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Stochastic Invariance and Aperiodic Control for Uncertain Constrained Systems

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Abstract

Uncertainties and constraints are present in most control systems. For example, robot motion planning and building climate regulation can be modeled as uncertain constrained systems. In this thesis, we develop mathematical and computational tools to analyze and synthesize controllers for such systems.

As our first contribution, we characterize when a set is a probabilistic controlled invariant set and we develop tools to compute such sets. A probabilistic controlled invariant set is a set within which the controller is able to keep the system state with a certain probability. It is a natural complement to the existing notion of robust controlled invariant sets. We provide iterative algorithms to compute a probabilistic controlled invariant set within a given set based on stochastic backward reachability. We prove that these algorithms are computationally tractable and converge in a finite number of iterations. The computational tools are demonstrated on examples of motion planning, climate regulation, and model predictive control.

As our second contribution, we address the control design problem for uncertain constrained systems with aperiodic sensing and actuation. Firstly, we propose a stochastic self-triggered model predictive control algorithm for linear systems subject to exogenous disturbances and probabilistic constraints. We prove that probabilistic constraint satisfaction, recursive feasibility, and closed-loop stability can be guaranteed. The control algorithm is computationally tractable as we are able to reformulate the problem into a quadratic program. Secondly, we develop a robust self-triggered control algorithm for time-varying and uncertain systems with constraints based on reachability analysis. In the particular case when there is no uncertainty, the design leads to a control system requiring minimum number of samples over finite time horizon. Furthermore, when the plant is linear and the constraints are polyhedral, we prove that the previous algorithms can be reformulated as mixed integer linear programs. The method is applied to a motion planning problem with temporal constraints.

Sammanfattning

Osäkerheter och begränsningar återfinns i de flesta reglersystem. Exempelvis kan planering av robotrörelser och reglering av inomhusklimat modelleras som osäkra begränsade system. I denna avhandling utvecklar vi matematiska och beräkningsmässiga verktyg för att analysera och syntetisera styrenheter för sådana system.

Som första bidrag karaktäriserar vi när en mängd är en probabilistiskt reglerad invariant mängd och utvecklar verktyg för att beräkna sådana mängder. En probabilistiskt reglerad invariant mängd är en mängd inom vilken regulatorn kan hålla tillståndet med en viss sannolikhet. Det är ett naturligt komplement till det befintliga begreppet robusta reglerade invarianta mängder. Vi tillhandahåller iterativa algoritmer för att beräkna en probabilistiskt reglerad invariant mängd inom en given mängd baserat på stokastisk bakåtriktad uppnåelighet. Vi bevisar att dessa algoritmer är beräkningsmässigt hanterbara och konvergerar inom ett finit antal iterationer. Beräkningsverktygen demonstreras med exempel inom rörelseplanering, klimatreglering och modellprediktiv reglering.

Som andra bidrag behandlar vi reglerdesign för osäkra begränsade system med aperiodisk mätning och aktivering. För det första föreslår vi en stokastisk självutlösande modellprediktiv reglering algoritm för linjära system som utsätts för exogena störningar och probabilistiska bivillkor. Vi bevisar att uppfyllande av probabilistiska bivillkor, rekursiv genomförbarhet och stabilitet för det återkopplade systemet kan garanteras. Regleralgoritmen är beräkningsmässigt hanterbar då vi kan omformulera problemet som ett kvadratiskt program. För det andra utvecklar vi en robust självutlösande regler-algoritm för tidsvarierande och osäkra begränsade system baserad på uppnåelighets-analys. I specialfallet när det inte finns någon osäkerhet leder konstruktionen till ett reglersystem som kräver minimalt antal sampel över finit tidshorisont. När systemet är linjärt och bivillkoren är polyhedriska visar vi också att de tidigare algoritmerna kan omformuleras som blandade linjära heltalsprogram. Metoden tillämpas på ett rörelseplanerings problem med temporala bivillkor.

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Yulong Gao
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Contents

Chapter 1

Introduction

Uncertainties and constraints are present in most control systems [1–5]. For example, plants are usually corrupted by external disturbances and accompanied by model errors [6]; the control inputs always have saturated values and the system states are generally required to stay within certain ranges. Many practical applications, such as robot motion planning [7], portfolio investment [8], and building climate regulation [9], can be modeled as uncertain constrained systems. A challenging task is how to stabilize such systems, i.e., design a controller which enforces the system state to an invariant region and ensures the constraint satisfactions.

Since it is often impossible to stabilize uncertain systems exactly to a set-point (e.g., an equilibrium), a first question is the characterization of an invariant region. In general, such a region is called a controlled invariant set [2, 3, 10]. Within this set, the states can be maintained by some admissible control inputs. Robust controlled invariant sets (RCISs) are defined for control systems with bounded external disturbances and address the invariance under any realization of the disturbance [11–13]. If the uncertainties are with known probability distributions instead, one interesting question is how to compute a stable region where the state can be kept with a required probability. In this thesis, we introduce probabilistic controlled invariant sets (PCISs) to deal with such situations.

This thesis also considers how to handle networked control systems [14, 15] with aperiodic sensing and actuation. These systems are a particular class of uncertain constrained systems, where the limited network recourses pose essential problems [16, 17]. We provide mathematical tools to design aperiodic controller which can tradeoff the control performance and the communication cost.

The remaining of this chapter is organized as follows. Section 1.1 provides the motivations of this thesis. Section 1.2 gives the mathematical models of the uncertain constrained systems. Section 1.3 formalizes the problems that will be studied. Section 1.4 reviews some related literature. Section 1.5 gives the outline of the thesis.

1.1 Motivation

In this section, we motivate the importance of studying stochastic invariance and aperiodic control for uncertain constrained systems through one example and a discussion of other application areas..

Example

A mobile robot example is shown in Figure 1.1. A mobile robot (gray) moves in a room, where there are some moving obstacles (blue and green robots) and static obstacles (black walls), and is controlled by a remote computer via a shared network. The robot encounters some constraints resulting from the actuation saturation (i.e., control input constraints) and the closed environment (i.e., state constraints). It is reasonable to assume that the robot control is exposed to random noises and uncertainties.

Consider the problem of computing a safe region for the robot. A PCIS is a region within which the controller is able to keep the robot with a certain probability. This region will depend on the stochastic uncertainty of the motion controller together with external influences. Chapter 3 investigates a similar scenario and illustrates how to model the motion of mobile robot and compute the PCIS for motion planning tasks.

Consider next the problem of remotely controlling the robot to fulfill some temporal tasks. For example, the robot is required to move from its initial position (the black dot) to the region $\mathbb{X}_1$ within the deadline N_1 and then move from $\mathbb{X}_1$ to $\mathbb{X}_2$ within the deadline N_2. Because of the limited communication bandwidth, it is natural to design an aperiodic controller, which can complete the control task and guarantee the constraint satisfactions under severe communication restriction. The state and control input trajectories under such an aperiodic control is shown in Figure 1.2. Chapter 5 describes more details on such aperiodic control design.

Application areas

Stochastic invariance and aperiodic control are important in many more areas than motion planning. Here we discuss three such areas:

- *Safety-critical control*. Safety is vital in many control systems [18–20]. In [21, 22], an air traffic management system is modeled as a stochastic hybrid control system. To resolve potential conflicts between aircrafts, an approach is proposed to compute the minimal probability of reaching unsafe regions [22]. Each aircraft is expected to stay in the safe region with some prescribed probability level. After computing the safe region, an aperiodic control protocol between the aircraft and the operation center can efficiently reduce computation burden and save communication resources when controlling the aircraft moving in the safe region.

- *Stochastic model predictive control (MPC)*. In MPC, an invariant set is in general imposed for terminal set to ensure recursive feasibility and stability [4, 5]. Stochastic MPC is used in controlling a stochastic system with probabilistic constraints [23, 24].

