2019 IEEE 58th Conference on Decision and Control (CDC)
Palais des Congrès et des Expositions Nice Acropolis
Nice, France, December 11-13, 2019

Synthesis for controllability and observability of logical control networks

Kuize Zhang and Karl Henrik Johansson

***Abstract*— Finite-state systems have applications in systems biology, formal verification and synthesis problems of infinite-state (hybrid) systems, etc. As deterministic finite-state systems, logical control networks (LCNs) consist of a finite number of nodes and their update states, where these nodes can be in a finite number of states. In this paper, we investigate the synthesis problem for controllability and observability of LCNs by state feedback under the semitensor product framework. We show that state feedback can never enforce controllability of an LCN, but sometimes can enforce its observability. We prove that for an LCN Σ and another LCN Σ' obtained by feeding a state-feedback controller into Σ, (1) if Σ is controllable, then Σ' can be either controllable or not; (2) if Σ is not controllable, then Σ' is not controllable either; (3) if Σ is observable, then Σ' can be either observable or not; (4) if Σ is not observable, Σ' can also be observable or not.**

I. Introduction

Finite-state systems have applications in many areas such as formal verification and synthesis problems of infinite-state (hybrid) systems [17], [5], systems biology [1], etc.

As special deterministic finite-state systems such that all nodes can be only in one of two states, Boolean control networks (BCNs) were proposed to describe genetic regulatory networks [13], [12]. In a BCN, nodes can be in one of two discrete states "1" and "0", which represent a gene state "on" and "off", respectively. Every node updates its state according to a Boolean function of the network node states. Although a BCN is a simplified model of a genetic regulatory network, they can be used to characterize many important phenomena of biological systems, e.g., cell cycles [10], cell apoptosis [16]. Hence the study on BCNs has been paid wide attention [14], [3], [4], [23].

A logical control network (LCN) is also a deterministic finite-state system that naturally extends a BCN in the sense that its nodes can be in one of a finite number (but not necessarily 2) of states [24]. From the practical point of view, LCNs can be used to describe more systems than BCNs. However, under the semitensor product (STP) framework, they have the same algebraic form [24], and hence they can be dealt with by using the same method. In this paper, we focus on LCNs.

In 2007, Akutsu et al. [2] proved that it is NP-hard to verify whether a BCN is controllable in the number of nodes (hence there exists no polynomial-time algorithm for determining controllability of BCNs unless P=NP), and pointed out that "One of the major goals of systems biology is to develop a control theory for complex biological systems". Since then, especially since a control-theoretic framework for BCNs based on the STP of matrices (proposed by Cheng [6] in 2001) was established by Cheng et al. [7] in 2009, the study on control problems in the area of BCNs has drawn vast attention, e.g., controllability [7], [25], observability [7], [11], [22], [15], just to name a few.

Among many control properties, *controllability* and *observability* are the most fundamental ones. The former implies that an arbitrary given state of a system can be steered to an arbitrary given state by some input sequence. The latter implies that the initial state can be determined by a sufficiently long input sequence and the corresponding output sequence. The importance of controllability of BCNs can be found in [2] and observability in [16], etc. Lack of these properties makes a system lose many good behaviors. So, it is important to investigate how to enforce controllability and observability.

Since the verification problem for controllability or observability of (infinite-state) hybrid systems is rather difficult and it is possible that both properties are undecidable, if one can construct an LCN as a finite abstraction that (bi)simulates a given hybrid system in the sense of preserving controllability or observability, then one can verify controllability or observability for the hybrid system by verifying them over the LCN. An attempt of using a similar scheme to verify opacity of (infinite-state) transition systems can be found in [21]. Related results on using finite abstractions to do verification or synthesis for infinite-state systems can be found in [19], [9], etc.

As for the synthesis problem, it is known that for linear control systems, controllability is not affected by state feedback, but observability may be affected by state feedback. However, both properties may be affected by state feedback for nonlinear control systems and hybrid systems. Again by using a simulation-based method, if one can construct an LCN as a finite abstraction that (bi)simulates a given unobservable hybrid system in the sense of preserving observability, then one can first try to find a state-feedback controller to make the obtained unobservable LCN observable, and then refine the obtained controller into the original hybrid system so as to make the original hybrid system observable. Here we do not mention controllability because in the sequel we will prove that state feedback will not enforce controllability for LCNs. That is, if an LCN is not controllable, then no state-feedback controller can make it controllable.

The main contributions of this paper are as follows: Let Σ be an LCN and Σ' an LCN obtained by feeding a state-feedback controller into Σ.

This work was supported by Knut and Alice Wallenberg Foundation, Swedish Foundation for Strategic Research, and Swedish Research Council.

