On the Existence of Executions of Hybrid Automata

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Abstract

Necessary and sufficient conditions for hybrid automata to be non-blocking and deterministic (local existence and uniqueness of executions, respectively) are developed. The problem of global existence of executions is discussed in the context of Zeno hybrid automata, that is, hybrid automata that can exhibit infinitely many discrete transitions in finite time.

1 Introduction

Despite a great deal of recent activity in the area of hybrid systems, the study of fundamental properties, such as the existence and uniqueness of executions, has been somewhat overlooked. These properties are important for analysis, controller synthesis and simulation, which in the absence of general analysis methodologies plays an important role in the study of hybrid systems. Existence and uniqueness properties have been formally studied only for special classes of systems [1, 2, 3].

In this paper we try to formalize some basic properties of executions for a general class of hybrid automata. In addition to the usual technical conditions for existence of solutions for continuous dynamical systems, conditions are introduced to prevent blocking and non-determinism. We also discuss briefly the Zeno phenomenon, where the executions of the hybrid system exhibit an infinite number of discrete transitions in finite time. We illustrate by means of examples that Zeno executions are closely related to chattering, that sometimes arises in optimal control [4] and in variable structure systems [5].

We start by giving formal definitions of hybrid automata and their executions in Section 2. Results on existence and uniqueness of executions are derived in Section 3. Section 4 presents some examples of Zeno hybrid automata, highlighting various aspects of the Zeno phenomenon.

2 Hybrid Automata

Let W be a finite collection of variables and let $\mathbf{W}$ denote the set of valuations of W, i.e. the set of all possible assignments of the variables in W. We refer to variables whose set of valuations is countable as *discrete* and to variables whose set of valuations is a subset of Euclidean space as *continuous*. We assume that Euclidean space is given the Euclidean metric topology, whereas countable and finite sets are given the discrete topology. Subsets of a topological space are given the subset topology and products of topological spaces are given the product topology. For a subset U of a topological space we use U^c to denote its complement, $|U|$ its cardinality, and 2^U the set of all subsets of U.

We are interested in hybrid phenomena, that involve both continuous and discrete dynamics. A hybrid time trajectory encodes the set of times over which the evolution of the system will be defined.

Definition 1 (Hybrid Time Trajectory) *A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite or infinite sequence of intervals such that $I_i = [\tau_i, \tau_i']$ for $i < N$, if $N < \infty$, $I_N = [\tau_N, \tau_N']$ or $I_N = [\tau_N, \tau_N')$, and for all i, $\tau_i \leq \tau_i' = \tau_{i+1}$.*

The interpretation is that τ_i are the "times" at which discrete transitions take place. Notice that discrete transitions are assumed to be instantaneous and that multiple discrete transitions may take place at the same time, since it is possible for $\tau_i = \tau_{i+1}$. We denote by $\mathcal{T}$ the set of all hybrid time trajectories. Note that hybrid time trajectories can extend to "infinity" if τ is an infinite sequence or if it is a finite sequence ending with an interval of the form $[\tau_N, \infty)$. For $t \in \mathbb{R}$ and $\tau \in \mathcal{T}$ we use $t \in \tau$ as a shorthand notation for "there exists a

j such that $t \in [\tau_j, \tau_j'] \in \tau$". For a topological space K and $\tau \in \mathcal{T}$, we use $k : \tau \to K$ as a shorthand notation for a map assigning values from K to every element of every interval of τ. For a collection of variables W, we denote by $\mathrm{Hyb}(W)$ the set of all *hybrid trajectories* of W, defined as $\mathrm{Hyb}(W) = \{(\tau, w) \mid \tau \in \mathcal{T} \text{ and } w : \tau \to \mathbf{W}\}$. We say (τ, w) with $\tau = \{I_i\}_{i=0}^N$ is a *prefix* of (τ', w') with $\tau' = \{I_i'\}_{i=0}^{N'}$ (write $(\tau, w) \leq (\tau', w')$) if $N \leq N'$, $I_i = I_i'$ for all $i = 0, \ldots, N-1$, $I_N \subseteq I_N'$ and $w(t) = w'(t)$ for all $t \in \tau$. We say (τ, w) is a *strict prefix* of (τ', w') (write $(\tau, w) < (\tau', w')$) if $(\tau, w) \leq (\tau', w')$ and $(\tau, w) \neq (\tau', w')$. Note that the prefix relation defines a partial order on the set of executions.

A hybrid automaton provides a formal way of restricting the set of hybrid trajectories of a collection of discrete and continuous variables. The following definitions are based on [6, 7].