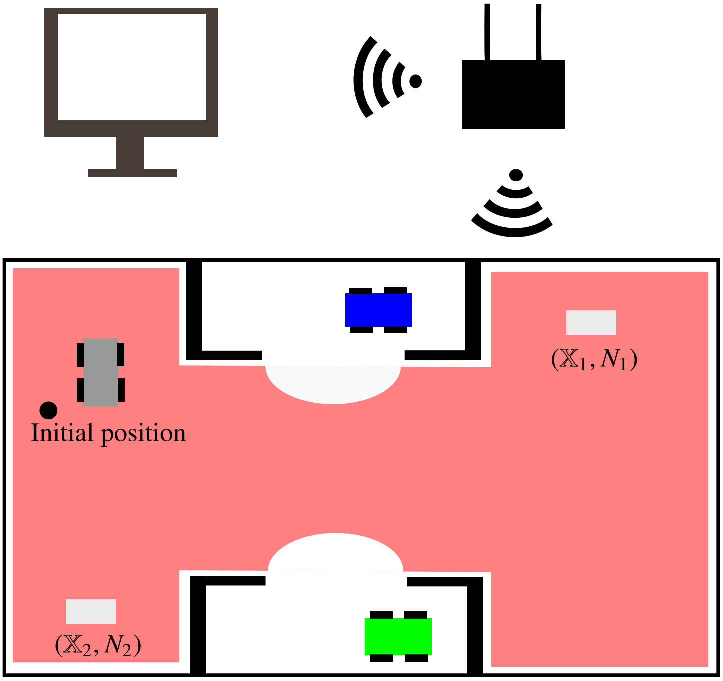


Figure 1.1: Motion of a mobile robot.

Stochastic invariance not only guarantees probabilistic constraint satisfactions but also helps to characterize stochastic stability. Compared with existing methods based on either RCISs as terminal sets [25] or no terminal sets [26], taking a stochastic invariant set as terminal set can possibly mitigate the conservatism of these methods by enlarging the domain of contraction. Furthermore, given a (not necessarily stochastic) terminal set, an aperiodic implementation of stochastic MPC can tradeoff the control performance and the communication usage.

- *Markov decision processes (MDPs).* MDPs are widely used in many control applications such as motion planning [27]. In [28,29], constraints are imposed on the state probability density function of an MDP under control. Probabilistic invariance, as developed in this thesis, can be used for such control systems to characterize the invariant region in the state space, while aperiodic control provides a more communication-efficient strategy to keep the system operating in its invariant region.

1.2 Mathematical Modeling

In this section, we provides two mathematical models that will be used in this thesis.

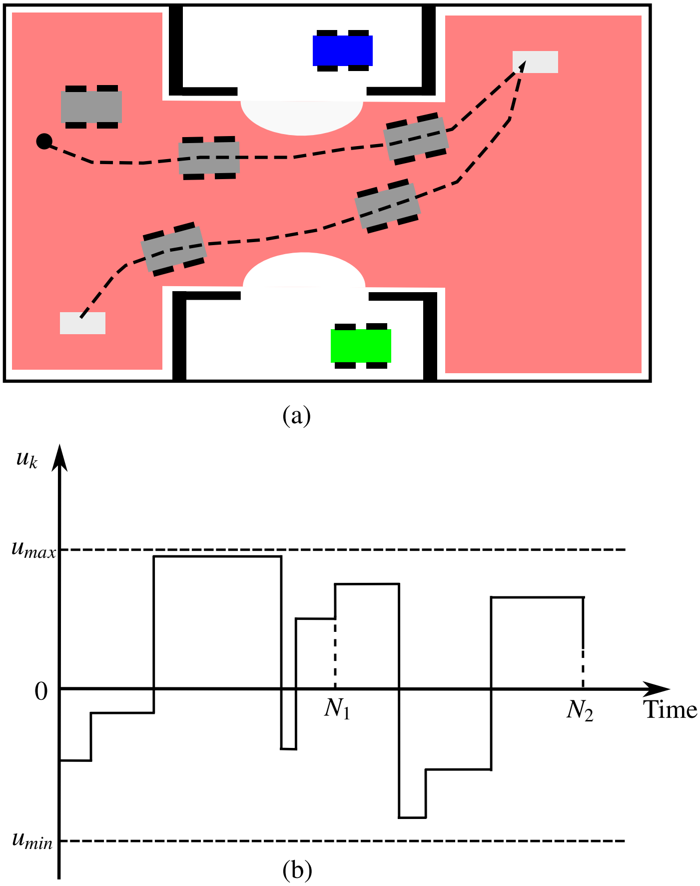


Figure 1.2: The robot trajectory (a) under aperiodic control input (b).

Uncertain Constrained Control Systems

We consider a discrete-time control system with additive disturbance

$$x_{k+1} = f(x_k, u_k) + w_k, \tag{1.1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ the control input, $w_k \in \mathbb{R}^{n_w}$ the disturbance, and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. In general, the disturbance w_k belongs to a given set $\mathbb{W} \subset \mathbb{R}^{n_x}$. At each time instant k, the control input u_k is constrained by a compact set $\mathbb{U} \subset \mathbb{R}^{n_u}$ and the state x_k is constrained by a compact set $\mathbb{X} \subset \mathbb{R}^{n_x}$.

If the set $\mathbb{W}$ is compact and the probability distribution of w_k is unknown, a robust controller can be designed by taking into account the worst case. If the disturbance w_k is with a known probability distribution, the system is a stochastic control system and a controller can be designed by making use of the probabilistic model.

Stochastic Control Systems

We also consider stochastic control systems described by a triple $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$, where

- $\mathbb{X}$ is a state space endowed with a Borel σ-algebra $\mathcal{B}(\mathbb{X})$;
- $\mathbb{U}$ is a compact control space endowed with a Borel σ-algebra $\mathcal{B}(\mathbb{U})$;
- $T : \mathbb{X}\times\mathbb{U} \to \mathbb{R}$ is a Borel-measurable stochastic kernel given $\mathbb{X}\times\mathbb{U}$, which assigns to each $x \in \mathbb{X}$ and $u \in \mathbb{U}$ a probability measure on the Borel space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$: $T(\cdot|x, u)$.

This model is often called a controlled Markov process [30]. It includes a quite large class of stochastic control systems. We remark that for the system (1.1), if w_k, $\forall k \in \mathbb{N}$, are independent and identically distributed (i.i.d.) and are with density function $g : \mathbb{R}^{n_x} \to \mathbb{R}$, it can be represented as a triple $\mathcal{S} = \{\mathbb{X}, \mathbb{U}, T\}$ with

$$\begin{cases} \mathbb{X} = \mathbb{R}^{n_x}, \\ \mathbb{U} \subset \mathbb{R}^{n_u}, \\ T(\mathbb{A}|x, u) = \int_{\mathbb{A}} g(y - f(x, u))dy. \end{cases}$$

1.3 Problem Formulation

In this thesis, we address two key problems for uncertain constrained systems. The first one is the problem of computing PCISs while the second one is the problem of designing self-triggered control schemes.

Problem 1: computation of a PCIS

Consider a stochastic control system described by a triple $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$. We say that a set $\mathbb{Q} \subset \mathbb{X}$ is an ϵ-PCIS if for any $x_0 \in \mathbb{Q}$, there exists control inputs $(u_0, u_1, \ldots)$ such that the generating trajectory $(x_0, x_1, \ldots)$ can be kept into $\mathbb{Q}$ with probability at least ϵ, where $0 \leq \epsilon \leq 1$ is a prescribed number. The problem of computing an ϵ-PCIS is that given a set $\mathbb{Q} \subset \mathbb{X}$ and a required probability level ϵ, find a subset $\tilde{\mathbb{Q}} \subseteq \mathbb{Q}$ such that $\tilde{\mathbb{Q}}$ is an ϵ-PCIS. An intuitive way to compute an ϵ-PCIS is to iteratively compute the stochastic backward reachable set, which for a given set $\mathbb{Q} \subset \mathbb{X}$, is defined by

$$\mathbb{X}^*_\epsilon(\mathbb{Q}) = \{x_0 \in \mathbb{Q} \mid \exists u_k \in \mathbb{U}, \forall k, Pr\{\forall k, x_k \in \mathbb{Q}\} \geq \epsilon\}. \tag{1.2}$$

The algorithm below is shown for computing an ϵ-PCIS.

In Chapter 3, we develop algorithms to compute PCISs by adapting Algorithm 1.1 for different model classes. We prove that these algorithms are computationally tractable and converge in a finite number of iterations.

Algorithm 1.1 Computing an ϵ-PCIS within $\mathbb{Q}$

1: Initialize $i = 0$ and $\mathbb{P}_i = \mathbb{Q}$.
2: Compute the stochastic backward reachable set of $\mathbb{P}_i$, i.e., $\mathbb{X}^*_\epsilon(\mathbb{P}_i)$ and set $\mathbb{P}_{i+1} = \mathbb{X}^*_\epsilon(\mathbb{P}_i)$.
3: Verify whether the resulting set $\mathbb{P}_{i+1}$ is an ϵ-PCIS.
4: If yes, stop. Else, set $i = i + 1$ and go to step 2.