K. Zhang and K. Johansson are with School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, 10044 Stockholm, Sweden {kuzhan, kallej}@kth.se.

1) We prove that state feedback will not enforce controllability of LCNs. In detail, if Σ is controllable, then Σ' can be either controllable or uncontrollable; if Σ is uncontrollable, then Σ' can only be uncontrollable.
2) We prove that state feedback sometimes can enforce observability of LCNs. If Σ is observable, then Σ' can be either observable or unobservable; if Σ is unobservable, then Σ' can also be either observable or unobservable.

The remainder of this paper is organized as follows. Section II introduces preliminaries of the paper, i.e., LCNs with their algebraic form under the STP framework, basic verification methods for controllability and observability of LCNs. Section III presents the main results of the paper: state feedback cannot enforce controllability of LCNs, but can enforce their observability. Section IV is a short conclusion.

II. Preliminaries

A. The semitensor product of matrices

The following notations are necessary in this paper.

- 2^A: power set of set A
- $\mathbb{Z}_+$: set of positive integers
- $\mathbb{N}$: set of natural numbers
- $\mathbb{R}_{\geq 0}$: set of nonnegative real numbers
- $\mathbb{R}^n$: set of n-length real column vectors
- $\mathbb{R}^{m\times n}$: set of real $m\times n$ real matrices
- $\mathcal{D}_k$: set $\{0, \frac{1}{k-1}, ..., 1\}$
- δ_n^i: i-th column of the identity matrix I_n
- $\mathbf{1}_k$: $\sum_{i=1}^k \delta_k^i$
- Δ_n: set $\{\delta_n^1, \dots, \delta_n^n\}$ ($\Delta := \Delta_2$)
- $[m, n]$: $\{m, m+1, ..., n\}$, where $m, n \in \mathbb{N}$ and $m \leq n$
- $\delta_n[i_1, \dots, i_s]$: logical matrix $[\delta_n^{i_1}, \dots, \delta_n^{i_s}]$, where $i_1, \dots, i_s \in [1, n]$
- $\mathcal{L}_{n\times s}$: set of $n\times s$ logical matrices
- $\text{Col}(A)$: set of columns of matrix A
- A^T: transpose of matrix A
- $\text{im}(A)$: image space of matrix A
- $A_1 \oplus A_2 \oplus \cdots \oplus A_n$: $\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$

Definition 2.1 ([8]): Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, and $\alpha = \text{lcm}(n, p)$ be the least common multiple of n and p. The STP of A and B is defined as

$$A \ltimes B = \left(A \otimes I_{\frac{\alpha}{n}}\right)\left(B \otimes I_{\frac{\alpha}{p}}\right),$$

where $\otimes$ denotes the Kronecker product.

From this definition, it is easy to see that the conventional product of matrices is a particular case of the STP, since if $n = p$ then $A \ltimes B = AB$. Since the STP keeps most properties of the conventional product [8], e.g., the associative law, the distributive law, etc., we usually omit the symbol "$\ltimes$" hereinafter.

Next we introduce some concepts and properties related to the STP of matrices that will be used later.

Definition 2.2: The swap matrix, $W_{[m,n]}$, is an $mn \times mn$ matrix defined by

$$W_{[m,n]} := \left[I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \dots, I_n \otimes \delta_m^m\right].$$

Proposition 2.3 ([8]): Let $W_{[m,n]}$ be a swap matrix, $P \in \mathbb{R}^m$, and $Q \in \mathbb{R}^n$. Then

$$W_{[m,n]}^T = W_{[m,n]}^{-1} = W_{[n,m]}, \tag{1}$$

$$W_{[m,n]}PQ = QP, \tag{2}$$

$$P^T Q^T W_{[m,n]} = Q^T P^T. \tag{3}$$

Definition 2.4: The matrix $M_{k_r} = \delta_k^1 \oplus \cdots \oplus \delta_k^k$ is called the *power-reducing matrix*. Particularly, we denote $M_{2_r} := M_r$.

By definition, the following propositions hold.

Proposition 2.5 ([8]): For power-reducing matrix M_{k_r}, we have

$$P^2 = M_{k_r} P$$

for each $P \in \Delta_k$.