Definition 2 (Hybrid Automaton) *A hybrid automaton H is a collection $H = (Q, X, \mathrm{Init}, f, I, E, G, R)$, where Q is a finite collection of discrete variables, X is a finite collection of continuous variables with $\mathbf{X} = \mathbb{R}^n$ and*

- $\mathrm{Init} \subseteq \mathbf{Q} \times \mathbf{X}$ *is a set of* initial states*;*
- $f : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}^n$ *is a* vector field*;*
- $I : \mathbf{Q} \to 2^{\mathbf{X}}$ *is an* invariant set *for each $q \in \mathbf{Q}$;*
- $E \subset \mathbf{Q} \times \mathbf{Q}$ *is a collection of* edges*;*
- $G : E \to 2^{\mathbf{X}}$ *is a* guard *for each edge; and*
- $R : E \times \mathbf{X} \to 2^{\mathbf{X}}$ *is a* reset relation *for each edge.*

We refer to $(q, x) \in \mathbf{Q} \times \mathbf{X}$ as the state of H. Pictorially, a hybrid automaton is represented by a directed graph with vertices $\mathbf{Q}$ and edges E. With each vertex $q \in \mathbf{Q}$, we associate a set of initial conditions $\{x \in \mathbf{X} \mid (q, x) \in \mathrm{Init}\}$, a vector field $f(q, x)$ and an invariant $I(q)$. With each edge $e \in E$, we associate a guard $G(e)$ and a reset relation $R(e, x)$.

Definition 3 (Execution) *An execution χ of a hybrid automaton H is hybrid trajectory $\chi = (\tau, q, x) \in \mathrm{Hyb}(Q \cup X)$ satisfying*

- initial condition*: $\big(q(\tau_0), x(\tau_0)\big) \in \mathrm{Init}$;*
- continuous evolution*: for all i with $\tau_i < \tau_i'$, $q(\cdot)$ is constant, $x(\cdot)$ is a solution to the differential equation $dx/dt = f(q, x)$ over $[\tau_i, \tau_i']$[1], and for all $t \in [\tau_i, \tau_i')$, $x(t) \in I(q(t))$; and,*
- discrete evolution*: for all i, $\big(q(\tau_i'), q(\tau_{i+1})\big) = e \in E$, $x(\tau_i') \in G(e)$, and $x(\tau_{i+1}) \in R\big(e, x(\tau_i')\big)$.*

Unlike conventional continuous dynamical systems, the interpretation is that a hybrid automaton *accepts* (as opposed to *generates*) executions. An execution (τ, q, x) is called *finite*, if τ is a finite sequence ending with a closed interval, *infinite*, if τ is an infinite sequence, or if $\sum_{i=0}^{\infty}(\tau_i' - \tau_i) = \infty$, *Zeno*, if it is infinite but $\sum_{i=0}^{\infty}(\tau_i' - \tau_i) < \infty$, and *maximal*, if it is not a strict prefix of any other execution of H. For an infinite execution we define the Zeno time as $\tau_\infty = \sum_{i=0}^{\infty}(\tau_i' - \tau_i)$. Clearly, $\tau_\infty < \infty$ if and only if the execution is Zeno. We use $\mathcal{H}_{(q_0, x_0)}$ to denote the set of all executions of H with initial condition $(q_0, x_0) \in \mathrm{Init}$, $\mathcal{H}_{(q_0, x_0)}^M$ the set of all maximal executions, and $\mathcal{H}_{(q_0, x_0)}^\infty$ the set of all infinite executions. Since an infinite execution can not be a strict prefix of any other execution, $\mathcal{H}_{(q_0, x_0)}^\infty \subseteq \mathcal{H}_{(q_0, x_0)}^M$. Notice that $\mathcal{H}_{(q_0, x_0)}$ is *prefix closed*, that is if $\chi \in \mathcal{H}_{(q_0, x_0)}$ then $\chi' \in \mathcal{H}_{(q_0, x_0)}$ for all $\chi' \leq \chi$.

[1] The solution is considered in the sense of Caratheodory, and may be defined over over $[\tau_i, \tau_i')$, if τ ends in a right open interval and $i = N$.

To simplify the statement of subsequent results we introduce the following assumption.

Assumption 1 *f is globally Lipschitz in its second argument. $(q, q') \in E$ if and only if $G(q, q') \neq \emptyset$ and $x \in G(q, q')$ if and only if $R(q, q', x) \neq \emptyset$.*

The Lipschitz assumption is standard. The rest of Assumption 1 be made without loss of generality. It can be shown that for every hybrid automaton, H, there exists a hybrid automaton, $\widehat{H}$, with an identical set of executions, that satisfies the assumption.

