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Stability Analysis of Discrete-time Systems with Poisson-distributed Delays

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Abstract— **This paper is concerned with the stability analysis of linear discrete-time systems with poisson-distributed delays. Firstly, the exponential stability condition of system with poisson-distributed delays is derived when the corresponding system with the zero-delay or the system without the delayed term is asymptotically stable. Then, an augmented Lyapunov functional is suggested to handle the case that the corresponding system without the delay as well as the system without the delayed term are not necessary to be asymptotically stable. Furthermore, we show that the results can be further improved by formulating the system as a higher-order augmented one and applying the corresponding augmented Lyapunov functional. Finally, the efficiency of the proposed results is illustrated by some numerical examples.**

Keywords: Infinite delays, poisson-distributed delays, stabilizing delays, Lyapunov method.

I. INTRODUCTION

Systems with distributed delays are frequently encountered in modeling the physiological behavior, the traffic flow, the population dynamics, and the control over networks [9], [10]. In general, there are two main classes of distributed delays, namely: finite distributed delays and infinite distributed delays. A great number of results have been reported for the stability and control of systems with finite distributed delays, e.g., [1], [2], [3], [12] and the reference therein. For the case of infinite distributed delays, we refer to [8], [9], [13], where necessary and sufficient conditions for the stability of continuous-time systems with gamma-distributed delays were derived in the frequency domain. In the time domain, sufficient conditions for the stability of continuous-time systems with gamma-distributed delays were derived in [14] via appropriate Lyapunov functionals. Recently, the Lyapunov-based stability and passivity analysis for diffusion partial differential equations with infinite distributed delays were presented in [15].

Discrete-time systems with infinite distributed delays have been analyzed in the literature. For example, the synchronization problem for an array of coupled complex discrete-time networks with infinite distributed delays was investigated in [7]. The state feedback control was considered in [16] for discrete-time stochastic systems with infinite distributed delays and nonlinear disturbances. In [17] and [18], the robust H_∞ control problem was discussed for discrete-time fuzzy systems with infinite distributed delays. It should be pointed out that all the results reported in [7], [16], [17], [18] are concerned with constant kernel function. Moreover, when the corresponding system without the delay as well as the system without the delayed term are not asymptotically stable, the proposed methods in [7], [16], [17], [18] are not applicable. It is well-known that poisson distribution is widespread in queuing theory [4]. In [11], the experimental data on the arrivals of pulses in indoor environments revealed that each cluster's time-delay is poisson-distributed, see also [5] for more explanations.

In the present paper, we consider linear discrete-time systems with poisson-distributed delays. The objective is to derive sufficient exponential stability conditions for the system via appropriate Lyapunov functionals. It is allowed that the corresponding system without the delay as well as the system without the delayed term are not asymptotically stable. Thus, the considered infinite distributed delays with a gap in the paper have stabilizing effects. We derive the results by transforming the system to an augmented one and applying augmented Lyapunov functionals [12], [14]. Due to the effect of poisson-distributed delays, the augmented system contains not only distributed but also discrete delays. This is different from the continuous-time counterpart in [14] for the general case of gamma-distributed delays, where only distributed delays were included in the resulting augmented system. Furthermore, we show that the results can be further improved by formulating the system as a higher-order augmented one and applying the corresponding augmented Lyapunov functional.

The structure of the paper is organized as follows. Section II presents the systems with poisson-distributed delays and the summation inequalities that will be employed. The exponential stability of systems with poisson-distributed delays is studied in Section III when the corresponding system without the delay or the system without the delayed term is asymptotically stable. Section IV shows the exponential stability conditions of systems with poisson-distributed delays when the corresponding system with the zero-delay as well as the system without the delayed term are allowed to be not asymptotically stable. Section V illustrates the efficiency of the presented approach with some examples. Finally, the conclusions and the further work are stated in Section VI.

Notations: The notations used throughout the paper are standard. The superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. $P \succ 0$ ($P \succeq 0$) means that P is positive definite (positive semi-

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definite). $*$ denotes the term that is induced by symmetry and I represents the unit matrix of appropriate dimensions. The symbol $\mathbb{Z}^+$ denotes the set of non-negative integers.

II. SYSTEM DESCRIPTION

Consider the following linear discrete-time system with poisson-distributed delays:

$$x(k+1) = Ax(k) + A_1 \sum_{\tau=0}^{+\infty} p(\tau)x(k-\tau), \quad k \in \mathbb{Z}^+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, the system matrices A and A_1 are constant with appropriate dimensions. The initial condition is given as $\mathrm{col}\{x(0), x(-1), x(-2), \dots\} = \mathrm{col}\{\phi(0), \phi(-1), \phi(-2), \dots\}$. The function $p(\theta)$ is a poisson distribution with a gap $h \in \mathbb{Z}^+$:

$$p(\theta) = \begin{cases} \frac{e^{-\lambda}\lambda^{\theta-h}}{(\theta-h)!} & \theta \geq h, \\ 0 & \theta < h. \end{cases}$$