Problem 2: self-triggered control design

A self-triggered control scheme is shown in Figure 1.3. In this scheme, we aim to jointly design a controller and an aperiodic communication protocol, which can achieve the control task, guarantee the constraint satisfactions, and efficiently utilize network resources.

Consider the uncertain constrained control system (1.1). To reduce the amount of communication, the sensor only measures and transmits the state x_{k_i} to the controller at the sampling instants $k_i \in \mathbb{N}$, $i \in \mathbb{N}$, which evolve as

$$k_{i+1} = k_i + M_i \tag{1.3}$$

with $k_0 = 0$. The inter-sampling time M_i is determined by a self-triggering mechanism based on the state x_{k_i} at the sampling instant k_i. An illustration of the difference between the periodically-triggered control scheme and the self-triggered control scheme is shown in Figure 1.4. The self-triggered control problem at sampling instant k_i can be formulated as the following multiple objective optimization problem:

$$\min_{\mathbf{u}, M_i} \quad \{J(x_{k_i}, \mathbf{u}), -M_i\} \tag{1.4a}$$

subject to

$$k_{i+1} = k_i + M_i, \tag{1.4b}$$

$$\forall j \in \mathbb{N}_{[k_i, k_{i+1}-1]} : \tag{1.4c}$$

$$u_j \in \mathbb{U}, \tag{1.4d}$$

$$\forall w_j \in \mathbb{W} : \quad \begin{cases} x_{j+1} = f(x_j, u_j) + w_j, \\ x_{j+1} \in \mathbb{X}, \end{cases} \tag{1.4e}$$

where the control sequence $\mathbf{u} = (u_j)_{j=k_i}^{k_{i+1}-1}$ and $J(x_{k_i}, \mathbf{u})$ is the cost function. The objective function of this optimization problem aims to minimize the cost function and maximize the inter-sampling time. Note that this optimization problem is infinite-dimensional since there are an infinite number of state constraints (1.4e). We address the computational tractability of this problem in Chapters 4 and 5.

In Chapter 4, we integrate self-triggered control and stochastic MPC for linear systems subject to probabilistic constraints. We assume that the optimal control sequence $\mathbf{u}^* = (u_j^*)_{j=k_i}^{k_{i+1}-1}$ can be transmitted via a communication network. The multiple objective optimization problem is solved by a layered framework. The lower layer solves a standard quadratic program indexed with the inter-sampling time while the upper layer maximizes

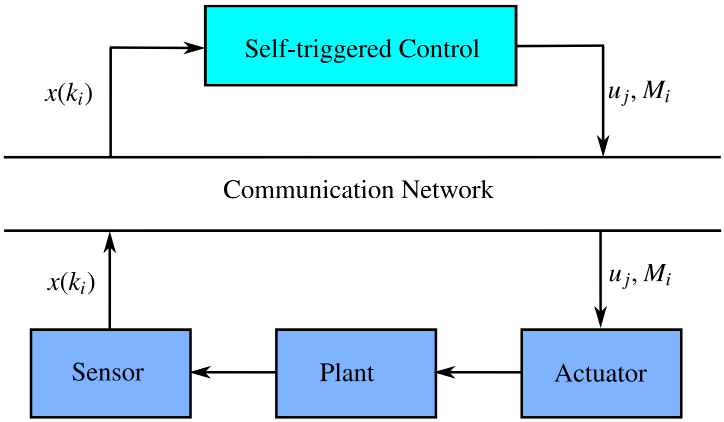


Figure 1.3: The self-triggered control framework.

the inter-sampling time subject to some constraints on the optimal cost function of the lower layer.

In Chapter 5, we investigate robust self-triggered control based on reachability analysis for time-varying and uncertain systems with constraints. We assume that only one control input and the corresponding inter-sampling time can be transmitted via the network. That is, all elements in the control sequence $\mathbf{u} = (u_j)_{j=k_i}^{k_{i+1}-1}$ are equal with each other. In order to provide a geometric insight of self-triggered control, we ignore the cost function and just seek the maximal inter-sampling time. The optimization problem is reformulated as a tractable integer program.

1.4 Related Work

Controlled invariant sets

Controlled invariant sets have been widely studied in the literature [2, 3, 10]. For example, invariant sets are often used to determine the terminal constraints in MPC as they guarantee recursive feasibility of the MPC problem.

In general, robust invariance is customized for dynamical systems with bounded uncertainties. There are lots of iterative approaches focusing on the computation of RCISs. One essential component in these approaches is to compute the robust backward reachable set, in which each state can be steered to the current set by an admissible input for all possible uncertainties [11–13].

Controlled invariant sets have recently been extended to stochastic systems. In [31], the target set used to define the stabilization in probability serves as an embryo of PCIS. A formal description of PCIS is provided in [32] for nonlinear systems by using reachability analysis and it is later applied to portfolio optimization [33]. Another definition of probabilistic invariance originates from stochastic MPC [34] and captures one-step

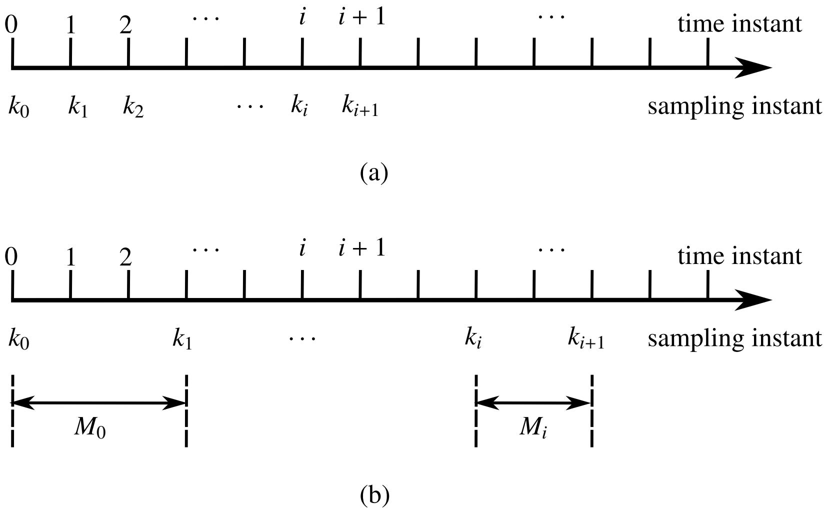


Figure 1.4: The periodically-triggered control scheme (a) and the self-triggered control scheme (b).

ahead invariance. In [34], an ellipsoidal approximation is given for linear systems with specific uncertainty structure. Similar invariant sets are recently used in [35] to construct a convex lifting function for linear stochastic control systems. A definition of a probabilistic invariant set is proposed in [36, 37] for linear stochastic systems without control inputs. This definition captures the inclusion of the state in probability for each time instant. A recent work [38] explores the correspondence between probabilistic and robust invariant sets for linear systems. In [36,37], polyhedral probabilistic invariant sets are approximated by using Chebyshev's inequality for linear systems with Gaussian noise. The recursive satisfaction is usually computationally intractable for general stochastic control systems.

Aperiodic control of constrained systems

Event-triggered and self-triggered control are two specific types of aperiodic control schemes [39, 40]. Self-triggered control scheme can determine the next update time in advance based on the information at the current sampling instant while event-triggered control scheme requires the continuous monitoring of the system states. In addition, self-triggered control scheme permits the shut-down of the sensors between two updates to reduce the energy utilization. This thesis focuses on the self-triggered control design.

Some combinations between MPC strategies and event-triggered control have been provided in [41–44]. In [42], an event-triggered MPC is proposed to reduce both communication and computational effort for discrete-time linear systems subject to input and state constraints as well as exogenous disturbances. In [44], the average sampling rate is explored for robust event-triggered MPC.

Some developments of self-triggered MPC are available. In [45], the authors presented a self-triggered model predictive controller for constrained linear time-invariant systems. As the unconstrained version of [45], a self-triggered linear quadratic regulation (LQR) scheme was considered in [46] for unconstrained systems with stochastic disturbances. In [47], an approach similar to [45] was proposed, where the cost function is defined depending on the length of the inter-sampling time, while in [45] the inter-sampling time is maximized subject to constraints on the cost function. Besides these works focussing on linear systems, a self-triggered nonlinear MPC approach for the case of nonlinear systems was presented in [48], in which a way was proposed to adaptively select sampling time intervals. It is worth noting that [46] does not address the constrained case, while [45], [47], and [48] consider the cases with constraints but without the uncertainty.