Proposition 2.6 ([8]): Let $A \in \mathbb{R}^{m\times n}$ and $z \in \mathbb{R}^t$. Then

$$A \ltimes z^T = z^T \ltimes (I_t \otimes A),$$
$$z \ltimes A = (I_t \otimes A) \ltimes z.$$

B. Logical control networks and their algebraic form

In this paper, we investigate the following LCN with n state nodes, m input nodes, and q output nodes:

$$\begin{aligned} x_1(t+1) &= f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ x_2(t+1) &= f_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ &\vdots \\ x_n(t+1) &= f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ y_1(t) &= h_1(x_1(t), \dots, x_n(t)), \\ y_2(t) &= h_2(x_1(t), \dots, x_n(t)), \\ &\vdots \\ y_q(t) &= h_q(x_1(t), \dots, x_n(t)), \end{aligned} \tag{4}$$

where $t \in \mathbb{N}$ denote discrete time steps; $x_i(t) \in \mathcal{D}_{n_i}$, $u_j(t) \in \mathcal{D}_{m_j}$, and $y_k(t) \in \mathcal{D}_{q_k}$ denote values of state node x_i, input node u_j, and output node y_k at time step t, respectively, $i \in [1, n]$, $j \in [1, m]$, $k \in [1, q]$; $\prod_{i=1}^n n_i =: N$; $\prod_{j=1}^m m_i =: M$; $\prod_{k=1}^q q_i =: Q$; $f_i : \mathcal{D}_{MN} \to \mathcal{D}_{n_i}$ and $h_k : \mathcal{D}_N \to \mathcal{D}_{q_k}$ are mappings, $i \in [1, n]$, $k \in [1, q]$.

When $n_1 = \cdots = n_n = m_1 = \cdots = m_m = q_1 = \cdots = q_q = 2$, Eqn. (4) reduces to a BCN.

Eqn. (4) can be represented in the compact form

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{5}$$

where $t \in \mathbb{N}$; $x(t) \in \mathcal{D}_N$, $u(t) \in \mathcal{D}_M$, and $y(t) \in \mathcal{D}_Q$ stand for the state, input, and output of the LCN at time step t; $f : \mathcal{D}_{NM} \to \mathcal{D}_N$ and $h : \mathcal{D}_N \to \mathcal{D}_Q$ are mappings.

Next we give the algebraic form of (5) under the STP framework, the detailed transformation can be found in [8].

For each $n \in \mathbb{Z}_+$ greater than 1, we identify $\delta_n^i \sim \frac{n-i}{n-1}$, $i \in [1, n]$. Then Eqn. (5) can be represented as

$$\begin{aligned} \tilde{x}(t+1) &= L\tilde{x}(t)\tilde{u}(t) = [L_1, \dots, L_N]\tilde{x}(t)\tilde{u}(t), \\ \tilde{y}(t) &= H\tilde{x}(t), \end{aligned} \tag{6}$$

where $t \in \mathbb{N}$; $\tilde{x}(t) \in \Delta_N$, $\tilde{u}(t) \in \Delta_M$, $\tilde{y}(t) \in \Delta_Q$; $L \in \mathcal{L}_{N \times NM}$ and $H \in \mathcal{L}_{Q \times N}$ are called the *structure matrices*, $L_i \in \mathcal{L}_{N \times M}$, $i \in [1, N]$.

C. Preliminaries for controllability

In this subsection we briefly introduce a controllability test criterion. Consider the *state transition graph* $(\mathcal{V}, \mathcal{E})$ of LCN (6), where $\mathcal{V} = \Delta_N$ is the vertex set, the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is defined as follows: for all states δ_N^i and δ_N^j, where $i, j \in [1, N]$, $(\delta_N^i, \delta_N^j) \in \mathcal{E}$ (i.e., there exists an edge from δ_N^i to δ_N^j) if and only if there exists an input δ_M^l with $l \in [1, M]$ such that $\delta_N^j = L\delta_N^i\delta_M^l$.

By Proposition 2.3, the adjacent matrix of the state transition graph of LCN (6) is $LW_{[M,N]}\mathbf{1}_M =: \mathcal{A} = (a_{ij})_{i,j \in [1,N]} \in \mathbb{R}^{N \times N}$, where $a_{ij} > 0$ if and only if $(\delta_N^j, \delta_N^i) \in \mathcal{E}$. The matrix $\mathcal{A}$ can be obtained by Eqn. (6) as

$$L\tilde{x}(t)\tilde{u}(t) = LW_{[M,N]}\tilde{u}(t)\tilde{x}(t) \tag{7}$$

when $\tilde{u}$ runs all over Δ_M.

Definition 2.7 ([25]): An LCN (6) is called *controllable* if for all states $x_0, x_d \in \Delta_N$, if $x(0) = x_0$, then $x(l) = x_d$ for some $l \in \mathbb{Z}_+$ and some input sequence $u(0)u(1)\dots u(l-1)$, where $u(j) \in \Delta_M$, $j = 0, 1, \dots, l-1$. An LCN is called *uncontrollable* if it is not controllable.

By an observation to the state transition graph, we see the following result.

Proposition 2.8: An LCN (6) is controllable if and only if the its state transition graph is strongly connected, i.e., for all vertices v_1 and v_2, there exists a path from v_1 to v_2.

