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Optimized Rate Allocation for
State Feedback Control over Noisy Channels

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Abstract— **Optimal rate allocation in a networked control system with highly limited communication resources is instrumental to achieve satisfactory overall performance. In this paper, we propose a rate allocation technique for state feedback control in linear dynamic systems over a noisy channel. Our method consists of two steps: (*i*) the overall distortion is expressed as a function of rates at all time instants by means of high-rate quantization theory, and (*ii*) a constrained optimization problem to minimize the overall distortion is solved. We show that a non-uniform quantization is in general the best strategy for state feedback control over noisy channels. Monte Carlo simulations illustrate the proposed scheme, which is shown to have good performance compared to arbitrarily selected rate allocations.**

I. INTRODUCTION

Networked control systems based on limited sensor and actuator information have attracted increasing attention during the past decade. In these systems, it is important to encode the sensor measurements before sending them to the controller using a few bits, because of the limited information that can be transmitted using battery-powered devices. However, the distortion introduced by the quantization should not reduce the performance of the controller. Hence, optimizing the rate allocation is important to overcome the limited communication resources and to achieve a better overall performance.

The effect of low-rate feedback quantization on the overall performance of a control system has been studied by a number of researchers, e.g., [1], [2], [3] and the references therein. The problem of optimizing encoder–controller mappings to improve performance of control over finite-rate channels, with or without transmission errors, has been addressed in, e.g., [4], [3], [5]. How to assign bits among the elements of the state vector of the plant, while imposing a constraint on the number of bits over time, can be found in e.g., [6], [7]. In these works, it has been assumed that bits (rates) are evenly distributed to sensor measurements over time. However, owing to the non-stationarity of the state observations, an even distribution of bits to sensor measurements is often not efficient for networked control. Hence, it is natural to expect considerable gains by employing a non-uniform allocation of rates.

In this paper, we extend our previous work of optimizing the rate allocation for *state estimation* [8] to solve the rate allocation problem for state feedback control. How to achieve the optimal rate allocation in control systems is a challenging task. One main obstacle is the lack of tractable distortion functions, which we need to use as objective functions for the rate optimization problem. The optimization problem is often non-convex and non-linear, which implies that it is difficult to compute the optimal solution.

The main contribution of this paper is a novel method for rate allocation for state feedback control of a linear system over a noisy channel. By resorting to an approximation based on high-rate quantization theory, we are able to derive a computationally feasible scheme that seeks to minimize the overall distortion over a finite time horizon. The resulting rate allocation is not necessarily evenly distributed. Practical considerations concerning integer rate constraints and the accuracy of the high-rate approximation are discussed and illustrated through numerical examples. Overall good performance of our method is shown by numerical simulations.

The problem we are addressing here is related to classical rate allocation problems in communications [9], [10], and high-rate quantization theory [9], [11], [12]. We also contribute to rate allocation based on high-rate theory by studying a general class of quantizers, while previous work has often focused on the special case of optimized quantizers. For example in [13], the problem is studied in the context of transform codes, where the objective function is convex, and the optimal solution can be derived in a closed-form. However, in our setting we will show that the overall distortion is a non-convex function of the rates, which makes it more difficult the computation of the optimal solution.

The remainder of the paper is organized as follows. In Section II, the overall system is described and the rate allocation problem is formulated. Some useful results on high-rate quantization theory are given in Section III, which are then used in Section IV, where we solve the rate constrained optimization problem. Here we should mention that due to limited space, in this paper the main results are stated without proof. The proofs can be found in the full version of the paper, [14]. Section V is devoted to practical issues such as the non-negativity and integer nature of the rates. Finally, numerical simulations are carried out in Section VI to demonstrate the performance of the proposed bit-rate allocation scheme.

II. PROBLEM FORMULATION

The goal of this work is to arrive at a rate allocation scheme for state feedback control of a dynamic system over a noisy channel. We consider a scalar system with full state observation. The plant is governed by the linear equation
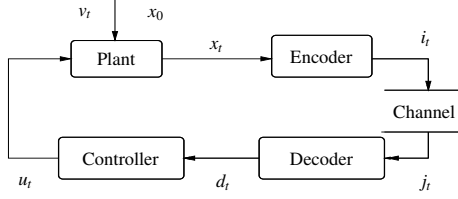
Fig. 1. A block-diagram of a closed-loop control system with a communication channel.