Further results of the self-triggered MPC have been reported for systems subject to both constraints and the uncertainty. Based on ideas from robust control-invariant sets, a self-triggered scheme was devised in [49] for linear systems subject to additive disturbances. However, it did not provide the analysis of stability and performance. In [50], a robust self-triggered MPC algorithm was presented for constrained linear systems subject to bounded additive disturbances, which employed the tube-based MPC methods in [51] to guarantee robust constraint satisfaction and followed the principles of [45] to obtain an a priori determination of the next sampling instant. Note that in [50], all constraint parameters of the optimization problem at each sampling instant depend only on the maximal inter-sampling time, which has the drawback of leading to a conservative region of attraction of the MPC scheme. Inspired by a more advanced tube-based MPC method in [52], the authors of [53] proposed a novel robust self-triggered MPC to alleviate the conservatism in [50]. Different from the predicted sets in [50] parametrized by a translation only, an additional scaling factor in the state-space was also introduced in [53] to define the predicted sets, thereby leading to a larger feasible region of the MPC scheme. A recent robust self-triggered MPC method was also presented in [54] for the same problem as in [50, 53]. By combining with the self-triggering mechanism in [53], the focus of [54] lay in extending the tube-based MPC method in [55] to describe the uncertainty in the prediction in the self-triggered setup.

Other than MPC, only a few work address the self-triggered control for constrained systems. For example, the focus of [56–58] is on the design of an event-triggered controller when the systems are subject to actuator saturation. One recent work [59] provides a contractive set-based approach to design self-triggered control for linear deterministic constrained systems.

1.5 Thesis outline and contributions

The rest of the thesis is organized as follows.

Chapter 2: Preliminaries

In Chapter 2, we provide some preliminary results that will be used in this thesis.

Chapter 3: Probabilistic controlled invariant sets

In Chapter 3, we investigate stochastic invariance for control systems by PCISs. We propose two definitions: finite- and infinite-horizon PCISs, and explore their relation to robust control invariant sets. We design iterative algorithms to compute the PCIS within a given set. For systems with discrete state and control spaces, the computations of the finite- and infinite-horizon PCISs at each iteration are based on linear programming and mixed integer linear programming, respectively. The algorithms are computationally tractable and terminate in a finite number of steps. For systems with continuous state and control spaces, we show how to discretize the spaces and prove the convergence of the approximation when computing the finite-horizon PCISs. In addition, it is shown that the infinite-horizon PCIS can be alternatively computed by the stochastic backward reachable set from the robust control invariant set contained in it.

The covered material is based on the following contribution.

- Y. Gao, K. H. Johansson, and L. Xie, "On probabilistic controlled invariant set," *Submitted to IEEE Transaction on Automatic Control*.

Chapter 4: Stochastic self-triggered model predictive control

In Chapter 4, we propose a stochastic self-triggered MPC algorithm for linear systems subject to exogenous disturbances and probabilistic constraints. The main idea behind the self-triggered framework is that at each sampling instant, an optimization problem is solved to determine both the next sampling instant and the control inputs to be applied between the two sampling instants. Although the self-triggered implementation achieves communication reduction, the control commands are necessarily applied in open-loop until the next sampling instant. To guarantee probabilistic constraint satisfaction, necessary and sufficient conditions are derived on the nominal systems by using the information on the distribution of the disturbances explicitly. Moreover, based on a tailored terminal set, a multi-step open-loop MPC optimization problem with infinite prediction horizon is transformed into a tractable quadratic programming problem with guaranteed recursive feasibility. The closed-loop system is shown to be stable.

The covered material is based on the following contribution.

- L. Dai*, Y. Gao*, L. Xie, K. H. Johansson, and Y. Xia, "Stochastic self-triggered model predictive control for linear systems with probabilistic constraints," *Automatica*, vol. 92, pp. 9-17, 2018.

*: Equal contribution of the authors.

Chapter 5: Robust self-triggered control via reachability analysis

In Chapter 5, we develop a robust self-triggered control algorithm for time-varying and uncertain systems with constraints based on reachability analysis. The resulting piecewise constant control inputs achieve communication reduction and guarantee constraint satisfactions. In the particular case when there is no uncertainty, we propose a control design with minimum number of samplings over finite time horizon. Furthermore, when the plant is linear and the constraints are polyhedral, we prove that the previous algorithms can be reformulated as computationally tractable mixed integer linear programs.

The covered material is based on the following contribution.

- Y. Gao, P. Yu, D. V. Dimarogonas, K. H. Johansson, and L. Xie, “Robust self-triggered control for time-varying and uncertain constrained systems via reachability analysis,” *Submitted to Automatica*.

Chapter 6: Conclusions and future research

In Chapter 6, we present a summary of the results, and discuss directions for future research.

Contributions not covered in the thesis

The following publications by the author are not covered in the thesis:

- Y. Gao, S. Wu, K. H. Johansson, L. Shi, and L. Xie, “Stochastic optimal control of dynamic queue systems: a probabilistic perspective,” *in Proceedings of 15th International Conference on Control, Automation, Robotics and Vision*, 2018.

- Y. Gao, M. Jafarian, K. H. Johansson, and L. Xie, “Distributed freeway ramp metering: optimization on flow speed,” *in Proceedings of IEEE Conference on Decision and Control*, 2017.

Chapter 2

Preliminaries

In this chapter, we provide notations and preliminaries that are used in the remaining parts of this thesis.

2.1 Notation

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{R}$ denote the set of real numbers. For some $q, s \in \mathbb{N}$ and $q < s$, let $\mathbb{N}_{\geq q}$, $\mathbb{N}_{>q}$, $\mathbb{N}_{\leq q}$, $\mathbb{N}_{<q}$, and $\mathbb{N}_{[q,s]}$ denote the sets $\{r \in \mathbb{N} \mid r \geq q\}$, $\{r \in \mathbb{N} \mid r > q\}$, $\{r \in \mathbb{N} \mid r \leq q\}$, $\{r \in \mathbb{N} \mid r < q\}$, and $\{r \in \mathbb{N} \mid q \leq r \leq s\}$, respectively. Let I denote an identity matrix. A matrix or vector of ones or zeors with appropriate dimension is denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively. When $\leq$, $\geq$, $<$, $>$, and $|\cdot|$ are applied to vectors, they are interpreted element-wise. Let x_k denote the value of variable x at time instant k, $x_{k+i|k}$ a prediction i steps ahead from time k.

Let Pr denote the probability measure, $\mathbb{E}$ the expectation, and $\mathbb{E}_k$ the conditional expectation of a random variable given the state at time k. Given a topological space $\mathbb{X}$, $\mathcal{B}(\mathbb{X})$ denotes the Borel σ-algebra of this space.

For a vector $x \in \mathbb{R}^n$, define $||x||_\infty = \max_i |x_i|$. For $W \in \mathbb{R}^{n\times n}$, $W \succ 0$ means that W is symmetric and positive definite. For $x \in \mathbb{R}^n$ and $W \succ 0$, $||x||_W^2 \triangleq x^T W x$. For $x_i \in \mathbb{R}^n$, $i \in \mathbb{N}$, define $\sum_{i=a}^{b} x_i = \mathbf{0}$ if $a > b$.