The gap h can be interpreted e.g., in the network as the minimal propagation delay, which is always strictly positive. The mean value of p is $\lambda + h$. Due to the fact that

$$\begin{aligned} \sum_{\tau=0}^{+\infty} p(\tau)x(k-\tau) &= \sum_{\tau=h}^{+\infty} p(\tau)x(k-\tau) \\ &= \sum_{\theta=0}^{+\infty} p(\theta+h)x(k-\theta-h), \end{aligned}$$

we arrive at the equivalent system to (1) as follows:

$$x(k+1) = Ax(k) + A_1 \sum_{\tau=0}^{+\infty} \mathrm{P}(\tau)x(k-\tau-h), \; k \in \mathbb{Z}^+, \tag{2}$$

where $\mathrm{P}(\tau) = \frac{e^{-\lambda}\lambda^{\tau}}{\tau!}$. Moreover, some elementary calculus shows that for scalar $0 < \delta \leq 1$

$$\begin{aligned} &\sum_{i=0}^{+\infty} \delta^{-i-h}\mathrm{P}(i) = \delta^{-h}e^{(\delta^{-1}-1)\lambda} \triangleq \mathrm{p}_{0\delta}, \\ &\sum_{i=0}^{+\infty} \delta^{-i-h}(i+h)\mathrm{P}(i) = \delta^{-h}e^{(\delta^{-1}-1)\lambda}(\lambda\delta^{-1}+h) \triangleq \mathrm{p}_{1\delta}, \\ &\mathrm{p}_{11} \triangleq \mathrm{p}_{1\delta}|_{\delta=1} = \lambda + h. \end{aligned}$$

The derivation of stability conditions for system (2) is based on the summation inequalities with infinite sequences formulated in the following lemma.

Lemma 1 *Given an $n \times n$ matrix $R \succ 0$, an integer $h \geq 0$, scalar functions $M(i) \in \mathbb{R}$, $\alpha(i) \in \mathbb{R}^+ \backslash \{0\}$, and a vector function $x(i) \in \mathbb{R}^n$ such that the series concerned are convergent. Then the inequality*

$$\begin{aligned} &\sum_{i=0}^{+\infty} \alpha(i)|M(i)|x^T(i)Rx(i) \\ &\quad \geq M_0^{-1}\Big[\sum_{i=0}^{+\infty} M(i)x(i)\Big]^T R \Big[\sum_{i=0}^{+\infty} M(i)x(i)\Big], \end{aligned} \tag{3}$$

and its double summation extension

$$\begin{aligned} &\sum_{i=0}^{+\infty} \sum_{j=k-i-h}^{k-1} \alpha(i)|M(i)|x^T(j)Rx(j) \\ &\quad \geq M_{1h}^{-1}\Big[\sum_{i=0}^{+\infty} \sum_{j=k-i-h}^{k-1} M(i)x(j)\Big]^T R \\ &\quad \times \Big[\sum_{i=0}^{+\infty} \sum_{j=k-i-h}^{k-1} M(i)x(j)\Big], \end{aligned} \tag{4}$$

hold, where

$$\begin{aligned} M_0 &= \sum_{i=0}^{+\infty} \alpha^{-1}(i)|M(i)|, \\ M_{1h} &= \sum_{i=0}^{+\infty} \alpha^{-1}(i)(i+h)|M(i)|. \end{aligned} \tag{5}$$

Proof: The proofs of (3) and (4) follow from those in [14] by involving sums instead of integral. Since $R \succ 0$, application of Schur complements implies that the following holds

$$\begin{bmatrix} \alpha(i)|M(i)|x^T(i)Rx(i) & x^T(i)M(i) \\ * & \alpha^{-1}(i)|M(i)|R^{-1} \end{bmatrix} \succeq 0 \tag{6}$$

for any $i \in [0, +\infty]$, $i \in \mathbb{Z}^+$. Summation of (6) from 0 to $+\infty$ leads to

$$\begin{bmatrix} \sum_{i=0}^{+\infty}\alpha(i)|M(i)|x^T(i)Rx(i) & \sum_{i=0}^{+\infty} x^T(i)M(i) \\ * & M_0R^{-1} \end{bmatrix} \succeq 0.$$

By Schur complements, the above matrix inequality yields (3). Furthermore, double summation of

$$\begin{bmatrix} \alpha(i)|M(i)|x^T(j)Rx(j) & x^T(j)M(i) \\ * & \alpha^{-1}(i)|M(i)|R^{-1} \end{bmatrix} \succeq 0$$

from $k-i-h$ to $k-1$ in j and from 0 to $+\infty$ in i, where

$$\sum_{i=0}^{+\infty} \sum_{j=k-i-h}^{k-1} \alpha^{-1}(i)|M(i)| = M_{1h}$$

and Schur complements ensure that the inequality (4) holds.

In the sequel, the summation inequalities (3) and (4) with infinite sequences play an important role in the stability problem of discrete-time systems with poisson-distributed delays.

