_i$ for all $i = 0, \ldots, N-1$, and $I_N \subseteq J_N$. For a hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$, we define $\langle\tau\rangle$ as the set $\{0, 1, \ldots, N\}$ if N is finite and $\{0, 1, \ldots\}$ if $N = \infty$ and $|\tau| = \sum_{i \in \langle\tau\rangle} (\tau_i' - \tau_i)$.

Definition II.3 (Execution): An execution of a hybrid automaton H is a collection $\chi = (\tau, q, x)$, where τ is a hybrid time trajectory, $q\colon \langle\tau\rangle \to \mathbf{Q}$ is a map, and $x = \{x^i\colon i \in \langle\tau\rangle\}$ is a collection of differentiable maps $x^i\colon I_i \to \mathbf{X}$, such that

- $(q(0), x^0(0)) \in \mathrm{Init}$;

[1] The domain is sometimes called the invariant set, especially in the hybrid system literature in computer science.

- for all $t \in [\tau_i, \tau_i')$, $\dot{x}^i(t) = f\big(q(i), x^i(t)\big)$ and $x^i(t) \in D(q(i))$;
- for all $i \in \langle\tau\rangle \setminus \{N\}$, $e = (q(i), q(i+1)) \in E$, $x^i(\tau_i') \in G(e)$, and $x^{i+1}(\tau_{i+1}) \in R(e, x^i(\tau_i'))$.

We say that a hybrid automaton H *accepts* an execution χ if χ fulfils the conditions of Definition II.3. For an execution $\chi = (\tau, q, x)$, we use $(q_0, x_0) = \big(q(\tau_0), x^0(\tau_0)\big)$ to denote the initial state. We say that an execution, $\chi = (\tau, q, x)$, of H is a prefix of another execution, $\hat{\chi} = (\hat{\tau}, \hat{q}, \hat{x})$, of H (write $\chi \sqsubseteq \hat{\chi}$), if $\tau \sqsubseteq \hat{\tau}$ and for all $i \in \langle\tau\rangle$ and all $t \in I_i$, $\big(q(i), x^i(t)\big) = \big(\hat{q}(i), \hat{x}^i(t)\big)$. We say χ is a strict prefix of $\hat{\chi}$ (write $\chi \sqsubset \hat{\chi}$), if $\chi \sqsubseteq \hat{\chi}$ and $\chi \neq \hat{\chi}$. An execution of H is called *maximal* if it is not a strict prefix of any other execution of H. An execution is called *finite* if τ is a finite sequence ending with a compact interval, and *infinite* if τ is either an infinite sequence, or if $|\tau| = \infty$. An execution is called *Zeno* if it is infinite but $|\tau| < \infty$, or, equivalently, if it takes an infinite number of discrete transitions in a finite amount of time. It is easy to see that, under our definitions, the transition times τ_i of a Zeno execution converge to some finite accumulation point from the left. In other words, the definition of an execution precludes the situation where the transition times have a right accumulation point. A discussion of this situation can be found in [9] and [30].

We use $\mathcal{E}_H(q_0, x_0)$ to denote the set of all executions of H with initial condition $(q_0, x_0) \in \text{Init}$, $\mathcal{E}_H^M(q_0, x_0)$ to denote the set of all maximal executions, $\mathcal{E}_H^*(q_0, x_0)$ to denote the set of all finite executions, and $\mathcal{E}_H^\infty(q_0, x_0)$ to denote the set of all infinite executions. We use $\mathcal{E}_H$ to denote the union of $\mathcal{E}_H(q_0, x_0)$ over all $(q_0, x_0) \in \text{Init}$.

C. Reachability

The well-posedness and stability results developed in subsequent sections involve arguments about the set of states reachable by a hybrid automaton and the set of states from which continuous evolution is impossible. We briefly review these concepts. The set of states reachable by H, Reach_H, is defined as

$$\text{Reach}_H = \Big\{(\hat{q}, \hat{x}) \in \mathbf{Q} \times \mathbf{X} : \exists \left(\{[\tau_i, \tau_i']\}_{i=0}^N, q, x\right) \in \mathcal{E}_H^*, \big(q(N), x^N(\tau_N')\big) = (\hat{q}, \hat{x})\Big\}.$$

Clearly, $\text{Init} \subseteq \text{Reach}_H$, since we may choose $N = 0$ and $\tau_0' = \tau_0$. Since states outside Reach_H will never be visited by H we can effectively restrict our attention only to states in Reach_H.

The set of states from which continuous evolution is impossible is given by

$$\text{Out}_H = \{(q, x) \in \mathbf{Q} \times \mathbf{X} : \forall \varepsilon > 0, \exists t \in [0, \epsilon), \psi(t, q, x) \notin D(q)\}.$$

For certain classes of hybrid automata the computation of Out_H is straightforward, using geometric control tools [17]. In some cases, Reach_H can be computed (or approximated) using induction arguments along the length of the system executions (see, for example, [17] and [29]). In general, however, the exact computation of Reach_H and Out_H may be very complicated.

Definition II.3 does not require the state to remain in the domain. This assumption often turns out to be implicit in models of physical systems, where the domains are typically used to encode physical constraints that all executions of the system must satisfy. Let

$$\text{Dom}_H = \bigcup_{q \in \mathbf{Q}} \{q\} \times D(q) \subseteq \mathbf{Q} \times \mathbf{X}.$$

The following assumption makes the statement of some of the results somewhat simpler.

Assumption II.2: The sets Init and Dom_H are closed. Moreover, $\text{Reach}_H = \text{Init} \subseteq \text{Dom}_H$.

Assumption II.2 will not be imposed as a standing assumption, an explicit statement will be included whenever it is invoked.

D. Invariant Sets and Stability

We recall some standard concepts from dynamical system theory and their extensions to hybrid automata. For a more thorough discussion the reader is referred to [19], [20], and [22].

Definition II.4 (Invariant Set): A set $M \subseteq \text{Reach}_H$ is called invariant if for all $(q_0, x_0) \in M$, all $(\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$, all $i \in \langle\tau\rangle$, and all $t \in I_i$, $\big(q(i), x^i(t)\big) \in M$.

Here, we use $\mathcal{E}_H(q_0, x_0)$ to denote the set of all triples (τ, q, x) starting at (q_0, x_0) and satisfying the second and third conditions of Definition II.3, even if $(q_0, x_0) \notin \text{Init}$. This abuse of notation will be later resolved under Assumption II.2.

Definition II.5 (Stable Invariant Set): An invariant set M is called stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $(q_0, x_0) \in \text{Reach}_H$ with $d((q_0, x_0), M) < \delta$, all $(\tau, q, x) \in \mathcal{E}_H(q_0, x_0)$, and all $i \in \langle\tau\rangle$, $t \in I_i$, $d((q(i), x^i(t)), M) < \epsilon$. M is called asymptotically stable if it is stable, and in addition there exists $\Delta > 0$ such that for all $(q_0, x_0) \in \text{Reach}_H$ with $d((q_0, x_0), M) < \Delta$ and all $(\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$, $\lim_{t \to |\tau|} d((q(i), x^i(t)), M) = 0$.

A very common and useful type of invariant set is an equilibrium point. For hybrid automata, the following generalization of the notion of an equilibrium has been used in the literature.

Definition II.6 (Equilibrium): A point $x_* \in \mathbf{X}$ is an equilibrium of H if

1) $x_* \in \overline{D(q)}$ for some $q \in \mathbf{Q}$, implies that $f(q, x_*) = 0$;
2) $x_* \in G(e)$ for some $e \in E$, implies that $R(e, x_*) = \{x_*\}$.

An equilibrium is called *isolated* if it has a neighborhood in $\mathbf{X}$ which contains no other equilibria. It is easy to show that if x_* is an equilibrium, then the set $\mathbf{Q} \times \{x_*\} \cap \text{Reach}_H$ is invariant. We say that the equilibrium x_* is (asymptotically) stable if the invariant set $\mathbf{Q} \times \{x_*\} \cap \text{Reach}_H$ is (asymptotically) stable.

The asymptotic behavior of an infinite execution is captured by its ω-limit set.

Definition II.7 (ω-Limit Set): A point $(\hat{q}, \hat{x}) \in \mathbf{Q} \times \mathbf{X}$ is an ω-limit point of an infinite execution $(\tau, q, x) \in \mathcal{E}_H^\infty$, if there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle\tau\rangle$ such that as $n \to \infty$, $\theta_n \to |\tau|$ and $(q(i_n), x^{i_n}(\theta_n)) \to (\hat{q}, \hat{x})$. The ω-limit set, $S_\chi \subseteq \mathbf{Q} \times \mathbf{X}$, of $\chi \in \mathcal{E}_H^\infty$ is the set of all ω-limit points of χ.

It is easy to see that, under Assumption II.2, all ω-limit points are reachable.

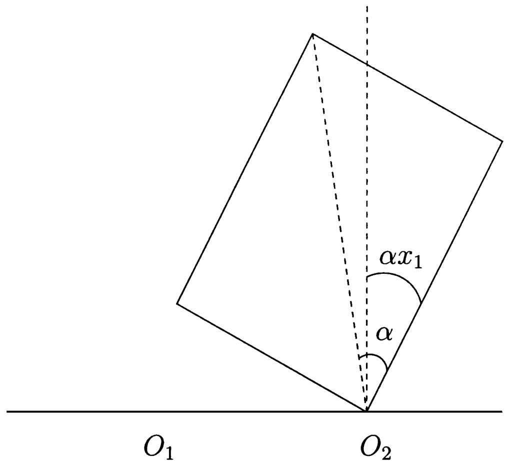

Fig. 1. Rocking block system.

E. Example: Rocking Block

The rocking block system (Fig. 1) has been studied extensively in the dynamics literature as a model for the rocking and toppling motion of rigid bodies (nuclear reactors, electrical transformers, and even tombstones) during earthquakes. In the presence of periodic excitation, the system turns out to have very complicated and in some cases chaotic dynamics. The formulation we use here comes from [31]. We assume that the rocking motion is small enough so that the block does not topple, but we remove the external excitation term (used in [31] to model earthquake forces) to make the system autonomous. Under these assumptions, the rocking block can be in one of two modes, leaning to the left, or leaning to the right. We assume that the block does not slip, therefore, when leaning to the left it rotates about pivot O_1 and when leaning to the right it rotates about pivot O_2. The continuous state of the system consists of the angle that the block makes with the vertical (measured here as a fraction of the angle made by the diagonal to simplify the equations) and the angular velocity. We assume that a fraction, r, of the angular velocity is lost every time the flat side hits the ground and the block switches from one pivot to the other.

It is relatively straightforward to write a hybrid automaton $\mathrm{RB} = (Q, X, f, \mathrm{Init}, D, E, G, R)$ to model this system. To capture the two modes, we set $Q = \{q\}$ with $\mathbf{Q} = \{\mathrm{Left}, \mathrm{Right}\}$. We also let $X = \{x_1, x_2\}$ and $\mathbf{X} = \mathbb{R}^2$, where x_1 represents the angle the block makes with the vertical (as a fraction of α) and x_2 represents the block's angular velocity. After normalizing some of the constants by rescaling time, the continuous dynamics simplify to

$$f(\mathrm{Left}, x) = \begin{bmatrix} x_2 \\ \alpha^{-1}\sin(\alpha(1+x_1)) \end{bmatrix}$$
$$f(\mathrm{Right}, x) = \begin{bmatrix} x_2 \\ -\alpha^{-1}\sin(\alpha(1-x_1)) \end{bmatrix}.$$

The domains over which each of these vector fields is valid are

$$D(\mathrm{Left}) = \{x \in \mathbb{R}^2 : x_1 \leq 0\}$$
$$D(\mathrm{Right}) = \{x \in \mathbb{R}^2 : x_1 \geq 0\}.$$

To ensure that the block does not topple (so that the two discrete state model remains valid), we impose a restriction on the energy initially present in the system

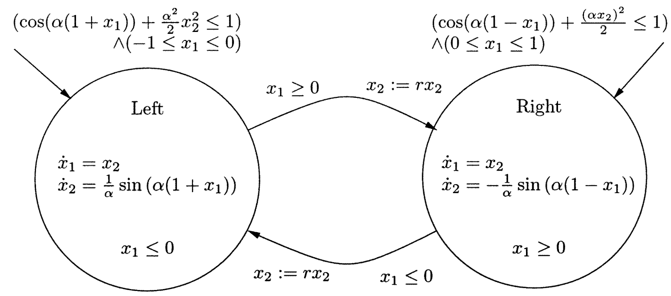

Fig. 2. Directed graph representation of rocking block automaton.

$$\begin{aligned}\mathrm{Init} = &\{\mathrm{Left}\} \times \Big\{x \in \mathbb{R}^2 : [-1 \leq x_1 \leq 0] \\ &\wedge \left[\cos(\alpha(1+x_1)) + \frac{(\alpha x_2)^2}{2} \leq 1\right]\Big\} \\ &\cup \{\mathrm{Right}\} \times \Big\{x \in \mathbb{R}^2 : [0 \leq x_1 \leq 1] \\ &\wedge \left[\cos(\alpha(1-x_1)) + \frac{(\alpha x_2)^2}{2} \leq 1\right]\Big\}.\end{aligned}$$

There are two possible discrete transitions $E = \{(\mathrm{Left}, \mathrm{Right}), (\mathrm{Right}, \mathrm{Left})\}$. The transitions can take place whenever x belongs to the guards

$$G(\mathrm{Left}, \mathrm{Right}) = \{x \in \mathbb{R}^2 : (x_1 = 0) \wedge (x_2 \geq 0)\}$$
$$G(\mathrm{Right}, \mathrm{Left}) = \{x \in \mathbb{R}^2 : (x_1 = 0) \wedge (x_2 \leq 0)\}.$$

Whenever a transition takes place a fraction of the block's energy is lost, according to

$$R(\mathrm{Left}, \mathrm{Right}, x) = R(\mathrm{Right}, \mathrm{Left}, x) = \left\{\begin{bmatrix} x_1 \\ rx_2 \end{bmatrix}\right\}$$

with $r \in [0, 1]$. The rocking block hybrid automaton is shown in Fig. 2, in the intuitive directed graph notation. An example of an execution is shown in Fig. 3.

