Stability interval for time-varying delay systems

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Abstract— We investigate the stability analysis of linear time-delay systems. The time-delay is assumed to be a time-varying continuous function belonging to an interval (possibly excluding zero) with a bound on its derivative. To this end, we propose to use the quadratic separation framework to assess the intervals on the delay that preserves the stability. Nevertheless, to take the time-varying nature of the delay into account, the quadratic separation principle has to be extended to cope with the general case of time-varying operators. The key idea lies in rewording the delay system as a feedback interconnection consisting of operators that characterize it. The original feature of this contribution is to design a set of additional auxiliary operators that enhance the system modelling and reduce the conservatism of the methodology. Then, separation conditions lead to linear matrix inequality conditions which can be efficiently solved with available semi-definite programming algorithms. The paper concludes with illustrative academic examples.

I. INTRODUCTION

Time-delay systems and their stability have been intensively studied since several decades. The reasons are not only the challenging theoretical issues of this problem, but also because the dead-time effects are often met in applied problems [1]. Indeed, many processes include dead-time phenomena such as biology, chemistry, economics, as well as population dynamics. Furthermore, in communication networks or networked control systems, delays are inherent to data transportation, propagation time as well as processing time and are often the origin of performances and stability degradation.

In the case of constant delay and unperturbed linear systems, efficient criteria based on roots location [2], [3] allow to find all the stability regions with respect to the value of the delay. For the case of uncertain linear systems, i.e. for proving the robust stability, the problem has been partially solved, either by using Lyapunov functionals [4], [5], [6] or robustness tools (small gain theory [4], quadratic separation [7], [8]). All resulting stability conditions are based on convex optimization (Linear Matrix Inequality framework) and allow to conclude on stability region with respect to the delay and/or the uncertainties. For time-varying delays, the results are much more scarce and are mainly based on Lyapunov-Krasovskii [9], [10], [11], [12], [13] or IQCs/quadratic separation [14], [15]. Besides, all these latter methodologies often require, explicitly or implicitly, the delay-free system to be stable which is a rather important restriction.

This paper aims at going further in providing an efficient delay range stability condition even if the delay-free system

is unstable. More precisely, we propose to construct criteria based on an extension of the quadratic separation principle [7], [16], already developed for several delay-dependent conditions. Criteria are then derived and expressed in terms of Linear Matrix Inequalities (LMIs) which may be solved efficiently with Semi-Definite Programming (SDP) solvers.

The derivation of the proposed results is based on redundant system modelling. Indeed, based on known interactions between delays, their variations and derivatives, redundant equations are introduced to construct a new modelling of the delay systems. To this end, an augmented state is considered which is composed of the original state vector and its derivatives. Then, a suitable interconnection modelling is proposed, improved with the use of auxiliary operators that emphasize the relationships. At last, a delay range stability condition (the delay h is belonging to a prescribed interval $[h_{min}, h_{max}]$) is also introduced. This condition is able to detect pockets of stability even in case of unstable delay-free systems.

Notations: Throughout the paper, the following notations are used. The set of L_2^n consists of all measurable functions $f : \mathbb{R}^+ \to \mathbb{C}^n$ such that the following norm $\|f\|_{L_2} = \left(\int_0^\infty (f^*(t)f(t)) \right)^{1/2} dt < \infty$. When context allows it, the superscript n of the dimension will be omitted. The set L_{2e}^n denotes the extended set of L_2^n which consists of the functions whose time truncation lies in L_2^n. For two symmetric matrices, A and B, $A > (\geq) B$ means that $A - B$ is (semi-) positive definite. A^T denotes the transpose of A. 1_n and $0_{m \times n}$ denote respectively the identity matrix of size n and null matrix of size $m \times n$. If the context allows it, the dimensions of these matrices are often omitted. $\text{diag}(A, B, C)$ stands for the block diagonal matrix: $\text{diag}(A, B, C) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$.

