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Comparison between sampled-data control, deadband control and model-based event-triggered control

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Abstract: This paper investigates the stability as well as communication properties of sampled-data control and two event-triggered control schemes: deadband control and model-based event-triggered control. After proposing a uniform representation, these schemes are compared by deriving approximation error bounds with respect to the behavior of the continuous-time state-feedback loop and by specifying lower bounds on the minimum inter-sampling times. The results show that, under the conditions derived in this paper, the model-based approach guarantees the best stability and communication properties which is also demonstrated by a numerical example.

Keywords: Event-triggered control, Model-based control, Sampled-data control, Stability, Networked control systems

1. INTRODUCTION

1.1 Event-triggered control

Traditionally, continuous controllers are implemented on digital hardware by sampling continuous-time signals at equidistant instants of time (sampled-data control). One of the main reasons for applying this approach is a well established theory for the analysis and the design of sampled-data systems. However, to meet certain application requirements, for example the reduction of computational power, energy consumption or the information exchange in networked control systems (Nair et al. [2007]), event-triggered sampling has been investigated as a suitable alternative to the time-driven paradigm. Event-triggered control aims at adapting the communication among the components of the feedback loop to the current needs. By reducing the information exchange to the minimum communication that is necessary to ensure the required system performance, an overload of the digital communication network should be avoided.

In the literature, there are several approaches which show that event-triggered control is able to significantly reduce the computational and communication effort while only slightly degrading the control performance, see e.g. Anta and Tabuada [2010], Årzén [1999], Åström and Bernhardsson [2002], Cervin and Henningsson [2008], Heemels et al. [2008], Wang and Lemmon [2009]. Although the basic idea of all these approaches is the same, namely, to sample a system only if certain event conditions are satisfied, their implementations vary. However, as an event usually forces a jump in certain continuous state variables, event-triggered control can be generally characterized as a hybrid, in particular, impulsive system (Lunze and Lamnabhi-Lagarrigue [2009], Donkers and Heemels [2010]).

This paper considers two different implementations which have been published as deadband control and model-based event-triggered control and compares these schemes with sampled-data control by using a continuous-time control loop as a common reference system.

Deadband control has been proposed by Otanez et al. [2002]. Here, the event generating mechanism compares the current plant state $\boldsymbol{x}(t)$ with the plant state $\boldsymbol{x}(t_k)$ sent over the network at the previous event time t_k. A communication is invoked, whenever the difference $\boldsymbol{x}(t) - \boldsymbol{x}(t_k)$ reaches an event threshold $\bar{e}$:

$$\|\boldsymbol{x}(t) - \boldsymbol{x}(t_k)\| = \bar{e}. \qquad (1)$$

At the new event time $t := t_{k+1}$, the current measurement $\boldsymbol{x}(t)$ is transmitted and a new deadband is established around the value $\boldsymbol{x}(t)$. A similar scheme has been studied by Donkers and Heemels [2010], where besides a deadband for the output $\boldsymbol{y}(t)$, an additional deadband for the input $\boldsymbol{u}(t)$ has been used.

Åström [2008], Lunze and Lehmann [2010], and Garcia and Antsaklis [2011] presented a model-based approach to event-triggered control. In contrast to the previous scheme, the event generator evaluates the current plant state $\boldsymbol{x}(t)$ in comparison with the state $\boldsymbol{x}_{\rm s}(t)$ that a model of the continuous-time state-feedback loop has according to

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)\| = \bar{e} \qquad (2)$$

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with $t := t_k$. Thus, an event does not indicate a large evolution of the system variables but a large deviation of the plant state from a desired reference behavior.

1.2 Contribution of this paper

In order to elaborate the relations between sampled-data control and the two event-triggered control schemes considered, this paper

(1) proposes a uniform representation of sampled-data control (SDC), deadband control (DBC) and model-based event-triggered control (MBETC), and
(2) uses this representation to derive and compare the respective bounds on the approximation error with respect to the behavior of the continuous-time state-feedback loop and on the minimum inter-sampling time (Theorems 1, 2, 3 and Corollary 4).

The paper is organized as follows. The uniform representation is proposed in Sec. 2. Section 3 presents the analytical results which are evaluated in Sec. 4 by means of a numerical example.