3 Local Existence and Uniqueness

We provide conditions to characterize the following classes of automata.

Definition 4 (Non-Blocking and Deterministic) *A hybrid automaton H is called non-blocking if $\mathcal{H}_{(q_0, x_0)}^\infty$ is non-empty for all $(q_0, x_0) \in \mathrm{Init}$. It is called deterministic if $\mathcal{H}_{(q_0, x_0)}^M$ contains at most one element for all $(q_0, x_0) \in \mathrm{Init}$.*

Subsequent results involve the set of states that can be "reached" by H, and the set of states from which continuous evolution is impossible. A state $(\widehat{q}, \widehat{x}) \in \mathbf{Q} \times \mathbf{X}$ is called *reachable* by H if there exists a finite execution (τ, q, x) with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^N$ and $(q(\tau_N'), x(\tau_N')) = (\widehat{q}, \widehat{x})$. We use $\mathrm{Reach}(H) \subseteq \mathbf{Q} \times \mathbf{X}$ to denote the set of states reachable by H.

To characterize the set of states from which continuous evolution is impossible, consider $q \in \mathbf{Q}$ and, for some $\epsilon > 0$ small enough[2], the solution, $x(\cdot) : [0, \epsilon) \to \mathbf{X}$ to the differential equation

$$\frac{dx}{dt} = f(q, x) \quad \text{with} \quad x(0) = x^0. \tag{1}$$

[2] Such ϵ exists by the Lipschitz assumption on f in Definition 2.

Consider $\mathrm{Out} : \mathbf{Q} \rightarrow 2^{\mathbf{X}}$ defined by

$$\mathrm{Out}(q) = \left\{x^0 \in \mathbf{X} \mid \forall \epsilon > 0 \; \exists t \in [0, \epsilon), x(t) \notin I(q)\right\},$$

Note that $I(q)^c \subseteq \mathrm{Out}(q)$. If $I(q)$ is a closed set, $\mathrm{Out}(q)$ may also contain pieces of its boundary.

Lemma 1 *A hybrid automaton H is non-blocking if for all $(q, x) \in \mathrm{Reach}(H)$ with $x \in \mathrm{Out}(q)$, there exists $(q, q') \in E$ with $x \in G(q, q')$.*

Proof: Consider an initial state $(q_0, x_0) \in \mathrm{Init}$ and assume, for the sake of contradiction, that there does not exist an infinite execution starting at (q_0, x_0). Let $\chi = (\tau, q, x)$ denote a maximal execution starting at (q_0, x_0), and note that τ is a finite sequence.

First consider the case $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N')$. Let $(q_N, x_N) = \lim_{t \rightarrow \tau_N'}(q(t), x(t))$. Note that, by the definition of execution and a standard existence argument for continuous dynamical systems, the limit exists and χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N}$, $\widehat{q}(\tau_N') = q_N$, and $\widehat{x}(\tau_N') = x_N$. This contradicts the maximality of χ.

Now consider the case $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N}$, and let $(q_N, x_N) = (q(\tau_N'), x(\tau_N'))$. Clearly, $(q_N, x_N) \in \mathrm{Reach}(H)$. If $x_N \notin \mathrm{Out}(q_N)$, then there exists $\epsilon > 0$ such that χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N' + \epsilon)$ by continuous evolution. If, on the other hand, $x_N \in \mathrm{Out}(q_N)$, then, there exists $(q', x') \in \mathbf{Q} \times \mathbf{X}$ such that $(q_N, q') \in E$, $x_N \in G(q_N, q')$ and $x' \in R(q_N, q', x_N)$ (by Assumption 1). Therefore, χ can be extended to $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N+1}$, $\tau_{N+1} = \tau_{N+1}' = \tau_N'$, $q(\tau_{N+1}) = q'$, $x(\tau_{N+1}) = x'$ by a discrete transition. In both cases the maximality of χ is contradicted. ∎

The conditions of Lemma 1 are tight, in the sense that blocking automata that violate the conditions exist, but are not necessary, in the sense that not all automata that violate them are blocking. However:

Lemma 2 *A deterministic hybrid automaton is non-blocking if and only if the conditions of Lemma 1 are satisfied.*

Proof: Consider a deterministic hybrid automaton H that violates the conditions of Lemma 1, that is, there exists $(q', x') \in \mathrm{Reach}(H)$ such that $x' \in \mathrm{Out}(q')$, but there is no $\widehat{q}' \in \mathbf{Q}$ with $(q', \widehat{q}') \in E$, $x' \in G(q', \widehat{q}')$. Since $(q', x') \in \mathrm{Reach}(H)$, there exists $(q_0, x_0) \in \mathrm{Init}$ and a finite execution, $\chi = (\tau, q, x) \in \mathcal{H}_{(q_0, x_0)}$ such that $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N}$ and $(q', x') = (q(\tau_N'), x(\tau_N'))$.

