Optimal stopping for updating controls

*** An earlier version appears in the proceedings of the IWSM 2009 [18].**

Maben Rabi and Karl Henrik Johansson

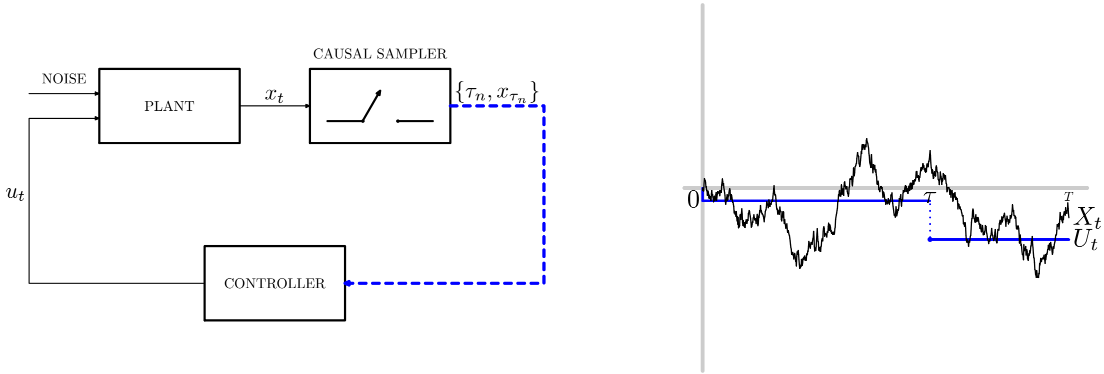
1. Control under limited communications

We study a problem of sampled-data control for an Itô diffusion process. In particular, we allow the sampling of the state process to be *event-triggered* and place a bound on the sampling rate. Such limits arise in *Networked control systems* where, the sensors, the controller and the actuators may be connected to each other over data networks. Although these networked control loops suffer from degraded communication, they offer several cost savings and the advantages of fast installation, easy maintenance, deployment in mobile environments etc. (see [**1**] for an overview).

In practical situations such as in wireless industrial control, frequently, it is the link from sensors to controllers that have restricted rates. Consider such a loop consisting of three components: the plant, a sensor and a controller (see fig. 1(a)). The link from the sensor to the controller is over a data network and so, its communication rate is limited. The other links in the loop have no such limits and can be considered ideal. Assume that the sensor obtains perfect and continuous measurements of the state process. Because of the rate limit, continuous transmission of the sensor observations to the controller is impossible; only a sampled version of the observations can be forwarded to the controller. The control process is hence generated with some compressed information about the state process. It is generated based on the timed sequence of samples received, and, is forced to be piecewise deterministic. The traditional solution is to sample sensor observations periodically choosing a sampling rate equalling the average communication rate permitted by the channel. Here, we pursue the more efficient strategy of letting the sensor choose sampling times to be stopping times w.r.t. its observations. The rate of arrival of the stopping times then should be no greater than the rate permitted by the channel.

This pursuit boils down to designing two elements namely, an *Event detector* at the sensor end which applies a set of stopping rules to generate samples, and, a *Control generator* inside the controller node which produces piece-wise deterministic waveforms derived from the sequence of received samples. In mathematical terms, we get a sequential decision problem for a two person team where, the control signal, and, sampling times are decision variables. Moreover, this decision problem has a non-standard information pattern with the control values and the stopping times being chosen based on different information.

1.1. Updating control waveforms at stopping times. We study the finite horizon version of the problem. The communication constraint is a hard limit on the number of times inside the finite horizon when the control waveform is modified based upon samples from the sensor. On the horizon $[0, T]$, consider the controlled scalar Diffusion



(a) Control with limited samples (b) Zero order hold Control

FIGURE 1. Subfig.(a) is the block diagram for control based on limited samples. The dashed blue link represents a channel with a restricted communication rate. Subfig.(b) shows an important subclass of allowed controls namely, the Zero order hold with piece-wise constant waveforms which jump only at sample times.

process x_t which obeys the SDE:

$$dx_t = \mu(x_t)\,dt + \sigma(x_t)\,dB_t + \kappa(x_t)\,u_t dt, \tag{1}$$

where, B_t is a standard scalar Brownian motion process, the functions μ, σ, and, κ satisfy the usual Lipschitz and regularity conditions ensuring existence and uniqueness of the state process x_t. The control signal is as usual required to be measurable w.r.t. the filtration $\mathcal{F}_t^x$. Because the control is actually generated based on compressed information regarding the state, it has a further restriction in the manner of its generation. This is explained subsequently.