1.3 Preliminaries

The plant is given by the linear state-space representation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \quad (3)$$

where the state $\boldsymbol{x} \in \mathbb{R}^n$ is assumed to be measurable. The input is denoted by $\boldsymbol{u} \in \mathbb{R}^m$ and the unknown disturbance by $\boldsymbol{d} \in \mathbb{R}^l$, which is assumed to be bounded according to

$$\|\boldsymbol{d}(t)\| \leq d_{\max},$$

where $\|\cdot\|$ is an arbitrary vector norm or the induced matrix norm. With the state-feedback controller $\boldsymbol{u}(t) = -\boldsymbol{K}_{\mathrm{CT}}\boldsymbol{x}(t)$, the continuous-time state-feedback loop

$$\dot{\boldsymbol{x}}_{\mathrm{CT}}(t) = \underbrace{(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_{\mathrm{CT}})}_{\bar{\boldsymbol{A}}}\boldsymbol{x}_{\mathrm{CT}}(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \quad (4)$$

results, where $\boldsymbol{x}_{\mathrm{CT}}(t)$ denotes the state of the continuous-time control loop. If $\bar{\boldsymbol{A}}$ is Hurwitz, the state $\boldsymbol{x}_{\mathrm{CT}}(t)$ is bounded according to $\|\boldsymbol{x}_{\mathrm{CT}}(t)\| \leq x_{\mathrm{CTmax}}$ with

$$x_{\mathrm{CTmax}} = \max_t \left\| \mathrm{e}^{\bar{\boldsymbol{A}}t} \right\| \cdot \|\boldsymbol{x}_0\| + \int_0^\infty \left\| \mathrm{e}^{\bar{\boldsymbol{A}}\tau}\boldsymbol{E} \right\| \mathrm{d}\tau d_{\max}. \quad (5)$$

The discrete-time model of plant (3) subject to the disturbance $\boldsymbol{d}(t) = \boldsymbol{d}(k) = \bar{\boldsymbol{d}}$ is given by

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_{\mathrm{D}}\boldsymbol{x}(k) + \boldsymbol{B}_{\mathrm{D}}\boldsymbol{u}(k) + \boldsymbol{E}_{\mathrm{D}}\boldsymbol{d}(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

with $t_k = kT_{\mathrm{s}}$, T_{s} the fixed sampling period, and

$$\boldsymbol{A}_{\mathrm{D}} = \mathrm{e}^{\boldsymbol{A}T_{\mathrm{s}}}, \quad \boldsymbol{B}_{\mathrm{D}} = \int_0^{T_{\mathrm{s}}} \mathrm{e}^{\boldsymbol{A}\alpha}\boldsymbol{B}\,\mathrm{d}\alpha, \quad \boldsymbol{E}_{\mathrm{D}} = \int_0^{T_{\mathrm{s}}} \mathrm{e}^{\boldsymbol{A}\alpha}\boldsymbol{E}\,\mathrm{d}\alpha.$$

Applying controller $\boldsymbol{u}(k) = -\boldsymbol{K}_{\mathrm{DT}}\boldsymbol{x}(k)$, the closed-loop system

$$\boldsymbol{x}_{\mathrm{DT}}(k+1) = \underbrace{(\boldsymbol{A}_{\mathrm{D}} - \boldsymbol{B}_{\mathrm{D}}\boldsymbol{K}_{\mathrm{DT}})}_{\bar{\boldsymbol{A}}_{\mathrm{D}}}\boldsymbol{x}_{\mathrm{DT}}(k) + \boldsymbol{E}_{\mathrm{D}}\boldsymbol{d}(k) \quad (6)$$

results ($\boldsymbol{x}_{\mathrm{DT}}(0) = \boldsymbol{x}_0$) with $\boldsymbol{x}_{\mathrm{DT}}(k)$ denoting the state of the discrete-time control loop.

In the following, it is assumed that $\boldsymbol{K} = \boldsymbol{K}_{\mathrm{CT}} = \boldsymbol{K}_{\mathrm{DT}}$ holds and $\boldsymbol{K}$ is designed so that the continuous-time control loop (4) is stable ($\bar{\boldsymbol{A}}$ is Hurwitz) which likewise holds for the discrete-time control loop (6). Moreover, the continuous-time state-feedback loop (4) is used as a reference system to evaluate the sampled-data and the event-triggered control schemes and, therefore, should have the desired disturbance attenuation properties.

2. UNIFORM REPRESENTATION FOR SYSTEMS WITH SAMPLING

2.1 Structure

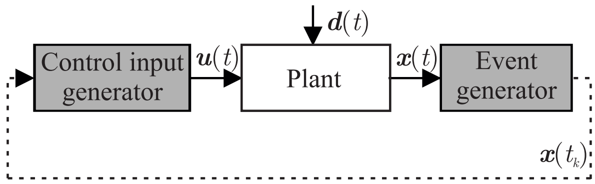

Fig. 1. Uniform structure

In order to compare sampled-data control, deadband control and model-based event-triggered control, this section firstly proposes a uniform representation for these three schemes. The structure is depicted in Fig. 1 which consists of the

- the plant,
- the event generator (EG) which invokes a sampling event whenever certain event conditions (see e.g. Eqs. (1), (2)) are satisfied,
- and the control input generator (CIG) which produces the control input $\boldsymbol{u}(t)$ by means of a model description of the controller function.