III. STABILITY IN THE CASE THAT A OR $A + A_1$ IS SCHUR STABLE

Consider system (2). It is assumed that A or $A + A_1$ is Schur stable. Our stability analysis will be based on the following discrete-time Lyapunov functional:

$$\begin{aligned} &V(k) = x^T(k)Wx(k) + V_{G_1}(k) + V_{H_1}(k), \\ &V_{G_1}(k) = \sum_{i=0}^{+\infty} \sum_{s=k-i-h}^{k-1} \delta^{k-s-1}\mathrm{P}(i)x^T(s)G_1x(s), \\ &V_{H_1}(k) = \sum_{i=0}^{+\infty} \sum_{j=1}^{i+h} \sum_{s=k-j}^{k-1} \delta^{k-s-1}\mathrm{P}(i)\eta_1^T(s)H_1\eta_1(s), \end{aligned} \tag{7}$$

where $0 < \delta < 1$, $W \succ 0$, $G_1 \succ 0$, $H_1 \succ 0$, and

$$\eta_1(k) = x(k+1) - x(k). \tag{8}$$

Remark 1 *The term $V_{G_1}(k)$ "compensates" the delayed term in (2) when A is Schur stable. The term $V_{H_1}(k)$ extends the triple integrals of [14] to discrete case. It "compensates" the summation term in*

$$\begin{aligned} x(k+1) &= (A + A_1)x(k) \\ &\quad + A_1 \sum_{\tau=0}^{+\infty} \mathrm{P}(\tau)[x(k-\tau-h) - x(k)], \; k \in \mathbb{Z}^+, \end{aligned}$$

and allows us to derive stability conditions of system (2) when $A + A_1$ is Schur stable whereas A is not. Moreover, the $V_{H_1}(k)$ term also improves the results provided that A is Schur stable.

By the standard arguments for discrete-time systems, we arrive at the following linear matrix inequality (LMI) condition for exponential stability of (2):

Proposition 1 *Given scalars $\lambda > 0$, $0 < \delta < 1$ and an integer $h \geq 0$, assume that there exist $n \times n$ positive definite matrices W, G_1 and H_1, such that the following LMI holds:*

$$\Xi = \Sigma + F_0^T W F_0 - \mathrm{p}_{1\delta}^{-1} F_{12}^T H_1 F_{12} + \mathrm{p}_{11} F_{01}^T H_1 F_{01} \prec 0, \tag{9}$$

where

$$\Sigma = \mathrm{diag}\{G_1 - \delta W, -\mathrm{p}_{0\delta}^{-1} G_1\},$$
$$F_0 = [A \;\; A_1],\;\; F_{01} = [A - I \;\; A_1],\;\; F_{12} = [I \;\; -I].$$

Then the system (2) is exponentially stable with the decay rate $\sqrt{\delta}$.

Proof: Denote

$$\begin{array}{l} f(k) = \sum_{\tau=0}^{+\infty} \mathrm{P}(\tau) x(k-\tau-h), \\ \xi(k) = \mathrm{col}\{x(k), f(k)\}, \quad k \in \mathbb{Z}^+. \end{array} \tag{10}$$

Taking difference of $V(k)$ along (2) leads to

$$\begin{array}{l} \Delta V(k) = V(k+1) - \delta V(k) \\ \leq \xi^T(k) F_0^T W F_0 \xi(k) + x^T(k)(G_1 - \delta W) x(k) \\ \quad + \mathrm{p}_{11} \eta_1^T(k) H_1 \eta(k) \\ \quad - \sum_{i=0}^{+\infty} \delta^{i+h} \mathrm{P}(i) x^T(k-i-h) G_1 x(k-i-h) \\ \quad - \sum_{i=0}^{+\infty} \sum_{s=k-i-h}^{k-1} \delta^{i+h} \mathrm{P}(i) \eta_1^T(s) H_1 \eta_1(s). \end{array} \tag{11}$$

Applying further the inequalities (3) and (4), we obtain

$$\begin{array}{l} -\sum_{i=0}^{+\infty} \delta^{i+h} \mathrm{P}(i) x^T(k-i-h) G_1 x(k-i-h) \\ \leq -\mathrm{p}_{0\delta}^{-1} f^T(k) G_1 f(k) \end{array} \tag{12}$$

and

$$\begin{array}{l} -\sum_{i=0}^{+\infty} \sum_{s=k-i-h}^{k-1} \delta^{i+h} \mathrm{P}(i) \eta_1^T(s) H_1 \eta_1(s) \\ \leq -\mathrm{p}_{1\delta}^{-1} \Big[\sum_{i=0}^{+\infty} \sum_{s=k-i-h}^{k-1} \mathrm{P}(i) \eta_1(s)\Big]^T H_1 \\ \quad \times \Big[\sum_{i=0}^{+\infty} \sum_{s=k-i-h}^{k-1} \mathrm{P}(i) \eta_1(s)\Big] \\ = -\mathrm{p}_{1\delta}^{-1} [x(k) - f(k)]^T H_1 [x(k) - f(k)] \\ = -\mathrm{p}_{1\delta}^{-1} \xi^T(k) F_{12}^T H_1 F_{12} \xi(k). \end{array} \tag{13}$$

Then (11)–(13) yield $\Delta V(k) = V(k+1) - \delta V(k) \leq \xi^T(k) \Xi \xi(k) \leq 0$ if LMI (9) holds. Due to the fact that

$$\lambda_{min}(W)|x(k)|^2 \leq V(k) \leq \delta^k V(0),\;\; V(0) \leq \beta \|\phi\|_c^2,$$

where $\beta > 0$ is a scalar and $\|\phi\|_c = \sup_{s=0,-1,-2,\ldots} |\phi(s)|$, the system (2) is exponentially stable with the decay rate $\sqrt{\delta}$ for given scalars $\lambda > 0$ and $h \geq 0$.