Based on its observations, the sensor chooses up to N sampling times $\{\tau_1, \tau_2, \dots \tau_N\}$. The controller observes its own output continuously and the discrete sequence of sampling times and sample values passed on from the sensor. Based on its observations, the controller updates the control waveform at the sampling times generated by the sensor. Thus an admissible control process must be measurable w.r.t. the filtration generated by the output of the sensor.

The goal of the control system designer is to keep the state as close to the origin as possible while using no more than N samples. We adopt the standard squared error distortion as the control performance measure. Hence, given the initial value x_0, the objective is to pick the sequence of N stopping times $\{\tau\}_{i=1}^N$ and the control policy $\mathcal{U}$ so that the following aggregate control error is minimized:

$$J_N\left(x_0, \mathcal{U}, \{\tau\}_{i=1}^N\right) = \mathbb{E}\left[\int_0^T x_s^2 ds \,\middle|\, x_0\right]. \tag{2}$$

1.2. Relation to previous works. The problem of choosing the time instants to sample sensor measurements has received early attention in the literature. Kushner [**9**] studied the deterministic choice of measurement times in a discrete-time, finite-horizon LQG optimal control problem. He found that controls can be chosen like in a conventional LQG control problem with fixed deterministic sampling instants. Using this fact, he shows that the best (deterministic) sampling schedule can be found by solving a nonlinear optimization problem. Skafidas & Nerode [**22**] also adopt the same setting but allow the sensor measurement times to be chosen online by the controller. Their conclusion is that the optimal choice of measurement times can be made offline. Their simplified scheduling problem is the same as Kushner's deterministic one.

The *Sensor scheduling* problem is a generalization of the problem of choosing measurement times which has been studied for estimation, detection and control tasks [**13, 12, 3, 26**]. The problem asks for online schedules of when to gather measurements from different sensors available; in some setups, at most one sensor can be scheduled at a time. However, the information pattern for this problem is the same as the the works of Kushner and of Skafidas & Nerode. The notable fact about this information pattern is that the flow of data from sensors to their recipients namely, controllers estimators etc., is regulated by the recipient. Such sensor sampling is of the "pull" type. an alternative is the "push" type of sampling where sensor itself regulates the flow of its data. When only one sensor is available, it has more information than the controller and hence, its conclusions on when to communicate its measurements can potentially improve performance.

Åström & Bernhardsson [**2**] treat a minimum variance control problem with the push type of sampling. The control consists of impulses which reset the state to the origin, but there is an upper limit on the rate at which impulses can be applied. Under such a constraint the design asks for a schedule of the application times for the impulses. For scalar Gaussian diffusions, they perform explicit calculations to show that the application of impulses triggered by fixed levels is more efficient than periodic application. In the setting of discrete time, Imer & Basar [**7**] study the problem of efficiently using a limited number of discrete-time impulses. For a finite-horizon LQG optimal control problem, they use backward dynamic programming to show that time-varying thresholds are optimal.

For the regular control problem for which the control waveform is piecewise continuous and is of bounded energy and magnitudes, the work of Borkar & Varaiya [**5**] provides a suitable starting point. They have studied *Finitely switched systems* which are controlled Itô processes that are time-discretized using the instants the process freshly crosses a prescribed set of levels. This framework has been adopted by [**16, 17**] for studying *Level-triggered control* where, the control signal is updated at the times of crossings of fixed levels.

A deterministic analysis of level-triggered control is carried out in the works [**6, 23, 25**] using techniques from Hybrid systems theory. An interesting choice of time-varying levels is made in [**24**] again for a deterministic system. The drawback with these approaches is that they cannot prescribe the optimal or near-optimal design of events. They do help us analyze a given set of event-based rules. A treatment of the Level-triggered control of linear diffusions using Rice's formula [**21, 20**] is presented in [**11**].

For the finite-horizon optimal control problems, it is quite easy to see that using fixed levels for triggering samples is not optimal. In our paper, we give a procedure for the optimal design of events which turn out to be fresh-crossings of time-varying boundaries. The results presented here extend earlier work [**19**] by the authors.