$$x_{t+1} = ax_t + u_t + v_t, \quad (1)$$

where $x_t, u_t, v_t \in \mathbb{R}$. The initial state x_0 and the process noise v_t are mutually independent. They are i.i.d. zero-mean Gaussian with variances $\sigma_{x_0}^2$ and σ_v^2. Following the block diagram in Fig. 1, we describe each system component in detail. The state measurement x_t is encoded and transmitted to the controller and decoder units through a noisy channel. The encoder is time-varying and memoryless,

$$i_t = f_t(x_t) \in \left\{0, \ldots, 2^{R_t} - 1\right\}. \quad (2)$$

The rate R_t is a non-negative integer. The index i_t will be mapped to a binary codeword before being fed to a binary channel. The mapping from an index to a codeword is commonly referred to as the *index assignment* (IA). Unlike in the error-free scenario where all IA's perform equally well, in the presence of channel errors different IA's have different impact on the system performance. Finding the optimal IA is a combinatorial problem which is known to be NP-hard [15]. Therefore, in this paper, we average out the dependence on a specific IA by randomization. At each transmission, a random assignment is generated and revealed to the encoder and decoder. Previous work that assumed a random IA to facilitate further analysis includes [16], [12].

Throughout the paper, the overall *noisy channel* is composed by the combination of the random IA and a binary symmetric channel (BSC). The channel is completely specified by the symbol transition probabilities $\Pr(j_t|i_t)$. At bit level, the channel is characterized by the crossover probability $\varepsilon = \Pr(0|1) = \Pr(1|0)$ of the BSC, while the overall symbol error probability $\Pr(j_t|i_t)$ is determined by both ε and the randomized IA, according to

$$\Pr(j_t|i_t) = \begin{cases} \alpha(R_t), & j_t \neq i_t, \\ 1 - (2^{R_t} - 1)\alpha(R_t), & j_t = i_t, \end{cases} \quad (3)$$

(cf., [16]), where $\alpha(R_t) \triangleq (1 - (1 - \varepsilon)^{R_t})/(2^{R_t} - 1)$ is obtained by averaging over all possible index assignments. For this channel, all symbol errors are equally probable.

At the receiver side, the decoder takes the current channel output as the input, and generates an output,

$$d_t = D_t(j_t) \in \mathbb{R}, \quad (4)$$

where D_t is a deterministic function. The estimate d_t can take one of 2^{R_t} values. Finally, the control u_t is determined by the decoded symbol, $u_t = g_t(d_t) \in \mathbb{R}$, where $g_t(\cdot)$ is the control law. We will be more specific about $g_t(\cdot)$ after the presentation of the control objective, which is the minimization of the expected value of the cost $J_{tot}(\mathbf{R})$, $\mathbf{R} = \{R_0, \ldots, R_{T-1}\}$,

$$J_{tot}(\mathbf{R}) = \sum_{t=1}^{T} J_t(\mathbf{R}_{t-1}) = \sum_{t=1}^{T} \left(x_t^2 + \rho u_{t-1}^2\right), \quad \rho \geq 0, \quad (5)$$

subject to a rate constraint $\sum_{t=0}^{T-1} R_t \leq R_{tot}$, with R_{tot} denoting the total rate. We refer to $\mathbf{R}$ as *the bit-rate allocation*. In (5), the instantaneous cost J_t depends on the previous rates $\mathbf{R}_{t-1} = \{R_0, \ldots, R_{t-1}\}$. The function (5) is the linear quadratic (LQ) cost from classical stochastic control [17], where ρ is the importance factor of the control input with respect to the state. The cost (5) can be minimized by minimizing the state variance at all time instances, with a power constraint on the control signal. The implicit relation of the cost $\mathbf{E}\{J_{tot}(\mathbf{R})\}$ and the allocation $\mathbf{R}$ is determined by the channel and coding–control scheme.

Throughout this paper, the control is a linear function of the decoded symbol d_t,

$$u_t = \ell_t d_t. \quad (6)$$

If the estimate d_t is close to the true state x_t then the classical linear quadratic Gaussian (LQG) theory is expected to give good results, even though it does not account for channel errors and quantization distortion. Accordingly, it is reasonable to use ℓ_t given by the classical LQG theory, i.e.,

$$\ell_t \triangleq -\frac{a\phi_{t+1}}{\phi_{t+1} + \rho}, \quad \phi_t = 1 + \frac{a^2\phi_{t+1}\rho}{\phi_{t+1} + \rho}, \quad \phi_T = 1. \quad (7)$$

Problem 1 below specifies the rate allocation problems studied in this paper.

Problem 1: Given the linear plant (1), the discrete memoryless channel (3), the memoryless encoder–decoder pair (2) and (4), the control law (6)–(7), find the optimal bit-rate allocation $\mathbf{R}$ minimizing the expected cost of (5), subject to the total bits constraint:

$$\min_{\mathbf{R}} \ \mathbf{E}\{J_{tot}(\mathbf{R})\}, \quad \text{s.t.} \sum_{t=0}^{T-1} R_t \leq R_{tot}.$$

From (5), one can show that

$$\mathbf{E}\{J_{tot}\} = \mathbf{E}\left\{(\phi_0 - 1)x_0^2 + \sum_{t=0}^{T-1}\phi_{t+1}v_t^2 + \sum_{t=0}^{T-1}(\phi_{t+1} + \rho)(-x_t\ell_t + u_t)^2\right\}. \quad (8)$$