The Minkowski sum of two sets is denoted by $\mathbb{A} \oplus \mathbb{B} = \{a + b \mid \forall a \in \mathbb{A}, \forall b \in \mathbb{B}\}$. The Minkowski difference of two sets is denoted by $\mathbb{A} \ominus \mathbb{B} = \{c \mid \forall b \in \mathbb{B}, c + b \in \mathbb{A}\}$. For two sets $\mathbb{X}$ and $\mathbb{Y}$, $\mathbb{X} \setminus \mathbb{Y} = \{x \mid x \in \mathbb{X}, x \notin \mathbb{Y}\}$. For a polyhedron $\mathbb{P} = \{x \in \mathbb{R}^n \mid Px \leq p\}$, define $||\mathbb{P}||_\infty = \max_{x\in\mathbb{P}}\{||Px - p||_\infty\}$. For $\mathbb{X} \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, define $A^{-1}\mathbb{X} = \{x \in \mathbb{R}^n \mid Ax \in \mathbb{X}\}$. For $x_l \in \mathbb{R}^n$, $l \in \mathbb{N}$, define $\sum_{l=k}^{j} x_l = \mathbf{0}$ if $k > j$. For $\mathbb{X}_l \subseteq \mathbb{R}^n$ and $A_l \in \mathbb{R}^{n\times n}$, $l \in \mathbb{N}$, define

$$\bigoplus_{l=k}^{j} \mathbb{X}_l = \begin{cases} \emptyset, \ k > j, \\ \mathbb{X}_k \oplus \mathbb{X}_{k+1} \oplus \ldots \mathbb{X}_j, \ k \leq j, \end{cases}$$

$$\prod_{l=k}^{j} A_l = \begin{cases} I, \ k > j, \\ A_j A_{j-1} \ldots A_k, \ k \leq j. \end{cases}$$

Given two $\mathbb{X}$ and $\tilde{\mathbb{X}}$, two indicator functions are, respectively, defined as

$$\mathbb{1}_{\mathbb{X}}(x) = \begin{cases} 1, & x \in \mathbb{X}, \\ 0, & x \notin \mathbb{X}, \end{cases} \quad \text{and } \mathbb{1}_{\mathbb{X}}(\tilde{\mathbb{X}}) = \begin{cases} 1, & \tilde{\mathbb{X}} \subseteq \mathbb{X}, \\ 0, & \tilde{\mathbb{X}} \nsubseteq \mathbb{X}. \end{cases}$$

2.2 Stochastic Optimal Control

Markov Policy

Consider a stochastic control system described by a triple $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$ as in Section 1.2.

Definition 2.1. *Given a Polish space $\mathbb{Y}$, a subset $\mathbb{A}$ in this space is universally measurable if it is measurable with respect to every complete probability measure on $\mathbb{Y}$ that measures all Borel sets in $\mathcal{B}(\mathbb{Y})$.*

Definition 2.2. *A function $\mu : \mathbb{Y} \to \mathbb{W}$ is universally measurable if $\mu^{-1}(\mathbb{A})$ is universally measurable in $\mathbb{Y}$ for every $\mathbb{A} \in \mathcal{B}(\mathbb{W})$.*

Let the finite horizon be N. We define a Markov policy as follows.

Definition 2.3. *(Markov Policy) A Markov policy μ for system $\mathcal{S}$ is a sequence $\mu = (\mu_0, \mu_1, \ldots, \mu_{N-1})$ of universally measurable maps*

$$\mu_k : \mathbb{X} \to \mathbb{U}, \forall k \in \mathbb{N}_{[0,N-1]}.$$

Remark 2.1. *As stated in [60, 61], the condition of universal measurability is weaker than the condition of Borel measurability for showing the existence of a solution to a stochastic optimal problem. Roughly speaking, this is because the projections of measurable sets are analytic sets and analytic sets are universally measurable but not always Borel measurable [61, 62].*

When extending the finite horizon to the infinite horizon, we define a stationary policy.

Definition 2.4. *(Stationary Policy) A Markov policy $\mu \in \mathcal{M}$ is said to be stationary if $\mu = (\bar{\mu}, \bar{\mu}, \ldots)$ with $\bar{\mu} : \mathbb{X} \to \mathbb{U}$ universally measurable.*

Let $\mathcal{M}$ denote the set of Markov policies. In the following, we recall two stochastic optimal control problems which aim at maximizing the probability of staying one set.

Finite-horizon stochastic optimal control

Consider a Borel set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$. Given an initial state $x_0 \in \mathbb{X}$ and a Markov policy $\mu \in \mathcal{M}$, an execution is a sequence of states $(x_0, x_1, \ldots, x_N)$. Introduce the probability with which the state x_k will remain within $\mathbb{Q}$ for all $k \in \mathbb{N}_{[0,N]}$:

$$p^{\mu}_{N,\mathbb{Q}}(x_0) = Pr\{\forall k \in \mathbb{N}_{[0,N]}, x_k \in \mathbb{Q}\}.$$

Let $p^*_{N,\mathbb{Q}}(x) = \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}}(x)$. The next theorem shows that this finite-horizon stochastic optimal control problem can be solved via a dynamic program (DP).

Theorem 2.1. *[60] Define value functions $V^*_{k,\mathbb{Q}} : \mathbb{X} \to [0,1], k = 0, 1, \ldots, N$, in the backward recursion:*

$$V^*_{k,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}} \mathbb{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y) T(dy|x,u), x \in \mathbb{X}, \tag{2.1}$$

with initialization

$$V^*_{N,\mathbb{Q}}(x) = 1, x \in \mathbb{Q}. \tag{2.2}$$

Then, $V^*_{0,\mathbb{Q}}(x) = p^*_{N,\mathbb{Q}}(x)$, $\forall x \in \mathbb{Q}$. The optimal Markov policy $\mu^*_{\mathbb{Q}} = (\mu^*_{0,\mathbb{Q}}, \mu^*_{1,\mathbb{Q}}, \ldots, \mu^*_{N-1,\mathbb{Q}})$ is given by

$$\mu^*_{k,\mathbb{Q}}(x) = \arg\sup_{u \in \mathbb{U}} \mathbb{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y) T(dy|x,u), x \in \mathbb{Q}, k \in \mathbb{N}_{[0,N-1]}.$$

In particular, one sufficient condition for the existence of $\mu^*_{\mathbb{Q}}$ is that for all $x \in \mathbb{Q}$, $\lambda \in \mathbb{R}$, and $k \in \mathbb{N}_{[0,N-1]}$, the set $\mathbb{U}_k(x, \lambda) = \{u \in \mathbb{U} \mid \int_{\mathbb{X}} V^*_{k+1,\mathbb{Q}}(y) T(dy|x,u) \geq \lambda\}$ is compact.

Remark 2.2. *Here, $V^*_{k,\mathbb{Q}}(x)$ denotes the maximum probability with which starting from an initial state $x \in \mathbb{Q}$, the states remain in $\mathbb{Q}$ over the time interval $\mathbb{N}_{[k,N]}$.*

Infinite-horizon stochastic optimal control

Consider a Borel set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$. Given an initial state $x_0 \in \mathbb{X}$ and a stationary policy $\mu \in \mathcal{M}$, an execution is denoted by a sequence of states $(x_0, x_1, \ldots)$. We introduce the probability with which the state x_k will remain within $\mathbb{Q}$ for all $k \in \mathbb{N}$:

$$p^{\mu}_{\infty,\mathbb{Q}}(x_0) = Pr\{\forall k \in \mathbb{N}, x_k \in \mathbb{Q}\}.$$

Let $p^*_{\infty,\mathbb{Q}}(x_0) = \sup_{\mu \in \mathcal{M}} p^{\mu}_{\infty,\mathbb{Q}}(x_0)$. The next theorem gives a way to solve this infinite-horizon stochastic optimal control problem.

Theorem 2.2. *[60] Define the value function $G^*_{k,\mathbb{Q}} : \mathbb{X} \to [0,1], k \in \mathbb{N}$, in the forward recursion:*

$$G^*_{k+1,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}} \mathbb{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} G^*_{k,\mathbb{Q}}(y) T(dy|x,u), x \in \mathbb{X}, \tag{2.3}$$

initialized with

$$G^*_{0,\mathbb{Q}}(x) = 1, x \in \mathbb{Q}. \tag{2.4}$$

Suppose that there exists a $\bar{k} \geq 0$ such that the set $\mathbb{U}_k(x, \lambda) = \{u \in \mathbb{U} \mid \int_{\mathbb{X}} G^*_{k,\mathbb{Q}}(y) T(dy|x,u) \geq \lambda\}$ is compact for all $x \in \mathbb{Q}$, $\lambda \in \mathbb{R}$, and $k \in \mathbb{N}_{\geq \bar{k}}$. Then, the limitation $G^*_{\infty,\mathbb{Q}}(x)$ exists and satisfies

$$G^*_{\infty,\mathbb{Q}}(x) = \sup_{u \in \mathbb{U}} \mathbb{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} G^*_{\infty,\mathbb{Q}}(y) T(dy|x,u)), \tag{2.5}$$

and $G^*_{\infty,\mathbb{Q}}(x) = p^*_{\infty,\mathbb{Q}}(x)$ *for all* $x \in \mathbb{Q}$*. Furthermore, there exists an optimal stationary policy* $\mu^* = (\bar{\mu}^*, \bar{\mu}^*, \ldots)$ *given by*

$$\bar{\mu}^*_{\mathbb{Q}}(x) = \arg\sup_{u \in \mathbb{U}} \mathbb{1}_{\mathbb{Q}}(x) \int_{\mathbb{Q}} G^*_{\infty,\mathbb{Q}}(y) T(dy|x,u), x \in \mathbb{Q}.$$

Remark 2.3. *Here,* $G^*_{k,\mathbb{Q}}(x)$ *denotes the maximum probability with which starting from an initial state* $x \in \mathbb{Q}$*, the states remain in* $\mathbb{Q}$ *over the time interval* $\mathbb{N}_{[0,k]}$*.*

2.3 Stochastic MPC

Different from robust MPC, stochastic MPC makes full use of the stochastic characteristics of the uncertainties to deal with probabilistic constraints, which allows constraint violations to occur with a prespecified probability level. Probabilistic constraints enable stochastic MPC to directly trade off the constraint satisfaction and the control performance, and therefore alleviate the inherent conservatism of robust MPC. In what follows, we introduce the general formulation of stochastic MPC in more details.