We first show that χ is maximal. Assume first that there exists $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N + \epsilon)$ for some $\epsilon > 0$. This would violate the assumption that $x' \in \mathrm{Out}(q')$. Next assume that there exists $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ with $\widehat{\tau} = \tau[\tau_{N+1}, \tau_{N+1}']$ with $\tau_{N+1} = \tau_N'$. This requires that the execution can be extended beyond (q', x') by a discrete transition, that is there exists $(\widehat{q}', \widehat{x}') \in \mathbf{Q}$ such that $(q', \widehat{q}') \in E$, $x' \in G(q', \widehat{q}')$ and $\widehat{x}' \in R(q', \widehat{q}', x')$. This would also contradict our original assumptions.

Now assume, for the sake of contradiction that H is non-blocking. Then, there exists $\chi' \in \mathcal{H}_{(q_0, x_0)}^{\infty} \subseteq \mathcal{H}_{(q_0, x_0)}^{M}$. But $\chi \neq \chi'$ (as the former is finite and the latter infinite), therefore $\mathcal{H}_{(q_0, x_0)}^{M} \supseteq \{\chi, \chi'\}$. This contradicts the assumption that H is deterministic. ∎

Lemma 3 *A hybrid automaton is deterministic if and only if for all $(q, x) \in \mathrm{Reach}(H)$, $x \in \bigcup_{(q, q') \in E} G(q, q')$ implies $x \in \mathrm{Out}(q)$, $(q, q') \in E$ and $(q, q'') \in E$ with $q' \neq q''$ imply $x \notin G(q, q') \cap G(q, q'')$, and $(q, q') \in E$ and $x \in G(q, q')$ imply $|R(q, q', x)| \leq 1$.*

Proof: For the "if" part, assume, for the sake of contradiction, that there exists an initial state $(q_0, x_0) \in \mathrm{Init}$ and two maximal executions $\chi = (\tau, q, x)$ and $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ starting at (q_0, x_0) with $\chi \neq \widehat{\chi}$. Let $\psi = (\rho, p, y) \in \mathcal{H}_{(q_0, x_0)}$ denote the maximal common prefix of χ and $\widehat{\chi}$. Such a prefix exists as the executions start at the same initial state. Moreover, ψ is not infinite, as $\chi \neq \widehat{\chi}$. Therefore, as in the proof of Lemma 1, ρ can be assumed to be of the form $\rho = \{[\rho_i, \rho_i']\}_{i=0}^{N}$, as otherwise the maximality of ψ would be contradicted by an existence and uniqueness argument of the continuous solution along f. Let $(q_N, x_N) = (q(\rho_N'), x(\rho_N')) = (\widehat{q}(\rho_N'), \widehat{x}(\rho_N'))$. Clearly, $(q_N, x_N) \in \mathrm{Reach}(H)$. We distinguish the following cases:

Case 1: $\rho_N' \notin \{\tau_i'\}$ and $\rho_N' \notin \{\widehat{\tau}_i'\}$, i.e., ρ_N' is not a time when a discrete transition takes place in either χ or $\widehat{\chi}$. Then, by a standard existence and uniqueness argument for continuous systems, there exists $\epsilon > 0$ such that the prefixes of χ and $\widehat{\chi}$ are defined over $\widehat{\rho} = \{[\rho_i, \rho_i']\}_{i=0}^{N-1}[\rho_N, \rho_N' + \epsilon)$ and are identical. This contradicts the maximality of ψ.

Case 2: $\rho_N' \in \{\tau_i'\}$ and $\rho_N' \notin \{\widehat{\tau}_i'\}$, i.e., ρ_N' is a time when a discrete transition takes place in χ but not in $\widehat{\chi}$. The fact that a discrete transition takes place from (q_N, x_N) in χ indicates that there exists $q' \in \mathbf{Q}$ such that $(q_N, q') \in E$ and $x_N \in G(q_N, q')$. The fact that no discrete transition takes place from (q_N, x_N) in $\widehat{\chi}$ indicates that there exists $\epsilon > 0$ such that $\widehat{\chi}$ is defined over $\widehat{\rho} = \{[\rho_i, \rho_i']\}_{i=0}^{N-1}[\rho_N, \rho_N' + \epsilon)$, therefore $x_N \notin \mathrm{Out}(q)$. This contradicts the first lemma condition.