Introduce as well the truncation operator $\mathbb{P}_T$ such that:
$$\mathbb{P}_T(f) = f_T = \begin{cases} f(t), & t \leq T, \\ 0, & t > T. \end{cases}$$

II. PRELIMINARIES

A. Problem statement

Let consider the following time-varying delay system:
$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - h(t)) & \forall t \geq 0, \\ x(t) = \phi(t) & \forall t \in [-h_{max}, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, ϕ is the initial condition and $A, A_d \in \mathbb{R}^{n \times n}$ are constant matrices. The delay h is time-varying and the following constraints are assumed

$$h(t) \in [h_{min}, h_{max}] \quad \text{and} \quad |\dot{h}(t)| \leq d, \quad (2)$$

where h_{min}, h_{max} and d are given constant scalars.

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B. Stability analysis via quadratic separation

Coming from robust control theory, the quadratic separation provides a fruitful framework to address the stability issue of non-linear and uncertain systems [7], [16]. Recent studies [8] have shown that such a framework allows to reduce significantly the conservatism of the stability analysis of time-delay systems with constant delay. Nevertheless, the delay being time-varying, the previous results [8], restricted to time-invariant systems, cannot be applied directly and should be extended to handle time-varying operators. To this end, based on the inner product and the L_{2e} space, a suitable theorem is proposed.

Let consider the interconnection defined by Figure 1 where $\mathcal{E}$ and $\mathcal{A}$ are two, real valued, possibly non-square matrices and ∇ is a linear operator from L_{2e} to L_{2e}. For simplicity of notations, we assume in the present paper that $\mathcal{E}$ is full column rank. Assuming the well-posedness, we are interested in looking for conditions that ensure the stability of the interconnection.
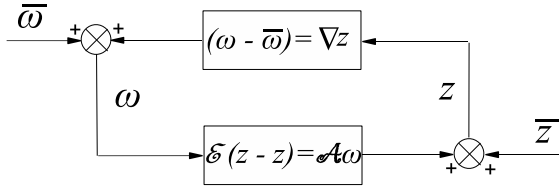

Fig. 1. Feedback system.

Theorem 1: The interconnected system of Figure 1 is stable if there exists a symmetric matrix $\Theta = \Theta'$ satisfying both conditions

$$\left[\begin{array}{cc} \mathcal{E} & -\mathcal{A} \end{array}\right]^{\perp'} \Theta \left[\begin{array}{cc} \mathcal{E} & -\mathcal{A} \end{array}\right]^{\perp} > 0 \qquad (3)$$

$$\forall u \in L_{2e}, \forall T > 0, \ \left\langle \left[\begin{array}{c} 1 \\ \mathbb{P}_T \nabla \end{array}\right] u_T, \Theta \left[\begin{array}{c} 1 \\ \mathbb{P}_T \nabla \end{array}\right] u_T \right\rangle \le 0 \qquad (4)$$

Proof: Inspired from [7], the proof is detailed in [15] and [17]. ∎

This result suggests that the proof of stability include two conditions: a matrix inequality (3) related to the lower bloc and a inner product (4) that states a quadratic constraint (Integral Quadratic Constraint) on the upper one. Basically, inequality (4) which forms an integral quadratic constraint, is built from definitions and informations on different operators which compose the matrix ∇. Then, the other one (3) provides the stability condition of the interconnection.

C. Defining operators

Toward modeling delay system as an interconnected system such as illustrated in Figure 1, it is required to define appropriate operators. Define the integral operator

$$\mathcal{I}: \quad \begin{array}{l} L_{2e} \to L_{2e}, \\ x(t) \to \int_0^t x(\theta) d\theta, \end{array} \qquad (5)$$

and the delay operator (or shift operator)

$$\mathcal{D}: \quad \begin{array}{l} L_{2e} \to L_{2e}, \\ x(t) \to x(t-h), \end{array} \qquad (6)$$

which constitute the fundamental elementary operators to describe a delay system. The related integral quadratic constraints are introduced in the following two lemmas. These latters will be helpful to construct inequality (4) and to derive then stability criteria for linear systems with time-varying delays in the next section.