Remark 2 *If A is Schur stable, Lyapunov functional (7) with $H_1 = 0$ can be applied to yield the following simple but conservative LMI condition:*

$$\begin{bmatrix} G_1 + A^T W A - \delta W & A^T W A_1 \\ * & -\mathrm{p}_{0\delta}^{-1} G_1 + A_1^T W A_1 \end{bmatrix} \prec 0.$$

IV. Stability in the Case That A and $A + A_1$ Are Allowed to Be Not Schur Stable

In this section, we will derive LMI conditions for the exponential stability of system (2), where A and $A + A_1$ may not be Schur stable. This means that the considered infinite distributed delays with a gap have stabilizing effects. We provide two sufficient stability conditions, the efficiency of which is illustrated by the given examples below.

A. Condition I: an augmented system

From (10), it follows that system (2) can be transformed into the following augmented one:

$$\begin{array}{l} x(k+1) = Ax(k) + A_1 f(k), \\ f(k+1) = \sum_{\tau=0}^{+\infty} \frac{e^{-\lambda}\lambda^\tau}{\tau!} x(k+1-\tau-h) \\ \quad = e^{-\lambda} x(k+1-h) + \sum_{\tau=1}^{+\infty} \frac{e^{-\lambda}\lambda^\tau}{\tau!} x(k+1-\tau-h) \\ \quad = e^{-\lambda} x(k+1-h) + \sum_{\tau=0}^{+\infty} \frac{e^{-\lambda}\lambda^{\tau+1}}{(\tau+1)!} x(k-\tau-h) \\ \quad = e^{-\lambda} A x(k-h) + e^{-\lambda} A_1 f(k-h) \\ \qquad + \sum_{\tau=0}^{+\infty} \mathrm{Q}(\tau) x(k-\tau-h), \end{array} \tag{14}$$

where $\mathrm{Q}(\tau) = \frac{e^{-\lambda}\lambda^{\tau+1}}{(\tau+1)!}$.

Remark 3 *The stability of system (14) implies the stability of system (2), but not vice versa. For a positive integer h, the effect of poisson-distributed delays leads to the augmented system (14) that contains one term $\sum_{\tau=0}^{+\infty} \mathrm{Q}(\tau) x(k-\tau-h)$ corresponding to distributed delays, and two terms $e^{-\lambda} A x(k-h)$, $e^{-\lambda} A_1 f(k-h)$ with discrete delays. This is different from the continuous-time counterpart in [14] for the general case of gamma-distributed delays, where only distributed delays were included in the resulting augmented system.*

Simple computation shows that

$$\begin{array}{l} \sum_{i=0}^{+\infty} \delta^{-i-h} \mathrm{Q}(i) = \delta^{1-h} e^{-\lambda}(e^{\lambda/\delta} - 1) \triangleq \mathrm{q}_{0\delta}, \\ \sum_{i=0}^{+\infty} \delta^{-i-h} (i+h) \mathrm{Q}(i) \\ = \delta^{-h} e^{-\lambda} [\lambda e^{\lambda/\delta} + \delta(h-1)(e^{\lambda/\delta} - 1)] \triangleq \mathrm{q}_{1\delta}, \\ \mathrm{q}_{01} \triangleq \mathrm{q}_{0\delta}|_{\delta=1} = 1 - e^{-\lambda}, \\ \mathrm{q}_{11} \triangleq \mathrm{q}_{1\delta}|_{\delta=1} = \lambda + (h-1)(1 - e^{-\lambda}). \end{array} \tag{15}$$

Consider system (14) with both distributed and discrete delays. We suggest the following discrete-time Lyapunov functional:

$$\begin{array}{l} \hat{V}(k) = \xi^T(k) \hat{W} \xi(k) + \sum_{i=1}^2 \Big[V_{G_i}(k) + V_{H_i}(k) + V_{S_i}(k)\Big], \\ V_{G_2}(k) = \sum_{i=0}^{+\infty} \sum_{s=k-i-h}^{k-1} \delta^{k-s-1} \mathrm{Q}(i) x^T(s) G_2 x(s), \\ V_{H_2}(k) = \sum_{i=0}^{+\infty} \sum_{j=1}^{i+h} \sum_{s=k-j}^{k-1} \delta^{k-s-1} \mathrm{Q}(i) \eta_1^T(s) H_2 \eta_1(s), \\ V_{S_1}(k) = \sum_{s=k-h}^{k-1} \delta^{k-s-1} x^T(s) S_1 x(s) \\ \qquad + h \sum_{j=-h}^{-1} \sum_{s=k+j}^{k-1} \delta^{k-s-1} \eta_1^T(s) R_1 \eta_1(s), \\ V_{S_2}(k) = \sum_{s=k-h}^{k-1} \delta^{k-s-1} f^T(s) S_2 f(s) \\ \qquad + h \sum_{j=-h}^{-1} \sum_{s=k+j}^{k-1} \delta^{k-s-1} \eta_2^T(s) R_2 \eta_2(s), \end{array} \tag{16}$$

where the terms $V_{G_1}(k)$ and $V_{H_1}(k)$ are defined in (7), $\xi(k)$ and $\eta_1(k)$ are given in (10) and (8), respectively, $0 < \delta < 1$, $G_2 \succ 0$, $H_2 \succ 0$, $S_i \succ 0$, $R_i \succ 0$, $i = 1, 2$, and,

$$\hat{W} = \begin{bmatrix} P & Q \\ * & Z \end{bmatrix} \succ 0, \tag{17}$$

and

$$\eta_2(k) = f(k+1) - f(k). \tag{18}$$

Here the last two terms $V_{S_1}(k)$ and $V_{S_2}(k)$ are added to "compensate", respectively, the delayed terms $x(k-h)$ and $f(k-h)$ of (14). Therefore, for system (14) with $h = 0$, $V_{S_1}(k)$ and $V_{S_2}(k)$ are not necessary. We derive conditions to guarantee exponential stability of system (14) in the following proposition:

Proposition 2 *Given scalars $\lambda > 0$, $0 < \delta < 1$ and an integer $h \geq 0$, let there exist $n \times n$ positive definite matrices P, Z, G_i, H_i, S_i, R_i, $i = 1, 2$, and an $n \times n$ matrix Q such that (17) and the following LMI are satisfied:*

$$\begin{aligned}\hat{\Xi} &= \hat{\Sigma}+\hat{F}_0^T\hat{W}\hat{F}_0-\delta\hat{F}_1^T\hat{W}\hat{F}_1 - \mathrm{p}_{1\delta}^{-1}\hat{F}_{12}^T H_1\hat{F}_{12}\\ &+\hat{F}_{01}^T[\mathrm{p}_{11}H_1 + \mathrm{q}_{11}H_2 + h^2R_1]\hat{F}_{01} + h^2\hat{F}_{02}^T R_2\hat{F}_{02}\\ &-\mathrm{q}_{1\delta}^{-1}\hat{F}_{15}^T H_2\hat{F}_{15} - \delta^h\hat{F}_{13}^T R_1\hat{F}_{13} - \delta^h\hat{F}_{24}^T R_2\hat{F}_{24} \prec 0,\end{aligned} \tag{19}$$

where $\hat{\Sigma} = \mathrm{diag}\{S_1 + G_1 + \mathrm{q}_{01}G_2, -\mathrm{p}_{0\delta}^{-1}G_1 + S_2, -\delta^h S_1, -\delta^h S_2, -\mathrm{q}_{0\delta}^{-1}G_2\}$,

$$\hat{F}_0 = \begin{bmatrix} A & A_1 & 0 & 0 & 0\\ 0 & 0 & e^{-\lambda}A & e^{-\lambda}A_1 & I\end{bmatrix},$$
$$\hat{F}_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\end{bmatrix},\ \hat{F}_{01} = [A-I\ A_1\ 0\ 0\ 0],$$
$$\hat{F}_{02} = [0\ \ -I\ \ e^{-\lambda}A\ \ e^{-\lambda}A_1\ \ I],\ \hat{F}_{12} = [I\ \ -I\ 0\ 0\ 0],$$
$$\hat{F}_{13} = [I\ 0\ \ -I\ 0\ 0],\ \hat{F}_{15} = [\mathrm{q}_{01}I\ 0\ 0\ 0\ \ -I],$$
$$\hat{F}_{24} = [0\ I\ 0\ \ -I\ 0].$$

Then the system (14) is exponentially stable with the decay rate $\sqrt{\delta}$.

Proof: Define

$$\upsilon(k) = \textstyle\sum_{\tau=0}^{+\infty}\mathrm{Q}(\tau)x(k-\tau-h) \tag{20}$$

and $\hat{\xi}(k) = \mathrm{col}\{x(k), f(k), x(k-h), f(k-h), \upsilon(k)\}$. By taking difference of $\hat{V}(k)$ along (14) and applying Jensen inequality with finite sequences (see e.g., [6]), we have

$$\begin{aligned}&\xi^T(k+1)\hat{W}\xi(k+1) - \delta\xi^T(k)\hat{W}\xi(k)\\ &= \hat{\xi}^T(k)[\hat{F}_0^T\hat{W}\hat{F}_0 - \delta\hat{F}_1^T\hat{W}\hat{F}_1]\hat{\xi}(k)\end{aligned} \tag{21}$$

and

$$\begin{aligned}&\textstyle\sum_{i=1}^2 \Big[V_{G_i}(k+1) + V_{H_i}(k+1) + V_{S_i}(k+1)\\ &\qquad -\delta V_{G_i}(k) - \delta V_{H_i}(k) - \delta V_{S_i}(k)\Big]\\ &\leq \hat{\xi}^T(k)\Big[\hat{\Sigma} + \hat{F}_{01}^T[\mathrm{p}_{11}H_1 + \mathrm{q}_{11}H_2 + h^2R_1]\hat{F}_{01}\\ &\qquad -\mathrm{p}_{1\delta}^{-1}\hat{F}_{12}^T H_1\hat{F}_{12} + h^2\hat{F}_{02}^T R_2\hat{F}_{02}\\ &\qquad -\delta^h\hat{F}_{13}^T R_1\hat{F}_{13} - \delta^h\hat{F}_{24}^T R_2\hat{F}_{24}\Big]\hat{\xi}(k)\\ &\quad +\mathrm{q}_{0\delta}^{-1}\upsilon^T(k)G_2\upsilon(k)\\ &\quad -\textstyle\sum_{i=0}^{+\infty}\delta^{i+h}\mathrm{Q}(i)x^T(k-i-h)G_2x(k-i-h)\\ &\quad -\textstyle\sum_{i=0}^{+\infty}\sum_{s=k-i-h}^{k-1}\delta^{i+h}\mathrm{Q}(i)\eta_1^T(s)H_2\eta_1(s).\end{aligned} \tag{22}$$