Only at event times (sampling times) t_k $(k = 0, 1, 2, ...)$ determined by the event generator, the state information $\boldsymbol{x}(t_k)$ is sent from the event generator towards the control input generator. This is indicated by the dashed lines whereas the solid lines indicate continuous-time signals.

Second, the behavior of each scheme is described by an impulsive system of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{E}\boldsymbol{d}(t) \qquad \text{if } h(\boldsymbol{x}, t) \neq 0 \quad (7)$$

$$\boldsymbol{x}(t_k^+) = \boldsymbol{G}\boldsymbol{x}(t_k) \qquad \text{if } h(\boldsymbol{x}, t) = 0. \quad (8)$$

The flow equation (7) holds between the event times t_k and the jump equation (8) describes the state jumps which occur whenever the event condition $h(\boldsymbol{x}, t) = 0$, which implicitly defines the event times t_k, is satisfied. Here, $\boldsymbol{x}(t_k^+)$ is used to indicate the jump of the state $\boldsymbol{x}(t)$ by defining the limit from above at event time t_k according to $\boldsymbol{x}(t_k^+) = \lim_{s \downarrow t_k} \boldsymbol{x}(s)$.

2.2 Sampled-data control

Sampled-data control with periodic sampling is the traditional control implementation on digital hardware. It generally consists of the plant, a sampler, a controller and a zero-order hold. In order to meet the structural requirements shown in Fig. 1, the controller and the zero-order hold have to be merged in the control input generator. The components can be described as follows.

Control input generator. Between two consecutive sampling times ($t \in [t_k, t_{k+1})$), the control input generator produces the control input $\boldsymbol{u}(t)$ by means of the following controller model

$$\dot{\boldsymbol{x}}_{\mathrm{s}}(t) = \boldsymbol{0}, \quad \boldsymbol{x}_{\mathrm{s}}(t_k^+) = \boldsymbol{x}(t_k) \tag{9}$$

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_{\mathrm{s}}(t) \tag{10}$$

with $\boldsymbol{x}_{\mathrm{s}}$ the state of the control input generator. At event time t_k, the state $\boldsymbol{x}_{\mathrm{s}}$ is updated with the current plant state $\boldsymbol{x}(t_k)$ which is communicated over the network by the event generator. As the model state $\boldsymbol{x}_{\mathrm{s}}(t)$ is held constant between two consecutive events, the control input $\boldsymbol{u}(t)$ is constant as well.

Event generator. For periodic sampling, a clock invokes a sampling whenever

$$t - t_k = T_{\mathrm{s}} \tag{11}$$

holds ($t > t_k$). The time t, at which this happens, denotes the new event time t_{k+1}. This is depicted in Fig. 2, where the dashed line indicates the behavior of the model state $\boldsymbol{x}_{\mathrm{s}}(t)$. The model state is updated at the sampling instants t_k ($k = 0, 1, 2, ...$) with the current state information $\boldsymbol{x}(t_k)$ and is held constant between two consecutive sampling times.

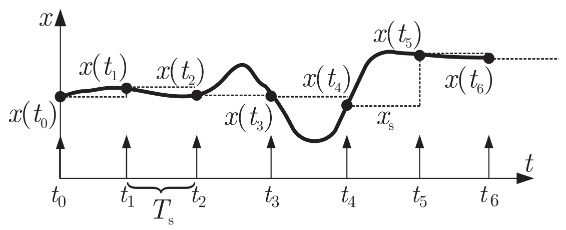

Fig. 2. Periodic sampling

Behavior of sampled-data control. In the time interval $[t_k, t_{k+1})$, plant (3) and control input generator (9), (10) lead to the state-space model

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_{\mathrm{s}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_{\mathrm{s}}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t)$$

$$\begin{pmatrix} \boldsymbol{x}(t_k) \\ \boldsymbol{x}_{\mathrm{s}}(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_k \\ \boldsymbol{x}_k \end{pmatrix},$$

where $\boldsymbol{x}_k$ is used to denote the plant state $\boldsymbol{x}$ at event time t_k. Next, this behavior is used to get an impulsive system description of the form (7), (8). By introducing the difference state

$$\boldsymbol{x}_\Delta(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t) \tag{12}$$

and by applying the state transformation

$$\begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_{\mathrm{s}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_{\mathrm{s}}(t) \end{pmatrix}, \tag{13}$$

the impulsive system model

$$\begin{pmatrix} \dot{\boldsymbol{x}}_\Delta(t) \\ \dot{\boldsymbol{x}}_{\mathrm{s}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_{\mathrm{s}}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t), \text{ if } t - t_k \neq T_{\mathrm{s}} \tag{14}$$

$$\begin{pmatrix} \boldsymbol{x}_\Delta(t_k^+) \\ \boldsymbol{x}_{\mathrm{s}}(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{x}_k \end{pmatrix}, \text{ if } t - t_{k-1} = T_{\mathrm{s}}$$

is obtained ($k = 0, 1, 2, ...$; $t_{-1} := -T_{\mathrm{s}}$) which holds for all times $t \geq 0$.