Lemma 1: An integral quadratic constraint for the operator $\mathcal{I}$ is given by the following inequality $\forall x \in L_{2e}^n$ and for any positive definite matrix P,

$$\left\langle \left[\begin{array}{c} 1_n \\ \mathbb{P}_T \mathcal{I} 1_n \end{array}\right] x_T, \left[\begin{array}{cc} 0 & -P \\ -P & 0 \end{array}\right] \left[\begin{array}{c} 1_n \\ \mathbb{P}_T \mathcal{I} 1_n \end{array}\right] x_T \right\rangle \le 0.$$

Proof: See [18]. ∎

The second step is to derive an integral constraint for the operator $\mathcal{D}$.

Lemma 2: An integral quadratic constraint for the operator $\mathcal{D}$ is given by the following inequality $\forall T > 0, \forall x \in L_{2e}^n$ and for any positive matrix Q,

$$\left\langle \left[\begin{array}{c} 1_n \\ \mathbb{P}_T \mathcal{D} 1_n \end{array}\right] x_T, \left[\begin{array}{cc} -Q & 0 \\ 0 & Q(1-\dot{h}) \end{array}\right] \left[\begin{array}{c} 1_n \\ \mathbb{P}_T \mathcal{D} 1_n \end{array}\right] x_T \right\rangle \le 0 \qquad (7)$$

Proof: See [18]. ∎

In the constant delay case, when looking at works dedicated to the robust analysis for time delay systems, another operator is also introduced and expressed as $\frac{1-e^{-hs}}{s}$, [14], [8]. This latter is usually embedded as a norm bounded uncertainty, considering that $\sup_w \left\| \frac{1-e^{-j\omega h}}{j\omega} \right\| \le h_{max}$. Following the same idea, we formulate now the time-varying counterpart.

Lemma 3: An integral quadratic constraint for the operator $\mathcal{F} = (1 - \mathcal{D}) \circ \mathcal{I}$ is given by the following inequality $\forall x \in L_{2e}^n$ and for a positive definite matrix R,

$$\left\langle \left[\begin{array}{c} 1_n \\ \mathbb{P}_T \mathcal{F} 1_n \end{array}\right] x_T, \left[\begin{array}{cc} -h_{max}^2 R & 0 \\ 0 & R \end{array}\right] \left[\begin{array}{c} 1_n \\ \mathbb{P}_T \mathcal{F} 1_n \end{array}\right] x_T \right\rangle \le 0,$$

where h_{max} is the upperbound on the delay $h(t)$.

Proof: See [14] or [15] for the same formulation. ∎

III. MAIN RESULTS

A. Stability of time-varying delay systems: methodology

To illustrate the idea of the methodology, let us reformulate the dynamic of the system (1) as suggested in Figure 1 on a simple case. As a first modelling, we take advantage of the three aforementioned operators. In these conditions, the system (1) can be described as the feedback

$$\underbrace{\left[\begin{array}{c} x(t) \\ x(t-h(t)) \\ x(t)-x(t-h) \end{array}\right]}_{w(t)} = \underbrace{\left[\begin{array}{ccc} \mathcal{I}1_n & & \\ & \mathcal{D}1_n & \\ & & \mathcal{F}1_n \end{array}\right]}_{\nabla} \underbrace{\left[\begin{array}{c} \dot{x}(t) \\ x(t) \\ \dot{x}(t) \end{array}\right]}_{z(t)} \qquad (8)$$

over the feedforward equation

$$
\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathcal{E}} z(t) = \underbrace{\begin{bmatrix} A & A_d & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}}_{\mathcal{A}} w(t). \quad (9)
$$