Example 1: Consider the following BCN

$$x(t+1) = Lx(t)u(t), \tag{8}$$

where $L = \delta_4[1, 1, 1, 1, 1, 2, 1, 2, 3, 3, 1, 1, 3, 4, 1, 2]$, $t \in \mathbb{N}$, $x(t), u(t) \in \Delta_4$.

The adjacent matrix of its state transition graph is

$$\mathcal{A} = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One sees that in the state transition graph of (8), there exists no path from δ_4^3 to δ_4^2, hence (8) is not controllable.

D. Preliminaries for observability

In [22], four types of observability were characterized for BCNs. In this paper, we are particularly interested in the linear type (first characterized in [11]), as if an LCN satisfies this observability property, it is very easy to recover the initial state by using an input sequence and the corresponding output sequence. Note that all results in [22] can be trivially extended to LCNs.

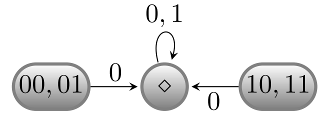

Fig. 1. Observability graph of BCN (9).

Definition 2.9: An LCN (5) is called *observable* if for all different initial states $x(0), x'(0) \in \mathcal{D}_N$, for each input sequence $u(0)u(1)\dots$, the corresponding output sequences $y(0)y(1)\dots$ and $y'(0)y'(1)\dots$ are different. An LCN is called *unobservable* if it is not observable.

We use a graph-theoretic method proposed in [22] to verify observability in what follows.

Definition 2.10: Consider an LCN (5). A triple $\mathcal{G}_o = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ is called its observability graph if $\mathcal{V}$ (elements of $\mathcal{V}$ are called vertices) equals $\{\{x, x'\} \in \mathcal{D}_N \times \mathcal{D}_N | h(x) = h(x')\}$[1], $\mathcal{E}$ (elements of $\mathcal{E}$ are called edges) equals $\{(\{x_1, x_1'\}, \{x_2, x_2'\}) \in \mathcal{V} \times \mathcal{V} |$there exists $u \in \mathcal{D}_M$ such that $f(x_1, u) = x_2$ and $f(x_1', u) = x_2'$, or, $f(x_1, u) = x_2'$ and $f(x_1', u) = x_2\}$, and the weight function $\mathcal{W} : \mathcal{E} \to 2^{\mathcal{D}_M}$ assigns to each edge $(\{x_1, x_1'\}, \{x_2, x_2'\}) \in \mathcal{E}$ a set $\{u \in \mathcal{D}_M | f(x_1, u) = x_2$ and $f(x_1', u) = x_2'$, or, $f(x_1, u) = x_2'$ and $f(x_1', u) = x_2\}$ of inputs. A vertex $\{x, x'\}$ is called diagonal if $x = x'$, and called non-diagonal otherwise.

Proposition 2.11 ([22]): An LCN (5) is not observable if and only if its observability graph has a non-diagonal vertex v and a cycle C such that there is a path from v to a vertex of C.

The *diagonal subgraph* of an observability graph is defined by all diagonal vertices of the observability graph and all edges between them. Similarly, the *non-diagonal subgraph* is defined by all non-diagonal vertices and all edges between them. Since in the diagonal subgraph, there must exist a cycle and each vertex will go to a cycle, we will denote the subgraph briefly by a symbol $\diamond$ when drawing an observability graph. Hence if there exists an edge from a non-diagonal vertex to a diagonal vertex, then the LCN is not observable.

Example 2 ([20]): Consider the following BCN

$$\begin{aligned} x_1(t+1) &= x_2(t) \wedge u(t), \\ x_2(t+1) &= \neg x_1(t) \vee u(t), \\ y(t) &= x_1(t), \end{aligned} \tag{9}$$

where $t = 0, 1, \dots$; $x_1(t), x_2(t), u(t), y(t)$ are Boolean variables (1 or 0); $\wedge, \vee$, and $\neg$ denote AND, OR, and NOT, respectively. The LCN is not observable (see Fig. 1) by Proposition 2.11.

III. Main results

In this section, we show our main results, i.e., the synthesis problem for controllability and observability of LCN (5) (or its algebraic form (6)) based on state feedback. Next we show the form of state-feedback LCNs.

[1]vertices are unordered state pairs, i.e., $\{x, x'\} = \{x', x\}$.