2.3 Deadband control

Deadband control as proposed by Otanez et al. [2002] uses the event condition (1) for invoking a communication between the event generator and the control input generator. Moreover, it uses a zero-order hold for producing the continuous-time input $\boldsymbol{u}(t)$ between two consecutive event times which leads to the following system description.

Control input generator. Due to applying a zero-order hold strategy, the control input generator can be described by the same controller model (9), (10) as applied by sampled-data control.

Event generator. Due to the fact that $\boldsymbol{x}(t_k) = \boldsymbol{x}_{\mathrm{s}}(t_k^+) = \boldsymbol{x}_{\mathrm{s}}(t)$ holds (Eq. (9)), event condition (1) can be rewritten:

$$\|\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{s}}(t)\| = \bar{e}. \tag{15}$$

The time t, at which this condition is satisfied, denotes the new event time t_{k+1}. Note that, in general, this sampling does not occur equidistantly in time as shown in Fig. 3. The corresponding deadbands, which depend on the state information $\boldsymbol{x}(t_k)$, are indicated by the grey regions.

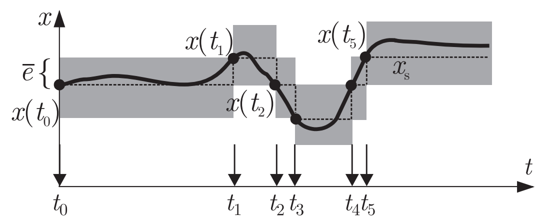

Fig. 3. Deadband sampling

Behavior of deadband control. The deadband control loop can be described by the impulsive system model

$$\begin{pmatrix} \dot{\boldsymbol{x}}_\Delta(t) \\ \dot{\boldsymbol{x}}_{\mathrm{s}}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \bar{\boldsymbol{A}} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_{\mathrm{s}}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t), \text{ if } \|\boldsymbol{x}_\Delta(t)\| \neq \bar{e} \tag{16}$$

$$\begin{pmatrix} \boldsymbol{x}_\Delta(t_k^+) \\ \boldsymbol{x}_{\mathrm{s}}(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{x}_k \end{pmatrix}, \text{ if } \|\boldsymbol{x}_\Delta(t)\| = \bar{e} \vee t_0 = 0,$$

where t_0 denotes the time instance of the initial event ($k = 0$).

2.4 Model-based event-triggered control

An alternative event-triggering scheme has been proposed by Lunze and Lehmann [2010] which extends deadband control by incorporating a more involved model in the control input generator. The next paragraphs summarize the components of this scheme.

Control input generator. The control input generator produces the control input $\boldsymbol{u}(t)$ by means of a model of the undisturbed continuous-time state-feedback loop (4) according to

$$\dot{\boldsymbol{x}}_{\mathrm{s}}(t) = \bar{\boldsymbol{A}}\boldsymbol{x}_{\mathrm{s}}(t), \quad \boldsymbol{x}_{\mathrm{s}}(t_k^+) = \boldsymbol{x}(t_k) \tag{17}$$

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}_{\mathrm{s}}(t), \quad t \in [t_k, t_{k+1}). \tag{18}$$

Event generator. The event generator determines the event times t_k $(k = 1, 2, 3, ...)$ by comparing the measured state $\boldsymbol{x}(t)$ with the model state $\boldsymbol{x}_{\rm s}(t)$, see Eq. (2). However, in contrast to deadband control, the model state $\boldsymbol{x}_{\rm s}(t)$ refers to a desired system behavior (Eq. (17)) which generally varies between two consecutive event times as depicted in Fig. 4 by the dashed lines.

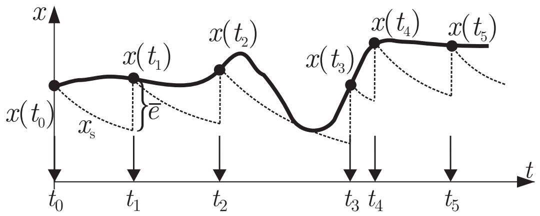

Fig. 4. Model-based event-triggered sampling

Note that event condition (2) requires a continuous-time access to the time-varying model state $\boldsymbol{x}_{\rm s}(t)$. In order not to send this information continuously from the control input generator to the event generator, the event generator has to include a copy of model (17). This makes the scheme computationally more demanding compared to deadband control which simply stores the previous state information $\boldsymbol{x}(t_k)$ until the next event time.