1.3. Contributions and outline of the paper. We focus on the single sampling problem where the sample budget N is exactly one. The multiple sampling problem is not much harder and it can be solved by recursively solving N problems of the single sampling type. We further specialize to the case of the controller which generates piecewise constant control waveforms, the so called *Zero order hold* waveforms (see Fig. 1(b)).

In section 2, we write down the person by person optimality conditions for the pair of control and sampling policies. We also derive a simpler expression for the cost which enables us to see that the optimal pair is such that the stopping rule for sampling is the first hitting time of a time-varying, double-sided barrier.

In section 3, we furnish an iterative algorithm for finding the optimal pair of policies. We also establish the convergence of the algorithm and the uniqueness of the resulting reduced cost. Section 3 applies this algorithm for the case of a Brownian Motion process where the convergence is found to be quite fast. Finally, in the last section, we conclude with some remarks about future work.

2. Properties of the optimal choice of stopping times and control updates

For the single stopping problem, the control waveform has the form:

$$u_t = \begin{cases} U_0\left(x_0, T\right) & \text{if } \ 0 \leq t < \tau, \\ U_1\left(x_\tau, \tau, T\right) & \text{if } \ \tau \leq t \leq T, \end{cases}$$

where, the switch time τ is a stopping time w.r.t. the x-process, one which is restricted to fall inside the interval $[0, T]$. Thus the set of decision variables is the triple: (U_0, U_1, τ). Because of the non-classical information pattern, we cannot use the existing results on combined stopping and control [**14, 8**]. We can however describe some properties of the optimal choice of policies.

We will first decompose the aggregate quadratic cost. Let $\Phi_U(t_2, t_1, x)$ denote the solution to the SDE 1 at time t_2 withe initial condition at time $t_1 \leq t_2$ being x, and, with the constant control input U over the interval $[t_1, t_2)$. Then,

$$\begin{aligned} J_N\left(x_0, \{U_0, U_1\}, \tau\right) &= \mathbb{E}\left[\int_0^\tau x_s^2 ds + \int_\tau^T x_s^2 ds\right] \ = \ \mathbb{E}\left[\int_0^\tau \Phi_{U_0}^2(s, 0, x_0)\, ds + \int_\tau^T \Phi_{U_1}^2(s, \tau, x_\tau)\, ds\right], \\ &= \mathbb{E}\left[\int_0^T \Phi_{U_0}^2(s, 0, x_0)\, ds - \int_\tau^T \Phi_{U_0}^2(s, 0, x_0)\, ds + \int_\tau^T \Phi_{U_1}^2(s, \tau, x_\tau)\, ds\right], \\ &= \int_0^T \mathbb{E}\left[\Phi_{U_0}^2(s, 0, x_0)\right] ds - \mathbb{E}\left[\int_\tau^T \left\{\Phi_{U_0}^2(s, 0, x_0) - \Phi_{U_1}^2(s, \tau, x_\tau)\right\} ds\right], \\ &= \int_0^T p_{U_0}(s, x_0)\, ds - \mathbb{E}\left[\int_\tau^T \left\{\Phi_{U_0}^2(s, 0, x_0) - \Phi_{U_1}^2(s, \tau, x_\tau)\right\} ds\right], \end{aligned}$$

where, $p_{U_0}(s, x_0)$ is the second moment of the state at time t, under the constant control value U_0. Denote the first intergral by $\alpha\left(x_0, U_0, T\right)$. Now, using iterated expectations, we get:

$$\begin{aligned} J_N\left(x_0, \{U_0, U_1\}, \tau\right) &= \alpha\left(x_0, U_0, T\right) - \mathbb{E}\left[\mathbb{E}\left[\int_\tau^T \left\{\Phi_{U_0}^2(s, 0, x_0) - \Phi_{U_1}^2(s, \tau, x_\tau)\right\} ds \,\middle|\, \tau, x_\tau\right]\right], \\ &= \alpha\left(x_0, U_0, T\right) - \mathbb{E}\left[\int_\tau^T \mathbb{E}\left[\Phi_{U_0}^2(s, 0, x_0) - \Phi_{U_1}^2(s, \tau, x_\tau) \,\middle|\, \tau, x_\tau\right] ds\right], \\ &= \alpha\left(x_0, U_0, T\right) - \mathbb{E}\left[\int_\tau^T \mathbb{E}\left[\Phi_{U_0}^2(s, \tau, x_\tau) - \Phi_{U_1}^2(s, \tau, x_\tau)\right] ds\right], \end{aligned}$$