The rocking block automaton possesses a number of interesting properties and will be used repeatedly throughout the paper to illustrate different points. We conclude this section by computing some of the quantities previously introduced ($\mathrm{Reach}_{\mathrm{RB}}$, $\mathrm{Out}_{\mathrm{RB}}$, etc.) that will be needed in subsequent derivations.

Clearly, the rocking block automaton satisfies Assumption II.1. If we let $\sigma(\mathrm{Left}, x) = -x_1$ and $\sigma(\mathrm{Right}, x) = x_1$, then $D(q) = \{x \in \mathbf{X} : \sigma(q, x) \geq 0\}$. By definition, the set Out does not intersect the interior of the domain and always contains the complement of the domain. Therefore

$$\begin{aligned}\mathrm{Out}_{\mathrm{RB}} \supseteq &\{\mathrm{Left}\} \times \{x \in \mathbb{R}^2 : x_1 > 0\} \\ &\cup \{\mathrm{Right}\} \times \{x \in \mathbb{R}^2 : x_1 < 0\} \\ \mathrm{Out}_{\mathrm{RB}} \subseteq &\{\mathrm{Left}\} \times \{x \in \mathbb{R}^2 : x_1 \geq 0\} \\ &\cup \{\mathrm{Right}\} \times \{x \in \mathbb{R}^2 : x_1 \leq 0\}.\end{aligned}$$

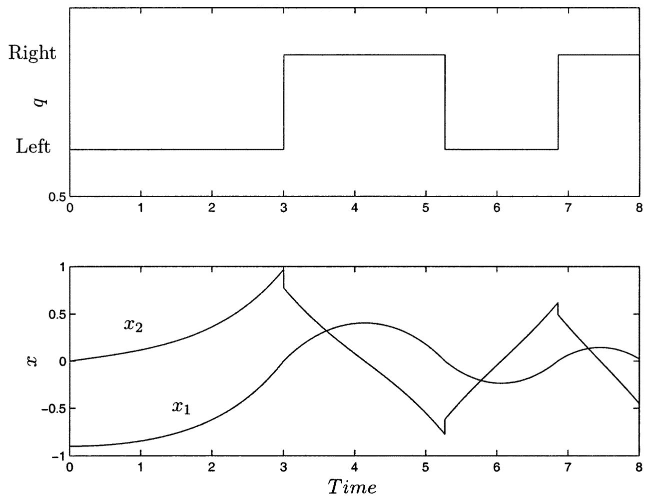

Fig. 3. Example of an execution, (τ, q, x) of the rocking block system with $\alpha = \pi/4$ and $r = 0.8$. τ consists of four intervals; roughly $I_0 = [0, 3]$, $I_1 = [3, 5.27]$, $I_2 = [5.27, 6.85]$ and $I_3 = [6.85, 8]$. The discrete state $q: \{0, 1, 2, 3\} \rightarrow \{\text{Left}, \text{Right}\}$ is shown in the upper plot, and the continuous state $x = \{x^i: I_i \rightarrow \mathbf{X}: i \in \{0, 1, 2, 3\}\}$ is shown in the lower plot.

The only question is what happens on the boundary $x_1 = 0$. Notice that $L_f\sigma(\text{Left}, x) = -x_2$ and $L_f\sigma(\text{Right}, x) = x_2$. If $q = \text{Left}$, $x_1 = 0$ and $x_2 > 0$, then $\sigma(q, x) = 0$ and $L_f\sigma(q, x) < 0$. A simple Taylor expansion argument reveals that continuous evolution in the domain is impossible from such a state. A similar argument for the case $q = \text{Right}$ shows that Out_{RB} contains the set

$$\{\text{Left}\} \times \{x \in \mathbb{R}^2: [x_1 > 0] \vee [(x_1 = 0) \wedge (x_2 > 0)]\}$$
$$\cup \{\text{Right}\} \times \{x \in \mathbb{R}^2: [x_1 < 0] \vee [(x_1 = 0) \wedge (x_2 < 0)]\}.$$

For the case $x_1 = x_2 = 0$, we take a second Lie derivative $L_f^2\sigma(\text{Left}, x) = -\alpha^{-1}\sin(\alpha(1 + x_1))$ and $L_f^2\sigma(\text{Right}, x) = -\alpha^{-1}\sin(\alpha(1 - x_1))$. If $x_1 = x_2 = 0$, then $\sigma(q, x) = 0$, $L_f\sigma(q, x) = 0$ and $L_f^2\sigma(q, x) < 0$ (since the geometry of the problem requires that $\alpha \in (0, \pi/2)$). A Taylor expansion argument shows that Out_{RB} is equal to

$$\{\text{Left}\} \times \{x \in \mathbb{R}^2: [x_1 > 0] \vee [(x_1 = 0) \wedge (x_2 \geq 0)]\}$$
$$\cup \{\text{Right}\} \times \{x \in \mathbb{R}^2: [x_1 < 0] \vee [(x_1 = 0) \wedge (x_2 \leq 0)]\}.$$

A more formal and general discussion of this procedure for computing the set Out using Lie derivatives can be found in [17].

To compute the set Reach_{RB}, we show that the set Init is invariant. Then, $\text{Reach}_{\text{RB}} = \text{Init}$ because $\text{Init} \subseteq \text{Reach}_{\text{RB}}$. Init can be formally shown to be invariant using induction arguments similar to those in [17], [29], and [32]; we give a sketch of the argument as follows.

Discrete transitions do not affect x_1 and decrease the magnitude of x_2. Therefore, a discrete transition from a state in Init will always lead to another state in Init. For the continuous argument we restrict our attention to the case $q = \text{Left}$ (the argument for $q = \text{Right}$ is similar). We can leave Init along continuous evolution, if we reach a state where either $x_1 > 0$, $x_1 < -1$, or $\cos(\alpha(1 + x_1)) + (\alpha x_2)^2/2 > 1$. To reach $x_1 > 0$, we first need to go through a state where $x_1 = 0$ and $\dot{x}_1 = x_2 \geq 0$. But such a state is in Out_{RB}, and therefore further continuous evolution is impossible. To reach $x_1 < -1$, we first need to reach a state where $x_1 = -1$. Since in Init, we have $\cos(\alpha(1 + x_1)) + (\alpha x_2)^2/2 \leq 1$ and thus $x_2 = 0$, it follows that $\dot{x}_1 = x_2 = 0$ and $\dot{x}_2 = \alpha^{-1}\sin(\alpha(1 + x_1)) = 0$. Finally, differentiating the function $\cos(\alpha(1 + x_1)) + (\alpha x_2)^2/2$ along the vector field $f(\text{Left}, x)$ leads to

$$-\alpha\sin(\alpha(1 + x_1))x_2 + \alpha\sin(\alpha(1 + x_1))x_2 = 0.$$

The aforementioned argument also shows that *RB* satisfies Assumption II.2.

The rocking block automaton has two equilibria, $(-1, 0)$ and $(1, 0)$ (more equilibria would be possible if the block toppled, but they are not reachable). An argument similar to the one given previously shows that the set $\mathbf{Q} \times \{(0, 0)\}$ is invariant. However, $(0, 0)$ is not an equilibrium, since it violates the first condition of Definition II.6.

III. Existence, Uniqueness and Continuity

A. Existence and Uniqueness

Definition III.1 (Nonblocking Hybrid Automaton): A hybrid automaton H is called nonblocking if $\mathcal{E}_H^\infty(q_0, x_0)$ is nonempty for all $(q_0, x_0) \in \text{Init}$.

Definition III.2 (Deterministic Hybrid Automaton): A hybrid automaton H is called deterministic if $\mathcal{E}_H^M(q_0, x_0)$ contains at most one element for all $(q_0, x_0) \in \text{Init}$.

Intuitively, a hybrid automaton is nonblocking if for all reachable states for which continuous evolution is impossible a discrete transition is possible. This fact is stated more formally in the following lemma.

Lemma III.1: A hybrid automaton H is nonblocking if for all $(q, x) \in \text{Reach}_H \cap \text{Out}_H$, there exists $(q, q') \in E$ such that $x \in G(q, q')$. A deterministic hybrid automaton H is nonblocking if and only if for all $(q, x) \in \text{Reach}_H \cap \text{Out}_H$, there exists $(q, q') \in E$ such that $x \in G(q, q')$.

Proof: Consider an initial state $(q_0, x_0) \in \text{Init}$ and assume, for the sake of contradiction, that there does not exist an infinite execution starting at (q_0, x_0). Let $\chi = (\tau, q, x)$ denote a maximal execution starting at (q_0, x_0), and note that τ is a finite sequence.

First, consider the case $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N')$ and let $(q_N, x_N) = \lim_{t \rightarrow \tau_N'}(q(N), x^N(t))$. Note that, by the definition of execution and a standard existence argument for continuous dynamical systems, the limit exists and χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^N$, $\widehat{q}(N) = q_N$, and $\widehat{x}^N(\tau_N') = x_N$. This contradicts the assumption that χ is maximal.

Now, consider the case $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$, and let $(q_N, x_N) = (q(N), x^N(\tau_N'))$. Clearly, $(q_N, x_N) \in \text{Reach}_H$. If $(q_N, x_N) \notin \text{Out}_H$, then there exists $\epsilon > 0$ such that χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N' + \epsilon)$, by continuous evolution. If, on the other hand $(q_N, x_N) \in \text{Out}_H$, then by assumption there exists $(q', x') \in \mathbf{Q} \times \mathbf{X}$ such that $(q_N, q') \in E$, $x_N \in G(q_N, q')$ and $x' \in R(q_N, q', x_N)$. Therefore, χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N+1}$, $\tau_{N+1} = \tau_{N+1}' = \tau_N'$, $q(N + 1) = q'$, $x^{N+1}(\tau_{N+1}) = x'$ by a discrete transition. In both cases the assumption that χ is maximal is contradicted.

This argument also establishes the "if" of the second part. For the "only if," consider a deterministic hybrid automaton that violates the conditions, i.e., there exists $(q',x') \in \mathrm{Reach}_H$ such that $(q',x') \in \mathrm{Out}_H$, but there is no $\widehat{q}' \in \mathbf{Q}$ with $(q',\widehat{q}') \in E$ and $x' \in G(q',\widehat{q}')$. Since $(q',x') \in \mathrm{Reach}_H$, there exists $(q_0,x_0) \in \mathrm{Init}$ and a finite execution, $\chi = (\tau,q,x) \in \mathcal{E}_H(q_0,x_0)$ such that $\tau = \{[\tau_i,\tau_i']\}_{i=0}^N$ and $(q',x') = (q(\tau_N'), x(\tau_N'))$.