Then, for applying Theorem 1, we have to find a separator Θ that fulfills both inequalities (3)-(4). Note that combining the three constraints related to the different operators (stated by the lemmas in Section II-C), a global (conservative) constraint on ∇ is deduced. Hence, the matrix

$$
\Theta = \begin{bmatrix} 0 & 0 & 0 & -P & 0 & 0 \\ 0 & -Q & 0 & 0 & 0 & 0 \\ 0 & 0 & -h_{max}^2 R & 0 & 0 & 0 \\ -P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q(1-\dot{h}(t)) & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{bmatrix}
$$
$$(10)$$

where P, Q and R are $n \times n$ positive definite matrices, satisfies the inequality (4). Then, it remains to assess the other one, which forms the stability criterion. Eventually, we can conclude that the interconnected system (8)-(9) (and therefore the system (1)) is stable if the matrix inequality (3), with $\mathcal{E}$, $\mathcal{A}$ and Θ defined as (9) and (10), holds. Because of the occurrences of h_{max} and $\dot{h}(t)$ in the criterion, it is refered as *delay and rate dependent*. Setting $\dot{h}(t) = d$ in the separator, the condition becomes a single LMI that can be easily solved via SDP programming.

Remark 1: It has been shown in [15] that the above criterion, based on the three well-known operators, provides the same results in terms of conservativeness than several classical results of the literature [5], [19]. Indeed, such a particular choice of operators and separator amounts to choosing a Lyapunov-Krasovskii functional candidate of the form:

$$
V(x_t) = x_t^T(0)Px_t(0) + \int_{-h(t)}^{0} x_t^T(\theta)Qx_t(\theta)d\theta
$$
$$
+ \int_{t-h_m}^{0}\int_{\theta}^{0} \dot{x}_t^T(s)R\dot{x}_t(s)dsd\theta.
$$

More generally, some interesting papers have emphasized the existing links between the Lyapunov method and the robust analysis [11], [20], [21].

Remark 2: A further simple criterion can be derived removing the third operator $\mathcal{F}$ from ∇ and considering only the minimal elementary operators (the integrator and the delay) required to describe a time-varying delay system. In that case, the stability condition would be *independent of the delay* because no information on the size of $h(t)$ (for instance, h_{max}) would be available in the matrices $\mathcal{E}$, $\mathcal{A}$ and Θ. However, it remains a *rate dependent* condition where a bound on $\dot{h}$ is required.

Remark 3: Because the inequality in the Lemma 2 imposes a constraint on the delay variation $\dot{h}(t)$, a *rate independent* condition can be obtained if the system (1) is modeled only through the first and the third operators. Of course, the matrices $\mathcal{E}$ and $\mathcal{A}$ of the implicit equation have to be appropriately designed so as to describe the original system and to link each component of the internal signals ($w(t)$ and $z(t)$).

In the next sections, we investigate new modellings of the delayed dynamic via the introduction of extra operators. The objective is twice: on one hand we expect to reduce the conservatism of the stability analysis, on the other hand, we want to take into account some informations on the delay (upperbound, interval). Throughout the paper we will observe the following procedure:

(a) *Rewrite the delay system (1) as an interconnected feedback.*

(b) *Embed the integrator, the delay and other auxiliary operators into the matrix ∇.*

(c) *Construct integral quadratic constraints for each operator and thus for ∇. Deduce the separator Θ.*

(d) *Obtain Linear Matrix Inequalities.*

B. Model extension

Previous works [22] and [8], [23] have shown that redundant system modelling (for linear uncertain systems and constant delay systems, respectively) may increase the relevancy of the stability analysis. The rational behind this model extension is to provide some extra relations between the delay, its variations and the state. Using the derivative operator, an augmented state is constructed and is composed of the original state vector and its derivative. Then defining relationship between augmented state $\dot{x}$, $\ddot{x}$, the delay h and its derivative $\dot{h}$, an enhanced stability condition is provided. Differentiating the system (1), we get:

$$
\ddot{x}(t) = A\dot{x}(t) + (1 - \dot{h}(t))A_d\dot{x}(t - h(t)).
$$

Consider the artificially augmented system

$$
\begin{cases} \dot{x}(t) = Ax(t) + A_dx(t - h(t)), \\ \ddot{x}(t) = A\dot{x}(t) + (1 - \dot{h}(t))A_d\dot{x}(t - h(t)), \end{cases} \quad (11)
$$

so as to embed on the model extra informations. Introducing the augmented state

$$
\varsigma(t) = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}, \quad (12)
$$

and specifying the relationship between the two components of $\varsigma(t)$ with the equality $[0\ \ 1]\varsigma(\mathrm{t}) = [1\ \ 0]\varsigma(\mathrm{t})$, we have the new descriptor augmented system

$$
E\dot{\varsigma}(t) = \bar{A}\varsigma(t) + \bar{A}_d\varsigma(t - h(t)), \quad (13)
$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & 0 \\ 0 & A \\ 0 & 1 \end{bmatrix},$$

$$\bar{A}_d = \begin{bmatrix} A_d & 0 \\ 0 & (1 - \dot{h}(t))A_d \\ 0 & 0 \end{bmatrix}.$$

C. Delay range stability condition

Most of the papers from the literature focus on the so-called *delay dependent* stability analysis using the Lyapunov-Krasovskii method (see for example [5], [19], [9], [24]). Basically, a stable delay-free system is considered and the maximal value of the delay that preserves the stability is looked for. In this section, we propose to address the tricky case of the *delay range* condition where the delay belongs to an interval ($h(t) \in [h_{min}, h_{max}]$) and the system may be unstable for small delays (for some values $\subset [0, h_{min})$).

Considering the artificially augmented system (13), a new operator $\mathcal{H}$, which will be applied to the new signal $\ddot{x}$, may be introduced:

$$\mathcal{H} = \frac{\mathcal{I}^2 - \mathcal{D}\mathcal{I}^2 - h(t)\mathcal{I}}{h(t)}: \quad x(t) \to \frac{1}{h(t)} \int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} x(\theta)d\theta ds. \tag{14}$$

The following lemma gives a parameterized constraint on $\mathcal{H}$.

Lemma 4: An integral quadratic constraint for the operator $\mathcal{H}$ is given by the following inequality $\forall T > 0, \forall x \in L_{2e}^n, \forall S > 0$,

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T\mathcal{H}1_n \end{bmatrix} x_T, \begin{bmatrix} -\frac{h_{max}^2}{2}S & 0 \\ 0 & 2S \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T\mathcal{H}1_n \end{bmatrix} x_T \rangle \leq 0.$$

Proof:

$$\|\mathcal{H}x\|^2 = \frac{1}{h^2(t)} \left(\int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} x(\theta)d\theta ds \right)^T \left(\int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} x(\theta)d\theta ds \right)$$

Using Cauchy-Schwartz inequality and setting $\tilde{\mathcal{H}} = \mathcal{H}h(t)$, $\forall T > 0, \forall x \in L_{2e}^n$, we get the following inequality,

$$\|\tilde{\mathcal{H}}x\|^2 \leq \left(\int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} d\theta ds \right) \left(\int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} \|x(\theta)\|^2 d\theta ds \right)$$

$$\frac{\|\tilde{\mathcal{H}}x\|^2}{h^2(t)/2} \leq \int\limits_{t-h(t)}^{t} \int\limits_{s}^{t} \|x(\theta)\|^2 d\theta ds$$

$$\int\limits_0^{\infty} \frac{2}{h^2(t)} \|\tilde{\mathcal{H}}x\|^2 dt \leq \int\limits_0^{\infty} \int\limits_{-h_{max}}^0 \int\limits_s^0 \|x_t(\theta)\|^2 d\theta ds dt$$

$$\int\limits_0^{\infty} \frac{2}{h^2(t)} \|\tilde{\mathcal{H}}x\|^2 dt \leq \frac{h_{max}^2}{2} \int\limits_0^{\infty} \|x(t)\|^2 dt$$