Applying further the inequality (3), we obtain

$$\begin{aligned}&-\textstyle\sum_{i=0}^{+\infty}\delta^{i+h}\mathrm{Q}(i)x^T(k-i-h)G_2x(k-i-h)\\ &\leq -\mathrm{q}_{0\delta}^{-1}\upsilon^T(k)G_2\upsilon(k).\end{aligned} \tag{23}$$

Furthermore, the application of (4) leads to

$$\begin{aligned}&-\textstyle\sum_{i=0}^{+\infty}\sum_{s=k-i-h}^{k-1}\delta^{i+h}\mathrm{Q}(i)\eta_1^T(s)H_2\eta_1(s)\\ &\leq -\mathrm{q}_{1\delta}^{-1}\Big[\textstyle\sum_{i=0}^{+\infty}\sum_{s=k-i-h}^{k-1}\mathrm{Q}(i)\eta_1(s)\Big]^T H_2\\ &\quad\times\Big[\textstyle\sum_{i=0}^{+\infty}\sum_{s=k-i-h}^{k-1}\mathrm{Q}(i)\eta_1(s)\Big]\\ &= -\mathrm{q}_{1\delta}^{-1}[\mathrm{q}_{01}x(k) - \upsilon(k)]^T H_2[\mathrm{q}_{01}x(k) - \upsilon(k)]\\ &= -\mathrm{q}_{1\delta}^{-1}\hat{\xi}^T(k)\hat{F}_{15}^T H_2\hat{F}_{15}\hat{\xi}(k).\end{aligned} \tag{24}$$

Therefore, (21)–(24) yield $\Delta\hat{V}(k) = \hat{V}(k+1) - \delta\hat{V}(k) \leq \hat{\xi}^T(k)\hat{\Xi}\hat{\xi}(k)$. Then if LMI (19) holds for given scalars $\lambda > 0$ and $h \geq 0$, the system (14) is exponentially stable with the decay rate $\sqrt{\delta}$.

B. Condition II: a higher-order augmented system

For the case that A and $A + A_1$ are not necessary to be Schur stable, alternatively, we analyze the stability of system (2) by transforming (2) into the following higher-order augmented one:

$$\begin{aligned}x(k+1) &= Ax(k) + A_1f(k),\\ f(k+1) &= e^{-\lambda}Ax(k-h) + e^{-\lambda}A_1f(k-h) + \upsilon(k),\\ \upsilon(k+1) &= \lambda e^{-\lambda}Ax(k-h) + \lambda e^{-\lambda}A_1f(k-h) + g(k),\end{aligned} \tag{25}$$

where $f(k)$ and $\upsilon(k)$ are defined in (10) and (20), respectively, and

$$\begin{aligned}g(k) &= \textstyle\sum_{\tau=0}^{+\infty}\mathrm{U}(\tau)x(k-\tau-h),\\ \mathrm{U}(\tau) &= \tfrac{e^{-\lambda}\lambda^{\tau+2}}{(\tau+2)!}.\end{aligned}$$

Similar to (15), we have

$$\begin{aligned}&\textstyle\sum_{i=0}^{+\infty}\delta^{-i-h}\mathrm{U}(i) = \delta^{2-h}e^{-\lambda}(e^{\lambda/\delta} - \delta^{-1}\lambda - 1) \triangleq \mathrm{u}_{0\delta},\\ &\textstyle\sum_{i=0}^{+\infty}\delta^{-i-h}(i+h)\mathrm{U}(i)\\ &=\delta^{2-h}e^{-\lambda}[\delta^{-2}\lambda^2+(h-2+\delta^{-1}\lambda)(e^{\lambda/\delta}-\delta^{-1}\lambda-1)] \triangleq \mathrm{u}_{1\delta},\end{aligned}$$

and

$$\begin{aligned}\mathrm{u}_{01} &\triangleq \mathrm{u}_{0\delta}|_{\delta=1} = 1 - e^{-\lambda} - \lambda e^{-\lambda},\\ \mathrm{u}_{11} &\triangleq \mathrm{u}_{1\delta}|_{\delta=1} = \lambda - \lambda e^{-\lambda} + (h-2)(1 - e^{-\lambda} - \lambda e^{-\lambda}).\end{aligned}$$

For system (25), we introduce the following augmented Lyapunov functional:

$$\begin{aligned}\bar{V}(k) &= \bar{x}^T(k)\bar{W}\bar{x}(k)+\textstyle\sum_{i=1}^2\Big[V_{G_i}(k) + V_{H_i}(k) + V_{S_i}(k)\Big]\\ &\quad +V_{G_3}(k) + V_{H_3}(k)\end{aligned}$$

with

$$\begin{aligned}V_{G_3}(k) &= \textstyle\sum_{i=0}^{+\infty}\sum_{s=k-i-h}^{k-1}\delta^{k-s-1}\mathrm{U}(i)x^T(s)G_3x(s),\\ V_{H_3}(k) &= \textstyle\sum_{i=0}^{+\infty}\sum_{j=1}^{i+h}\sum_{s=k-j}^{k-1}\delta^{k-s-1}\mathrm{U}(i)\eta_1^T(s)H_3\eta_1(s),\end{aligned}$$

where the terms $V_{G_i}(k)$, $V_{H_i}(k)$ and $V_{S_i}(k)$, $i = 1, 2$, are defined in (16), $0 < \delta < 1$, $G_3 \succ 0$, $H_3 \succ 0$, $\bar{x}(k) = \mathrm{col}\{x(k),\ f(k),\ \upsilon(k)\}$, $\eta_1(k)$ is given by (8), and,

$$\bar{W} = \begin{bmatrix} P & Q_1 & Q_2\\ * & Z & Q_3\\ * & * & R\end{bmatrix} \succ 0. \tag{26}$$