A. State-feedback logical control networks

Consider an LCN (5). Let a *state-feedback controller* be

$$u(t) = g(x(t), v(t)), \tag{10}$$

where $v(t) \in \mathcal{D}_P$, $P = \prod_{l=1}^{p} p_l$ with each $p_l \in \mathbb{N}$ greater than 1 (corresponding to l new input nodes); or $P = 1$, which means that there is only one constant input; $g : \mathcal{D}_{NP} \to \mathcal{D}_M$ is a mapping. Equivalently, for an LCN (6), we can set a state-feedback controller to be

$$\tilde{u}(t) = G\tilde{x}(t)\tilde{v}(t) = [G_1, \ldots, G_N]\tilde{x}(t)\tilde{v}(t), \tag{11}$$

where $\tilde{v}(t) \in \Delta_P$, $G \in \mathcal{L}_{M\times NP}$ is called the structure matrix, $G_i \in \mathcal{L}_{M\times P}$, $i \in [1, N]$.

When $P = 1$ and g is the identity mapping, a state-feedback controller will not change the original LCN, and hence will not change controllability or observability of the LCN.

Substituting (10) into (5), we obtain a state-feedback LCN as

$$\begin{aligned} x(t+1) &= f(x(t), g(x(t), v(t))),\\ y(t) &= h(x(t)). \end{aligned} \tag{12}$$

Putting (11) into (6), we obtain the algebraic form of the state-feedback LCN (12) as

$$\begin{aligned} \tilde{x}(t+1) &= L\tilde{x}(t)G\tilde{x}(t)\tilde{v}(t),\\ \tilde{y}(t) &= H\tilde{x}(t). \end{aligned} \tag{13}$$

Proposition 3.1: Eqn (13) is equivalent to

$$\begin{aligned} \tilde{x}(t+1) &= [L_1G_1, ..., L_NG_N]\tilde{x}(t)\tilde{v}(t),\\ \tilde{y}(t) &= H\tilde{x}(t). \end{aligned} \tag{14}$$

Proof By Propositions 2.5 and 2.6, (13) can be rewritten as

$$\begin{aligned} \tilde{x}(t+1) &= L\tilde{x}(t)G\tilde{x}(t)\tilde{v}(t)\\ &= L(I_N \otimes G)M_{N_r}\tilde{x}(t)v(t)\\ &= L\begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix}\left(\begin{bmatrix} \delta_N^1 & & \\ & \ddots & \\ & & \delta_N^N \end{bmatrix} \otimes I_P\right)\\ &\quad \tilde{x}(t)\tilde{v}(t)\\ &= L\begin{bmatrix} G & & \\ & \ddots & \\ & & G \end{bmatrix}\begin{bmatrix} \delta_N^1 \otimes I_P & & \\ & \ddots & \\ & & \delta_N^N \otimes I_P \end{bmatrix}\\ &\quad \tilde{x}(t)\tilde{v}(t)\\ &= L\begin{bmatrix} G(\delta_N^1 \otimes I_P) & & \\ & \ddots & \\ & & G(\delta_N^N \otimes I_P) \end{bmatrix}\tilde{x}(t)\tilde{v}(t)\\ &= [L_1, \ldots, L_N]\begin{bmatrix} G_1 & & \\ & \ddots & \\ & & G_N \end{bmatrix}\tilde{x}(t)\tilde{v}(t)\\ &= [L_1G_1, ..., L_NG_N]\tilde{x}(t)\tilde{v}(t). \end{aligned}$$

□

Consider a newly obtained LCN (14), if $P = 1$, then the corresponding structure matrix $[L_1G_1, \ldots, L_NG_N]$ is square. However, generally the structure matrix is not necessarily square, hence the updating of states generally depends on the input $\tilde{v}(t)$.

B. Synthesis for controllability

In this subsection we characterize whether state feedback can enforce controllability of an LCN. Unfortunately, we will show that a state-feedback controller may make a controllable LCN uncontrollable, but never makes an uncontrollable LCN controllable.

Proposition 3.2: Consider an LCN (6) and a state-feedback controller (11). The adjacent matrices of the state transition graphs of (6) and the corresponding state-feedback LCN (14) are

$$\begin{aligned} \mathcal{A} &= [L_1\mathbf{1}_M, \ldots, L_N\mathbf{1}_M] \text{ and}\\ \mathcal{A}' &= [L_1G_1\mathbf{1}_P, \ldots, L_NG_N\mathbf{1}_P], \end{aligned}$$

respectively.