We first show that χ is maximal. Assume first that there exists $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N-1}[\tau_N,\tau_N'+\epsilon)$ for some $\epsilon > 0$. This would violate the assumption that $(q',x') \in \mathrm{Out}_H$. Next assume that there exists $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \tau[\tau_{N+1},\tau_{N+1}']$ with $\tau_{N+1} = \tau_N'$. This requires that the execution can be extended beyond (q',x') by a discrete transition, i.e., there exists $(\widehat{q}',\widehat{x}') \in \mathbf{Q}\times\mathbf{X}$ such that $(q',\widehat{q}') \in E$, $x' \in G(q',\widehat{q}')$ and $\widehat{x}' \in R(q',\widehat{q}',x')$. This would contradict our original assumptions. Overall, $\chi \in \mathcal{E}_H^M(q_0,x_0)$.

Now assume, for the sake of contradiction, that H is nonblocking. Then, there exists $\chi' \in \mathcal{E}_H^\infty(q_0,x_0)$. It is evident that $\chi' \in \mathcal{E}_H^M(q_0,x_0)$. But $\chi \neq \chi'$ (as the former is finite and the latter infinite), therefore, $\mathcal{E}_H^M(q_0,x_0) \supset \{\chi,\chi'\}$. This contradicts the assumption that H is deterministic. ■

The conditions of the lemma are tight, but not necessary unless the automaton is deterministic. If the conditions are violated, then there exists an execution that blocks. However, unless the automaton is deterministic, a nonblocking execution may also exist from the same initial state.

Intuitively, a hybrid automaton may be nondeterministic if either there is a choice between continuous evolution and a discrete transition, or if a discrete transition can lead to multiple destinations (under Assumption II.1, continuous evolution is unique). Lemma III.2 provides a formal statement of this fact.

Lemma III.2: A hybrid automaton H is deterministic if and only if for all $(q,x) \in \mathrm{Reach}_H$

1) if $x \in G(q,q')$ for some $(q,q') \in E$ then $(q,x) \in \mathrm{Out}_H$;
2) if $(q,q') \in E$ and $(q,q'') \in E$ with $q' \neq q''$, then $x \notin G(q,q') \cap G(q,q'')$;
3) if $(q,q') \in E$ and $x \in G(q,q')$, then $R(q,q',x)$ contains at most one element.

Proof: For the "if" part, assume, for the sake of contradiction, that there exists an initial state $(q_0,x_0) \in \mathrm{Init}$ and two maximal executions $\chi = (\tau,q,x)$ and $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ starting at (q_0,x_0) with $\chi \neq \widehat{\chi}$. Let $\bar{\chi} = (\bar{\tau},\bar{q},\bar{x}) \in \mathcal{E}_H(q_0,x_0)$ denote the maximal common prefix of χ and $\widehat{\chi}$. Such a prefix exists as the executions start at the same initial state. Moreover, $\bar{\chi}$ is not infinite, as $\chi \neq \widehat{\chi}$. As in the proof of Lemma III.1, $\bar{\tau}$ can be assumed to be of the form $\bar{\tau} = \{[\bar{\tau}_i,\bar{\tau}_i']\}_{i=0}^N$. Let $(q_N,x_N) = (q(N),x^N(\bar{\tau}_N')) = (\widehat{q}(N),\widehat{x}^N(\bar{\tau}_N'))$. Clearly, $(q_N,x_N) \in \mathrm{Reach}_H$. We distinguish the following four cases.

Case 1) $\bar{\tau}_N' \notin \{\tau_i'\}$ and $\bar{\tau}_N' \notin \{\widehat{\tau}_i'\}$, i.e., $\bar{\tau}_N'$ is not a time when a discrete transition takes place in either χ or $\widehat{\chi}$. Then, by the definition of execution and a standard existence and uniqueness argument for continuous dynamical systems, there exists $\epsilon > 0$ such that the prefixes of χ and $\widehat{\chi}$ are defined over $\{[\bar{\tau}_i,\bar{\tau}_i']\}_{i=0}^{N-1}[\bar{\tau}_N,\bar{\tau}_N'+\epsilon)$ and are identical. This contradicts the fact that $\bar{\chi}$ is maximal.

Case 2) $\bar{\tau}_N' \in \{\tau_i'\}$ and $\bar{\tau}_N' \notin \{\widehat{\tau}_i'\}$, i.e., $\bar{\tau}_N'$ is a time when a discrete transition takes place in χ but not in $\widehat{\chi}$. The fact that a discrete transition takes place from (q_N,x_N) in χ indicates that there exists $q' \in \mathbf{Q}$ such that $(q_N,q') \in E$ and $x_N \in G(q_N,q')$. The fact that no discrete transition takes place from (q_N,x_N) in $\widehat{\chi}$ indicates that there exists $\epsilon > 0$ such that $\widehat{\chi}$ is defined over $\{[\bar{\tau}_i,\bar{\tau}_i']\}_{i=0}^{N-1}[\bar{\tau}_N,\bar{\tau}_N'+\epsilon)$. A necessary condition for this is that $(q,x_N) \notin \mathrm{Out}_H$. This contradicts condition 1 of the lemma.

Case 3) $\bar{\tau}_N' \notin \{\tau_i'\}$ and $\bar{\tau}_N' \in \{\widehat{\tau}_i'\}$, symmetric to Case 2).

Case 4) $\bar{\tau}_N' \in \{\tau_i'\}$ and $\bar{\tau}_N' \in \{\widehat{\tau}_i'\}$, i.e., $\bar{\tau}_N'$ is a time when a discrete transition takes place in both χ and $\widehat{\chi}$. The fact that a discrete transition takes place from (q_N,x_N) in both χ and $\widehat{\chi}$ indicates that there exist (q',x') and $(\widehat{q}',\widehat{x}')$ such that $(q_N,q') \in E$, $(q_N,\widehat{q}') \in E$, $x_N \in G(q_N,q')$, $x_N \in G(q_N,\widehat{q}')$, $x' \in R(q_N,q',x_N)$, and $\widehat{x}' \in R(q_N,\widehat{q}',x_N)$. Note that by condition 2 of the lemma, $q' = \widehat{q}'$, hence, by condition 3, $x' = \widehat{x}'$. Therefore, the prefixes of χ and $\widehat{\chi}$ are defined over $\{[\bar{\tau}_i,\bar{\tau}_i']\}_{i=0}^N[\bar{\tau}_{N+1},\bar{\tau}_{N+1}']$, with $\bar{\tau}_{N+1} = \bar{\tau}_{N+1}' = \bar{\tau}_N'$, and are identical. This contradicts the fact that $\bar{\chi}$ is maximal and concludes the proof of the "if" part.

For the "only if" part, assume that there exists $(q',x') \in \mathrm{Reach}_H$ such that at least one of the conditions of the lemma is violated. Since $(q',x') \in \mathrm{Reach}_H$, there exists $(q_0,x_0) \in \mathrm{Init}$ and a finite execution, $\chi = (\tau,q,x) \in \mathcal{E}_H(q_0,x_0)$ such that $\tau = \{[\tau_i,\tau_i']\}_{i=0}^N$ and $(q',x') = (q(N),x^N(\tau_N'))$. If condition 1 is violated, then there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N-1}[\tau_N,\tau_N'+\epsilon)$, $\epsilon > 0$, and $\tilde{\tau} = \tau[\tau_{N+1},\tau_{N+1}']$, $\tau_{N+1} = \tau_N'$, such that $\chi \sqsubset \hat{\chi}$ and $\chi \sqsubset \tilde{\chi}$. If condition 2 is violated, there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1},\tau_{N+1}']$, $\tau_{N+1} = \tau_{N+1}' = \tau_N'$, and $\widehat{q}(N+1) \neq \tilde{q}(N)$, such that $\chi \sqsubset \hat{\chi}$, $\chi \sqsubset \tilde{\chi}$. Finally, if condition 3 is violated, then there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1},\tau_{N+1}']$, $\tau_{N+1} = \tau_{N+1}' = \tau_N'$, and $\widehat{x}^{N+1}(\tau_{N+1}) \neq \tilde{x}^{N+1}(\tau_{N+1})$, such that $\chi \sqsubset \hat{\chi}$, $\chi \sqsubset \tilde{\chi}$. In all three cases, let $\overline{\widehat{\chi}} \in \mathcal{E}_H^M(q_0,x_0)$ and $\overline{\tilde{\chi}} \in \mathcal{E}_H^M(q_0,x_0)$ denote maximal executions of which $\widehat{\chi}$ and $\tilde{\chi}$ are prefixes, respectively. Since $\widehat{\chi} \neq \tilde{\chi}$, it follows that $\overline{\widehat{\chi}} \neq \overline{\tilde{\chi}}$. Therefore, $\mathcal{E}_H^M(q_0,x_0)$ contains at least two elements and thus H is nondeterministic. ■

Combining Lemmas III.1 and III.2 leads to the following.

Theorem III.1 (Existence and Uniqueness): If a hybrid automaton H satisfies the conditions of Lemma III.1 and Lemma III.2, then it accepts a unique infinite execution for all $(q_0,x_0) \in \mathrm{Init}$.

It is worth noting that the set Reach_H is needed in Lemmas III.1 and III.2 only to make the conditions necessary. If, as in Theorem III.1, we are only interested to establish whether a hybrid automaton has infinite executions and whether they are unique, it suffices to check the conditions of the lemmas over any set containing Reach_H, for example, the entire state space $\mathbf{Q}\times\mathbf{X}$, or any invariant set containing Init.

Using Lemmas III.1 and III.2, it is easy to show that the rocking block automaton accepts a unique infinite execution for all initial states. Notice that the set $\mathrm{Reach}_{\mathrm{RB}} \cap \mathrm{Out}_{\mathrm{RB}}$ is equal to

$$\mathrm{Reach}_{\mathrm{RB}} \cap \big(\{\text{Left}\} \times G(\text{Left}, \text{Right}) \cup \{\text{Right}\} \times G(\text{Right}, \text{Left})\big).$$

This shows that RB is nonblocking. It also shows that it satisfies the first condition of Lemma III.2. Since there is only one transition defined for each discrete state and the reset relation is a function, the remaining conditions of Lemma III.2 are trivially satisfied. Therefore, RB is also deterministic.

Even if a hybrid automaton accepts infinite executions for all initial states, that does not necessarily mean that it accepts executions that extend over infinite time horizons. This may be the case if for some $(q_0, x_0) \in \mathrm{Init}$ all executions in $\mathcal{E}_H^\infty(q_0, x_0)$ are Zeno. For example, using arguments similar to those in [18], one can show that this is the case for the rocking block system, whenever $r \in [0, 1)$ (i.e., some energy gets dissipated at impact). In fact, for certain initial states there may not even be an execution with $|\tau| > 0$. In the rocking block system, this is the case when $x_0 = (0, 0)$. Zeno hybrid automata will not be studied further in this paper. The reader is referred to [33]–[35] for a discussion of Zeno systems from a computer science perspective and [12], [18], and [36]–[39] for a dynamical systems treatment.

B. Continuous Dependence on Initial Conditions

In general, the behavior of hybrid automata may change dramatically even for small changes in initial conditions. This fact is unavoidable if one wants to allow hybrid automata that are powerful enough to model realistic systems. However, discontinuous dependence on initial conditions may cause problems, both theoretical and practical, when one tries to simulate hybrid automata [14], [15], [36]. In this paper, we are interested in continuity with respect to initial conditions primarily as a tool in the study of ω-limit sets and their stability. For simplicity, we restrict our attention to hybrid automata satisfying Assumption II.2.

Definition III.3 (Continuous Hybrid Automaton): A hybrid automaton, H, satisfying Assumption II.2 is called continuous if for all finite executions $(\tau, q, x) \in \mathcal{E}_H^*(q_0, x_0)$ and all $\epsilon > 0$, there exists $\delta > 0$ such that for all $(\tilde{q}_0, \tilde{x}_0) \in \mathrm{Init}$ with $d((\tilde{q}_0, \tilde{x}_0), (q_0, x_0)) < \delta$ and for all maximal executions $\overline{\chi} \in \mathcal{E}_H^M(\tilde{q}_0, \tilde{x}_0)$ there exists a finite prefix $(\tilde{\tau}, \tilde{q}, \tilde{x}) \in \mathcal{E}_H^*(\tilde{q}_0, \tilde{x}_0)$ of $\overline{\chi}$ with $\langle\tilde{\tau}\rangle = \langle\tau\rangle = \{0, 1, \ldots, N\}$ that satisfies

1) $\big|\, |\tilde{\tau}| - |\tau| \,\big| < \epsilon$
2) $d((\tilde{q}(N), \tilde{x}^N(\tilde{\tau}'_N)), (q(N), x^N(\tau'_N))) < \epsilon$.