where, for a given, U_0 and τ, the integral $\int_\tau^T \mathbb{E}\left[\Phi_{U_0}^2(s, \tau, x_\tau) - \Phi_{U_1}^2(s, \tau, x_\tau)\right] ds$ can be minimized by a straightforward choice of U_1 via a standard optimal control problem for the integral cost. Let $U_1^*\left(x_\tau, \tau, T\right)$ denote an optimal choice obtained this way. Then the aggregate control distortion can be described as:

$$J_N\left(x_0, \{U_0, U_1\}, \tau\right) \stackrel{\Delta}{=} \alpha\left(x_0, T\right) - \mathbb{E}\left[\beta\left(x_0, U_0, \tau, T\right)\right], \tag{3}$$

where the function: $\beta\left(x_0, U_0, \tau, T\right) = \int_\tau^T \mathbb{E}\left[\Phi^2_{U_0}(s,\tau,x_\tau) - \Phi^2_{U_1^*(x_\tau,\tau,T)}(s,\tau,x_\tau)\right]$. Notice that we can first minimize $\mathbb{E}\left[\beta\right]$ over all possible τ and use it to find the optimal choice of U_0. Minimizing the expected value of the function β is an optimal stopping problem with a reward collected at the stopping time. Although the reward is time-varying, the problem in is standard form and can be converted into a time-homogeneous problem by adding time as a state variable. Its solution is the first time the state hits time-varying and double sided barriers, or, if the current time exceeds the length of the horizon [**15**].

2.1. Person-by-person optimality. A standard result from Team theory states that the optimal set of action policies should satisfy an equilibrium condition. Essentially, any optimal set of policies $\mathcal{SP}$ is such that, each person/agent employs a policy which is the best possible as long as the others employ the fixed policies as per $\mathcal{SP}$. This provides us with the following necessary conditions for optimality:

$$\begin{cases} \tau^*\left(x_0\right) & = \operatorname*{ess\,sup}_\tau \;\; \mathbb{E}\left[\beta\left(x_0, U_0^*\left(x_0\right), \tau, T\right)\right], \\ U_0^*\left(x_0\right) & = \inf_U \left\{ \alpha\left(x_0, U, T\right) - \mathbb{E}\left[\beta\left(x_0, U, \tau^*\left(x_0\right), T\right)\right] \right\}. \end{cases} \tag{4}$$

3. An iterative search procedure

The procedure we provide is inspired by the necessary conditions for optimality described earlier (eqn. 4). In this scheme, we find the optimal combination of policies for sampling and control through a possibly infinite sequence of policy iteration steps. In each round of the iteration, we execute two steps. In step one, we seek the best sampling policy for the previously derived control policy. This is an optimal stopping problem in standard form. In the second step, we utilize the newly computed sampling policy and seek the best control update policy that goes with it. The second step is thus an optimal control problem. This solution scheme leads to the optimal policies because of the team nature of the problem and because the system being controlled is linear. However, each round of the solution scheme is computationally intensive.

Iterative Search Algorithm	
Step 0: Initialize the algorithm	Set the counter $k := 0$. To initialize the policies, any admissible pair of initial control ν, and the stopping time θ would do. Use: $$\nu_0 := -Tx_0, \quad \theta_0 := \frac{T}{2}$$ Compute the cost: $$\eta_0 := \alpha\left(x_0, -Tx_0, T\right) - \beta\left(x_0, -Tx_0, \tfrac{T}{2}, T\right).$$ We expect improvements in the cost and need a threshold on the improvement of the cost in each round. For that purpose, pick a small non-negative number ϵ. If the improvement in the cost after an iteration is less than this threshold, the algorithm terminates.
Step 1: Improve the stopping rule	The improved stopping time is given by: $$\theta_{k+1} := \operatorname*{ess\,sup}_{\tau} \; \mathbb{E}\left[\beta\left(x_0, \nu_k, \tau, T\right)\right].$$
Step 2: Improve the initial control policy	The improved control level is given by: $$\nu_{k+1} := \inf_{U}\left\{\alpha\left(x_0, U, T\right) - \mathbb{E}\Big[\beta\left(x_0, U, \theta_{k+1}, T\right)\Big]\right\}.$$ And, the new lowered cost is: $$\eta_{k+1} := \alpha\left(x_0, \nu_{k+1}, T\right) - \mathbb{E}\Big[\beta\left(x_0, \nu_{k+1}, \theta_{k+1}, T\right)\Big].$$
Step 3: Decide on termination	**If** $\eta_{k+1} - \eta_k < \epsilon$, **then,** STOP. **Else,** Increase the counter: $k := k + 1$, and, **Go To** Step 1.