Behavior of model-based event-triggered control. In the time interval $[t_k, t_{k+1})$, plant (3) and control input generator (17), (18) result in the state-space model

$$\begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}_{\rm s}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{O} & \bar{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}_{\rm s}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t)$$

$$\begin{pmatrix} \boldsymbol{x}(t_k) \\ \boldsymbol{x}_{\rm s}(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}_k \\ \boldsymbol{x}_k \end{pmatrix}.$$

By applying state transformation (13), this model leads to the impulsive system of the form

$$\begin{pmatrix} \dot{\boldsymbol{x}}_\Delta(t) \\ \dot{\boldsymbol{x}}_{\rm s}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{O} & \bar{\boldsymbol{A}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_\Delta(t) \\ \boldsymbol{x}_{\rm s}(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{d}(t), \text{ if } \|\boldsymbol{x}_\Delta(t)\| \neq \bar{e}$$

$$\begin{pmatrix} \boldsymbol{x}_\Delta(t_k^+) \\ \boldsymbol{x}_{\rm s}(t_k^+) \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{x}_k \end{pmatrix}, \text{ if } \|\boldsymbol{x}_\Delta(t)\| = \bar{e} \vee t_0 = 0.$$

In contrast to the previous schemes, the state $\boldsymbol{x}_\Delta(t)$ is only affected by the exogenous disturbance $\boldsymbol{d}(t)$ and does not depend on the model state $\boldsymbol{x}_{\rm s}(t)$ of the control input generator.

2.5 Discussion

The previous investigation shows some remarkable analogies of the control schemes considered:

(1) Deadband control can be seen as an event-triggered realization of sampled-data control.
(2) Model-based event-triggered control extends deadband control by generating exponential inputs instead of piecewise constant input signals.

The components of these schemes are summarized in Tab. 1.

Table 1. Components of sampled-data control and the event-triggered control schemes

Scheme	CIG	EG
SDC	Eqs. (9), (10)	$t - t_k = T_{\rm s}$
DBC		$\|\boldsymbol{x}(t) - \boldsymbol{x}_{\rm s}(t)\| = \bar{e}$
MBETC	Eqs. (17), (18)	

3. STABILITY AND COMMUNICATION PROPERTIES

3.1 Comparison to the behavior of the continuous-time control loop

The stability analysis is carried out in this section by comparing the behavior of sampled-data control, deadband control and model-based event-triggered control with the behavior of the continuous-time state-feedback loop (4). The analysis exploits the fact that both the sampled-data and the event-triggered control schemes can be described by the state-space model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_{\rm s}(t) + \boldsymbol{E}\boldsymbol{d}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

(see Eqs. (3), (10)), where the model state $\boldsymbol{x}_{\rm s}(t)$ is generated in different ways by the respective control input generator (Tab. 1). Using relation (12), the model can be rewritten as

$$\dot{\boldsymbol{x}}(t) = \bar{\boldsymbol{A}}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_\Delta(t) + \boldsymbol{E}\boldsymbol{d}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \quad (19)$$

Hence, the difference

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\rm CT}(t)$$

between the state $\boldsymbol{x}(t)$ of control loop (19) and the state $\boldsymbol{x}_{\rm CT}(t)$ of the continuous-time state-feedback loop (4) is given by

$$\dot{\boldsymbol{e}}(t) = \dot{\boldsymbol{x}}(t) - \dot{\boldsymbol{x}}_{\rm CT}(t) = \bar{\boldsymbol{A}}\boldsymbol{e}(t) + \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_\Delta(t), \quad \boldsymbol{e}(0) = \boldsymbol{0}.$$

Since $\bar{\boldsymbol{A}}$ is assumed to be Hurwitz, the approximation error $\boldsymbol{e}(t)$ is bounded according to

$$\|\boldsymbol{e}(t)\| = \left\| \int_0^t {\rm e}^{\bar{\boldsymbol{A}}(t-\tau)} \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}_\Delta(\tau)\,{\rm d}\tau \right\| \leq \int_0^\infty \left\| {\rm e}^{\bar{\boldsymbol{A}}\tau}\boldsymbol{B}\boldsymbol{K} \right\| {\rm d}\tau \cdot \max_t \|\boldsymbol{x}_\Delta(t)\| \quad (20)$$

if the difference state $\boldsymbol{x}_\Delta(t)$ is bounded by

$$\max_t \|\boldsymbol{x}_\Delta(t)\| \leq x_{\Delta\max}. \quad (21)$$

Consequently, as the continuous-time state-feedback loop is assumed to be stable, a bounded approximation error also implies the stability of the sampled control loops. In the next sections, the upper bound $x_{\Delta\max}$ is derived for the three schemes considered.