Case 3: $\rho_N' \notin \{\tau_i'\}$ and $\rho_N' \in \{\widehat{\tau}_i'\}$, symmetric to 2.

Case 4: $\rho_N' \in \{\tau_i'\}$ and $\rho_N' \in \{\widehat{\tau}_i'\}$, i.e., ρ_N' is a time when a discrete transition takes place in both χ and $\widehat{\chi}$. The fact that a discrete transition takes place from

(q_N, x_N) in both χ and $\widehat{\chi}$ indicates that there exist (q', x') and $(\widehat{q}', \widehat{x}')$ such that $(q_N, q') \in E$, $(q_N, \widehat{q}') \in E$, $x_N \in G(q_N, q')$, $x_N \in G(q_N, \widehat{q}')$, $x' \in R(q_N, q', x_N)$, and $\widehat{x}' \in R(q_N, \widehat{q}', x_N)$. Note that by condition 2 of the lemma, $q' = \widehat{q}'$, hence, by condition 3, $x' = \widehat{x}'$. Therefore, the prefixes of χ and $\widehat{\chi}$ are defined over $\widehat{\rho} = \{[\rho_i, \rho'_i]\}_{i=0}^N [\rho_{N+1}, \rho'_{N+1}]$, with $\rho_{N+1} = \rho'_{N+1} = \rho'_N$, and are identical. This contradicts the maximality of ψ and concludes the proof of the "if" part.

For the "only if" part, assume that there exists $(q', x') \in \text{Reach}(H)$ such that at least one of the conditions of the lemma is violated. Since $(q', x') \in \text{Reach}(H)$, there exists $(q_0, x_0) \in \text{Init}$ and a finite execution, $\chi = (\tau, q, x) \in \mathcal{H}_{(q_0,x_0)}$ such that $\tau = \{[\tau_i, \tau'_i]\}_{i=0}^N$ and $(q', x') = (q(\tau'_N), x(\tau'_N))$. If condition 1 is violated, then there exists $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \{[\tau_i, \tau'_i]\}_{i=0}^{N-1}[\tau_N, \tau_N + \epsilon)$, $\epsilon > 0$, and $\tilde{\tau} = \tau[\tau_{N+1}, \tau'_{N+1}]$, $\tau_{N+1} = \tau'_N$, such that $\chi < \widehat{\chi}$ and $\chi < \tilde{\chi}$. If condition 2 is violated, there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1}, \tau_{N+1}]$, $\tau_{N+1} = \tau'_N$, and $\widehat{q}(\tau_{N+1}) \neq \tilde{q}(\tau_{N+1})$, such that $\chi < \widehat{\chi}$, $\chi < \tilde{\chi}$. Finally, if condition 3 is violated, then there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1}, \tau'_{N+1}]$, $\tau_{N+1} = \tau'_N$, and $\widehat{x}(\tau_{N+1}) \neq \tilde{x}(\tau_{N+1})$, such that $\chi < \widehat{\chi}$, $\chi < \tilde{\chi}$. In all three cases, let $\overline{\widehat{\chi}} \in \mathcal{H}^M_{(q_0,x_0)}$ and $\overline{\tilde{\chi}} \in \mathcal{H}^M_{(q_0,x_0)}$ denote maximal executions of which $\widehat{\chi}$ and $\tilde{\chi}$ are prefixes respectively. Since $\widehat{\chi} \neq \tilde{\chi}$, it follows that $\overline{\widehat{\chi}} \neq \overline{\tilde{\chi}}$. Therefore $|\mathcal{H}^M_{(q_0,x_0)}| \geq 2$ and thus H is non-deterministic. ∎

Theorem 1 (Existence and Uniqueness) *If a hybrid automaton satisfies the conditions of Lemmas 1 and 3, then it accepts a unique infinite execution for all* $(q_0, x_0) \in \text{Init}$.

Proof: If the hybrid automaton satisfies Lemma 1, then $|\mathcal{H}^\infty_{(q_0,x_0)}| \geq 1$, for all $(q_0, x_0) \in \text{Init}$. If it satisfies Lemma 3, then $|\mathcal{H}^M_{(q_0,x_0)}| \leq 1$, for all $(q_0, x_0) \in \text{Init}$. But $\mathcal{H}^\infty_{(q_0,x_0)} \subseteq \mathcal{H}^M_{(q_0,x_0)}$, therefore, $1 \leq |\mathcal{H}^\infty_{(q_0,x_0)}| \leq |\mathcal{H}^M_{(q_0,x_0)}| \leq 1$ and $|\mathcal{H}^\infty_{(q_0,x_0)}| = |\mathcal{H}^M_{(q_0,x_0)}| = 1$. ∎

The conditions of the theorem are sufficient and tight. They require one to compute the set $\text{Out}(q)$ for $q \in \mathbf{Q}$. We list some cases for which this computation is straightforward; in general one may have to make use of more powerful tools from viability theory [8].