Remark 4 *Differently from the continuous-time counterpart in [14] for the general case of gamma-distributed delays, to analyze stability of the higher-order augmented system (25) for poisson-distributed delays, two additional terms $V_{G_3}(k)$ and $V_{H_3}(k)$ are necessary to "compensate" the term $g(k) = \sum_{\tau=0}^{+\infty}\mathrm{U}(\tau)x(k-\tau-h)$ of (25).*

Following the proof of Proposition 2, we derive the following condition for exponential stability of system (25):

Proposition 3 *Given scalars $\lambda > 0$, $0 < \delta < 1$ and an integer $h \geq 0$, let there exist $n \times n$ positive definite matrices*

TABLE I

COMPLEXITY OF DIFFERENT STABILITY CONDITIONS

Method	Decision variables	Number and order of LMIs
Proposition 1	$1.5n^2+1.5n$	one of $3n\times 3n$
Proposition 2	$6n^2+5n$	one of $7n\times 7n$ one of $2n\times 2n$
Proposition 3	$9.5n^2+6.5n$	one of $8n\times 8n$ one of $3n\times 3n$

P, Z, R, S_i, R_i, $i=1,2$, G_j, H_j, $j=1,2,3$, and $n\times n$ matrices Q_j, $j=1,2,3$, such that (26) and the following LMI are feasible:

$$\begin{aligned}\bar{\Xi} &= \bar{\Sigma} + \bar{F}_0^T \bar{W} \bar{F}_0 - \delta \bar{F}_1^T \bar{W} \bar{F}_1 - \mathrm{p}_{1\delta}^{-1} \bar{F}_{12}^T H_1 \bar{F}_{12} \\ &+ \bar{F}_{01}^T[\mathrm{p}_{11}H_1 + \mathrm{q}_{11}H_2 + \mathrm{u}_{11}H_3 + h^2 R_1]\bar{F}_{01} \\ &+ h^2 \bar{F}_{02}^T R_2 \bar{F}_{02} - \mathrm{q}_{1\delta}^{-1} \bar{F}_{15}^T H_2 \bar{F}_{15} - \mathrm{u}_{1\delta}^{-1} \bar{F}_{16}^T H_3 \bar{F}_{16} \\ &- \delta^h \bar{F}_{13}^T R_1 \bar{F}_{13} - \delta^h \bar{F}_{24}^T R_2 \bar{F}_{24} \prec 0,\end{aligned} \tag{27}$$

where $\bar{\Sigma} = \mathrm{diag}\{S_1 + G_1 + \mathrm{q}_{01}G_2 + \mathrm{u}_{01}G_3, -\mathrm{p}_{0\delta}^{-1}G_1 + S_2, -\delta^h S_1, -\delta^h S_2, -\mathrm{q}_{0\delta}^{-1}G_2, -\mathrm{u}_{0\delta}^{-1}G_3\}$,

$$\bar{F}_0 = \begin{bmatrix} A & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\lambda}A & e^{-\lambda}A_1 & I & 0 \\ 0 & 0 & \lambda e^{-\lambda}A & \lambda e^{-\lambda}A_1 & 0 & I \end{bmatrix},$$

$$\bar{F}_1 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}, \ \bar{F}_{01} = [A-I \ A_1 \ 0 \ 0 \ 0 \ 0],$$

$$\bar{F}_{02} = [0 \ -I \ \ e^{-\lambda}A \ \ e^{-\lambda}A_1 \ \ I \ 0], \ \bar{F}_{12} = [I \ -I \ 0 \ 0 \ 0 \ 0],$$
$$\bar{F}_{13} = [I \ 0 \ -I \ 0 \ 0 \ 0], \ \bar{F}_{15} = [\mathrm{q}_{01}I \ 0 \ 0 \ 0 \ -I \ 0],$$
$$\bar{F}_{16} = [\mathrm{u}_{01}I \ 0 \ 0 \ 0 \ 0 \ -I], \quad \bar{F}_{24} = [0 \ I \ 0 \ -I \ 0 \ 0].$$

Then the system (25) is exponentially stable with the decay rate $\sqrt{\delta}$.

Remark 5 *Compare the number of scalar decision variables and the resulting LMIs. See Table I for the complexity of different stability conditions. Note that Proposition 3 achieves less conservative results than Propositions 1 and 2 on account of computational complexity (see examples below).*

Remark 6 *The system (2) with poisson-distributed delays can be also transformed into an augmented system with respect to the state* $\mathrm{col}\{x(k), f(k), \upsilon(k), g(k)\}$. *By virtue of corresponding augmented Lyapunov functional, the achieved less conservative results suffer from more decision variables and higher order LMIs.*

V. ILLUSTRATIVE EXAMPLES

In this section, we provide two examples to demonstrate the efficiency of stability conditions.