3.2 Sampled-data control

Theorem 1. Under the assumption $\|\bar{\boldsymbol{A}}_{\rm D}\| < 1$, the difference $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\rm CT}(t)$ between the state $\boldsymbol{x}(t)$ of the sampled-data control loop (3), (9)–(11) and the state $\boldsymbol{x}_{\rm CT}(t)$ of the continuous-time state-feedback loop (4) is bounded from above by

$$\|\boldsymbol{e}(t)\| \leq e_{\max,{\rm SDC}} = x_{\Delta\max} \cdot \int_0^\infty \left\| {\rm e}^{\bar{\boldsymbol{A}}\tau}\boldsymbol{B}\boldsymbol{K} \right\| {\rm d}\tau \quad (22)$$

with

$$x_{\Delta\max} = \int_0^{T_s} \left\| e^{\boldsymbol{A}\tau} \right\| d\tau \cdot (\|\bar{\boldsymbol{A}}\| x_{\mathrm{DTmax}} + \|\boldsymbol{E}\| d_{\max}) \quad (23)$$

and

$$x_{\mathrm{DTmax}} = \max_k \left\| \bar{\boldsymbol{A}}_{\mathrm{D}}^k \right\| \cdot \|\boldsymbol{x}_0\| + \frac{\|\boldsymbol{E}_{\mathrm{D}}\| \, d_{\max}}{1 - \|\bar{\boldsymbol{A}}_{\mathrm{D}}\|}.$$

Proof. Assuming that $\|\bar{\boldsymbol{A}}_{\mathrm{D}}\| < 1$ holds, an upper bound x_{DTmax} of the state $\boldsymbol{x}(t_k)$ can be determined by means of the discrete-time evolution of $\boldsymbol{x}_{\mathrm{DT}}(k)$ given by

$$\begin{aligned} \|\boldsymbol{x}_{\mathrm{DT}}(k)\| &= \left\| \bar{\boldsymbol{A}}_{\mathrm{D}}^k \boldsymbol{x}_0 + \sum_{j=0}^{k-1} \bar{\boldsymbol{A}}_{\mathrm{D}}^{k-1-j} \boldsymbol{E}_{\mathrm{D}} \bar{\boldsymbol{d}} \right\| \\ &\le \max_k \left\| \bar{\boldsymbol{A}}_{\mathrm{D}}^k \right\| \cdot \|\boldsymbol{x}_0\| + \sum_{j=0}^{\infty} \|\bar{\boldsymbol{A}}_{\mathrm{D}}\|^j \|\boldsymbol{E}_{\mathrm{D}}\| d_{\max} \\ &= \max_k \left\| \bar{\boldsymbol{A}}_{\mathrm{D}}^k \right\| \cdot \|\boldsymbol{x}_0\| + \frac{\|\boldsymbol{E}_{\mathrm{D}}\| \, d_{\max}}{1 - \|\bar{\boldsymbol{A}}_{\mathrm{D}}\|} = x_{\mathrm{DTmax}} \end{aligned}$$

(Eq. (6)). Using this state bound, an upper bound on $\max_t \|\boldsymbol{x}_\Delta(t)\|$ can be determined by considering the solution of Eq. (14) leading to

$$\max_t \|\boldsymbol{x}_\Delta(t)\| = \max_t \left\| \int_{t_k}^{t} e^{\boldsymbol{A}(t-\tau)} (\bar{\boldsymbol{A}} \boldsymbol{x}(t_k) + \boldsymbol{E} \boldsymbol{d}(\tau)) \, d\tau \right\|$$

which can be overapproximated by Eq. (23). □

Note that the upper bound on the approximation error depends implicitly on the disturbance $\boldsymbol{d}(t)$ and the sampling period T_s since the bound $x_{\Delta\max}$ depends on these variables. The theorem shows that the sampled-data controller is capable of emulating the continuous-time state feedback with arbitrary precision by accordingly choosing the sampling period T_s.

3.3 Deadband control

Using deadband control, the state $\boldsymbol{x}_\Delta(t)$ is bounded according to event condition (15):

$$\|\boldsymbol{x}_\Delta(t)\| \le \bar{e} = x_{\Delta\max}.$$

However, in contrast to sampled-data control which operates with a fixed inter-sampling time T_s (Eq. (11)), a minimum inter-sampling time

$$T_{\min} = \min_k \{t_{k+1} - t_k\} = \arg\min_t \{\|\boldsymbol{x}_\Delta(t)\| = \bar{e}\} \quad (24)$$

has to be specified when dealing with event-triggered control schemes.