Consider a discrete-time control system of the form

$$x_{k+1} = f(x_k, u_k) + w_k, \tag{2.6}$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_x}$, and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. Assume that w_k, $k \in \mathbb{N}$, are independent and identically distributed (i.i.d.) and the elements of w_k have zero mean. The distribution F of w_k is assumed to be known and continuous with a compact support $\mathbb{W}$. At each time instant k, the state x_k of system (2.6) is subject to n_c probabilistic constraints

$$Pr\{g_\ell(x_k) \leq 0\} \geq p_\ell,\ \ell \in \mathbb{N}_{[1,n_c]},\ k \in \mathbb{N}, \tag{2.7}$$

where $g_\ell : \mathbb{R}^{N_x} \to \mathbb{R}$, and $p_\ell \in [0, 1]$, and the control input u_k is constrained by a compact set $\mathbb{U} \subset \mathbb{R}^{n_u}$.

Given a finite horizon N, at time instant k, a sequence of predictive control inputs is defined by $\mathbf{u}_k = (u_{k|k}, u_{k+1|k}, \cdots, u_{k+N-1|k})$ and the corresponding sequence of predictive states is defined by $\mathbf{x}_k = (x_{k|k}, x_{k+1|k}, \cdots, x_{k+N|k})$, where

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}) + w_{k+i|k}, x_{k|k} = x_k, i \in \mathbb{N}_{\leq N-1}.$$

The objective function is

$$J(x_k, \mathbf{u}_k) = \mathbb{E}_k[\sum_{i=0}^{N-1}(\|x_{k+i|k}\|^2_Q + \|u_{k+i|k}\|^2_R) + \|x_{k+N|k}\|^2_P] \tag{2.8}$$

where $Q \succ 0$, $R \succ 0$, and $P \succ 0$ are the weighting matrices.

Given x_k, the stochastic MPC optimization problem can be formulated as

$$\min_{\mathbf{u}_k} J(x_k, \mathbf{u}_k) \tag{2.9a}$$

subject to

$$x_{k|k} = x_k, \tag{2.9b}$$

$$\forall i \in \mathbb{N}_{[0,N-1]} : \tag{2.9c}$$

$$\begin{cases} x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}) + w_{k+i|k}, \\ u_{k+i|k} \in \mathbb{U}, \\ w_{k+i|k} \in \mathbb{W}, w_{k+i|k} \sim F, \\ Pr\{g_\ell(x_{k+i|k} \leq 0 \mid x_k\} \geq p_\ell, \ell \in \mathbb{N}_{[1,n_c]}, \end{cases} \tag{2.9d}$$

$$x_{k+N|k} \in \mathbb{X}_f, \tag{2.9e}$$

where $\mathbb{X}_f$ is the terminal set to be designed. After solving this optimization problem, the first piece of the sequence of the optimal control inputs $\mathbf{u}_k^*$ is implemented, i.e., $u_k = u_{k|k}^*$.

Note that the probabilistic constraints restrict the tractability of the above optimization problem. Here, we summarize three main approaches on handling the probabilistic constraints in the literature.

(1) *Stochastic Tube Approaches*. Like tube-based MPC [63], stochastic tube is constructed by exploiting the distribution of the uncertainties and is deployed to guarantee recursive feasibility, closed-loop stability, and probabilistic constraint satisfaction. In [25], stochastic tube-based MPC is used for linear systems with additive and bounded disturbances. This stochastic tube is determined by tightening constraints, which requires large and complex offline computations. To improve the computational efficiency, the stochastic tube with fixed orientations but scalable cross sections is investigated in [23].

(2) *Affine Parameterization Approaches*. In [64], the feedback control input is defined in terms of an affine function of past disturbances. The decision variables in the optimization problem become the affine parameters. Thus, the probabilistic constraints are also parameterized. In [65], the Chebyshev–Cantelli inequality is used to handle the parameterized probabilistic constraints. The first drawback of affine parameterization approaches is the difficulty to ensure of the closed-loop stability. Another one is the conservatism associated with the approximations like the Chebyshev–Cantelli inequality.

(3) *Scenario-based Approaches*. Scenario optimization provides an explicit bound on the number of samples required to obtain a solution to the convex optimization problem that guarantees constraint satisfaction with a prespecified probability level [66]. In [67, 68], various scenario-based approaches have been considered for approximating the probabilistic constraints in MPC. However, the challenges of scenario-based MPC arise from both the high computation and the guarantees of recursive feasibility and closed-loop stability.

2.4 Reachability Analysis

The reachability problem is fundamental in systems and control and is critical for controllability and tracking problems [69–71]. In the following, we summarize some results on computing the backward reachable tube for constrained systems.

Consider a discrete-time dynamic system

$$x_{k+1} = f_k(x_k, u_k) + w_k, \tag{2.10}$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_x}$, and $f_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. The control input u_k at time k is constrained by a set $\mathbb{U}_k \subset \mathbb{R}^{n_u}$. The additive disturbance w_k at time instant k belongs to a compact set $\mathbb{W}_k \subset \mathbb{R}^{n_x}$. In addition, given a finite time horizon $N \in \mathbb{N}$, the system (2.6) is subject to a target tube, denoted by $\{(\mathbb{X}_k, k), k \in \mathbb{N}_{[1,N]}\}$, where $\mathbb{X}_k \subseteq \mathbb{R}^{n_x}$.

Definition 2.5. *(Reachability) The target tube* $\{(\mathbb{X}_k, k), k \in \mathbb{N}_{[1,N]}\}$ *of the system (2.10) is reachable from the initial state* $x_0 \in \mathbb{X}_0$ *if there exists a sequence of control inputs* $u_k \in \mathbb{U}_k$, $\forall k \in \mathbb{N}_{[0,N-1]}$, *such that the state* $x_k \in \mathbb{X}_k$, $\forall k \in \mathbb{N}_{[1,N]}$, *for all possible disturbance sequences* $w_k \in \mathbb{W}_k$, $\forall k \in \mathbb{N}_{[0,N-1]}$.

Let $\mathbb{X}_N^* = \mathbb{X}_N$. For $k \in \mathbb{N}_{[0,N-1]}$, the backward reachable set $\mathbb{X}_k^*$ for the system (2.10) is recursively computed by:

$$\mathbb{P}_k = \{z \in \mathbb{R}^{n_x} \mid \exists u_k \in \mathbb{U}_k, f_k(z, u_k) \oplus \mathbb{W}_k \subseteq \mathbb{X}_{k+1}^*\}, \tag{2.11a}$$

$$\mathbb{X}_k^* = \mathbb{P}_k \cap \mathbb{X}_k. \tag{2.11b}$$

Proposition 2.1. *[69] The target tube* $\{(\mathbb{X}_j, j), j \in \mathbb{N}_{[k+1,N]}\}$ *of the system (2.10) is reachable from* x_k *if and only if* $x_k \in \mathbb{P}_k$. *Furthermore, the target tube* $\{(\mathbb{X}_k, k), k \in \mathbb{N}_{[1,N]}\}$ *is reachable from the initial state* $x_0 \in \mathbb{X}_0$ *if and only if* $x_0 \in \mathbb{X}_0^*$.

The computation of backward reachable tube for control systems has attracted much attentions in the past decades. Many existing results focus on sets of pre-specified shapes, such as polytopes or hyperplanes [72]. In particular, for linear systems with polyhedra constraints, there are some geometry software packages which can efficiently implement set operations, such as projection, set difference, piecewise affine maps and their inverse, Minkowski sums, intersections, etc [73, 74]. Some recent research progresses have been reported for nonlinear systems. For example, the authors of [71] generalize the reachability of linear systems to piecewise affine systems with polygonal constraints and prove that the set can be computed by using polyhedral algebra and computational geometry software. Recently, a decomposition approach is proposed in [75] to compute the reachable set for a class of high dimensional nonlinear systems.