It follows that we can rewrite (5) as

$$J_{tot}(\mathbf{R}) = \sum_{t=0}^{T-1}(\phi_{t+1} + \rho)\ell_t^2(x_t - d_t)^2 = \sum_{t=0}^{T-1}\pi_t(x_t - d_t)^2, \quad (9)$$

with $\pi_t \triangleq (\phi_{t+1} + \rho)\ell_t^2$, and then the instantaneous cost is

$$\mathbf{E}\{J_t(\mathbf{R}_t)\} = \mathbf{E}\left\{\pi_t(x_t - d_t)^2\right\}. \quad (10)$$

One main difficulty of Problem 1 is that the cost function does not have an analytical expression of $\mathbf{R}$. In the next section, we propose an approximation of adequate accuracy, which will then be used for the solution of the optimization problem.

III. HIGH-RATE APPROXIMATION OF MSE

By (10), we note that the mean-squared error (MSE) is a key factor in the overall cost, therefore, a major challenge lies in deriving a useful expression for the MSE. In general, it is not possible to derive a closed-form expression, even in the case of simple uniform quantizers. Therefore, we resort to approximations based on *high-rate theory* [9] (for further details, we refer the reader to [16] and [12]). Roughly speaking, the high-rate assumption requires the probability distribution function (PDF) of the source to be approximately constant within a quantization cell. Let $\mathscr{P}(x)$ denote the PDF of the source x_t, which is zero-mean and with variance $\sigma_{x_t}^2$. Following [12], the MSE $\mathbf{E}\{(x_t - d_t)^2\}$ at high-rate can be approximated by the expression,

$$
\mathbf{E}\{(x_t - d_t)^2\} \approx 2^{R_t}\alpha(R_t)\sigma_{x_t}^2 + \varphi_t\alpha(R_t)\int_y y^2\lambda_t(y)\mathrm{d}y \\
+ \frac{G^{-2}}{3}\varphi_t^{-2}\int_x \lambda_t^{-2}(x)\mathscr{P}(x)\mathrm{d}x.
\tag{11}
$$

The constant G is the volume of a unit sphere. The function $\lambda_t(x)$ is referred to as the *quantizer point density function*. This function is used to specify a quantizer in terms of the density of the reconstruction points. It holds that $\lambda_t(x) \geq 0$, $\forall x$, and $\int \lambda_t(x)\mathrm{d}x = 1$, which resembles a probability density function. Finally, the parameter $1 \leq \varphi_t \leq 2^{R_t}$ specifies the number of codewords the encoder will chose. If the crossover probability ε is large, in order to protect against channel error, a good encoder may only use a part of the available codewords. In this paper, we consider only the encoder–decoders for which $\varphi_t = 2^{R_t}$.

Essentially, we are in need of a useful expression to describe the relation between the MSE and the rate R_t. Therefore, we observe that $2^{R_t}\alpha(R_t) \approx 1 - (1 - \varepsilon)^{R_t}$, and we rewrite (11) and introduce the high-rate approximation $\hat{J}_t(\beta_t, \kappa_t, R_t)$ as

$$
\mathbf{E}\{(x_t - d_t)^2\} \approx \hat{J}_t(\beta_t, \kappa_t, R_t) \triangleq \beta_t(1 - (1 - \varepsilon)^{R_t}) + \kappa_t 2^{-2R_t},
$$
$$
\beta_t \triangleq \sigma_{x_t}^2 + \int_y y^2\lambda_t(y)\mathrm{d}y, \quad \kappa_t \triangleq \frac{G^{-2}}{3}\int_x \lambda_t^{-2}(x)\mathscr{P}(x)\mathrm{d}x.
\tag{12}
$$

Such an expression of the distortion $\hat{J}_t$ is rather general for a large variety of quantizers, described in terms of the point density function, and derived under the high-rate assumption. For practical sources and quantizers, it holds that $0 < \beta_t < \infty$ and $0 < \kappa_t < \infty$, which is assumed throughout the paper. The distortion (12) has certain useful properties that will allow us to solve the rate allocation problem.

The crossover probability ε plays a significant role on the convexity of the objective function $\hat{J}_t$. When $\varepsilon = 0$, $\hat{J}_t$ is monotonically decreasing with respect to R_t. In fact, $\hat{J}_t$ is a convex function of R_t, for all $0 < \kappa_t < \infty$. On the other hand, for the general case of an arbitrary $\{\beta_t, \kappa_t\}$ pair, (12) is a quasi-convex function, as shown by the following lemma.

Lemma 1: The distortion function $\hat{J}_t = \beta_t(1 - (1 - \varepsilon)^{R_t}) + \kappa_t 2^{-2R_t}$, $0 < \beta_t < \infty$, $0 < \kappa_t < \infty$, is a quasi-convex function and has a unique global minimum.