Roughly speaking, H is continuous if two executions starting close to one another remain close to one another. If H is continuous, one can always choose δ such that finite executions starting within δ of one another go through the same initial sequence of discrete states. Arguing directly from the definition, one can show that the rocking block system is continuous.

The following theorem provides conditions under which a hybrid automaton is guaranteed to be continuous.

Theorem III.2 (Continuity With Initial Conditions): A hybrid automaton H satisfying Assumption II.2 is continuous if

1) H is deterministic;
2) for all $e = (q, q') \in E$, $G(e) \cap D(q)$ is an open subset of $\partial D(q)$;
3) for all $e \in E$, $R(e, \cdot)$ is a continuous function;
4) there exists a function $\sigma: \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$, differentiable in its second argument, such that $D(q) = \{x \in \mathbf{X} |\, \sigma(q, x) \geq 0\}$ for all $q \in \mathbf{Q}$;
5) for all $(q, x) \in \mathbf{Q} \times \mathbf{X}$ with $\sigma(q, x) = 0$, $L_f\sigma(q, x) \neq 0$.

Roughly speaking, conditions 4 and 5 are used to show that if from some initial state we can flow to a state from which a discrete transition is possible, then from all neighboring states we can do the same. This observation is summarized in the following lemma.

Lemma III.3: Consider a hybrid automaton, H, satisfying conditions 4 and 5 of Theorem III.2. Let $(\tau, q, x) \in \mathcal{E}_H^*(q_0, x_0)$ be a finite execution of H defined over a single interval $\tau = [0, \tau'_0]$ with $\tau'_0 > 0$ and $x^0(\tau'_0) \in \partial D(q_0)$. Then there exists a neighborhood W of x_0 in $D(q_0)$ and a differentiable function $T: W \to \mathbb{R}^+$, such that for all $y \in W$

1) $\psi(T(y), q_0, y) \in \partial D(q_0)$;
2) $\psi(t, q_0, y) \in D(q_0)^o$, for all $t \in (0, T(y))$;
3) $\Psi: W \to \partial D(q_0)$, defined by $\Psi(y) = \psi(T(y), q_0, y)$, is continuous.

Recall that a neighborhood of x_0 in $D(q_0)$ is a set of the form $U \cap D(q_0)$ where U is a neighborhood of x_0 in $\mathbb{R}^n$.

Proof of Lemma III.3: Since $\tau'_0 > 0$, the state $(q_0, x^0(\tau'_0))$ is reached from (q_0, x_0) along continuous evolution. We drop the superscript on x to simplify the notation.

To show 1), recall that, by the definition of an execution, $x(t) \in D(q_0)$ for all $t \in [\tau_0, \tau'_0)$. Since $x(\tau'_0) \in \partial D(q_0)$, $\sigma(q_0, x(\tau'_0)) = 0$. The function $\sigma(q_0, \psi(\cdot, q_0, \cdot))$ is differentiable in its first argument (t) and continuous in its second argument (x) in a neighborhood of (τ'_0, x_0) in $\mathbb{R}^+ \times \mathbb{R}^n$. Moreover

$$\frac{\partial}{\partial t}\sigma(q_0, \psi(t, q_0, x))\bigg|_{(t,x)=(\tau'_0, x_0)} = L_f\sigma(q_0, x(\tau'_0)) \neq 0$$

(by condition 5 of Theorem III.2). By the implicit function theorem for nonsmooth functions (see [40, Th. 3.3.6]), there exists a neighborhood $\Omega \subseteq \mathbb{R}^+$ of τ'_0 and a neighborhood $U \subseteq \mathbb{R}^n$ of x_0, such that for each $y \in U$ the equation $\sigma(q_0, \psi(t, q_0, y)) = 0$ has a unique solution $t \in \Omega$. Furthermore, this solution is given by $t = T(y)$, where T is a continuous mapping from U to Ω and $\psi(T(y), q_0, y) \in \partial D(q_0)$. For part 1), we can choose $W = U \cap D(q_0)$.

To show 2) assume, for the sake of contradiction, that for all neighborhoods W of x_0 in $D(q_0)$ such that $W \subseteq U \cap D(q_0)$, there exists $y \in W$ and $t \in (0, T(y))$ such that $\psi(t, q_0, y)) \notin D(q_0)^\circ$. Let y_k be a sequence of such y converging to x_0 and define

$$t_k = \sup\{t \in [0, T(y_k)]\colon \forall\, t' \in [0, t], \sigma(q_0, \psi(t', q_0, y_k)) \geq 0\}.$$

Take a Taylor expansion of $\sigma(q_0, \psi(t, q_0, y_k))$ about $t = t_k$

$$\sigma(q_0, \psi(t, q_0, y_k)) = \sigma(q_0, \psi(t_k, q_0, y_k)) + L_f\sigma(q_0, \psi(t_k, q_0, y_k))(t - t_k)$$

up to $O((t-t_k)^2)$. By definition, $\sigma(q_0, \psi(t, q_0, y_k)) \geq 0$ for $t - t_k < 0$ and $\sigma(q_0, \psi(t, q_0, y_k)) < 0$ for some $t - t_k > 0$ arbitrarily small. Therefore, $\sigma(q_0, \psi(t_k, q_0, y_k)) = 0$ (by continuity of σ) and $L_f\sigma(q_0, \psi(t_k, q_0, y_k)) < 0$. The fact that $\sigma(q_0, \psi(t_k, q_0, y_k)) = 0$ also implies that $t_k \in [0, \tau_0'] \setminus \Omega$, otherwise the uniqueness of the implicit function theorem would be contradicted.

Consider a subsequence of y_k (also denoted by y_k for simplicity) such that t_k converges to some $t_0 \in [0, \tau_0'] \setminus \Omega$. By continuity of ψ

$$\lim_{k\to\infty} \psi(t_k, q_0, y_k) = \psi(t_0, q_0, x_0) = x(t_0).$$

Therefore, $\sigma(q_0, x(t_0)) = 0$ and $L_f\sigma(q_0, x(t_0)) \leq 0$. Together with condition 5 of Theorem III.2 this implies that $L_f\sigma(q_0, x(t_0) < 0$. As before, the Taylor expansion of $\sigma(q_0, \psi(t, q_0, x_0))$ about t_0 implies that $\sigma(q_0, \psi(t, q_0, x_0)) < 0$ for $t > t_0$ small enough. Since $t_0 \in [0, \tau_0'] \setminus \Omega$, this contradicts the fact that $x(t) \in D(q_0)$ for all $t \in [0, \tau_0')$.

To show 3), recall that, since $\psi(\cdot, q_0, \cdot)$ is continuous in both arguments, for all $\epsilon > 0$ there exists $\delta_1 > 0$, such that for all t with $|t - T(x_0)| < \delta_1$ and all $y \in W$ with $\|y - x_0\| < \delta_1$, $\|\psi(t, q_0, x_0) - \psi(T(x_0), q_0, x_0)\| < \epsilon$ and $\|\psi(T(y), q_0, y) - \psi(T(y), q_0, x_0)\| < \epsilon$. By the continuity of T there exists some $\delta_2 > 0$ such that for all $y \in W$ with $\|y - x_0\| < \delta_2$, we have $|T(y) - T(x_0)| < \delta_1$. By setting $\delta = \min(\delta_1, \delta_2)$, it follows that for all $y \in W$ with $\|y - x_0\| < \delta$, $\|\mathbf{\Psi}(y) - \Psi(x_0)\| =< 2\epsilon$. ■

To complete the proof of Theorem III.2, conditions 1, 2, and 3 are used to piece together the intervals of continuous evolution.

Proof of Theorem III.2: Consider a finite execution $(\tau, q, x) \in \mathcal{E}_H^*(q_0, x_0)$ with $\tau = \{I_i\}_{i=0}^N$ and an $\epsilon > 0$. We construct a sequence of sets $\{W^0, V^0, \ldots, W^N, V^N\}$, where W^i is a neighborhood of $x^i(\tau_i)$ in $D(q(i))$ and V^i is a neighborhood of $x^i(\tau_i')$ in $D(q(i))$, such that the continuous evolution in $q(i)$ provides a continuous map from W^i to V^i and the reset $R(q(i), q(i+1), \cdot)$ provides continuous map from V^i to W^{i+1}. The notation is illustrated in Fig. 4.

Under the conditions of the theorem, the domain can not contain any isolated points. Indeed, assume there exists $\bar{x} \in D(q)$ and a neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ such that $U \cap D(q) = \{\bar{x}\}$. Then, $\sigma(q, \bar{x}) = 0$ and $\sigma(q, x) < 0$ for all $x \in U \setminus \{\bar{x}\}$. Therefore, $\sigma(q, \cdot)$ attains a local maximum at $\bar{x}$, and $\partial\sigma/\partial x(q, \bar{x}) = 0$. This, however, implies that $L_f\sigma(q, \bar{x}) = 0$, which is a contradiction.

The construction of W^i, V^i is recursive, starting with $i = N$. Define $V^N = \{x \in D(q(N)) : \|x - x^N(\tau_N')\| < \epsilon\}$. We distinguish the following three cases.

Case 1) $\tau_N' > \tau_N$ and $x^N(\tau_N') \in \partial D(q(N))$. By Lemma III.3, there exists a neighborhood, $W \subseteq D(q(N))$, of $x^N(\tau_N)$ and a differentiable function, $T^N : W \to \mathbb{R}^+$, such that for all $y \in W$, $\psi(T^N(y), q(N), y) \in \partial D(q(N))$ and $\psi(t, q(N), y) \in D(q(N))^o$ for all $t \in (0, T^N(y))$. As in Lemma III.3, define $\Psi^N : W \to \partial D(q(N))$ by $\Psi^N(y) = \psi(T^N(y), q(N), y)$. By the continuity of Ψ^N, there exists a neighborhood, $W^N \subseteq W$, of $x^N(\tau_N)$ such that $\Psi^N(W^N) \subseteq V^N$. Furthermore, all executions $\tilde{\chi}$ with $\tilde{x}^N(\tilde{\tau}_N) \in W^N$ fulfil $\tilde{x}^N(\tilde{\tau}_N + T^N(\tilde{x}^N(\tilde{\tau}_N))) \in V^N$.

Case 2) $\tau_N' > \tau_N$ and $x^N(\tau_N') \in D(q(N))^o$. Let $T^N(y) = \tau_N' - \tau_N$ for all y. Let $W \subseteq D(q(N))$ be a neighborhood of $x^N(\tau_N)$ such that for all $y \in W$ and $t \in (0, \tau_N' - \tau_N)$, $\psi(t, q(N), y) \in D(q(N))^o$. Such a neighborhood exists, because for all $t \in (0, \tau_N' - \tau_N)$, $\psi(t, q(N), x^N(\tau_N)) \in D(q(N))^o$ (cf., proof of Lemma III.3). Define a function $\Psi^N : W \to D(q(N))$ by $\Psi^N(y) = \psi(T^N(y), q(N), y)$. By continuous dependence of the solutions of the differential equation with respect to initial conditions, there exists a neighborhood $W^N \subseteq W$ of $x^N(\tau_N)$ such that both $\Psi^N(W^N) \subseteq V^N$ and all executions with $\tilde{x}^N(\tilde{\tau}_N) \in W^N$ satisfy $\tilde{x}^N(\tilde{\tau}_N + T^N(\tilde{x}^N(\tilde{\tau}_N))) \in V^N$.

Case 3) $\tau_N' = \tau_N$. Define T^N by $T^N(y) \equiv 0$, $W^N = V^N$ and Ψ^N the identity map. Clearly, $\Psi^N(W^N) = V^N$.

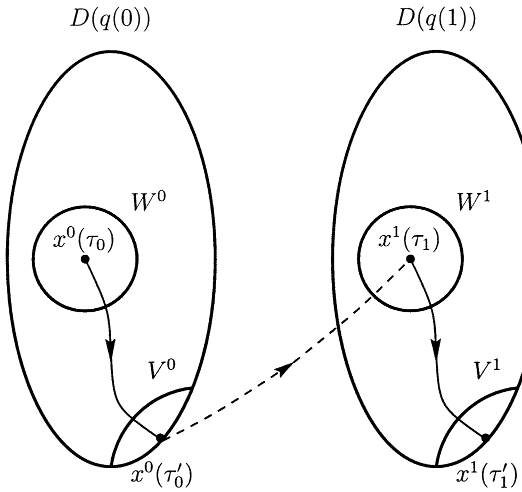

Fig. 4. Illustration of the proof of Theorem III.2 for $N = 1$ and Case 1).