Theorem 2. The deadband control loop (3), (9), (10), (15) has the following properties:

- The difference $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{CT}}(t)$ between the state $\boldsymbol{x}(t)$ of the deadband control loop and the state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous-time state-feedback loop (4) is bounded from above by

$$\|\boldsymbol{e}(t)\| \le e_{\max,\mathrm{DBC}} = \bar{e} \cdot \int_0^\infty \left\| e^{\bar{\boldsymbol{A}}\tau} \boldsymbol{BK} \right\| d\tau. \quad (25)$$

- The minimum inter-sampling time $T_{\min} \ge \bar{T}_{\mathrm{DBC}}$ is bounded from below by $\bar{T}_{\mathrm{DBC}}$ given by

$$\bar{T}_{\mathrm{DBC}} = \arg\min_t \left\{ \int_0^t \left\| e^{\boldsymbol{A}\tau} \right\| d\tau = \frac{\bar{e}}{\|\bar{\boldsymbol{A}}\|(x_{\mathrm{CTmax}} + e_{\max,\mathrm{DBC}}) + \|\boldsymbol{E}\| d_{\max}} \right\}. \quad (26)$$

Proof. The first property follows directly from Eq. (20) and the fact that event condition (15) holds.

The second property results from the fact that the minimum inter-sampling time is given by

$$T_{\min} = \arg\min_t \left\{ \left\| \int_{t_k}^t e^{\boldsymbol{A}(t-\tau)} (\bar{\boldsymbol{A}} \boldsymbol{x}(t_k) + \boldsymbol{E} \boldsymbol{d}(\tau)) d\tau \right\| = \bar{e} \right\}$$

(cf. Eqs. (16), (24)). If the upper bound

$$\begin{aligned} &\int_0^t \left\| e^{\boldsymbol{A}(t-\tau)} \right\| d\tau \cdot (\|\bar{\boldsymbol{A}}\| (x_{\mathrm{CTmax}} + e_{\max,\mathrm{DBC}}) + \|\boldsymbol{E}\| d_{\max}) \\ &\ge \left\| \int_{t_k}^t e^{\boldsymbol{A}(t-\tau)} (\bar{\boldsymbol{A}} \boldsymbol{x}(t_k) + \boldsymbol{E} \boldsymbol{d}(\tau)) \, d\tau \right\| \end{aligned}$$

is set to $\bar{e}$, with x_{CTmax} given by Eq. (5), then the lower bound $\bar{T}_{\mathrm{DBC}} \le T_{\min}$ can be obtained by Eq. (26). □

This theorem shows that deadband control is likewise able to mimic a continuous-time state feedback with arbitrary precision. However, instead of the sampling period T_s, the event threshold $\bar{e}$ has to be chosen accordingly.

Moreover, it shows how the communication depends on the disturbance magnitude $d_{\max}$ which contrasts with sampled-data control, where the sampling frequency is chosen with respect to the time constants of the plant. Note that the bound $\bar{T}_{\mathrm{DBC}}$ on the minimum inter-sampling time can also be chosen arbitrarily by accordingly choosing the event threshold $\bar{e}$.

3.4 Model-based event-triggered control

Theorem 3. (Lunze and Lehmann [2010]) The model-based event-triggered control loop (2), (3), (17), (18) has the following properties:

- The difference $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{CT}}(t)$ between the state $\boldsymbol{x}(t)$ of the model-based event-triggered control loop and the state $\boldsymbol{x}_{\mathrm{CT}}(t)$ of the continuous-time state-feedback loop (4) is bounded from above by

$$\|\boldsymbol{e}(t)\| \le e_{\max,\mathrm{MBETC}} = \bar{e} \cdot \int_0^\infty \left\| e^{\bar{\boldsymbol{A}}\tau} \boldsymbol{BK} \right\| d\tau. \quad (27)$$

- The minimum inter-sampling time $T_{\min}$ is bounded from below by $\bar{T}_{\mathrm{MBETC}}$ given by

$$\bar{T}_{\mathrm{MBETC}} = \arg\min_t \left\{ \int_0^t \left\| e^{\boldsymbol{A}\tau} \right\| d\tau = \frac{\bar{e}}{\|\boldsymbol{E}\| d_{\max}} \right\}. \quad (28)$$

Interestingly, the approximation error bounds (25) and (27) are identical. This shows that the approximation guarantee $e_{\max}$ obtained by using event-triggered control is not affected by the way of producing the control input $\boldsymbol{u}(t)$ between two consecutive event times and only depends on the event threshold $\bar{e}$.

However, the input generation affects the inter-sampling times. Comparing bound (28) with bound (26), it can be seen that deadband control always has a smaller bound on the minimum inter-sampling time than model-based event-triggered control.