Hence, we get

$$\int\limits_0^{\infty} \frac{2}{h^2(t)} \|\tilde{\mathcal{H}}x\|^2 - \frac{h_{max}^2}{2} \|x(t)\|^2 dt \leq 0$$

$$\int\limits_0^{\infty} 2\|\frac{\tilde{\mathcal{H}}x}{h(t)}\|^2 - \frac{h_{max}^2}{2} \|x(t)\|^2 dt \leq 0$$

which concludes the proof. ∎

In the same way, the former operator $\mathcal{F} = (1 - \mathcal{D})\mathcal{I}$ is slightly transformed as $\bar{\mathcal{F}} = \frac{1}{h(t)}\mathcal{F}$. The corresponding integral constraint is now expressed as follow.

Lemma 5: An integral quadratic constraint for the operator $\bar{\mathcal{F}}$ is given by the following inequality $\forall T > 0, \forall x \in L_{2e}^n, \forall R > 0$,

$$\langle \begin{bmatrix} 1_n \\ \mathbb{P}_T\bar{\mathcal{F}}1_n \end{bmatrix} x_T, \begin{bmatrix} -h_{max}R & 0 \\ 0 & hR \end{bmatrix} \begin{bmatrix} 1_n \\ \mathbb{P}_T\bar{\mathcal{F}}1_n \end{bmatrix} x_T \rangle \leq 0.$$

Proof: Omitted. ∎

Let us now model the augmented time-varying delay system (13) through the new set of operators:

$$\underbrace{\begin{bmatrix} \varsigma(t) \\ \varsigma_d(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}}_{w(t)} = \underbrace{\begin{bmatrix} \mathcal{I}1_{2n} & & & \\ & \mathcal{D}1_{2n} & & \\ & & \bar{\mathcal{F}}1_{2n} & \\ & & & \mathcal{H}1_n \end{bmatrix}}_{\nabla} \underbrace{\begin{bmatrix} \dot{\varsigma}(t) \\ \varsigma(t) \\ \dot{\varsigma}(t) \\ \ddot{x}(t) \end{bmatrix}}_{z(t)} \tag{15}$$

with

$$\varsigma_d(t) = \varsigma(t - h(t)),$$

$$w_1(t) = \frac{\varsigma(t) - \varsigma(t - h(t))}{h(t)},$$

$$w_2(t) = \dot{x}(t) - \frac{x(t) - x(t - h(t))}{h(t)} = E_1\varsigma(t) - E_2 w_1(t).$$

Matrices E_1 and E_2 are defined as $E_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ respectively. Then, according to the lemmas related to the different operators, a particular separator

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix}, \tag{16}$$

$$\Theta_{11} = \text{diag}\big(0_{2n}, -Q, -h_{max}R, -\frac{h_{max}^2}{2}S\big),$$

$$\Theta_{12} = \text{diag}\big(-P, 0_{5n}\big),$$

$$\Theta_{22} = \text{diag}\big(0_{2n}, (1 - \dot{h}(t))Q, h(t)R, 2S\big),$$

with some positive definite matrices $P, Q, R \in \mathbb{R}^{2n \times 2n}$ and $S \in \mathbb{R}^{n \times n}$, fulfils the requirement (4). Consequently, the stability of (13) (and thus (1)) will be proved if the condition

$$\xi^T(t)\Theta(h(t), \dot{h}(t))\xi(t) > 0 \tag{17}$$

such that $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix} \xi(t) = 0$ with $\xi = \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}$, is true. This condition is an equivalent formulation of (3). The condition (17) can again be rewritten as another equivalent condition

$$\psi^T(t)N^T(\dot{h}(t))\Theta(\dot{h}(t))N(\dot{h}(t))\psi(t) > 0, \tag{18}$$

with

$$\psi = \begin{bmatrix} x(t) \\ \varsigma(t-h(t)) \\ w_1(t) \\ w_2(t) \end{bmatrix}, \ s.t. \ S(h(t))\psi(t) = 0,$$