Next, let us define V^{N-1}. Let $e_i = (q(i), q(i+1)) \in E$ and notice that $x^{N-1}(\tau_{N-1}') \in G(e_{N-1})$. Since $R(e_{N-1}, \cdot)$ is continuous, there exists a neighborhood $V \subseteq D(q(N-1))$ of $x^{N-1}(\tau_{N-1}')$ such that $R(e_{N-1}, V \cap G(e_{N-1})) \subseteq W^N$. By condition 2 of the theorem, $G(e_{N-1}) \cap D(q(N-1))$ is an open subset of $\partial D(q(N-1))$, so there exists a neighborhood $V^{N-1} \subseteq V$ of $x^{N-1}(\tau_{N-1}')$ such that $V^{N-1} \cap \partial D(q(N-1)) \subseteq G(e_{N-1}) \cap D(q(N-1))$. Since H is deterministic, it follows that all executions with $\tilde{q}(N-1) = q(N-1)$ and $\tilde{x}^{N-1}(\tilde{\tau}_{N-1}') \in V^{N-1} \cap \partial D(q(N-1))$ satisfy $\tilde{x}^N(\tilde{\tau}_N) \in W^N$.

Next, define T^{N-1} and Ψ^{N-1} using Lemma III.3, as for Cases 1) and 3). There exists a neighborhood $W^{N-1} \subseteq D(q(N-1))$ of $x^{N-1}(\tau_{N-1})$ such that $\Psi^{N-1}(W^{N-1}) \subseteq V^{N-1} \cap \partial D(q(N-1))$. Moreover, all executions $\tilde{\chi}$ with $\tilde{x}^{N-1}(\tilde{\tau}_{N-1}) \in W^{N-1}$ satisfy $\tilde{x}^{N-1}(\tilde{\tau}_{N-1}') \in V^{N-1} \cap \partial D(q(N-1))$ and $\tilde{\tau}_{N-1}' = \tilde{\tau}_{N-1} + T^{N-1}(\tilde{x}^{N-1}(\tilde{\tau}_{N-1}))$. If $\tau_{N-1}' = \tau_{N-1}$, some executions close to χ may take an instantaneous transition from

$q(N-1)$ to $q(N)$ ($\tilde{\tau}'_{N-1} = \tilde{\tau}_{N-1}$) while others may have to flow for a while ($\tilde{\tau}'_{N-1} > \tilde{\tau}_{N-1}$) before they follow χ's transition from $q(N-1)$ to $q(N)$. In the former case T^{N-1} and Ψ^{N-1} can be defined as in Case 3), while in the latter they can be defined as in Case 1).

By induction, we can construct a sequence of sets $\{W^0, V^0, \ldots, W^N, V^N\}$ and continuous functions $T^i: W^i \to \mathbb{R}^+$ and $\Psi^i: W^i \to V^i$ for $i = 0, \ldots, N$. For $k = 1, \ldots, N$, define the function $\Phi^k: W^0 \to W^k$ recursively by $\Phi^0(\tilde{x}_0) = \tilde{x}_0$ and $\Phi^k(\tilde{x}_0) = R(e_{k-1}, \Psi^{k-1}(\Phi^{k-1}(\tilde{x}_0)))$. For $k = 0, \ldots, N$, define the function $\gamma^k: W^0 \to \mathbb{R}^+$ by

$$\gamma^k(\tilde{x}_0) = \sum_{\ell=0}^{k} T^\ell(\Phi^\ell(\tilde{x}_0)).$$

Then, $\Phi^k(\tilde{x}_0) = \tilde{x}^k(\tilde{\tau}_k)$ and $\gamma^k(\tilde{x}_0) = \tilde{\tau}'_k - \tilde{\tau}_0$ for the execution $\tilde{\chi} = (\tilde{\tau}, \tilde{q}, \tilde{x})$ with $(\tilde{q}_0, \tilde{x}_0) \in q_0 \times W^0$. The functions Φ^k and γ^k are continuous by construction. By the continuity of γ^N, there exists $\delta_1 > 0$ such that for all $\tilde{x}_0$ with $\|\tilde{x}_0 - x_0\| < \delta_1$, we have $|\gamma^N(\tilde{x}_0) - \gamma^N(x_0)| < \epsilon$, or, in other words, $|\sum_{i=0}^N (\tilde{\tau}'_i - \tilde{\tau}_i) - \sum_{i=0}^N (\tau'_i - \tau_i)| < \epsilon$. By the continuity of Ψ^N there exists $\delta_2 > 0$ such that for all $y \in W^N$ with $\|y - x^N(\tau_N)\| < \delta_2$, $\|\Psi^N(y) - x^N(\tau'_N)\| < \epsilon$. Hence, by the continuity of Φ^N, there exists $\delta_3 > 0$ such that for all $\tilde{x}_0 \in W^0$ with $\|\tilde{x}_0 - x_0\| < \delta_3$, $\|\Phi^N(\tilde{x}_0) - x^N(\tau_N)\| < \delta_2$. Since $\Psi^N(\Phi^N(\tilde{x}_0)) = \tilde{x}^N(\tilde{\tau}'_N)$, we have $\|\tilde{x}^N(\tilde{\tau}'_N) - x^N(\tau'_N)\| < \epsilon$. The proof is completed by setting $\delta = \min(\delta_1, \delta_3)$. ∎

It should be stressed that the conditions of Theorem III.2 are not tight. For example, the rocking block automaton RB is continuous, but does not satisfy conditions 2 and 5 of the theorem (the only point where the conditions fail is the origin of the continuous state space).

IV. Stability of Equilibria and Invariant Sets

A. *Extension of Lasalle's Principle*

One of the most useful extensions of Lyapunov's stability theorems for continuous dynamical systems is LaSalle's invariance principle [41]. LaSalle's invariance principle provides conditions for an invariant set to be attracting. Here, we extend this result to continuous hybrid automata. The theorem builds on the following proposition, which establishes some properties of ω-limit sets for deterministic, continuous hybrid automata.

Lemma IV.1: Let H be a deterministic, continuous hybrid automaton satisfying Assumption II.2. Consider an infinite execution $\chi = (\tau, q, x)$ and assume that there exists $C > 0$ such that for $i \in \langle\tau\rangle$ and all $t \in I_i$, $\|x^i(t)\| < C$. The ω-limit set S_χ of χ is nonempty, compact and invariant. Furthermore, for all $\epsilon > 0$ there exists $K \in \langle\tau\rangle$ such that $d((q(t), x^n(t)), S_\chi) < \epsilon$ for all $n > K$ and $t \in I_n$.

Proof: The proof is inspired by the corresponding proof for continuous dynamical systems (see, for example, [42]). To show that S_χ is not empty, recall that $\mathbf{Q} \times \mathbf{X}$ is a metric space. If x is bounded, χ is contained in a compact subset of that space. Therefore, it has a limit point, by the Bolzano–Weierstrass property [43]. Hence, $S_\chi \neq \emptyset$.

To show that S_χ is compact, it suffices to show that S_χ is closed, since x is assumed to be bounded. Consider an arbitrary

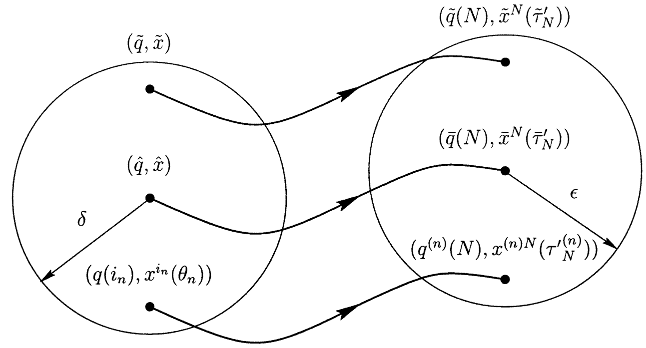

Fig. 5. Illustration of the proof of Lemma IV.1.

$(\hat{q}, \hat{x}) \in S_\chi^c$. Then there exists a neighborhood U of $(\hat{q}, \hat{x})$ and an $N > 0$, such that $(q(n), x^n(t)) \notin U$ for all $n > N$ and all $t \in I_n$. Therefore, $U \cap S_\chi = \emptyset$ and, since $(\hat{q}, \hat{x})$ is arbitrary, S_χ^c is open.

To show S_χ is invariant, consider an arbitrary $(\hat{q}, \hat{x}) \in S_\chi$, see Fig. 5. We need to show that for all $\bar{\chi} = (\bar{\tau}, \bar{q}, \bar{x}) \in \mathcal{E}_H(\hat{q}, \hat{x})$ with $\bar{\tau} = \{\bar{I}_i\}_{i=0}^N$ we have $(\bar{q}(N), \bar{x}^N(\bar{\tau}'_N)) \in S_\chi$. If there is no execution starting at $(\hat{q}, \hat{x})$ the property is trivially satisfied. Otherwise, note that since $(\hat{q}, \hat{x}) \in S_\chi$, there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in I_{i_n}$, $i_n \in \langle\tau\rangle$, such that as $n \to \infty$, $\theta_n \to |\tau|$ and $(q(i_n), x^{i_n}(\theta_n)) \to (\hat{q}, \hat{x})$.

Since H is continuous, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $(\tilde{q}, \tilde{x}) \in \mathrm{Reach}_H$ with $d((\tilde{q}, \tilde{x}), (\hat{q}, \hat{x})) < \delta$, every maximal execution starting at $(\tilde{q}, \tilde{x})$ has a finite prefix $\tilde{\chi} = (\tilde{\tau}, \tilde{q}, \tilde{x}) \in \mathcal{E}_H^*(\tilde{q}, \tilde{x})$ with $\tilde{\tau} = \{\tilde{I}_i\}_{i=0}^N$ satisfying $\||\tilde{\tau}| - |\bar{\tau}|\| < \epsilon$ and $d((\tilde{q}(N), \tilde{x}^N(\tilde{\tau}'_N)), (\bar{q}(N), \bar{x}^N(\bar{\tau}'_N))) < \epsilon$. Since $(q(i_n), x^{i_n}(\theta_n)) \to (\hat{q}, \hat{x})$ as $n \to \infty$, for this particular δ and for all n large enough, $d((q(i_n), x^{i_n}(\theta_n)), (\hat{q}, \hat{x})) < \delta$. Therefore, for n large enough, there exists a finite execution $\tilde{\chi}^{(n)} = (\tau^{(n)}, q^{(n)}, x^{(n)}) \in \mathcal{E}_H(q(i_n), x^{i_n}(\theta_n))$ with $\tau^{(n)} = \{I_i^{(n)}\}_{i=0}^N$, satisfying $||\tau^{(n)}| - |\bar{\tau}|| < \epsilon$ and $d((q^{(n)}(N), x^{(n)N}(\tau'^{(n)}_N)), (\bar{q}(N), \bar{x}^N(\bar{\tau}'_N))) < \epsilon$. By determinism, since χ passes through $(q(i_n), x^{i_n}(\theta_n))$, then it must also pass through $(q^{(n)}(N), x^{(n)N}(\tau'^{(n)}_N))$. Therefore, for any $\epsilon > 0$ there exists a point $(q^{(n)}(N), x^{(n)N}(\tau'^{(n)}_N))$ in χ within ϵ of $(\bar{q}(N), \bar{x}^N(\bar{\tau}'_N))$. In other words, $(\bar{q}(N), \bar{x}^N(\bar{\tau}'_N))$ is an accumulation point of χ and therefore $(\bar{q}(N), \bar{x}^N(\bar{\tau}'_N)) \in \mathcal{S}_\chi$.

For the last claim, assume, for the sake of contradiction, that there exists $\epsilon > 0$ such that for all $K > 0$, $d((q(n), x^n(t)), S_\chi) \geq \epsilon$ for some $n > K$ and $t \in I_n$. Then, there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in I_{i_n}$ and $i_n \in \langle\tau\rangle$ such that as $n \to \infty$, $\theta_n \to |\tau|$ and $d(q(i_n), x^{i_n}(\theta_n)), S_\chi) \geq \epsilon$. This sequence is bounded, therefore, by the Bolzano–Weierstrass property it has a limit point $(\hat{q}, \hat{x})$. Moreover, $(\hat{q}, \hat{x}) \in S_\chi$. But, by construction of the sequence, $d((\hat{q}, \hat{x}), S_\chi) \geq \epsilon$, which is a contradiction. ∎

Theorem IV.1 (Invariance Principle): Consider a nonblocking, deterministic and continuous hybrid automaton, H, satisfying Assumption II.2. Let $\Omega \subseteq \mathrm{Reach}_H$ be a compact invariant set and define $\Omega_1 = \Omega \cap \mathrm{Out}_H^c$ and $\Omega_2 = \Omega \cap \mathrm{Out}_H$. Assume there exists a continuous function $V: \Omega \to \mathbb{R}$, such that

1) for all $(q, x) \in \Omega_1$, V is continuously differentiable with respect to x and $L_f V(q, x) \leq 0$;

2) for all $(q,x) \in \Omega_2$, $e = (q,q') \in E$, with $x \in G(e)$, $V(q', R(e,x)) \le V(q,x)$.