Proposition 1 (Open Invariant) *If $I(q)$ is open, then* $\text{Out}(q) = I(q)^c$.

Proof: Recall that $I(q)^c \subseteq \text{Out}(q)$. Consider $x^0 \in I(q)$. By the continuity of the solution of (1) with respect to time and the fact that $I(q)$ is open, there exists $\epsilon > 0$ such that for all $t \in [0, \epsilon)$, $x(t) \in I(q)$. Therefore, $I(q) \subseteq \text{Out}(q)^c$. ∎

Next, assume that $I(q) = \{x \in \mathbf{X} \mid \sigma(q, x) \geq 0\}$ for some $\sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$. Assuming f and σ are sufficiently differentiable in x, we inductively define the *Lie derivatives* of σ along f, $L_f^m \sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$, $m \in \mathbb{N}$ by $L_f^0 \sigma(q, x) = \sigma(q, x)$ and for $m > 0$

$$L_f^m \sigma(q, x) = \left(\frac{\partial}{\partial x} L_f^{m-1} \sigma(q, x) \right) f(q, x).$$

The *pointwise relative degree* of σ with respect to f is defined as the function $n_{(\sigma,f)} : \mathbf{Q} \times \mathbf{X} \to \mathbb{N}$ given by

$$n_{(\sigma,f)}(q, x) = \min \left\{ m \in \mathbb{N} \mid L_f^m \sigma(q, x) \neq 0 \right\}.$$

Note that $n_{(\sigma,f)}(q, x) = 0$ if and only if $\sigma(q, x) \neq 0$.

Proposition 2 (Analytic Invariant) *If f and σ are analytic in x,* $\text{Out}(q) = \{x \in \mathbf{X} \mid L_f^{n_{(\sigma,f)}(q,x)} \sigma(q, x) < 0\}$.

Proof: Since f is analytic in x, the solution $x(t)$ of (1) is analytic as a function of t. Since σ is analytic in x, $\sigma(q, x(t))$ is also analytic as a function of t. Consider the Taylor expansion of $\sigma(q, x(t))$ about $t = 0$. By analyticity,

$$\sigma(q, x(t)) = \sigma(q, x^0) + L_f \sigma(q, x^0) t + L_f^2 \sigma(q, x^0) \frac{t^2}{2!} + \ldots$$

converges locally. If $n_{(\sigma,f)}(q, x^0) < \infty$, the first nonzero term in the expansion, $L_f^{n_{(\sigma,f)}(x)} \sigma(q, x^0)$, dominates the sum for t small enough. Therefore, if $L_f^{n_{(\sigma,f)}(q,x)} \sigma(q, x^0) < 0$, for all $\epsilon > 0$ there exists $t \in [0, \epsilon)$ such that $\sigma(q, x(t)) < 0$. Hence, $\{x \in \mathbf{X} \mid L_f^{n_{(\sigma,f)}(q,x)} \sigma(q, x) < 0\} \subseteq \text{Out}(q)$. If, on the other hand, $L_f^{n_{(\sigma,f)}(q,x)} \sigma(q, x^0) > 0$, then there exists $\epsilon > 0$ such that for all $t \in [0, \epsilon)$, $\sigma(q, x(t)) > 0$, and $x^0 \notin \text{Out}(q)$. Finally, if $n_{(\sigma,f)}(Q, x^0) = \infty$, the Taylor expansion is identically equal to zero and, locally, $\sigma(q, x(t)) = 0$ and $x(t) \in I(q)$. Therefore, $\text{Out}(q) \subseteq \{x \in \mathbf{X} \mid L_f^{n_{(\sigma,f)}(q,x)} \sigma(q, x) < 0\}$. ∎

The analyticity requirement can be relaxed somewhat, since the only part where it is crucial is when $n_{(\sigma,f)}(q, x^0) = \infty$. $n_{(\sigma,f)}(q, x) < \infty$ indicates that the vector field f is in a sense "transverse" to the boundary of the invariant set.

Proposition 3 (Transverse Invariant) *If σ and f are $m \geq 1$ times differentiable in x and $n_{(\sigma,f)}(q, x) < m$ for all $(q, x) \in \mathbf{Q} \times \mathbf{X}$, then* $\text{Out}(q) = \{x \in \mathbf{X} \mid L_f^{n_{(\sigma,f)}(q,x)} \sigma(q, x) < 0\}$.