As shown later, Lemma 1 is instrumental in solving the rate allocation problems. Here we introduce a class of $\hat{J}_t$ which can be written as

$$
\hat{J}_t = \sigma_{x_t}^2\left(\tilde{\beta}_t(1 - (1 - \varepsilon)^{R_t}) + \tilde{\kappa}_t 2^{-2R_t}\right) = \sigma_{x_t}^2\tilde{J}_t(\tilde{\beta}_t, \tilde{\kappa}_t, R_t),
\tag{13}
$$

with $\tilde{J}_t(\tilde{\beta}_t, \tilde{\kappa}_t, R_t) \triangleq \tilde{\beta}_t(1 - (1 - \varepsilon)^{R_t}) + \tilde{\kappa}_t 2^{-2R_t}$, where $0 < \tilde{\beta}_t < \infty$ and $0 < \tilde{\kappa}_t < \infty$ are independent of R_t and of $\sigma_{x_t}^2$. As will be shown later, this class of $\hat{J}_t$ is central to our solutions to the state feedback control problems. Owing to the fact that $\tilde{J}_t$ is a special case of $\hat{J}_t$, Lemma 1 applies directly to $\tilde{J}_t$.

Next, we use the uniform quantizer as an example to illustrate the use of (12) and (13). Consider a uniform quantizer, for which the step size $\Delta_t = 2v_t/2^{R_t}$ is a function of the quantizer range $[-v_t, v_t]$ and the rate R_t, then the point density function is $\lambda_t(x) = 1/(2v_t)$. If the source signal and the uniform quantizer have the same support $[-v_t, v_t]$, then a high-rate approximation of the MSE distortion according to (12) is given by

$$
\hat{J}_t = \left(\sigma_{x_t}^2 + \frac{v_t^2}{3}\right)\left(1 - (1 - \varepsilon)^{R_t}\right) + \frac{4}{3}v_t^2 G^{-2}2^{-2R_t}.
$$

Consider a zero-mean Gaussian source and if the distortion caused by the signals out of the quantizer support $[-v_t, v_t]$ is negligible, then a high-rate approximation of the MSE can use (13) with the following $\tilde{\beta}_t$ and $\tilde{\kappa}_t$,

$$
\tilde{\beta}_t = 1 + \frac{\left(Q^{-1}\left(\frac{1 - p_{v_t}}{2}\right)\right)^2}{3}, \quad \tilde{\kappa}_t = \frac{4}{3}G^{-2}\left(Q^{-1}\left(\frac{1 - p_{v_t}}{2}\right)\right)^2.
$$

where p_{v_t} denotes the probability that x_t is within the support of the quantizer, i.e., $p_{v_t} = \Pr(x_t \in [-v_t, v_t])$, and $Q^{-1}(\cdot)$ is the inverse function of the Q-function which is defined as $Q(x) \triangleq \int_x^\infty 1/\sqrt{2\pi}e^{-y^2/2}\mathrm{d}y$.

In the next Section, we show that the high rate approximation of MSE described in this section is instrumental to formulate a useful version of (9). By using these results, we propose a solution to Problem 1.

IV. RATE ALLOCATION FOR STATE FEEDBACK CONTROL

We notice that the terms $\mathbf{E}\{x_t^2\}$ and $\mathbf{E}\{(x_t - d_t)^2\}$ are essential to the instantaneous cost (10). In order to proceed, we will approximate the state x_t by a zero-mean Gaussian source, because the initial state and the process noise are zero-mean Gaussian. By imposing such a Gaussian approximation, we only need to estimate the variance, which we denote by $\hat{\sigma}_{x_t}^2$. The next challenge lies in the derivation of $\hat{\sigma}_{x_t}^2$. In order to facilitate the derivation of a tractable overall cost for optimization, we consider an upper bound for $\hat{\sigma}_{x_t}^2$ by simplifying the correlation between x_t and d_t, so that it holds

$$
\hat{\sigma}_{x_t}^2 = (A_t + B_t(\tilde{\beta}_t(1 - (1 - \varepsilon)^{R_{t-1}}) + \tilde{\kappa}_t 2^{-2R_{t-1}})\hat{\sigma}_{x_{t-1}}^2 + \sigma_v^2,
\tag{14}
$$

where $A_t > 0$ and $B_t > 0$ are terms independent of $\mathbf{R}_{t-1}$ and $\hat{\sigma}_{x_{t-1}}^2$. The following two cases are used to illustrate the utility and motivation of (14).