Proof Consider (6). Denote $L_i = [l_1^i, \ldots, l_M^i]$, $i \in [1, N]$. The adjacent matrix $\mathcal{A}$ of its state transition graph is

$$\begin{aligned} \mathcal{A} &= LW_{[M,N]}\mathbf{1}_M\\ &= [L_1, \ldots, L_N]W_{[M,N]}\mathbf{1}_M\\ &= \left[l_1^1, \ldots, l_M^1, l_1^2, \ldots, l_M^2, \ldots, l_1^N, \ldots, l_M^N\right] W_{[M,N]}\mathbf{1}_M\\ &= \left[l_1^1, \ldots, l_1^N, l_2^1, \ldots, l_2^N \ldots, l_M^1, \ldots, l_M^N\right] \mathbf{1}_M\\ &= \left[\sum_{i=1}^{M} l_i^1, \ldots, \sum_{i=1}^{M} l_i^N\right]\\ &= [L_1\mathbf{1}_M, \ldots, L_N\mathbf{1}_M]. \end{aligned} \tag{15}$$

Similarly we have the adjacent matrix $\mathcal{A}'$ of the state transition graph of (14) is

$$\mathcal{A}' = [L_1G_1\mathbf{1}_P, \ldots, L_NG_N\mathbf{1}_P]. \tag{16}$$

□

Theorem 3.3: Consider an LCN (6) and a state-feedback controller (11). If (6) is not controllable, then the corresponding state-feedback LCN (14) is not controllable either.

Proof Observe that in the adjacent matrices (15) and (16) of the state transition graphs of LCN (6) and the corresponding state-feedback LCN (14), for each $i \in [1, N]$, L_iG_i is obtained by rearranging several of columns of L_i (repeated use of columns of L_i is permitted), then $\text{Col}(L_iG_i) \subset \text{Col}(L_i)$. Hence for all $i, j \in [1, N]$, if the j-th entry of $L_iG_i\mathbf{1}_P$ is greater than 0, then the j-th entry of $L_i\mathbf{1}_M$ is also greater than 0. Hence although the two state transition graphs share the same vertex set, the edge set of the state transition graph for (14) is a subset of that of the state transition graph for (6).

Assume that an LCN (6) is not controllable. Then in its state transition graph, there exist states δ_N^k and δ_N^l such that there exists no path from δ_N^k to δ_N^l by Proposition 2.8. Hence in the state transition graph of (14), one also has that there exists no path from δ_N^k to δ_N^l. That is, (14) is not controllable by Proposition 2.8. □

Example 3: We give an example to show that a state-feedback controller can make a controllable LCN uncontrollable. Consider the following BCN

$$x(t+1) = Lx(t)u(t), \tag{17}$$

where $L = \delta_4[2, 2, 1, 3, 4, 4, 2, 2]$, $t \in \mathbb{N}$, $x(t) \in \Delta_4$, $u(t) \in \Delta$.

By Proposition 3.2, the adjacent matrix of its state transition graph is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

It is easy to see that the graph is strongly connected, then (17) is controllable.

Substituting the state-feedback controller

$$u(t) = Gx(t)v(t), \tag{18}$$

where $G = \delta_2[1, 2, 2, 2, 1, 2, 1, 2]$, $v(t) \in \Delta$, into (17), by Proposition 3.1, we obtain the state-feedback BCN

$$x(t+1) = \tilde{L}x(t)u(t), \tag{19}$$

where $\tilde{L} = \delta_4[2, 2, 3, 3, 4, 4, 2, 2]$.

Then by Proposition 3.2, the adjacent matrix of the state transition graph of (19) is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

The graph is not strongly connected, then by Proposition 2.8, (19) is not controllable.

Remark 1: Let us compare LCNs with linear control systems. It is known that state feedback will not affect controllability of linear control systems [18]. Consider a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{20}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$. Consider state-feedback controller

$$u(t) = Fx(t) + v(t), \tag{21}$$

where $F \in \mathbb{R}^{m\times n}$, $t \in \mathbb{R}_{\geq 0}$, $v(t) \in \mathbb{R}^m$. It is well known that [18]

$$\langle A|\,\mathrm{im}(B)\rangle = \langle A + BF|\,\mathrm{im}(B)\rangle,$$

where $\langle A|\,\mathrm{im}(B)\rangle = \mathrm{im}(B) + A\,\mathrm{im}(B) + \cdots + A^{n-1}\,\mathrm{im}(B)$ is the controllable subspace of (20), and $\langle A+BF|\,\mathrm{im}(B)\rangle$ is the controllable subspace of the state-feedback linear control system $\dot{x}(t) = (A + BF)x(t) + Bv(t)$.

Hence Example 3 shows an essential difference between LCNs and linear control systems from the perspective of controllability.

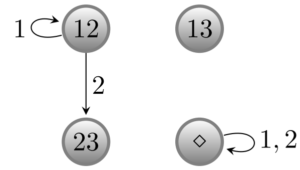

Fig. 2. Observability graph of BCN (17) with output function (22), where number ij in a circle denotes state pair $\{\delta_4^i, \delta_4^j\}$, weight i denotes input δ_2^i.

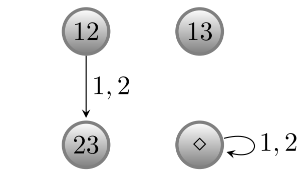

Fig. 3. Observability graph of BCN (19) with output function (22).