The conditions of Theorem 1 also involve $\text{Reach}(H)$, but do not necessarily require one to compute it explicitly; it suffices to show that the conditions hold in a set of states that contains $\text{Reach}(H)$. A set $S \subseteq \mathbf{Q} \times \mathbf{X}$ is called *invariant* if $\text{Reach}(H) \subseteq S$. Trivially $\mathbf{Q} \times \mathbf{X}$ is invariant; more interesting sets can be shown to be invariant by deductive arguments. For example, assume that $S = \{(q,x) \in \mathbf{Q} \times \mathbf{X} \mid s(q,x) \geq 0\}$, for some $s : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$, and that s and f are analytic in x. Let $\mathrm{Out}_S(q) = \{x \in \mathbf{X} \mid L_f^{n_{(\sigma,f)}(q,x)} \sigma(q,x) < 0\}$.

Proposition 4 *S is invariant if* Init $\subseteq S$, *and for all* $(q,x) \in S \cap \mathrm{Reach}(H)$, $x \in G(q,q')$ *implies* $\{q'\} \times R((q,q'),x) \subseteq S$ *and* $x \in \mathrm{Out}_S(q)$ *implies* $x \in \mathrm{Out}(q)$.

Proof: Consider an arbitrary execution $\chi = (\tau, q, x) \in \mathcal{H}_{(q_0,x_0)}$ and show that for all $t \in \tau$, $(q(t), x(t)) \in S$. By the first condition of the proposition, Init $\subseteq S$, therefore $(q_0, x_0) \in S$. Consider the finite prefix of χ defined over $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{M-1}[\tau_M, r]$ and assume $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}$.

First consider the case $r < \tau_M'$. Assume for the sake of contradiction that there exists $r' \in (r, \tau_M']$ such that $(q(r'), x(r')) \notin S$. Then, by continuity of the execution along continuous evolution and by the assumption that S is closed, there must exist $r'' \in (r, r')$ such that $x(r'') \in \mathrm{Out}_S(q(r''))$ (i.e. $s(q(r''), x(r'')) = 0$ and the vector field pointing "outside" S). Since $(q(r''), x(r''))$ is reachable (in particular, by the finite prefix of χ defined over $\{[\tau_i, \tau_i']\}_{i=0}^{M-1}[\tau_M, r'']$), by the third condition of the proposition, $x(r'') \in \mathrm{Out}(q(r''))$, which contradicts the assumption that $r'' < \tau_M'$. Therefore, if $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}$, then $(q(t), x(t)) \in S$ for all $t \in \{[\tau_i, \tau_i']\}_{i=0}^{M}$.

Finally, consider the case $r = \tau_M'$. Clearly, $(q(r), x(r))$ is reachable and $x(r) \in G(q(r), q(\tau_{M+1}))$. Therefore, by the second condition of the proposition, $(q(\tau_{M+1}), x(\tau_{M+1})) \in S$. Therefore, if $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}$, then $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}[\tau_{M+1}, \tau_{M+1}]$. The claim follows by induction. ■

Notice that Reach(H) again appears in the statement of Proposition 4. This allows us to build chains of invariant sets, $S_0 \supset S_1 \supset S_2 \supset \ldots$, starting with $S_0 = \mathbf{Q} \times \mathbf{X}$ and using the fact that S_i is invariant to prove that S_{i+1} is invariant.

4 Zeno Hybrid Automata

It remains to investigate whether executions can be extended over arbitrary time horizons. The Lipschitz assumption on the vector field f excludes the possibility of escape in finite time along continuous evolution. There is still, however, the possibility of Zeno executions. We illustrate the Zeno property through a number of examples. These are further analyzed in [6].

Example 1: (Non-Analytic System) Consider the hybrid automaton of Figure 1. The (unique) execution with initial state $(q_1, -1)$ exhibits an infinite number

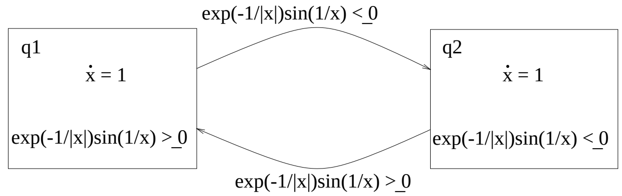

Figure 1: Non-analytic system

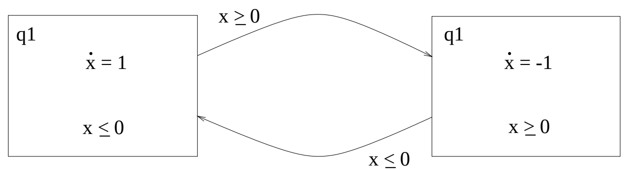

Figure 2: Chattering system

of discrete transitions by $\tau_\infty = \tau_0 + 1$. The reason is that the (non-analytic) function $e^{-1/|x|}\sin(1/x)$ has an infinite number of zeros in the finite interval $(-1, 0)$.