Case 1: *Consider the decoder* $d_{t-1} = \mathbf{E}\{x_{t-1}|j_{t-1}\}$, *for which the estimation error* $x_{t-1} - d_{t-1}$ *is uncorrelated with the estimate* d_{t-1}. *Therefore* (14) *becomes an exact expression, with*

$$A_t = a^2 + \ell_{t-1}^2 + 2a\ell_{t-1}, \quad B_t = -(\ell_{t-1}^2 + 2a\ell_{t-1}).$$

Case 2: *In general, we can write* $\mathbf{E}\{x_t^2\}$ *as*

$$\mathbf{E}\{x_t^2\} = \ell_{t-1}^2 \mathbf{E}\{(x_{t-1} - d_{t-1})^2\} + (a + \ell_{t-1})^2 \mathbf{E}\{x_{t-1}^2\}$$
$$- 2(a + \ell_{t-1})\ell_{t-1}\mathbf{E}\{x_{t-1}(x_{t-1} - d_{t-1})\} + \sigma_v^2.$$

The term $\mathbf{E}\{x_{t-1}(x_{t-1} - d_{t-1})\}$ *depends on the source, the quantizer and the channel. In the special case that*

$$\mathbf{E}\{x_{t-1}(x_{t-1} - d_{t-1})\} = \gamma(\varepsilon)\mathbf{E}\{x_{t-1}^2\},$$

where $\gamma(\varepsilon)$ *does not depend on* x_{t-1} *and* d_{t-1}, *then* $\hat{\sigma}_{x_t}^2$ *can be expressed in the form of* (14), *with*

$$A_t = (a + \ell_{t-1})^2 - 2(a + \ell_{t-1})\ell_{t-1}\gamma(\varepsilon), \quad B_t = \ell_{t-1}^2. \quad (15)$$

Based on the above Gaussian approximation, we arrive at the instantaneous cost

$$\mathbf{E}\{J_t(\mathbf{R}_t)\} = \pi_t \hat{\sigma}_{x_t}^2 (\tilde{\beta}_t(1 - (1 - \varepsilon)^{R_t}) + \tilde{\kappa}_t 2^{-2R_t}), \quad (16)$$

where $\hat{\sigma}_{x_t}^2$ can be recursively calculated according to (14). In practice, (16) can be applied generally to all systems in Section II by finding suitable A_t and B_t to approximate the true instantaneous costs. Therefore, the unconstrained and constrained rate allocation problems based on (16) are formulated as the following approximate versions of Problem 1.

Problem 2: Find $\mathbf{R}$ that solves the problem

$$\min_{\mathbf{R}} \quad \sum_{t=0}^{T-1} \mathbf{E}\{J_t(\mathbf{R}_t)\},$$

where $\mathbf{E}\{J_t(\mathbf{R}_t)\}$ is as given in (16).

Problem 3: Find $\mathbf{R}$ that solves the problem

$$\min_{\mathbf{R}} \quad \sum_{t=0}^{T-1} \mathbf{E}\{J_t(\mathbf{R}_t)\}, \quad \text{s.t.} \sum_{t=0}^{T-1} R_t \leq R_{tot},$$

where $\mathbf{E}\{J_t(\mathbf{R}_t)\}$ is given in (16).

By recursively replacing $\hat{\sigma}_{x_t}^2$ with $\hat{\sigma}_{x_{t-1}}^2$, back to $\hat{\sigma}_{x_0}^2 = \sigma_{x_0}^2$, we are able to represent $\hat{\sigma}_{x_t}^2$ by $\mathbf{R}_{t-1}$ and $\sigma_{x_0}^2$. In particular, $\mathbf{E}\{J_t(\mathbf{R}_t)\}$ is a sum of 2^t terms, as shown in Lemma 2,

Lemma 2: The cost (16) can be written as

$$\mathbf{E}\{J_t(\mathbf{R}_t)\} = \sum_{b_0=0}^{1} \cdots \sum_{b_{t-1}=0}^{1} \pi_t \Psi_t(\mathbf{b}_0^{t-1})\tilde{J}_t(\tilde{\beta}_t, \tilde{\kappa}_t, R_t),$$

where $b_s \in \{0, 1\}$, $0 \leq s \leq t - 1$, is a binary variable, and the term $\Psi_t(\mathbf{b}_0^{t-1})$ is

$$\Psi_t(\mathbf{b}_0^{t-1}) \triangleq \overline{B}\left(\prod_{s=\overline{s}+1}^{t-1} \overline{B}_s^{b_s}\left(\tilde{J}_s(\tilde{\beta}_s, \tilde{\kappa}_s, R_s)\right)^{b_s}\right), \quad (17)$$

with

$$\overline{B} \triangleq \begin{cases} \tau_{\overline{s}-1}, & \overline{s} > 0, \\ B_0 \sigma_{x_0}^2, & \overline{s} = 0, \end{cases} \quad (18)$$

where $\overline{s}$ is the smallest integer s for which $b_s = 1$, (in other words $b_s = 0$, $s < \overline{s}$), and $\tau_{\overline{s}-1}$ is calculated recursively as,

$$\tau_s = A_s \tau_{s-1} + \sigma_v^2, \quad \tau_0 = A_0 \sigma_{x_0}^2 + \sigma_v^2.$$