In general, reachability analysis leads to a large number of constraints with the increase of the dimension and horizon. In order to make the computation efficient, there have been a lot of results on inner approximations of backward reachable sets [76]. Note that according to Proposition 2.1, the inner approximations are still the sufficient conditions to guarantee the reachability.

Chapter 3

Probabilistic Controlled Invariant Sets

3.1 Introduction

This chapter investigates stochastic invariance for control systems by PCISs. Our results build on some existing work but improve them in several aspects. (**i**) The results in [32, 34, 36, 37] focus on some specific stochastic systems (e.g., linear or nonlinear systems) or on some specific stochastic disturbances (e.g., Gaussian noise or state-independent noise). In our model, general system dynamics and stochastic disturbances are considered. (**ii**) Different from [36, 37], the control inputs are incorporated in our invariant sets and these sets are defined based on the trajectory inclusion as in [32]. This type of definition allows us to verify and compute a PCIS in an iterative way. Comparisons between our invariant sets and a possible extension of probabilistic invariant set in [36, 37] to control systems are provided in Section 3.4. (**iii**) The stochastic reachability analysis studied in [60] provides us an important tool for maximizing the probability of staying in a set. Based on this, one focus of this chapter is on the computation of maximal PCIS within a set with a prescribed probability level, which is beyond the scope of [32, 60, 77]. In addition, note that the PCISs in this chapter are different from the maximal probabilistic safe sets in [60] (see Remark 3.1).

Recall that Algorithm 1.1 in Chapter 1 provides the basic procedures to compute the PCISs within a given set. However, some remarkable challenges in Algorithm 1.1 should be highlighted: (**i**) how to make it tractable to compute the stochastic backward reachable set, in particular for continuous spaces; (**ii**) how to mitigate the conservatism when characterizing the stochastic backward reachable set subject to the prescribed probability; (**iii**) how to guarantee the convergence of the iterations. These issues will be addressed in this chapter. The contributions are summarized as follows.

(1) We propose two novel definitions of PCIS: N-step ϵ-PCIS and infinite-horizon ϵ-PCIS. An N-step ϵ-PCIS is a set within which the state can stay for N steps with probability ϵ under some admissible controller while an infinite-horizon ϵ-PCIS is a set within which the state can stay forever with probability ϵ under some admissible controller. These invariant sets are different from the existing ones [34, 36], which address probabilistic set invariance at each time step. Our

definitions are applicable for general discrete-time stochastic control systems. We provide fundamental properties of PCISs and explore their relation to RCISs.

(2) We design iterative algorithms to compute the largest finite- and infinite-horizon PCIS within a given set for systems with discrete and continuous spaces. The PCIS computation is based on the stochastic backward reachable set. For finite state and control spaces, it is shown that at each iteration, the stochastic backward reachable set computation of N-step ϵ-PCIS can be reformulated as a linear program (LP) and the infinite-horizon ϵ-PCIS as a computationally tractable mixed-integer linear program (MILP). Furthermore, we prove that these algorithms terminate in a finite number of steps. For continuous state and control spaces, we present a discretization procedure. Under weaker assumptions than [78], we prove the convergence of such approximations for N-step ϵ-PCISs. The approximations generalize the case in [60], which only discretizes the state space for a given finite control space. In addition, another different method to compute the infinite-horizon ϵ-PCIS is provided based on the structure that an infinite-horizon PCIS always contain a RCIS..

The remainder of the chapter is organized as follows. Section 3.2 presents the definition, properties, computation algorithms of finite-horizon PCISs. Section 3.3 extends to the infinite-horizon case. In Section 3.4, we analyze the algorithm complexities and discuss the relation to the existing work. Numerical examples in Section 3.5 illustrate the effectiveness of our approach. Section 3.6 concludes this chapter. In addition, Section 2.2 provides some preliminaries that will be used in this chapter.

3.2 Finite-horizon PCIS

We consider a stochastic control system described by a triple $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$. We first define a finite-horizon ϵ-PCIS for system $\mathcal{S}$ and provide some properties of this set. Then, we explore how to compute the finite-horizon ϵ-PCIS with a given set in an iterative way.

Definition 3.1. *(N-step ϵ-PCIS) Given a confidence level $0 \leq \epsilon \leq 1$, a set $\mathbb{Q} \subseteq \mathbb{X}$ is an N-step ϵ-PCIS for system $\mathcal{S}$ if for any $x \in \mathbb{Q}$, there exists at least one Markov policy $\mu \in \mathcal{M}$ such that $p^{\mu}_{N,\mathbb{Q}}(x) \geq \epsilon$.*

According to Theorem 2.1, an N-step ϵ-PCIS can be verified by checking if $p^{*}_{N,\mathbb{Q}}(x) \geq \epsilon$ for every $x \in \mathbb{Q}$, as stated by the following proposition.

Proposition 3.1. *A set $\mathbb{Q} \subset \mathbb{X}$ is an N-step ϵ-PCIS for the system $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$ if the following conditions are satisfied:*

(i) $V^{*}_{0,\mathbb{Q}}(x) \geq \epsilon$ for all $x \in \mathbb{Q}$, where $V^{*}_{0,\mathbb{Q}}(x)$ is calculated by the DP (2.1) and (2.2);

(ii) for all $x \in \mathbb{X}$, $\lambda \in \mathbb{R}$, and $k \in \mathbb{N}_{[0,N-1]}$, the set $\mathbb{U}_k(x, \lambda) = \{u \in \mathbb{U} \mid \int_{\mathbb{X}} V^{*}_{k+1,\mathbb{Q}}(y) T(dy|x, u) \geq \lambda\}$ is compact.

An equivalent way to verify an N-step ϵ-PCIS is based on the following stochastic backward reachable set:

$$\begin{aligned}\mathbb{X}^*_{\epsilon,N}(\mathbb{Q}) &= \{x \in \mathbb{Q} \mid \exists \mu \in \mathcal{M}, p^{\mu}_{N,\mathbb{Q}}(x) \geq \epsilon\} \\ &= \{x \in \mathbb{Q} \mid \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}}(x) \geq \epsilon\} \\ &= \{x \in \mathbb{Q} \mid V^*_{0,\mathbb{Q}}(x) \geq \epsilon\}.\end{aligned}$$

Proposition 3.2. *A set $\mathbb{Q} \subseteq \mathbb{X}$ is an N-step ϵ-PCIS for the system $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$ if and only if $\mathbb{X}^*_{\epsilon,N}(\mathbb{Q}) = \mathbb{Q}$.*

Proof. Follow from the definition of $\mathbb{X}^*_{\epsilon,N}(\mathbb{Q})$. □

Corollary 3.1. *The state space $\mathbb{X}$ is an N-step ϵ-PCIS for any finite $N \in \mathbb{N}_{\geq 1}$ and any $0 \leq \epsilon \leq 1$.*

Remark 3.1. *The stochastic backward reachable set $\mathbb{X}^*_{\epsilon,N}(\mathbb{Q})$ is called the maximal probabilistic safe set in [60]. The N-step ϵ-PCIS $\mathbb{Q}$ in Definition 3.1 refines the maximal probabilistic safe set by requiring that for any initial state from $\mathbb{Q}$, the maximal probability of staying in $\mathbb{Q}$ is no less than ϵ, as in Proposition 3.2.*

Properties

In the following, some properties of N-step PCISs are presented.

Property 3.1. *Consider two Borel sets $\mathbb{Q}, \mathbb{P} \in \mathcal{B}(\mathbb{X})$ for system $\mathcal{S}$ with $\mathbb{Q} \subseteq \mathbb{P}$. For any $x \in \mathbb{Q}$, the following statements hold:*

(i) $p^{\mu}_{N,\mathbb{Q}}(x) \leq p^{\mu}_{\tilde{N},\mathbb{Q}}(x)$, $\forall \tilde{N} \leq N$ *and* $\forall \mu \in \mathcal{M}$;

(ii) $p^{\mu}_{N,\mathbb{Q}}(x) \leq p^{\mu}_{N,\mathbb{P}}(x)$, $\forall \mu \in \mathcal{M}$ *and* $\forall N \in \mathbb{N}$;

(iii) $\sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}}(x) \leq \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{P}}(x)$, $\forall N \in \mathbb{N}$.