Notice that in each execution of steps 1, 2, the cost may or may not be lowered but never increased. In practical implementation, the order of execution of the steps has no influence on the outcome of the algorithm. The order of steps we have given however, allows an interpretation of the procedure as policy iteration.

3.1. Equivalence to policy iteration.

THEOREM 1 (Global convergence of iterative search). *The iterative search procedure of equations converges for any feasible pair of initial policies and the limit of the policy iterates is unique.*

PROOF. Convergence of the iterations is easy to establish because at each step, the cost can only decrease or stay the same. Since the cost is lower bounded by zero, the sequence of iterates is necessarily convergent.

We discretize time and establish convergence and the uniqueness of the limit. We attack the continuous time problem using a continuity argument by making the time-discretizations infinitesimally fine. We use two key properties of the optimization problem to show that the iterates converge . The first fact is that the optimal stopping problem of step 1 of our algorithm has as its solution, the first hitting time of barriers subject to a time-out namely, the length of the horizon. This means that the search for a stopping time is the search for double-sided time-varying barriers. The latter search can be viewed as a problem of control up to an absorption time [**10, 4**] also called a stochastic shortest path problem. Such a conversion is in general not possible for stopping time problems.

But here, we use the finiteness of the horizon and the threshold crossing nature of the optimal stopping time to enable this conversion.

Consider the Markov process $\xi_n = \left(x_{[n]}, u_{[n]}\right)$. We stop this process when it is absorbed by a double sided barrier (of the sort that produce the optimal stopping time). We mandate that the upper and lower barriers are both equal to zero at the end time of the horizon T. This makes sure that all admissible trajectories are stopped at least by T.

We now specify the control sets. At time zero, $u_{[0]}$ can be chosen to be any real number. We also choose two levels (barriers Upp_n, Low_n with $Upp_n \geq Low_n$) to be applied to the state at the next discrete time step. This is a device which allows us to "control" the probability of getting absorbed at the next time tick. At all other times, the control set is such that $u_{[n]}$ cannot be influenced. But we are allowed to choose two real-valued levels to determine absorption at the next time-tick. In the penultimate and ultimate time-ticks, the control set is really the Null set. The dynamics of the discrete process ξ_k inherit the dynamics of the x and u processes. Notice that in our setup, u can be chosen arbitrarily at the zeroth time-tick but not influenced at all subsequently.

Why is this characterization useful to us ? Because, it casts the mapping: $(\nu_k, \{Upp_n(k), Low_n(k)\}) \to (\nu_{k+1}, \{Upp_n(k), Low_n(k)\})$ as policy iteration. In fact, our iterative procedure of carries out policy iteration by backwards dynamic programming. If we start with finite levels for the barriers at each time tick, the transition kernel of the Markov process ξ_k is always strictly contractive. From this, the existence of a limit and its uniqueness follow [**10, 4**].

Because the controlled process is continuous and the integrand of the cost is a C^2 function, as the time-discretizations get infinitesimally finer, the limiting policy and cost converge to the optimal one for the continuous time problem. □

4. Single stopping for controlling Brownian motion

Here, we utilize the results of the previous section for a controlled Brownian motion process:

$$dx_t = u_t dt + dB_t.$$

The optimal terminal control level U_1^* is the linear feedback law [**19**]:

$$U_1^* \left(x_\tau, T-\tau\right) = -\frac{3x_\tau}{2(T-\tau)}.$$

Then, the variables U_0 and τ can be chosen by minimizing the cost:

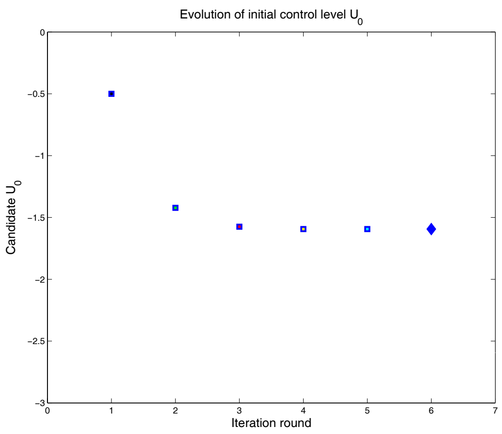
$$\tfrac{1}{T^2} J\left(x_0, \mathcal{U}, \tau\right) = \left(\tfrac{x_0}{\sqrt{T}}\right)^2 + \tfrac{1}{2} + \left(\frac{\frac{x_0}{\sqrt{T}}\sqrt{3}}{2} + \frac{U_0\sqrt{T}}{\sqrt{3}}\right)^2 - \mathbb{E}\left[\left(\frac{\frac{x_\tau}{\sqrt{T}}\sqrt{3}}{2} + \frac{U_0\sqrt{T}\left(1-\frac{\tau}{T}\right)}{\sqrt{3}}\right)^2 \left(1-\tfrac{\tau}{T}\right)\right].$$

This suggests that the length of the horizon does not really matter for the optimization problem. We can use a new time scale with time $s = \frac{t}{T}$ and also use the change of variables: $\bar{x} = \frac{x_0}{\sqrt{T}}$ and $\bar{u} = U_0\sqrt{T}$. In this time-scale, the length of the horizon is exactly one. Hence we will simply assume that $T = 1$.

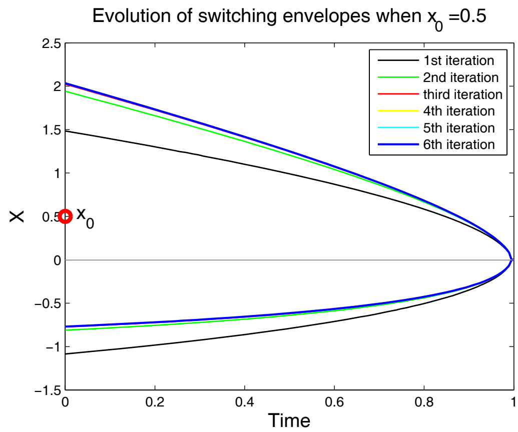
4.1. Case of zero initial condition.

If the initial value is zero, then, the optimal control and its performance can be explicitly computed in closed form [**19**]. Because of symmetry, U_0. The optimal stopping rule is the symmetric quadratic envelope:

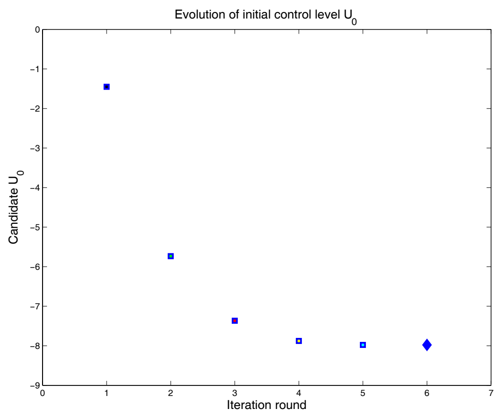
$$\tau^* = \inf\left\{ t \middle| x_t^2 \geq \sqrt{3}\left(T-t\right)\right\}.$$