Example 2: (Chattering System) Consider the hybrid automaton of Figure 2. It is easy to show that the (unique) execution starting in x_0 at reaches $x = 0$ in finite time $\tau_\infty = \tau_0 + |x_0|$ and takes an infinite number of transitions from then on, without any time progress.

Example 3: (Water Tank System) Consider the water tank system of [9] (Figure 3). x_i denotes the volume of water in Tank i, and $v_i > 0$ denotes the (constant) flow of water out of Tank i. Let w denote the constant flow of water into the system, dedicated exclusively to either Tank 1 or Tank 2 at each point in time. The goal is to keep the water volumes above r_1 and r_2, respectively, assuming that $x_1(0) > r_1$ and $x_2(0) > r_2$. This is to be achieved by a switching strategy that turns the inflow to Tank 1 whenever $x_1 \leq r_1$ and to Tank 2 whenever $x_2 \leq r_2$.

Using Theorem 1, one can show that the water tank automaton accepts a unique infinite execution for each initial state. One can also show that if $\max\{v_1, v_2\} < w < v_1 + v_2$, then the execution is Zeno with Zeno time

$$\tau_\infty = \tau_0 + \frac{x_1(\tau_0) + x_2(\tau_0) - r_1 - r_2}{v_1 + v_2 - w}.$$

Example 4: (Bouncing Ball System) The bouncing ball automaton (Figure 4) is a simplified model of an elastic ball that is bouncing on a level surface, loosing a fraction of its energy with each bounce. Let x_1 denote the altitude of the ball and x_2 its vertical speed. One can show that the bouncing ball automaton accepts a unique infinite, Zeno execution for each initial

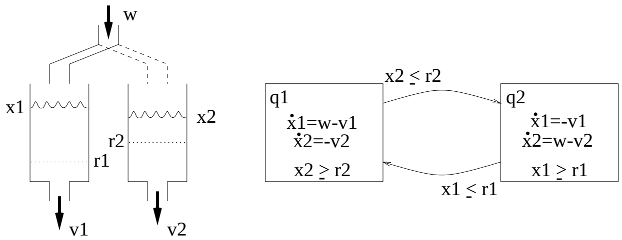

Figure 3: Water tank system

state, with Zeno time

$$\tau_\infty = \tau_0 + \frac{x_2(\tau_0) + \sqrt{x_2(\tau_0)^2 + 2gx_1(\tau_0)}}{g} + \frac{2x_2(\tau_0)}{g(1-c)}.$$

The four examples shed light on different aspects of the Zeno phenomenon. The first, shows how things can go wrong even with simple systems involving only infinitely differentiable functions. The second, is an instance of a differential equations with discontinuous right hand side (see [1] for a thorough treatment). In this example, an infinite number of transitions takes place at τ_∞, while there exists an interval $(\tau_\infty - \epsilon, \tau_\infty)$ with $\epsilon > 0$ that contains no discrete transitions; in the remaining examples, there are infinitely many transitions on any interval $(\tau_\infty - \epsilon, \tau_\infty)$. In the water tank example, x is continuous and converges as t tends to τ_∞. Finally, in the bouncing ball example the x is discontinuous (due to the non-trivial reset relation associated with the bounce), but still converges as t tends to τ_∞.

Regularization was proposed in [6] as a way of extending a Zeno execution beyond the Zeno time. It was, however, shown that in some cases different regularizations may lead to different extensions. This may be an undesirable property (especially from a simulation point of view) since it suggests that the model does not contain sufficient information to yield a unique solution.

5 Conclusions

Conditions were derived for local existence and uniqueness of executions for hybrid automata; the structure of the invariant in each discrete state was found to play a crucial role. Using examples, it was also shown that the Zeno phenomenon may in some cases prevent extension of the execution over infinite time horizons. Motivated by simulation problems for hybrid automata [10], we are currently investigating properties of Zeno executions, in an attempt to develop formal methods for extending them beyond the Zeno time.

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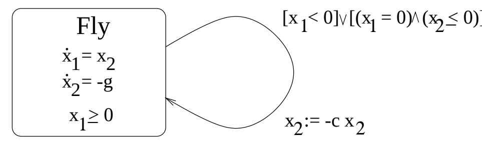

Figure 4: Bouncing ball system

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