The parameter $\overline{B}_s$ is determined by b_s, as

$$\overline{B}_s \triangleq \begin{cases} A_s, & b_s = 0, \\ B_s, & b_s = 1. \end{cases} \quad (19)$$

Lemma 2 is proved by direct calculations. We note that each $\mathbf{E}\{J_t(\mathbf{R}_t)\}$ is a sum of 2^t product terms, where all of them have the common terms π_t and $\tilde{J}_t(\tilde{\beta}_t, \tilde{\kappa}_t, R_t)$. Based on Lemma 2, we solve Problem 2 and Problem 3 as shown in Theorem 1.

Theorem 1: Suppose $\mathbf{R} \in \mathbb{R}^T$. The solution to Problem 3 is as follows:

1) If $R_{tot} \geq \sum_{t=0}^{T-1} R_t^\star$, where $\mathbf{R}^\star$ is the solution to

$$\begin{cases} 0 = \tilde{J}_0(\tilde{\beta}_0, \tilde{\kappa}_0, R_0^\star), \\ \quad\quad \vdots \\ 0 = \tilde{J}_{T-1}(\tilde{\beta}_{T-1}, \tilde{\kappa}_{T-1}, R_{T-1}^\star), \end{cases} \quad (20)$$

then $\mathbf{R}^\star$ also solves Problem 3.

2) If $R_{tot} < \sum_{t=0}^{T-1} R_t^\star$, where $\mathbf{R}^\star$ is the solution to (20), then the solution $\{\mathbf{R}, \theta\}$ to the system of equations

$$\begin{cases} -\sum_{s=t}^{T-1} \Psi_{t,s} = \theta, \ t = 0, \ldots, T-1, \\ \sum_{t=0}^{T-1} R_t = R_{tot}, \end{cases} \quad (21)$$

solves Problem 3, where θ is the associated Lagrange multiplier. The term $\Psi_{t,s}$ is defined as

$$\Psi_{t,s} \triangleq \sum_{b_0=0}^{1} \cdots \sum_{b_t=1}^{1} \cdots \sum_{b_s=0}^{1} \pi_s \overline{\Psi}(\mathbf{b}_0^s), \quad (22)$$

where b_k is a binary variable, π_t is as given by (9), and $\overline{\Psi}(\mathbf{b}_0^s)$ is

$$\overline{\Psi}(\mathbf{b}_0^s) \triangleq \overline{B}\left(\prod_{m=\overline{s}+1}^{s-1} \overline{B}_m^{b_m}\right)\left(\prod_{n=\overline{s}+1}^{s} C_n^{b_n}\right).$$

The terms $\overline{B}$ and $\overline{B}_m$ are previously given in (18)–(19), and C_n is

$$C_n \triangleq \begin{cases} \frac{\partial \tilde{J}_n}{\partial R_n}(\tilde{\beta}_n, \tilde{\kappa}_n, R_n), & n = t, \\ \tilde{J}_n(\tilde{\beta}_n, \tilde{\kappa}_n, R_n), & n \neq t, \end{cases}$$

where $\partial \tilde{J}_t / \partial R_t$ is the first order derivative of $\tilde{J}_t(\tilde{\beta}_t, \tilde{\kappa}_t, R_t)$ with respect to R_t.

The proof of Theorem 1 can be found in [14]. Here we comment the steps briefly. We prove the general case for noisy channels in three steps. First, we use Lemma 2 to write the overall objective as a function of R_t, $t = 0, \ldots, T-1$, explicitly. Then, we show that when $\varepsilon > 0$, the unconstrained rate allocation problem has a global minimum which solves the system of equations (20). This global minimum is the solution to the constrained rate allocation problem if the rate constraint fulfills. Finally, we show that the solution to the system of equations (21) solves the constrained rate allocation problem, if the rate constraint is violated by the global minimum.

The analogous result for the special case that $\varepsilon = 0$ is summarized in Proposition 1.
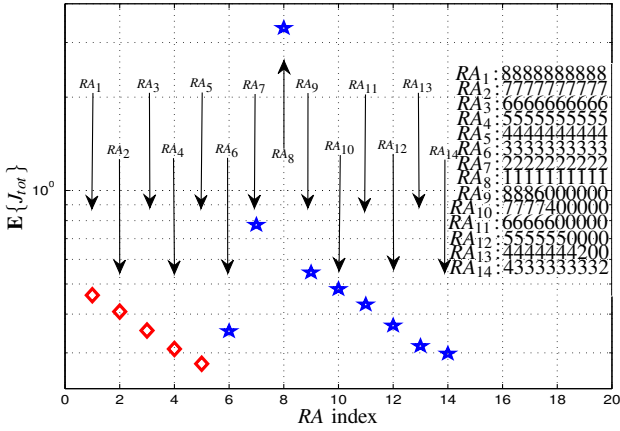
Fig. 2. Performance comparison among various rate allocations. The *x*-axis is associated to the allocation, whereas the *y*-axis is the overall distortion. A rate allocation scheme is described by an integer sequence, e.g., RA_1: 8888888888 means that the rate allocation RA_1 has $R_t = 8$, $t = 0, \ldots, 9$. Notice that schemes marked with a diamond do not satisfy the total rate constraint, i.e., $R_{tot} > 30$.