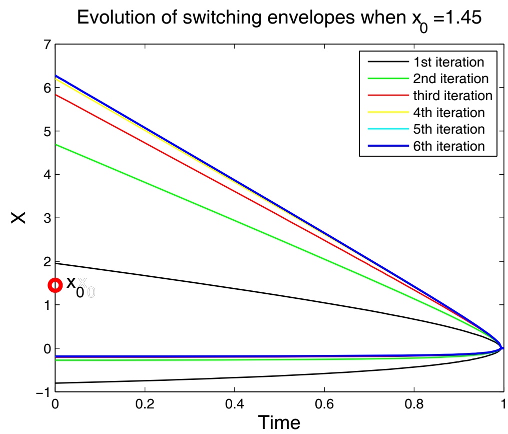
(a) Control level ν_k when $x_0 = 0.5$



(b) Envelopes for sampling time θ_k when $x_0 = 0.5$



(c) Control level ν_k when $x_0 = 1.45$

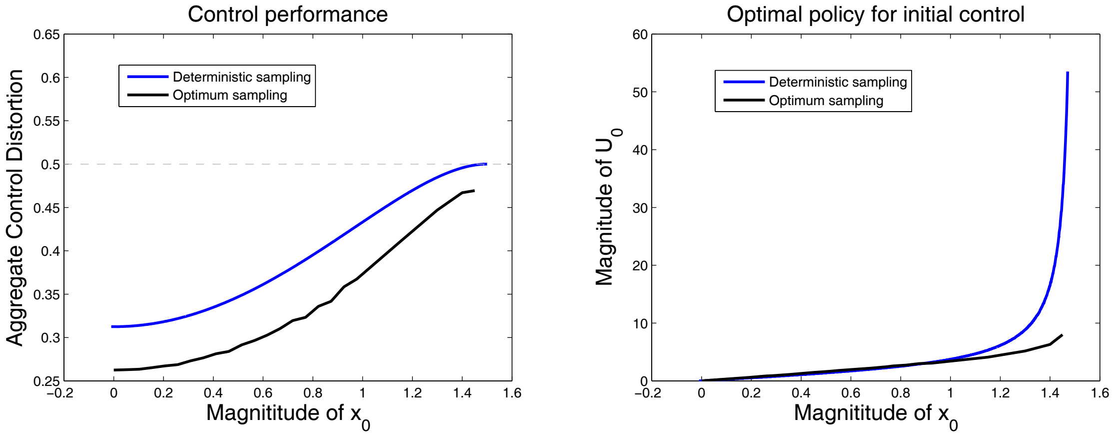


(d) Envelopes for sampling time θ_k when $x_0 = 1.45$

FIGURE 2. Subfigs (a,b) show the results of the iterative search procedure for a relatively modest value of 0.5 for x_0. The convergence to the optimal pair happens by the third round and the minimum aggregate distortion $J\left(0.5, \mathcal{U}^*\left(0.5\right), \tau^*\left(0.5\right)\right) = 0.29$. Subfigs (c,d) show the results for a value of $x_0 = 1.45$. This puts the optimal initial control effort "on the edge" of becoming an impulse. Here, the optimal pair is obtained at round five and the minimum aggregate distortion $J\left(1.45, \mathcal{U}^*\left(1.45\right), \tau^*\left(1.45\right)\right) = 0.47$.

The expected control performance cost incurred by the optimal switching scheme is then $\frac{3\sqrt{3}-1}{16}T^2$, while the cost of using deterministic switching is $\frac{5}{16}T^2$.

4.2. Starting away from zero. One admissible strategy is to have a very large magnitude initial control level U_0 to drive the state quickly towards zero and to switch off the control level when the state does reach zero. Since we do not penalize control effort directly, the cost of this strategy if not impacted by the magnitude of the initial condition. This cost is actually equal to 0.5. This means that for a given initial condition, if no admissible policy can incur a cost lesser than 0.5, the one we described will be optimal. This situation does occur for a range of initial values (See fig. 3).



(a) Relative performance (b) Initial controls

FIGURE 3. Subfig (a) shows the minimum aggregate control error $J(x_0, \mathcal{U}, \tau)$ for Brownian Motion with one allowed sample due to deterministic switching and due to optimal switching. When the initial value x_0 approaches 1.5, the cost approaches the critical value of 0.5. Subfig (b) shows the initial control policy mapping the initial value x_0 to the initial control level U_0. Notice that the optimal switching scheme is less aggressive than the deterministic one when x_0 approaches 1.5.

4.2.1. *Optimal sampling.* We determine the optimal initial control level and the optimal envelope (double-side barrier) by the iterative algorithm of defn. 1. Notice that for x_0 larger than 1.5, immediate resetting to zero is the best option.

5. Further remarks

It is important to extend the results of this paper to the general piece-wise deterministic class of control waveforms. Extension to the vector case and the partially observed case would also be useful. Finally, we need to examine the impact of different control performance measures.

6. Acknowledgements

We are grateful for useful discussions with Mikael Johansson of KTH. We are also thankful for financial support from the Swedish Research Council, the Swedish Foundation for Innovation Systems, and the European Commission.

References

[1] P. Antsaklis and J. Baillieul, *Control and communication challenges in networked real-time systems*, Proceedings of the IEEE, 95 (2007), pp. 9–28. Special issue on Technology of Networked Control Systems.

[2] K. J. Åström and B. Bernhardsson, *Comparison of Riemann and Lebesgue sampling for first order stochastic systems*, in Proceedings of the 41st IEEE conference on Decision and Control (Las Vegas NV, 2002), IEEE Control Systems Society, 2002, pp. 2011–2016.