C. Synthesis for observability

Unlike controllability, we next give an example to show that state feedback can enforce observability of an LCN.

Example 4: Consider BCN (17) with the output function

$$y(t) = \delta_2[1, 1, 1, 2]x(t). \tag{22}$$

Its observability graph is shown in Fig. 2. This graph shows that the BCN is not observable by Proposition 2.11.

Putting state-feedback controller (18) into (17), and consider the state-feedback BCN (19) with output function (22). Its observability graph is shown in Fig. 3. This graph shows that the BCN is observable also by Proposition 2.11.

Remark 2: Different from controllability, it is known that observability of linear control systems can be affected by state feedback. Consider system $\dot{x}_1(t) = x_2(t) + u(t)$, $\dot{x}_2(t) = x_2(t)$, $y(t) = x_1(t)$. This system is observable, because its observability matrix I_2 has rank 2. However, if we put state-feedback controller $u(t) = v(t) - x_2(t)$ into the system, we obtain an unobservable system, where its observability matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ has rank 1. By these two systems, one sees that state feedback can make an observable linear control system unobservable, and also can make an unobservable system observable.

Next we show that a state-feedback controller can make an observable LCN unobservable.

Example 5: Consider the LCN

$$\begin{aligned} x(t+1) &= \delta_3[1, 3, 3, 2, 1, 1]x(t)u(t), \\ y(t) &= \delta_2[1, 1, 2]x(t), \end{aligned} \tag{23}$$

where $t \in \mathbb{N}$, $x(t) \in \Delta_3$, $u(t), y(t) \in \Delta$.

The observability graph of (23) consists of vertex $\{\delta_3^1, \delta_3^2\}$ and the diagonal subgraph. Then by Proposition 2.11, the BCN is observable.

Putting state-feedback controller

$$u(t) = \delta_2[1, 2, 1]x(t)$$

into (23), by Proposition 3.1, we obtain a state-feedback LCN

$$\begin{aligned} x(t+1) &= \delta_3[1, 2, 1]x(t), \\ y(t) &= \delta_2[1, 1, 2]x(t). \end{aligned} \tag{24}$$

There is a self-loop on vertex $\{\delta_3^1, \delta_3^2\}$ in the observability graph of (24), then by Proposition 2.11, (24) is not observable.

Next we show that there exists an unobservable LCN such that no state-feedback controller can make it observable.

Example 6: Consider the BCN

$$x(t+1) = Lx(t)u(t), \tag{25}$$

where $L = \delta_4[1,1,1,1,1,1,2,3]$, $t \in \mathbb{N}$, $x(t) \in \Delta_4$, $u(t) \in \Delta$.

By Proposition 2.11, the BCN with output function (22) is not observable, since there exists a path $\{\delta_4^1, \delta_4^2\} \xrightarrow{\delta_2^1} \{\delta_4^1, \delta_4^1\} \xrightarrow{\delta_2^1} \{\delta_4^1, \delta_4^1\}$ in its observability graph.

Putting an arbitrary state-feedback controller $u(t) = Gx(t)v(t)$, where $G \in \mathcal{L}_{2\times 4P}$, $v(t) \in \Delta_P$, P is an arbitrary positive integer, into (25), by Proposition 3.1, we obtain state-feedback LCN

$$x(t+1) = \left[\delta_4^1 \otimes \mathbf{1}_P^T, \delta_4^1 \otimes \mathbf{1}_P^T, \delta_4^1 \otimes \mathbf{1}_P^T, L_4G_4\right] x(t)u(t), \tag{26}$$

where $L_4G_4 = \delta_4[i_1, \dots, i_P]$, $i_1, \dots, i_P \in [2,3]$.

The observability graph of (26) with output function (22) contains a path $\{\delta_4^1, \delta_4^2\} \xrightarrow{\delta_P^1} \{\delta_4^1, \delta_4^1\} \xrightarrow{\delta_P^1} \{\delta_4^1, \delta_4^1\}$, then the LCN is not observable by Proposition 2.11.

Based on the above discussion, we know that state feedback sometimes can enforce observability of an LCN, sometimes cannot.

IV. Conclusion

In this paper, we showed that state feedback cannot enforce controllability of a logical control network (LCN), but sometimes can enforce observability of an LCN, by using the semitensor product. Future works are to study how to verify whether observability of an LCN can be enforced by state feedback, how to design fast algorithms for synthesizing observability of an unobservable LCN when the LCN can be synthesized to be observable.

In order to make the obtained results be applied to the simulation-based method for synthesizing hybrid systems over their finite abstractions introduced in the Introduction, the further work is to generalize the obtained results to nondeterministic finite-transition systems, since usually nondeterministic finite-transition systems better simulate hybrid systems.