Define $S_1 = \{(q,x) \in \Omega_1 : L_f V(q,x) = 0\}$ and $S_2 = \{(q,x) \in \Omega_2 : \forall\ e = (q,q') \in E \text{ with } x \in G(e),\ V(q', R(e,x)) = V(q,x)\}$. Let M be the largest invariant subset of $S_1 \cup S_2$. Then, for all $(q_0, x_0) \in \Omega$ the execution $(\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$ approaches M as $t \to |\tau|$.

"Approaches" should be interpreted as

$$\lim_{t \to |\tau|} d((q(t), x^i(t)), M) = 0.$$

Since the class of invariant sets is closed under arbitrary unions, M, the unique largest invariant set contained in $S_1 \cup S_2$, exists. Note also that under the assumption that H is nonblocking and deterministic, for all $(q,x) \in \Omega_2$, there exists a unique $e = (q,q') \in E$, with $x \in G(e)$ and $R(e,x)$ contains a single element (Lemmas III.1 and III.2).

Proof: Consider an arbitrary state $(q_0, x_0) \in \Omega$ and let $\chi = (\tau, q, x) \in \mathcal{E}_H^\infty(q_0, x_0)$. Since Ω is invariant, $(q(i), x^i(t)) \in \Omega$ for all $i \in \langle\tau\rangle$ and $t \in I_i$. Since Ω is compact and V is continuous, $V(q(i), x^i(t))$ is bounded from below. Moreover, $V(q(i), x^i(t))$ is a nonincreasing function of $i \in \langle\tau\rangle$ and $t \in I_i$ (recall that τ is linearly ordered), therefore, the limit $c = \lim_{t \to |\tau|} V(q(i), x^i(t))$ exists.

Since Ω is bounded, x is bounded, and therefore the ω-limit set S_χ is nonempty by Lemma IV.1. Since Ω is closed, $S_\chi \subseteq \Omega$. By definition, for any $(\hat{q}, \hat{x}) \in S_\chi$, there exists a sequence $\{\theta_n\}_{n=0}^\infty$ with $\theta_n \in I_{i_n}$, $i_n \in \langle\tau\rangle$ such that as $n \to \infty$, $\theta_n \to |\tau|$ and $(q(i_n), x^{i_n}(\theta_n)) \to (\hat{q}, \hat{x})$. Moreover, $V(\hat{q}, \hat{x}) = V(\lim_{n\to\infty}(q(i_n), x^{i_n}(\theta_n))) = \lim_{n\to\infty} V(q(i_n), x^{i_n}(\theta_n)) = c$, by continuity of V. Since S_χ is invariant (Lemma IV.1), it follows that $L_f V(\hat{q}, \hat{x}) = 0$ if $(\hat{q}, \hat{x}) \notin \mathrm{Out}_H$, and $V(\hat{q}', R(\hat{e}, \hat{x})) = V(\hat{q}, \hat{x})$ if $(\hat{q}, \hat{x}) \in \mathrm{Out}_H$ and $\hat{e} = (\hat{q}, \hat{q}') \in E$. Therefore, $S_\chi \subseteq S_1 \cup S_2$, which implies that $S_\chi \subseteq M$ since S_χ is invariant and M is maximal. By Lemma IV.1, as $t \to |\tau|$, the execution χ approaches S_χ, and hence M. ■

We demonstrate the application of Theorem IV.1 on the rocking block system. Assume that the dissipation constant satisfies $r \in [0, 1)$. We have already established that RB is a nonblocking, deterministic, continuous hybrid automaton, and that it satisfies Assumption II.2. The argument used in Section II to show that Init is invariant reveals that the compact set

$$\begin{aligned}\Omega =& \{\text{Left}\} \times \left\{x \in \mathbb{R}^2 : [-1 \le x_1 \le 0] \wedge \left[\cos(\alpha(1+x_1)) + \frac{(\alpha x_2)^2}{2} \le \epsilon\right]\right\}\\ &\cup \{\text{Right}\} \times \left\{x \in \mathbb{R}^2 : [0 \le x_1 \le 1] \wedge \left[\cos(\alpha(1-x_1)) + \frac{(\alpha x_2)^2}{2} \le \epsilon\right]\right\}\end{aligned}$$

is an invariant subset of Reach_H for any $\epsilon \in (0,1)$. Recalling our earlier computation of $\mathrm{Out}_{\mathrm{RB}}$, we see that

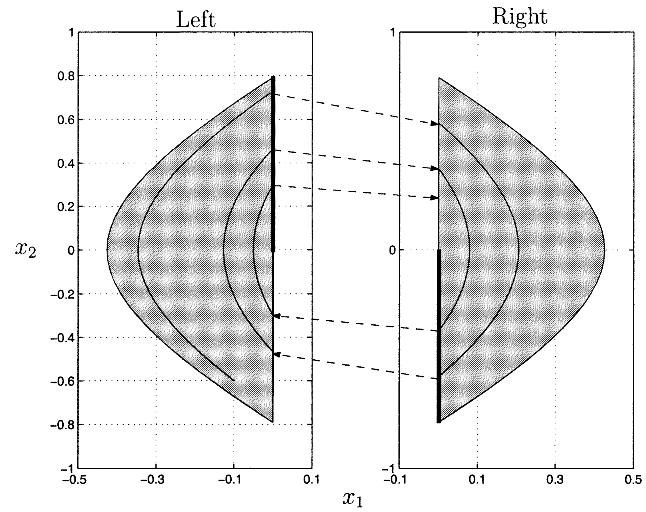

Fig. 6. Set Ω for the application of Theorem IV.1 to the rocking block system. Ω is the shaded region and Ω_2 thick part of its boundary. The dotted arrows indicate discrete transitions in the execution. The parameters used in the figure were $r = 0.8$, $\alpha = \pi/4$, $\epsilon = 0.9$, and $(q_0, x_0) = (\text{Left}, (-0.1, -0.6))$.

$$\begin{aligned}\Omega_1 =& \{\text{Left}\}\\ &\times \left\{x \in \mathbb{R}^2 : [x_1 < 0 \vee (x_1 = 0 \vee x_2 < 0)] \wedge \left[-1 \le x_1 \le 0\right] \wedge \left[\cos(\alpha) + \frac{(\alpha x_2)^2}{2} \le \epsilon\right]\right\}\\ &\cup \{\text{Right}\}\\ &\times \left\{x \in \mathbb{R}^2 : [x_1 > 0 \vee (x_1 = 0 \vee x_2 > 0)] \wedge \left[0 \le x_1 \le 1\right] \wedge \left[\cos(\alpha) + \frac{(\alpha x_2)^2}{2} \le \epsilon\right]\right\}\\ \Omega_2 =& \{\text{Left}\} \times \left\{x \in \mathbb{R}^2 :, [x_1 = 0] \wedge [x_2 \ge 0] \wedge \left[\cos(\alpha) + \frac{(\alpha x_2)^2}{2} \le \epsilon\right]\right\}\\ &\cup \{\text{Right}\} \times \left\{x \in \mathbb{R}^2 : [x_1 = 0] \wedge [x_2 \le 0] \wedge \left[\cos(\alpha) + \frac{(\alpha x_2)^2}{2} \le \epsilon\right]\right\}.\end{aligned}$$

These sets are shown in Fig. 6, together with an execution of the rocking block hybrid automaton starting in Ω.

As a Lyapunov function, we use

$$V(q,x) = \cos(\alpha(1 - |x_1|)) + \frac{(\alpha x_2)^2}{2}$$

which relates to the the energy of the system. As shown in Section II, $L_f V(q,x) = 0$ for all $(q,x) \in \Omega_1$. Therefore, V fulfils the first requirement of the theorem. To establish that V does not increase along discrete evolution, consider

$(q,x) \in \Omega_2$ and assume that $q = \text{Left}$ (the argument is similar if $q = \text{Right}$). Then, $V(q,x) = \cos(\alpha) + (\alpha x_2)^2/2$ and $V(\text{Right}, R(\text{Left}, \text{Right}, x)) \leq V(\text{Left}, x)$. Therefore, V satisfies the required conditions, and $S_1 = \Omega_1$

$$S_2 = \{\text{Left}, \text{Right}\} \times \{x \in \mathbb{R}^2 : (x_1 = 0) \wedge (x_2 = 0)\}.$$

As discussed in Section II, the set S_2 is invariant. Moreover, it is easy to see that executions starting at points $(q,x) \in S_1 \cup S_2$ with $x_1 \neq 0$ or $x_2 \neq 0$ will soon reach a point which is in Ω_2 but outside of $S_1 \cup S_2$. Therefore, the largest invariant set contained in $S_1 \cup S_2$ is $M = S_2$. By Theorem IV.1 all trajectories starting in Ω converge to $S_2 = \mathbf{Q} \times \{(0,0)\}$.

Since ϵ can be chosen arbitrarily in $(0,1)$, the interior of the set Init is in a sense the *domain of attraction* of the invariant set $\mathbf{Q} \times \{(0,0)\}$.

The conclusion of this example could also have been derived using the properties of Zeno executions established in [39]. The advantage of using LaSalle's principle is that it does not require one to integrate the differential equations and argue about their solutions, which is typically needed, for example, to establish that the system is Zeno. Recall also that, strictly speaking, $\mathbf{Q} \times \{(0,0)\}$ is not an equilibrium of the system, therefore most of the standard Lyapunov arguments for hybrid systems do not apply in this case.

B. Extension of Lyapunov's Indirect Method

In this section, we develop a method for determining stability of equilibria of hybrid automata using linearization. Roughly speaking, the procedure involves linearising all the relevant objects (vector fields, guards, and images of resets) in a neighborhood of the equilibrium, and combining the linearizations to get a number η representing the Lipschitz constant of the "going around the equilibrium once" map. If $\eta < 1$, the equilibrium is locally asymptotically stable. The novelty of our method is that it does not require integration of nonlinear vector fields and can deal with nonlinearities in the vector fields, the guards and the resets. Our approach can be viewed as a generalization of the work of [23] and [24], where systems with piecewise linear dynamics were considered. For simplicity, we focus on automata satisfying Assumption II.2 throughout.

For an equilibrium $x_* \in \mathbf{X}$, let

$$Q_* = \{q \in \mathbf{Q} : (q, x_*) \in \text{Reach}_H\}.$$

We develop stability conditions for the case where the states in Q_* are visited cyclically by all executions of H starting close enough to x_*. We first define some special classes of hybrid automata appropriate for stability analysis.

Definition IV.1 (Piecewise Smooth H.A.): A hybrid automaton is called piecewise smooth if its domains and guards are piecewise smooth manifolds with piecewise smooth boundary. The former must be of dimension n, the latter of dimension $n-1$. Furthermore, each guard is a submanifold of a domain, and for each $e \in E$, the union $\text{image}\, R^e = \cup_{x \in G(e)} R(e,x)$ is a piecewise smooth $(n-1)$ submanifold (with piecewise smooth boundary) of a domain.

The degree of smoothness will be fixed to C^∞ throughout this section.

Definition IV.2 (Almost Deterministic H.A.): A hybrid automaton is called almost deterministic if it satisfies

1) conditions 1 and 2 of Lemma III.2;
2) for all $e = (q,q') \in E$, the reset relation $R^e = R(e,\cdot)$ is a family of piecewise smooth homeomorphisms, i.e., there exists an index set $\mathcal{A}(e)$ such that $R^e(x) = \{R^e_\alpha(x) : \alpha \in \mathcal{A}(e)\}$ for all $x \in G(e)$, where $R^e_\alpha : G(e) \to D(q')$ is a piecewise smooth homeomorphism.

Recall that a submersion is a smooth map such that at every point its derivative is a surjective linear map. For a set $A \subseteq \mathbb{R}^n$ and $x \in A$, let $T_A(x)$ be the set of all vectors $v \in \mathbb{R}^n$ such that there exists a smooth curve $c : [0,1] \to A$ with $c(0) = x$ and $\dot{c}(0) = v$. Roughly speaking, $T_A(x)$ is the set of all directions pointing into A. In particular, if x is in the interior of A, then $T_A(x) = T_x A$, whereas if A is closed and x is a smooth point on its boundary, then $T_A(x)$ is a half space of $T_x A$. One can show that $T_A(x)$ is a cone in $\mathbb{R}^n$, i.e., $0 \in T_A(x)$ and $\lambda v \in T_A(x)$ for all $v \in T_A(x)$ and $\lambda \geq 0$. In fact, one can show that $T_A(x)$ is the same as the *contingent (tangent) cone* discussed in [40], [44].