[3] J. S. Baras and A. Bensoussan, *Optimal sensor scheduling in nonlinear filtering of diffusion processes*, SIAM J. Control Optim., 27 (1989), pp. 786–813.

[4] D. P. Bertsekas and J. N. Tsitsiklis, *An analysis of stochastic shortest path problems*, Math. Oper. Res., 16 (1991), pp. 580–595.

[5] V. S. Borkar and P. Varaiya, *Finite chain approximation for a continuous stochastic control problem*, IEEE Trans. Automat. Control, 26 (1981), pp. 466–470.

[6] R. W. Brockett and D. Liberzon, *Quantized feedback stabilization of linear systems*, IEEE Trans. Automat. Control, 45 (2000), pp. 1279–1289.
[7] O. Imer and T. Basar, *Optimal control with limited controls*, in Proceedings of the 2006 American Control Conference, June 2006, pp. 298–303.
[8] I. Karatzas and I.-M. Zamfirescu, *Martingale approach to stochastic control with discretionary stopping*, Appl. Math. Optim., 53 (2006), pp. 163–184.
[9] H. J. Kushner, *On the optimum timing of observations for linear control systems with unknown initial state*, IEEE Trans. Automatic Control, AC-9 (1964), pp. 144–150.
[10] ——, *Introduction to stochastic control*, Holt, Rinehart and Winston, Inc., New York, 1971.
[11] D. Liberzon and R. W. Brockett, *Nonlinear feedback systems perturbed by noise: steady-state probability distributions and optimal control*, IEEE Trans. Automat. Control, 45 (2000), pp. 1116–1130.
[12] R. Mehra, *Optimization of measurement schedules and sensor designs for linear dynamic systems*, Automatic Control, IEEE Transactions on, 21 (1976), pp. 55–64.
[13] I. Meier, L., J. Peschon, and R. Dressler, *Optimal control of measurement subsystems*, Automatic Control, IEEE Transactions on, 12 (1967), pp. 528–536.
[14] H. Morimoto, *Variational inequalities for combined control and stopping*, SIAM J. Control Optim., 42 (2003), pp. 686–708 (electronic).
[15] B. Øksendal, *Stochastic differential equations*, Universitext, Springer-Verlag, Berlin, sixth ed., 2003.
[16] M. Rabi, *Packet based Inference and Control*, PhD thesis, University of Maryland, College Park, Sep 2006.
[17] M. Rabi and J. Baras, *Level-triggered control of a scalar linear system*, in Proceedings of the 2007 Mediterranean Conference on Control & Automation., June 2007, pp. 1–6.
[18] M. Rabi and K. H. Johansson, *Optimal stopping for updating controls*, in Proccedings of the second International workshop on sequential methods, UTT, Troyes, France, June 2009.
[19] M. Rabi, K. H. Johansson, and M. Johansson, *Optimal stopping for event-triggered sensing and actuation*, in Proceedings of the 47th IEEE Conference on Decision and Control, December 2008.
[20] S. O. Rice, *Mathematical analysis of random noise*, Bell System Tech. J., 23 (1944), pp. 282–332.
[21] ——, *Mathematical analysis of random noise*, Bell System Tech. J., 24 (1945), pp. 46–156.
[22] E. Skafidas and A. Nerode, *Optimal measurement scheduling in linear quadratic gaussian control problems*, in Proceedings of the 1998 IEEE International Conference on Control Applications, 1998, vol. 2, September 1998, pp. 1225–1229.
[23] P. Tabuada, *Event-triggered real-time scheduling of stabilizing control tasks*, IEEE Transactions on Automatic Control, 52 (2007), pp. 1680–1685.
[24] X. Wang and M. Lemmon, *Event design in event-triggered feedback control systems*, in Proceedings of the IEEE, Dec. 2008, pp. 2105–2110.
[25] ——, *Self-triggered feedback control systems with finite-gain L_2 stability*, Automatic Control, IEEE Transactions on, 54 (2009), pp. 452–467.
[26] W. Wu and A. Arapostathis, *Optimal sensor querying: General markovian and lqg models with controlled observations*, Automatic Control, IEEE Transactions on, 53 (2008), pp. 1392–1405.

(Maben Rabi, and, Karl H. Johansson) Automatic Control lab, Royal Institute of Technology (KTH), Osquldas väg 10, plan 6, 100 44 Stockholm, Sweden.
E-mail address: firstname.lastname@ee.kth.se