$$S = \begin{bmatrix} A & -1 & A_d & -1h(t) & 0 & 0 \\ 1 & 0 & -1 & 0 & -1h(t) & 0 \\ A & 0 & A_d & 0 & -1 & -1 \end{bmatrix} \quad (19)$$

and

$$N = \begin{bmatrix} \begin{array}{ccc} AA & A_d(1-\dot{h}) & AAd \\ A & 0 & A_d \\ A & 0 & A_d \\ 1 & 0 & 0 \\ AA & A_d(1-\dot{h}) & AAd \\ A & 0 & A_d \\ AA & A_d(1-\dot{h}) & AAd \\ A & 0 & A_d \end{array} & 0_{8n \times 3n} \\ \hline \multicolumn{2}{c}{1_{6n}} \end{bmatrix}. \quad (20)$$

Invoking Finsler's lemma, condition (18) is equivalent to

$$N^T(\dot{h}(t))\Theta N(\dot{h}(t)) + XS(h(t)) + S^T(h(t))X^T > 0. \quad (21)$$

Some calculus show that $N^T(\dot{h}(t))\Theta N(\dot{h}(t))$ is affine, and thus convex, in h and $\dot{h}$. So the condition (21) has to be assessed on the 4 vertices of the polytop generated by the intervals on $h(t)$ and $\dot{h}(t)$. We are now in a position to state our main result.

Theorem 2: For given positive scalars d, h_{min} and h_{max}, if there exist positive definite matrices P, Q, $R \in \mathbb{R}^{2n \times 2n}$ and a positive definite matrix $S \in \mathbb{R}^{n \times n}$ and a matrix $X \in \mathbb{R}^{6n \times 3n}$, then the system (1) with a time varying delay constrained by (2) is asymptotically stable if the LMI (21) holds for $\dot{h}(t) = \{-d, d\}$ and $h(t) = \{h_{min}, h_{max}\}$.

IV. NUMERICAL EXAMPLES

A. First example: delay dependent case

Considering the following academic numerical example

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h(t)). \quad (22)$$

First, let us remark that the delay-free system is stable. Then, for various d, the maximal allowable delay, h_{max}, is computed. To demonstrate the effectiveness of our criterion, results are compared to few ones from the literature. All these papers, except [14], [18], use the Lyapunov theory in order to derive some stability analysis criteria for time delay systems. In [14], the stability problem is solved by a robust control approach: the IQC framework. The results are shown in Table I.

In [14] and [18], the delay is likened to some uncertain operators and appropriate weighting filters are used to bound it. Their methodologies provide very good results, however, they are restricted to time-delay system that are stable without delay. Although our Theorem 2 does not provide the best values, it shows interesting results in terms of conservatism

reduction compared to most well-known conditions extracted from the literature. Besides, the proposed theorem has been primarily designed to address the stability issue of systems with *interval delays*, which may be unstable for small delays (or without delays).

B. Second example: delay range case

Now, the system is such that $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = u(t)$. We aim at stabilizing the system using a static delayed output-feedback $u(t) = ky(t - h(t))$. Choosing $k = 1$, we get the following state space model:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-h(t)). \quad (23)$$

Among the possible choices of set of operators at the modeling step, only the one of Theorem 2 is able to find stability interval for delayed systems that are unstable without any delay. In order to assess the interval of the delay such that system (23) is stable, Theorem 2 is applied with given h_{min} and h_{max}. Then, a sliding window principle is performed to stretch the bounds. The following results are obtained in Table II.

This result is also illustrated in Figure 2. It shows the assessment of an interval on the delay for which the system (23) is stable (when $d = 1$). It can be noticed that the system is unstable for a small delay. Let us recall that the Theorem 2 ensures the stability of (23) for the entire interval $\forall h(t) \in [h_{min}, h_{max}]$ (via a sliding window) and is not a gridding based estimation.