For any cone $C \subseteq \mathbb{R}^n$, we say that C is k-dimensional if it contains a basis for $\mathbb{R}^k$ but does not contain a basis for $\mathbb{R}^{k+1}$. Given a linear map $F : \mathbb{R}^k \to \mathbb{R}^k$ and a k-dimensional cone C in $\mathbb{R}^k$, the *norm of F restricted to C*, is defined by $\|F|_C\| = \sup\{\|F(v)\| : v \in C, \|v\| = 1\}$. If λ is an eigenvalue of F with largest absolute value and C contains an eigenvector corresponding to λ, then $\|F|_C\| = |\lambda| = \|F\|$, where $\|F\|$ is the ordinary operator norm of F. This is always the case when $C = \mathbb{R}^k$, or when C contains a half-space of $\mathbb{R}^k$. However, in general, computing $\|F|_C\|$ may not be that simple; it may require one to solve a convex optimization problem, for example. If $C = T_A(x)$, the case of interest to us, where A is a piecewise smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, then there exist affine maps $L_1, \ldots, L_r : \mathbb{R}^n \to \mathbb{R}$ such that $C = \{v : L_1(v) \geq 0, \ldots, L_r(v) \geq 0\}$. In this case, $\|F|_C\|$ can be computed using the method of Lagrange multipliers as the maximum of the function $v \mapsto \|F(v)\|$, subject to constraints $L_1(v) \geq 0, \ldots, L_r(v) \geq 0, \|v\| = 1$. With this notation in place, we are now in a position to state the main result of this section.

Theorem IV.2 (Indirect Method): Let x_* be an isolated equilibrium of a nonblocking, piecewise smooth, and almost deterministic hybrid automaton, H, that satisfies Assumption II.2. Assume that the set Q_* can be ordered such that $Q_* = \{q_0, q_1, \ldots, q_l\}$ with $e_j = (q_j, q_{j+1}) \in E$, for $0 \leq j \leq l-1$ and $e_l = (q_l, q_0) \in E$. Assume also that there exists a neighborhood W of x_* in Reach_H such that for each $0 \leq j \leq l$, the following hold.

a) $x_* \in \partial A_j \cap \partial B_j$, where $A_j = \overline{\text{image}\, R^{e_{j-1}}} \cap W$ and $B_j = \overline{G(e_j)} \cap W$.
b) There exists a submersion $\phi_j : W \setminus (A_j \cap B_j) \to \mathbb{R}$ such that $\phi_j = a_j$ on $W \cap A_j \setminus B_j$ and $\phi_j = b_j$ on $W \cap B_j \setminus A_j$, for some numbers $a_j < b_j$.
c) There exists numbers m_j^-, m_j^+ such that for all $x \in W \setminus (A_j \cap B_j)$, $0 < m_j^- \leq L_{f(q_j,\cdot)} \phi_j(x) \leq m_j^+$.
d) There exists $\tau_j > 0$ such that $e^{\tau_j L_j}(T_{A_j}(x_*)) \subseteq T_{B_j}(x_*)$, where $L_j = Df(q_j, x_*)$.

Let $\mu_j = \|e^{\tau_j L_j}|_{T_{A_j}(x_*)}\|$, and $\sup_{\alpha \in \mathcal{A}(e_j)} \mathrm{Lip}_{x_*}(R_\alpha^{e_j}) = \nu_j$ and define $\eta_H(x_*) = \prod_{j=1}^{l} \mu_j \nu_j$. If $\eta_H(x_*) < 1$, then x_* is a locally asymptotically stable.

Remarks:

i) Note that, under Assumption IV.1, A_j and B_j are piecewise smooth manifolds of dimension $n-1$.

ii) Condition b says that $A_j \setminus B_j$ and $B_j \setminus A_j$ are level sets of the function ϕ_j, which measures the progress trajectories of the vector field make toward B_j, starting from A_j; their "speed" along $f(q_j, \cdot)$ is between m_j^- and m_j^+, as follows from c). Observe that ϕ_j is not defined at x_*.

iii) Condition d says that the time-τ_j map of the linearization of the flow at (q_j, x_*) maps $T_{A_j}(x_*)$ to $T_{B_j}(x_*)$. This means that B_j is reachable from A_j in a bounded amount of time by the flow of the linearized system.

iv) Note that no vector fields need to be integrated, in contrast to certain extensions of Lyapunov's direct method. Only the linearised dynamics at the equilibrium point are considered, see condition d.

v) Let H' be the *reverse* hybrid automaton to H obtained by reversing the time in H. If the dimension of each $D(q)$ is two, H' is nonblocking and deterministic, and $\eta_H(x_*) > 1$, then it is not difficult to see that x_* is unstable.

Theorem IV.2 is a direct consequence of the following two lemmas. Lemma IV.2 provides conditions to ensure that the states in Q_* are visited cyclically, while Lemma IV.3 gives an estimate of the "contraction" in the continuous state every time the discrete state traverses the entire cycle.

Lemma IV.2: Let f be a globally Lipschitz, smooth vector field on $\mathbb{R}^n$ with flow ψ, and assume x_* is an isolated equilibrium for f. Let U be a neighborhood of x_*. Suppose A and B are closed sets which are piecewise smooth $(n-1)$-dimensional submanifolds (with piecewise smooth boundary) of U, and assume the following hold.

a) $x_* \in \partial A \cap \partial B$.

b) There exists a submersion $\phi: U \setminus (A \cap B) \to \mathbb{R}$ and numbers $a < b$ such that $\phi = a$ on A and $\phi = b$ on B.

c) There exists numbers m_-, m_+ such that on $U \setminus (A \cap B)$, $0 < m_- \le L_f \phi \le m_+$.

d) There exists a number $\tau_* > 0$ such that $e^{\tau_* L}(T_A(x_*)) \subseteq T_B(x_*)$, where $L = Df(x_*)$.

Then, the following are true.

1) There exists a bounded function $\tau: A \setminus B \cup \{x_*\} \to \mathbb{R}^+$ such that $\psi(\tau(x), x) \in B$ and $\psi(t, x) \notin B$, for all $t < \tau(x)$.

2) The function $h: A \setminus B \cup \{x_*\} \to B$ defined by $h(x) = \psi(\tau(x), x)$ is Lipschitz continuous. The Lipschitz constant at x_* is $\mathrm{Lip}_{x_*} h = \|e^{\tau_* L}|_{T_A(x_*)}\|$.

Proof: Let us first show that for every $x \in A \setminus B$, the flow starting at x reaches B. Since $\phi(x) = a$, we have, for $t > 0$

$$\begin{aligned}\phi(\psi(t,x)) =& \phi(x) + \int_0^t (L_f\phi)(\psi(s,x))\, ds \\ \in& [a + m_- t, a + m_+ t].\end{aligned}$$

Since ϕ is continuous on $U \setminus (A \cap B)$, $a < b$, and $t \mapsto \phi(\psi(t,x))$ is strictly increasing, there exists a unique $\tau(x) \in [(b-\phi(x))/m_+, (b-\phi(x))/m_-]$ such that $\phi(\psi(\tau(x),x)) = b$ and $\phi(\psi(t,x)) < b$, for all $t \in [0, \tau(x))$. This shows that the forward f-orbit of x reaches B and that τ is a bounded function on $A \setminus B$. Namely

$$\frac{b-a}{m_+} \le \tau(x) \le \frac{b-a}{m_-}.$$

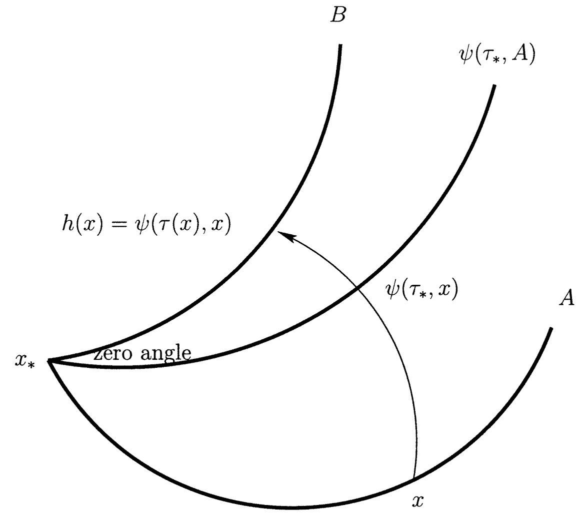

Fig. 7. Limit (1) of the proof of Lemma IV.2.

Next, let us show that $\tau(x) \to \tau_*$, as $x \to x_*$. Observe that, by d), $\psi(\tau_*, A)$ and B are tangent to each other at x_*. Since A and B are not necessarily smooth at x_*, by this, we mean $T_{\psi(\tau_*, A)}(x_*) = T_B(x_*)$. Therefore

$$\frac{\ell(\psi(\tau_*, x), \psi(\tau(x), x))}{d_B(\psi(\tau(x), x), x_*)} \to 0 \tag{1}$$

as $x \to x_*$, where $\ell(\psi(r,x), \psi(s,x))$ denotes the arc length of the indicated segment of the f-orbit of x (see Fig. 7). Observe that $f \cdot \nabla\phi = L_f\phi \ge m_- > 0$, so the angle between f and the level surfaces of ϕ is bounded away from zero. In particular, the angle between f and $A \setminus B$, and f and $B \setminus A$ is bounded away from zero. This implies $\ell(\psi(\tau_*, x), \psi(\tau(x), x)) \to 0$, as $x \to x_*$, so since $h(x) = \psi(\tau(x), x)$, for $x \in A \setminus B$ we obtain $|\phi(\psi(\tau_*, x)) - \phi(h(x))| \le m_+ \ell(\psi(\tau_*, x), h(x))$ which tends to 0 as $x \to x_*$. Hence

$$\begin{aligned} m_-|\tau(x) - \tau_*| \le& \left| \int_{\tau(x)}^{\tau_*} (L_f\phi)(\psi(t,x)\, dt \right| \\ =& |\phi(\psi(\tau_*, x)) - \phi(h(x))| \to 0 \end{aligned}$$

as $x \to x_*$. This shows that $\tau(x) \to \tau_*$.

It is a consequence of the implicit function theorem that τ and h are smooth functions on $A \setminus B$. Then, using the chain rule to differentiate $h(x) = \psi(\tau(x), x)$ with respect to x, we obtain

$$Dh(x)v = d\tau(v) f(h(x)) + D\psi(\tau(x), x)v \tag{2}$$

for all $x \in A \setminus B$ and $v \in T_A(x_*)$. Here Dh denotes the derivative of h as a map from $A \setminus B$ to B, and $d\tau$ is the differential (equivalently: the derivative) of the real valued function τ. Since τ is bounded, it follows that the map $x \mapsto \|D\psi(\tau(x), x)\|$ is bounded for $x \in U \cap A \setminus B$. To prove that h is Lipschitz, it remains to show that $x \mapsto \|d\tau(v) f(h(x))\|$ is bounded at x_*, with $\|v\| = 1$. In fact, we will show that $\|d\tau\, f(h(x))\|$ goes to zero as $x \to x_*$.

Let $g: V \to \mathbb{R}$ be a submersion defined on some neighborhood V of x_* such that g is constant on B and the number $\delta := \inf_{x \in V} \inf_{\|v\|=1} |d_x g(v)|$ is strictly positive. (Here $d_x g$ is the differential of g at x.) Such a function exists if we take V sufficiently small so that $B \cap V$ is smooth. That δ is strictly positive means that g has no critical points in V which follows from the fact that g is a submersion. Observe that δ is the minimal amount of "stretching" done by g. Also, if $x \in A \setminus B$ and $v \in T_A(x)$, then $Dh(x) \in T_{h(x)} B$, so $dg(Dh(x)v) = 0$.

Now, let $v \in T_A(x_*)$ be arbitrary. Take a smooth curve $c: [0,1] \to A$ such that $c(0) = x_*$ and $\dot{c}(0) = v$. Let $x_s = c(s)$ and $v_s = \dot{c}(s)$. Then, by (2) and d), we have

$$\begin{aligned} d\tau(v_s)(L_f g)(h(x_s)) &= -\,dg(D\psi(\tau(x_s), x_s)v_s) \\ &\to -\,dg(e^{\tau_* L} v) = 0 \end{aligned}$$

as $s \to 0$. Therefore

$$\begin{aligned} \|d\tau(v_s) f(h(x_s))\| &= |d\tau(v_s)(L_f g)(h(x_s))| \frac{\|f(h(x_s))\|}{|(L_f g)(h(x_s))|} \\ &\le |d\tau(v_s)(L_f g)(h(x_s))| \frac{1}{\delta} \to 0 \end{aligned}$$

as $s \to 0+$.