3.5 Summary

Corollary 4. The comparison of sampled-data control, deadband control and model-based event-triggered control brings about the following relations:

(1) Model-based event-triggered control is superior to deadband control in terms of providing a larger bound on the minimum inter-sampling time $\bar{T}_{\text{MBETC}} > \bar{T}_{\text{DBC}}$ while guaranteeing the same approximation error bound.
(2) Whenever the condition
$$\arg\min_t \left\{ \int_0^t \left\| \mathrm{e}^{\boldsymbol{A}\tau} \right\| \mathrm{d}\tau = \frac{e_{\max,\text{SDC}}}{\bar{h}\|\boldsymbol{E}\| d_{\max}} \right\} > T_{\text{s}}$$
is satisfied with
$$\bar{h} = \int_0^\infty \left\| \mathrm{e}^{\bar{\boldsymbol{A}}\tau} \boldsymbol{BK} \right\| \mathrm{d}\tau, \qquad (29)$$
model-based event-triggered control is superior to sampled-data control as it provides a larger bound on the minimum inter-sampling time $\bar{T}_{\text{MBETC}} > T_{\text{s}}$ while guaranteeing the same approximation error bound.

Proof. The first property follows directly by comparing Eqs. (25), (27) with Eqs. (26), (28).

The second property can be obtained by assuming that the sampling time T_{s} and the event threshold $\bar{e}$ have been chosen, so that $e_{\max,\text{SDC}} = e_{\max,\text{MBETC}}$ holds. Inserting Eq. (27) into relation (28) yields
$$\bar{T} = \arg\min_t \left\{ \int_0^t \left\| \mathrm{e}^{\boldsymbol{A}\tau} \right\| \mathrm{d}\tau = \frac{e_{\max,\text{SDC}}}{\bar{h}\|\boldsymbol{E}\| d_{\max}} \right\}$$
with $\bar{h}$ given by Eq. (29). As long as $\bar{T} > T_{\text{s}}$ holds, model-based event-triggered control guarantees a less frequent communication while guaranteeing the same approximation error bound. This analysis can be simply adopted to deadband control by replacing Eq. (28) by Eq. (26). □

4. NUMERICAL EXAMPLE

To illustrate the results, the stable scalar system
$$\dot{x}(t) = -0.5x(t) + u(t) + d(t) \qquad (30)$$
is considered. With the controller $u(t) = -1.5x(t)$, the continuous-time state-feedback loop $\dot{x}_{\text{CF}}(t) = -2x_{\text{CF}}(t) + d(t)$ results. The disturbance $d(t)$ affecting system (30) is assumed to be bounded according to $|d(t)| < d_{\max} = 1$, the sampling period T_{s} is set to $T_{\text{s}} = 0.25$ s and the event-threshold $\bar{e}$ has been varied in order to adapt the properties of the event-triggered control to the sampled-data situation.

Consider first the sampled-data control loop (3), (9)–(11), where Eqs. (22), (23) yield $|e(t)| < e_{\max,\text{SDC}} = 0.35$. Using event-triggered control, the same approximation error bound results for the event threshold $\bar{e} = 0.47$ (see Eqs. (25), (27)). However, the resulting minimum inter-sampling times differ, i.e. $\bar{T}_{\text{DBC}} = 0.18$ s, $\bar{T}_{\text{MBETC}} = 0.54$ s. This shows that model-based event-triggered control has the same bound on the maximum approximation error as sampled-data control while guaranteeing a larger bound on the minimum-inter sampling time. In contrast, deadband control might even cause more sampling events compared to sampled-data control.

Similar results are obtained by considering the same bounds on the minimum inter-sampling time, i.e. $T_{\text{s}} = \bar{T}_{\text{DBC}} = \bar{T}_{\text{MBETC}} = 0.25$ s, which are obtained by using the event thresholds $\bar{e}_{\text{DBC}} = 0.72$, $\bar{e}_{\text{MBETC}} = 0.24$. The results are summarized in Tab. 2 which bring about that, in this example, model-based event-triggered control guarantees the best bounds on the approximation error and the minimum inter-sampling time whereas the worst guarantees are provided by deadband control.

Table 2. Approximation error bounds and minimum inter-sampling times

Sampling scheme	$\bar{e}$	$e_{\max}$	$\bar{T}$
SDC	–	0.35	0.25 s
DBC	0.47	0.35	0.18 s
DBC	0.72	0.54	0.25 s
MBETC	0.47	0.35	0.54 s
MBETC	0.24	0.18	0.25 s

5. CONCLUSIONS

This paper was focussed on analyzing and comparing sampled-data control, deadband control and model-based event-triggered control. The analysis was carried out by, firstly, proposing a uniform representation of these schemes and by, secondly, deriving stability and communication properties for each scheme. The results show that model-based event-triggered control always provides better guarantees than deadband control. This likewise holds compared to sampled-data control as long as certain conditions are satisfied.

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