At last, varying the output feedback gain, the Theorem 2 allows to assess an inner (conservative) region of stability w.r.t k and $h(t)$ (for $d = 1$). It thus provides a set of values of k that ensures a stabilizing delayed output feedback for the LTI system $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = u(t)$ (see Figure 3).

d	0	0.1	0.2	0.5	0.8	1
[5]	4.472	3.604	3.033	2.008	1.364	0.999
[19]	4.472	3.604	3.033	2.008	1.364	-
[11]	1.632	1.632	1.632	1.632	1.632	1.632
[14]	6.117	4.714	3.807	2.280	1.608	1.360
[9]	4.472	3.605	3.039	2.043	1.492	1.345
[18]	6.117	4.794	3.995	2.682	1.957	1.602
[13]	4.476	3.611	3.047	2.072	1.590	1.529
Theorem 2	5.120	4.081	3.448	2.528	2.152	1.991

TABLE I

THE MAXIMAL ALLOWABLE DELAYS h_{max} FOR SYSTEM (22)

TABLE II

INTERVAL OF STABILIZING DELAYS FOR SYSTEM (23)

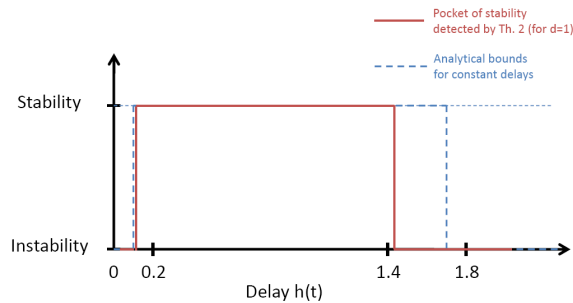|	h_{min}	h_{max}
$d = 0$	0.102	1.424
$d = 0.1$	0.102	1.424
$d = 0.2$	0.103	1.423
$d = 0.5$	0.104	1.421
$d = 0.8$	0.105	1.419
$d = 1$	0.105	1.418
analytical (constant case)	0.10016826	1.7178

Fig. 2. Detection of a pocket of stability for the system (23) (case $d = 1$).

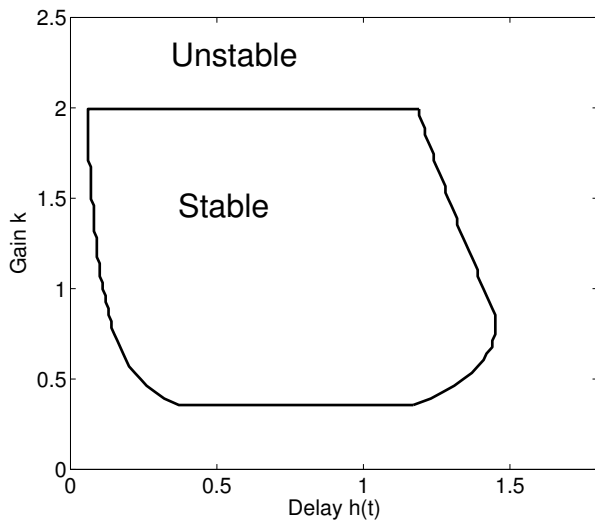

Fig. 3. Stability region of $\ddot{y}(t) - 0.1\dot{y}(t) + 2y(t) = ky(t - h(t))$ w.r.t. k and $h(t)$ (for $d = 1$).

V. Conclusion

In this paper, the problem of the delay dependent stability analysis of a time varying delay system has been studied by means of quadratic separation. Inspired from previous work on time delay systems with constant delay [8], stability criteria for time varying delay system are provided. Based on this first result, and using an augmented state, new modelling of time delay systems are introduced which emphasizes the relation between $\dot{h}$ and signals $\dot{x}$ and $\ddot{x}$. The resulting criteria are then expressed in terms of a convex optimization problem with LMI constraints, allowing the use of efficient solvers. Finally, a numerical example shows that these methods reduced conservatism and improved the maximal allowable delay.

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