It still remains to show that h is Lipschitz at x_* and that its Lipschitz constant there equals $\|e^{\tau_* L}|_{T_A(x_*)}\|$.

Let $x \in A$ be an arbitrary point close to x_*. Recall that $d_B(h(x), h(x_*))$ equals the infimum of the lengths of all curves $\gamma: [0,1] \to B$ which connect $h(x_*)$ and $h(x)$. For every such curve γ, we get a unique curve $c = h^{-1} \circ \gamma$ in A which connects x_* and x. Moreover, $\dot{\gamma}(0) \in T_B(x_*)$ and $\dot{c}(0) \in T_A(x_*)$. Furthermore

$$\begin{aligned} \ell(\gamma) &= \int_0^1 \|Dh(c(s))\dot{c}(s)\|\, ds \\ &\le \sup_{0<s<1} \|Dh(c(s))|_{T_A(c(s))}\|\, \ell(c). \end{aligned}$$

Taking the infimum over γ, we get

$$d_B(h(x), h(x_*)) \le \limsup_{A \ni x \to x_*} \|Dh(x)|_{T_A(x)}\|\, d_A(x, x_*).$$

Since $\limsup_{A \ni x \to x_*} \|Dh(x)|_{T_A(x)}\| = \|e^{\tau_* L}|_{T_A(x_*)}\|$, this yields $\mathrm{Lip}_{x_*}(h) \le \|e^{\tau_* L}|_{T_A(x_*)}\|$. To prove the reverse inequality, let $v_* \in T_A(x_*)$ be a unit vector such that $\|e^{\tau_* L}|_{T_A(x_*)}\|$ is realized at v_*, i.e., $\|e^{\tau_* L}|_{T_A(x_*)}\| = \|e^{\tau_* L} v_*\|$. Choose a curve c_* in A such that $c_*(0) = x_*$, $\dot{c}_*(0) = v_*$, and $h \circ c_*$ minimizes length in B between any two of its points.

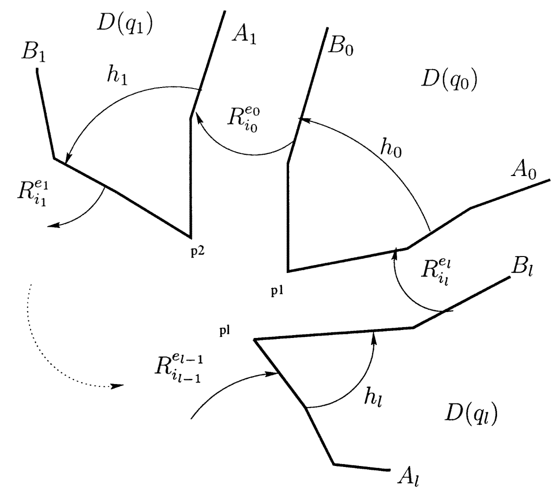

Fig. 8. Illustration of Lemma IV.3.

Then, it can be seen without difficulty that

$$\begin{aligned} \mathrm{Lip}_{x_*}(h) &\ge \lim_{s \to 0+} \frac{d_B(h(c_*(s)), h(x_*))}{d_A(c_*(s), x_*)} \\ &\ge \lim_{s \to 0+} \frac{\int_0^s \|Dh(c_*(t))\dot{c}_*(t)\|\, dt}{\int_0^s \|\dot{c}_*(t)\|\, dt} = \|e^{\tau_* L} v_*\|. \end{aligned}$$

■

Lemma IV.3: Let x_* be an isolated equilibrium such that $(q_j, x_*) \in \mathrm{Reach}_H$, for $0 \le j \le l$. Suppose that there exists a neighborhood W of x_* in Reach_H such that for every $x \in W$, every $0 \le j \le l$, and every infinite execution $(\tau, q, x) \in \mathcal{E}_H^\infty(q_j, x)$ the sequence, q of discrete states goes cyclically through the ordered set $\{q_j, q_{j+1}, \ldots, q_l, q_0, \ldots, q_{j-1}\}$.

Let A_j's and B_j's be as in Theorem IV.2. For each j, and for $x \in A_j \setminus B_j$, define $\tau_j(x) = \min\{t \ge 0: \psi(t, q_j, x) \in B_j\}$, and assume that τ_j is a bounded above function for all j in some neighborhood of x_* in $A_j \setminus B_j$. Define the map $h_j: (A_j \setminus B_j) \cup \{x_*\} \to B_j$ by $h(x) = \psi(\tau_j(x), q_j, x)$. Let $\mu_j = \mathrm{Lip}_{x_*}(h_j)$, $\sup_{\alpha \in \mathcal{A}(e_j)} \mathrm{Lip}_{x_*}(R_\alpha^{e_j}) = \nu_j$ and define $\eta = \prod_{j=1}^{l} \mu_j \nu_j$. If $\eta < 1$, then x_* is locally asymptotically stable.

Proof: Fix a j, with $0 \le j \le l$, and consider $x \in A_j$. Since the system in not necessarily deterministic, there may exist several executions starting at (q_j, x) which return to A_j; we analyze each one of them separately in the following way. For each discrete transition e_k, take from the reset relation R^{e_k} any reset map $R_{\alpha_k}^{e_k}$, with $\alpha_k \in \mathcal{A}(e_k)$. The first-return map for A_j is

$$P_j^\alpha = R_{\alpha_{j-1}}^{e_{j-1}} \circ h_{j-1} \circ \cdots \circ R_{\alpha_0}^{e_0} \circ h_0 \circ R_{\alpha_l}^{e_l} \circ h_l \circ \cdots \circ R_{\alpha_j}^{e_j} \circ h_j$$

where $\alpha = (\alpha_0, \ldots, \alpha_l) \in \mathcal{A}(e_0) \times \cdots \times \mathcal{A}(e_l)$. Then $P_j^\alpha(x_*) = x_*$ and $\mathrm{Lip}_{x_*}(P_j^\alpha) \le \eta < 1$. Therefore, there exists a ball V_j around x_* in A_j such that for all $x \in V_j$, $\|P_j^\alpha(x) - P_j^\alpha(x_*)\| \le \eta \|x - x_*\|$. This is clearly true for all $0 \le j \le l$ and all α as above. Therefore, if we follow any execution starting at (q_j, x), where $x \in V_j$, each time it returns to A_j it is by a factor of $\eta < 1$ closer to x_* than it was the previous time see Fig. 8). Thus, x_* is locally asymptotically stable. ■

Example (Stable Equilibrium in Three Dimensions): Let c be a positive constant and define a hybrid automaton H with $\mathbf{Q} = \{q_1, q_2\}$, $\mathbf{X} = \mathbb{R}^3$

$$\begin{aligned} D(q_1) =& \{(x,y,z): x \geq 0,\ y \geq x^2,\ z \in \mathbb{R}\} \\ & \cup \{(x,y,x): x \leq 0,\ y \geq -x(x-c),\ z \in \mathbb{R}\} \\ D(q_2) =& \overline{\mathbb{R}^3 - D(q_1)} \\ G(q_1,q_2) =& \{(x,y,z) \in D(q_1): y = x^2\} \\ G(q_2,q_1) =& \{(x,y,z) \in D(q_2): y = -x(x-c)\} \\ f(q_1,x,y,z) =& (-x-y, x-y, -\lambda_1 z)^T \\ f(q_2,x,y,z) =& (x-y, x+y, \lambda_2 z)^T \end{aligned}$$

where $0 < \lambda_2 \leq 1 \leq \lambda_1$. Assume the resets are the identity map. Observe that 0 is a sink for f_{q_1} and a source for f_{q_2}. It is lengthy but not difficult to check that $\eta_H(0) = e^{-2\gamma}$, where $\gamma = \arctan c$. Hence, 0 is asymptotically stable for H. Notice that, even though both vector fields are linear in this case, a linearization argument is still necessary, because the boundaries of the guards and the domains are given by nonlinear functions that need to be linearized at 0.

Example (Theorem IV.2 Inconclusive): Again, let $\mathbf{Q} = \{q_1, q_2\}$, $\mathbf{X} = \mathbb{R}^3$

$$\begin{aligned} D(q_1) =& [0,\infty) \times [0,\infty) \times \mathbb{R} \\ D(q_2) =& \overline{\mathbb{R}^2 - [0,\infty) \times [0,\infty)} \times \mathbb{R} \\ G(q_1,q_2) =& \{(x,y,z) \in D(q_1): x = 0\} \\ G(q_2,q_1) =& \{(x,y,z) \in D(q_2): y = 0\} \\ f(q_1,x,y,z) =& (x-y, x+y, -\lambda_1 z)^T \\ f(q_2,x,y,z) =& (-x-y, x-y, \lambda_2 z)^T \end{aligned}$$

where $\lambda_1, \lambda_2 > 0$. The resets are again the identity map. The trajectories of f_{q_1} are spirals around the z-axis that increase in radius and converge to the xy-plane. The trajectories of f_{q_2} are also spirals around the z-axis, but they decrease in radius and diverge from the xy-plane. It is not difficult to check that, with notation from Theorem IV.2, $\mu_1 = e^{\pi\lambda_1/2}$, $\mu_2 = e^{3\pi\lambda_2/2}$, so $\eta_H(0) > 1$ and the theorem is inconclusive.

It is worth noting that if the flows are decoupled into their xy- and z-components, one can observe a small amount of contraction around 0 in the flows of both f_{q_1} and f_{q_2}: the flow of f_{q_1} restricted to $G(q_2, q_1)$ contracts in the z direction (and expands in xy direction) whereas the flow of f_{q_2} on $G(q_1, q_2)$ contracts in the xy direction (and expands in the z direction). Some analysis shows that if $\lambda_1 > 3\lambda_2$, then the small contraction turns out to be sufficient to guarantee asymptotic stability of the equilibrium 0. The conditions of Theorem IV.2 are too conservative however to capture this contraction.

V. Conclusion

Hybrid automata were studied from a dynamical systems perspective. Basic properties of this class of systems, such as well-posedness and stability, were discussed. The main results were conditions for existence and uniqueness of executions, continuity with respect to initial conditions and stability of equilibria and invariant sets. We conclude the paper with a brief discussion of some open problems, which are topics for our ongoing work: Zeno hybrid systems, composition, and multi-domain modeling.

An execution of a hybrid system may exhibit infinitely many discrete jumps in finite time. This is a truly hybrid phenomenon, in the sense that it requires the interaction between continuous and discrete behavior and can not appear in purely continuous or discrete systems. A systematic investigation of the dynamical properties of Zeno hybrid systems has started only recently, e.g., [12], [18], and [36]–[39]. This work indicates that there are interesting connections between the Zeno problem and chattering arising in optimal control and in variable structure systems. For examples, see the extensive literature on the Fuller phenomenon in optimal control [45], [46].

The results in this paper deal with autonomous systems. The introduction of control variables and the formalization of composition of hybrid automata are crucial extensions. The issue of composition is particularly important, since hybrid systems frequently arise in the modeling of complex and heterogeneous systems. For such systems one would like to be able to model the different parts of the system independently, compose the individual models to form larger entities, and deduce properties of the composite models from properties of the individual components. All these steps have to be performed in a consistent, formal way if one is to guarantee correctness and safety properties for the overall system.

This process of modeling complex, heterogeneous systems, is further complicated by the need to employ a number of different modeling languages, each designed to operate within a different domain. Developing a proper interface between these modeling languages is important. In this paper, such an interface was defined in Section II between continuous systems (specified as ordinary differential equations) and discrete systems (specified as a finite state machine). Other constellations of practical interest are under investigation.

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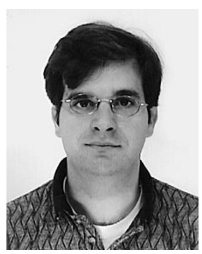

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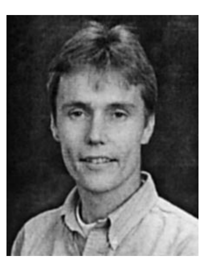

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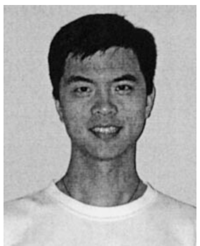

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