Proposition 1: Suppose $\mathbf{R} \in \mathbb{R}^T$. A solution $\{\mathbf{R}, \theta\}$ to Problem 3 for an noise-free channel ($\varepsilon = 0$) is

$$\begin{cases} 2 \sum_{s=t}^{T-1} \sum_{b_0=0}^{1} \cdots \sum_{b_t=1}^{1} \cdots \sum_{b_{s-1}=0}^{1} \Psi_s(\mathbf{b}_0^{s-1}) = \theta, \ t = 0, \ldots, T-1, \\ \sum_{t=0}^{T-1} R_t = R_{tot}. \end{cases}$$

where θ is the Lagrange multiplier, and

$$\Psi_t(\mathbf{b}_0^{t-1}) \triangleq \pi_t \overline{B} \left(\prod_{s=\overline{s}+1}^{t-1} \overline{B}_s{}^{b_s} \right) \left(\prod_{m=\overline{s}+1}^{t} \tilde{\kappa}_m^{b_m} \right) 2^{-2 \left(\sum_{n=0}^{t-1} b_n R_n + R_t \right)}.$$

The rest of the notations are referred to Lemma 2 and Theorem 1.

V. PRACTICAL CONSIDERATIONS

In this section we deal with the assumption of Theorem 1 that R_t is allowed to be real and negative. In practice R_t is integer-valued and positive. If Problem 2 and Problem 3 give negative rates, then we set them to zero, which is equivalent to excluding the corresponding instantaneous distortions from the overall distortion. Then, we resolve Problem 2 and Problem 3 with respect to the new overall distortion. The proposed algorithms in Section IV result in real-valued rates. As a simple approach, we round the solutions to the nearest integer. A more sophisticate rounding algorithm can be formulated as a binary optimization problem, where the rounded rate $\tilde{R}_t$ is related to the real-valued rate R_t as, $\tilde{R}_t = b_t \lceil R_t \rceil + (1 - b_t) \lfloor R_t \rfloor$, $b_t \in \{0, 1\}$, where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the rounding upwards and downwards to the nearest integer, respectively. We optimize the rounding by finding the binary sequence $\mathbf{b}_0^{T-1}$ which minimizes $\sum_{t=0}^{T-1} \mathbf{E} \left\{ \mathbf{J}_t(\tilde{R}_t) \right\}$, subject to $\sum_{t=0}^{T-1} \tilde{R}_t \leq R_{tot}$. A solution to the optimization problem can always be obtained by applying exhaustive search or combinatorial algorithms [18].

VI. NUMERICAL EXPERIMENTS

In this section, we present numerical experiments performed to verify performance of the proposed bit-rate allocation algorithm for state feedback control. Let us first address some issues common for all experiments in this paper. The optimized rate allocation is the one obtained by applying Theorem 1 and the binary rounding algorithm from Section V. In particular, we optimize the rate allocation by means of the objective function (16) of Problem 3. The overall performance is on the other hand evaluated in terms of the objective function (5) of Problem 1, which is achieved numerically. We use time-varying uniform quantizers where the quantizer range is related to the estimated signal variance as $v_t = 4\hat{\sigma}_{x_t}$ and the distortion caused by the signals outside the support of the quantizer is negligible. Moreover, (14) and (15) are utilized, where we let $\gamma(\varepsilon)$ be a linear function heuristically obtained by numerical experiments. We remark that $\gamma(\varepsilon)$ is interfluent in the optimization, since it is a common multiplicative constant in the cost function.