A. Example 1

Consider the linear discrete-time system (1) with

$$A = \begin{bmatrix} 1.05 & 0.01 \\ 0.1 & 1.05 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} -0.15 & 0 \\ 0.1 & -0.1 \end{bmatrix}.$$

Here $A + A_1$ is Schur stable whereas A is not. For the values of h given in Table II, by applying Propositions 1, 2 and 3 with $\delta = 1$, we obtain the maximum allowable values of λ that guarantee the asymptotic stability (see Table II). For $\delta = 1$ the stability region in the (λ, h) plane that preserves the asymptotic stability is depicted in Figs. 1–3 by using Propositions 1, 2 and 3, respectively. The simulation results in Figs. 1–3 verify that Proposition 2 with the number $\{6n^2 + 5n\}_{n=2} = 34$ of variables induces more dense stability region than Proposition 1 with $\{1.5n^2 + 1.5n\}_{n=2} = 9$ variables, but guarantees sparser stability region than Proposition 3 with the number $\{9.5n^2 + 6.5n\}_{n=2} = 51$ of variables.

TABLE II

EXAMPLE 1: MAXIMUM ALLOWABLE VALUES OF λ FOR DIFFERENT h

$[\max \lambda] \setminus h$	0	1	2	8
Proposition 1	8.25	7.25	6.25	0.25
Proposition 2	8.41	7.41	6.41	0.41
Proposition 3	8.55	7.55	6.55	0.55

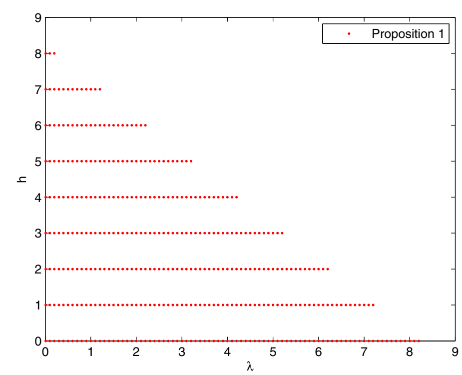

Fig. 1. Example 1: stability region by Proposition 1 with $\delta = 1$

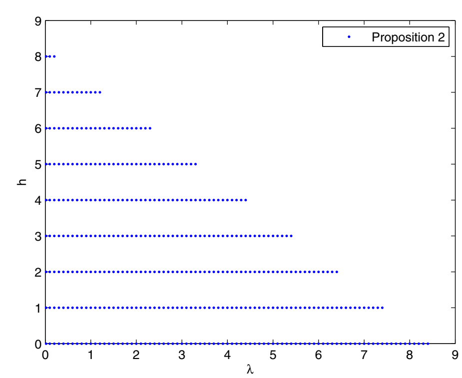

Fig. 2. Example 1: stability region by Proposition 2 with $\delta = 1$

B. Example 2

Consider the linear discrete-time system (1) with the coefficient matrices:

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} -0.5 & 0.8 \\ 0.5 & -0.2 \end{bmatrix}.$$

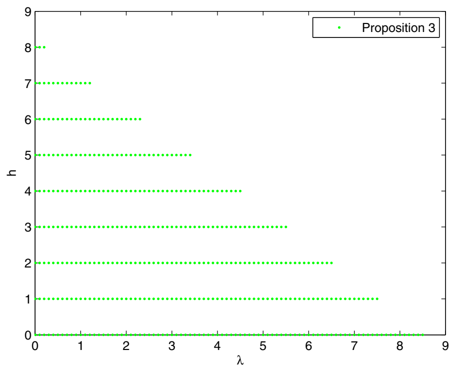

Fig. 3. Example 1: stability region by Proposition 3 with $\delta = 1$

It is worth mentioning that in this system both A and $A + A_1$ are not Schur stable. Thus, Proposition 1 is not applicable. For $h = 0$ the allowable values of λ that guarantee the asymptotic stability of the system by Propositions 2 and 3 with $\delta = 1$, are shown in Fig. 4. Therefore, the delays have stabilizing effects. It is observed that in comparison with Proposition 2, Proposition 3 improves the results but at the price of $\{3.5n^2 + 1.5n\}_{n=2} = 17$ additional decision variables.

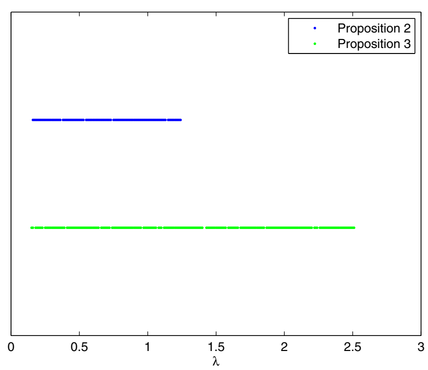

Fig. 4. Example 2: allowable values of λ by Propositions 2 and 3 with $h = 0$ and $\delta = 1$

VI. Conclusions

In this paper, we have presented LMI conditions for exponential stability of linear discrete-time systems with poisson-distributed delays. Especially, by formulating the system as an augmented one we have handled the case of stabilizing delay, i.e., the corresponding system with the zero-delay as well as the system without the delayed term are not necessary to be asymptotically stable. Extensions of the proposed direct Lyapunov approach to networked control systems will be interesting topics for future research.

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