References

[1] T. Akutsu. *Algorithms for Analysis, Inference, and Control of Boolean Networks*. World Scietific, 2018.

[2] T. Akutsu, M. Hayashida, W. K. Ching, and M. K. Ng. Control of Boolean networks: Hardness results and algorithms for tree structured networks. *Journal of Theoretical Biology*, 244(4):670–679, 2007.

[3] T. Akutsu, S. Miyano, and S. Kuhara. Inferring qualitative relations in genetic networks and metabolic pathways. *Bioinformatics*, 16 (8):727–734, 2000.

[4] R. Albert and A.-L. Barabási. Dynamics of complex systems: Scaling laws for the period of Boolean networks. *Phys. Rev. Lett.*, 84:5660–5663, Jun 2000.

[5] C. Belta, B. Yordanov, and E. A. Gol. *Formal Methods for Discrete-Time Dynamical Systems*. Springer International Publishing AG, 2017.

[6] D. Cheng. Semi-tensor product of matrices and its application to Morgen's problem. *Science in China Series : Information Sciences*, 44(3):195–212, 2001.

[7] D. Cheng and H. Qi. Controllability and observability of Boolean control networks. *Automatica*, 45(7):1659–1667, 2009.

[8] D. Cheng, H. Qi, and Z. Li. *Analysis and Control of Boolean Networks: A Semi-tensor Product Approach*. Springer-Verlag London, 2011.

[9] S. Coogan, M. Arcak, and C. Belta. Finite state abstraction and formal methods for traffic flow networks. In *2016 American Control Conference (ACC)*, pages 864–879, July 2016.

[10] A. Fauré, A. Naldi, C. Chaouiya, and D. Thieffry. Dynamical analysis of a generic Boolean model for the control of the mammalian cell cycle. *Bioinformatics*, 22(14):e124, 2006.

[11] E. Fornasini and M. E. Valcher. Observability, reconstructibility and state observers of Boolean control networks. *IEEE Transactions on Automatic Control*, 58(6):1390–1401, June 2013.

[12] T. Ideker, T. Galitski, and L. Hood. A new approach to decoding life: systems biology. *Annual Review of Genomics and Human Genetics*, 2:343–372, 2001.

[13] S. A. Kauffman. Metabolic stability and epigenesis in randomly constructed genetic nets. *Journal of Theoretical Biology*, 22(3):437–467, 1969.

[14] H. Kitano. Systems biology: A brief overview. *Science*, 295:1662–1664, 2002.

[15] R. Li, M. Yang, and T. Chu. Controllability and observability of Boolean networks arising from biology. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 25(2):023104–(1–15), 2015.

[16] S. Sridharan, R. Layek, A. Datta, and J Venkatraj. Boolean modeling and fault diagnosis in oxidative stress response. *BMC Genomics*, 13(Suppl 6), S4:1–16, 2012.

[17] P. Tabuada. *Verification and Control of Hybrid Systems: A Symbolic Approach*. Springer Publishing Company, Incorporated, 1st edition, 2009.

[18] W. M. Wonham. *Linear Multivariable Control: a Geometric Approach, 3rd Ed.* Springer-Verlag New York, 1985.

[19] M. Zamani, P. Mohajerin Esfahani, R. Majumdar, A. Abate, and J. Lygeros. Symbolic control of stochastic systems via approximately bisimilar finite bstractions. *IEEE Transactions on Automatic Control*, 59(12):3135–3150, 2014.

[20] K. Zhang and K. H. Johansson. Efficient observability verification for large-scale Boolean control networks. In *2018 37th Chinese Control Conference (CCC)*, pages 560–567, July 2018.

[21] K. Zhang, X. Yin, and M. Zamani. Opacity of nondeterministic transition systems: A (bi)simulation relation approach. *IEEE Transactions on Automatic Control*, conditionally accepted, 2018. https://arxiv.org/abs/1802.03321.

[22] K. Zhang and L. Zhang. Observability of Boolean control networks: A unified approach based on finite automata. *IEEE Transactions on Automatic Control*, 61(9):2733–2738, Sept 2016.

[23] Y. Zhao, J. Kim, and M. Filippone. Aggregation algorithm towards large-scale Boolean network analysis. *IEEE Transactions on Automatic Control*, 58 (8):1976–1985, 2013.

[24] Y. Zhao, Z. Li, and D. Cheng. Optimal control of logical control networks. *IEEE Transactions on Automatic Control*, 56(8):1766–1776, Aug 2011.

[25] Y. Zhao, H. Qi, and D. Cheng. Input-state incidence matrix of Boolean control networks and its applications. *Systems & Control Letters*, 59(12):767–774, 2010.