First we demonstrate the performance of the proposed scheme by comparing it with several other allocations. The system parameters are chosen in the interest of demonstrating non-uniform rate allocations, in particular, the system setup is: $a = 0.5$, $\rho = 2$, $T = 10$, $R_{tot} = 30$, $\varepsilon = 0.005$, $\sigma_{x_0}^2 = 10$, and $\sigma_v^2 = 0.1$. The simulated costs are obtained by averaging over 100 IA's and each IA 150 000 samples. In Fig. 2, we compare the optimized allocation scheme, denoted by RA_{14}, which was obtained by the method proposed in this paper, with 13 other schemes, denoted by RA_1–RA_{13}. Especially, the scheme RA_5 was achieved with our method by solving the unconstrained rate allocation problem. Performance shown in Fig. 2 is measured by the overall cost (5). Regarding this optimized allocation, R_t is fairly evenly distributed over t, and compared with the uniform allocation RA_6, the performance improvement is evident. The uniform allocations RA_1–RA_8 have a time-invariant rate from 8 bits to 1 bit. Among these allocations, RA_8, for which $R_t = 1$, $\forall t$, has the worst performance, while RA_5, for which $R_t = 4$, $\forall t$, has the best performance. In fact, based on our analysis, $\tilde{\beta}_t = \tilde{\beta}$, $\tilde{\kappa}_t = \tilde{\kappa}$, and the solution to Problem 2 is $R_t^\star = 4$. In the presence of the channel errors, more bits can sometimes do more harm than good. This is consistent with the simulation result that RA_5 is superior to allocations that are assigned more than 4 bits for every t, cf., RA_1–RA_4. The allocations RA_9–RA_{13} are used to represent the strategies that more bits are assigned to the initial states. Obviously, this strategy is not efficient in the current example because of the following facts. First, as discussed, the additional bits exceeding the critical point do more harm than good. Second, the degradation caused by reducing one bit at a lower rate is more significant than the improvement along with adding one bit at a higher rate.

The next example shows the impact of ρ, a parameter which plays a role of regulating the power of control signal. More precisely, the magnitude of the control signal decreases as ρ increases. That is to say, a large ρ yields in average small-valued controls, and consequently, a slow state
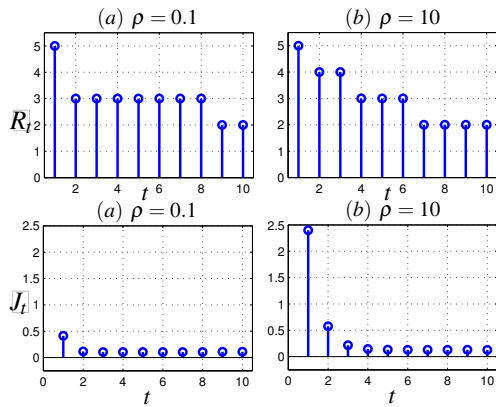
Fig. 3. Performance comparison between $\rho = 0.1$ and $\rho = 10$.

response and a high steady-state level. In this experiment, the system setup is: $a = 0.5$, $T = 10$, $R_{tot} = 30$, $\varepsilon = 0.001$, $\sigma_{x_0}^2 = 10$ and $\sigma_v^2 = 0.1$. The quantizer is still the time-varying uniform quantizer. In Fig. 3, the optimized rate allocations are demonstrated for two ρ-values. By applying Theorem 1, we have that the global minimum to the rate unconstrained problem is $R_t^\star = R^\star = 5$. This is consistent with Fig. 3, where there is no R_t larger than 5. When ρ is small, for example $\rho = 0.1$, large-valued controls are allowed and the steady state is quickly reached. As ρ increases, only small-valued controls are allowed and it takes longer time to reach the steady state. This explains what we see from Fig. 3 that more bits are needed in the initial states when ρ is large. The simulated instantaneous costs for $\rho = 0.1$ and $\rho = 10$ are depicted in the same figure.

Since we formulated a useful overall objective function by a number of approximations and simplifications, certain performance degradation is expected. However, the proposed algorithms are still able to provide a satisfactory solution, because what really matters is often the ratios among the costs at all time instants. In addition, even though the high-rate assumption requires the source PDF to be approximately constant over one quantization cell, however, the quantization works fairly well in practice for low rates such as 3, 4 bits. On the other hand, we may say generally that the accuracy decreases when the rate approaches 0. That is to say, at low rates the proposed rate allocation algorithm does not work as well as in the high-rate region, attributed to all high-rate approximations made in the calculation. The worst case occurs at $R_t = 0$, where the estimation error given by (12) is even worse than $\mathbf{E}\left\{x_t^2\right\} = \sigma_x^2$, obtained by setting $d_t = 0$. The Gaussian approximation becomes also inaccurate as the rate decreases. The problem becomes more serious for unstable systems because errors accumulate as time goes on. As the rate increases, the problem of accuracy is quickly resolved.

VII. CONCLUSION

In this paper, we formulated a problem to optimally assign totally R_{tot} bits to T time units for control over noisy channels. First, we approximated the overall distortion

function by means of high-rate quantization theory. Second, we showed that the unconstrained optimization problem has a global minimum, which solves the rate allocation problem if such a global minimum does not violate the rate constraint. On the other hand, if the global minimum violates the rate constraint, then we solved the rate constrained optimization problem by means of Lagrangian duality for non-convex non-linear problems. Finally, numerical simulations showed good performance of the proposed scheme. By studying the rate allocation problem, we see that in the presence of the channel errors, the encoder–controller mapping is instrumental to achieve satisfactory overall performance.

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