Proof. The results (i) and (ii) follow from the definition. For (iii), let us denote $\mu_1^* = \arg\sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}}(x)$ and $\mu_2^* = \arg\sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{P}}(x)$. Then, we have

$$\sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}}(x) = p^{\mu_1^*}_{N,\mathbb{Q}}(x) \leq p^{\mu_1^*}_{N,\mathbb{P}}(x) \leq p^{\mu_2^*}_{N,\mathbb{P}}(x) = \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{P}}(x).$$

The proof is completed. □

Property 3.2. *If $\mathbb{Q} \subseteq \mathbb{X}$ is an N-step ϵ-PCIS for system $\mathcal{S}$, it is also an $\tilde{N}$-step $\tilde{\epsilon}$-PCIS for any $\tilde{N} \in \mathbb{N}_{[0,N]}$ and $0 \leq \tilde{\epsilon} \leq \epsilon$.*

Proof. Follow from the definition of N-step ϵ-PCIS. □

Property 3.3. *Consider a collection of Borel sets $\mathbb{Q}_i \in \mathcal{B}(\mathbb{X})$, $i = 1, \ldots, r$. If each $\mathbb{Q}_i$ is an N_i-step ϵ_i-PCIS for system $\mathbb{S}$, then the union $\bigcup_{i=1}^{r} \mathbb{Q}_i$ is an N-step ϵ-PCIS, where $N = \min\{N_i, i = 1, \ldots, r\}$ and $\epsilon = \min\{\epsilon_i, i = 1, \ldots, r\}$.*

Proof. It suffices to consider $r = 2$. For any $x \in \mathbb{Q}_1 \cup \mathbb{Q}_2$, we have that either $x \in \mathbb{Q}_1$ or $x \in \mathbb{Q}_2$. From Properties 3.1 and 3.2, we have:

$$\forall x \in \mathbb{Q}_1,\ \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}_1 \cup \mathbb{Q}_2}(x) \geq \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}_1}(x) \geq \sup_{\mu \in \mathcal{M}} p^{\mu}_{N_1,\mathbb{Q}_1}(x) \geq \epsilon_1 \geq \min\{\epsilon_1, \epsilon_2\},$$
$$\forall x \in \mathbb{Q}_2,\ \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}_1 \cup \mathbb{Q}_2}(x) \geq \sup_{\mu \in \mathcal{M}} p^{\mu}_{N,\mathbb{Q}_2}(x) \geq \sup_{\mu \in \mathcal{M}} p^{\mu}_{N_2,\mathbb{Q}_2}(x) \geq \epsilon_2 \geq \min\{\epsilon_1, \epsilon_2\}.$$

Then, we complete the proof. □

Remark 3.2. *The finite-horizon PCISs are closed under union. In general, they are not closed under intersection, i.e., the intersection of two PCISs is not necessarily a PCIS. The reason is that the corresponding control policies of two invariant sets may be different. This is different from the property of probabilistic invariant sets in [36], which does not involve control inputs.*

Finite-horizon PCIS computation

A general procedure to compute the N-step ϵ-PCIS within a given set $\mathbb{Q} \subset \mathbb{X}$ is presented by Algorithm 3.1, which is initialized by setting $\mathbb{P}_0 = \mathbb{Q}$. Then, we compute $V^*_{0,\mathbb{P}_i}(x)$, $\forall x \in \mathbb{P}_i$, by the DP (2.1) and (2.2). We further update the set $\mathbb{P}_{i+1} = \mathbb{X}^*_{\epsilon,N}(\mathbb{P}_i)$, which is a stochastic backward reachable set within $\mathbb{P}_i$ with respect to a finite horizon N and a probability level ϵ. According to Propositions 3.1 and 3.2, the algorithm terminates when $\mathbb{P}_{i+1} = \mathbb{P}_i$.

Algorithm 3.1 N-step ϵ-PCIS

1: Initialize $i = 0$ and $\mathbb{P}_i = \mathbb{Q}$.
2: Compute $V^*_{0,\mathbb{P}_i}(x)$ for all $x \in \mathbb{P}_i$.
3: Compute the set $\mathbb{P}_{i+1} = \mathbb{X}^*_{\epsilon,N}(\mathbb{P}_i, \mathbb{P}_i)$.
4: If $\mathbb{P}_{i+1} = \mathbb{P}_i$, stop. Else, set $i = i + 1$ and go to step 2.

Theorem 3.1. *Consider a Borel set $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$. If Algorithm 3.1 converges to a nonempty set, this set is the maximal N-step PCIS within $\mathbb{Q}$.*

Proof. According to the definition of $\mathbb{X}^*_{\epsilon,N}$, we have $\mathbb{P}_{i+1} \subseteq \mathbb{P}_i$, $\forall i \geq 0$. Since Algorithm 3.1 converges to a nonempty set, $\mathbb{P}_\infty = \lim_{i \to \infty} \mathbb{P}_i$ exists and satisfies $\mathbb{P}_\infty = \mathbb{X}^*_{\epsilon,N}(\mathbb{P}_\infty)$. Based on the fixed-point theory, we conclude that $\mathbb{P}_\infty$ is the maximal N-step PCIS within $\mathbb{Q}$. □

The computational tractability of Algorithm 3.1 depends on the computation of $V^*_{0,\mathbb{P}_i}(x)$ in the DP (2.1) and (2.2). When the state space is continuous, it is in general impossible to compute $V^*_{0,\mathbb{P}_i}(x)$ for each $x \in \mathbb{P}_i$. Next, we investigate how to compute or approximately compute $V^*_{0,\mathbb{P}_i}(x)$ for discrete spaces and continuous spaces, respectively.

Discrete state and control action spaces

If the state and control spaces are both finite sets, denote by $\mathbb{U}_x$ the set of the admissible control actions for each $x \in \mathbb{X}$. Assume that $\mathbb{U}_x$ is nonempty for each $x \in \mathbb{X}$. The stochastic kernel $T(\cdot|x, u)$ is specified as $T(y|x, u)$, which denotes the transition probability from the state $x \in \mathbb{X}$ and the control action $u \in \mathbb{U}_x$ to the state $y \in \mathbb{X}$. For any $x \in \mathbb{X}$ and $u \in \mathbb{U}_x$, $\sum_{y\in\mathbb{X}} T(y|x, u) = 1$.

In this case, according to Theorem 1 of [79], we can exactly compute $V^*_{0,\mathbb{P}_i}(x)$ via an LP. Moreover, the existence of the optimal Markov policy can be always guaranteed.

Proposition 3.3. *Given any set $\mathbb{P}_i \subset \mathbb{X}$ and any positive real number $\alpha_k(x)$, $\forall x \in \mathbb{X}$ and $\forall k \in \mathbb{N}_{[0,N]}$, the value functions $V^*_{k,\mathbb{P}_i}$ in (2.1)–(2.2) can be obtained by solving an LP:*

$$\min \sum_{k=0}^{N} \sum_{x\in\mathbb{P}_i} \alpha_k(x) v_k(x) \tag{3.1a}$$

$$\textit{subject to } \forall x \in \mathbb{P}_i$$

$$\forall k \in \mathbb{N}_{[0,N-1]} :$$

$$v_k(x) \geq \sum_{y\in\mathbb{P}_i} v_{k+1}(y) T(y|x, u), \forall u \in \mathbb{U}_x, \tag{3.1b}$$

$$v_N(x) \geq 1, \tag{3.1c}$$

$$\forall k \in \mathbb{N}_{[0,N]} : v_k(x) \in \mathbb{R}, \tag{3.1d}$$

which gives $V^*_{k,\mathbb{P}_i}(x) = v^*_k(x)$, $\forall x \in \mathbb{P}_i$ and $\forall k \in \mathbb{N}_{[0,N]}$.

Proof. Please refer to [79] for the proof. □

Corollary 3.2. *The optimal Markov policy $\mu^* = (\mu^*_0, \mu^*_1, \ldots, \mu^*_{N-1})$ can be characterized by the optimal solution to the dual problem of the LP (3.1):*

$$\max \sum_{x\in\mathbb{P}_i} q_N(x) \tag{3.2a}$$

$$\textit{subject to } \forall x \in \mathbb{P}_i$$

$$\sum_{u\in\mathbb{U}_x} q_0(x, u) \leq \alpha_0(x), \tag{3.2b}$$

$$\forall k \in \mathbb{N}_{[1,N-1]} :$$

$$\sum_{u\in\mathbb{U}_x} q_k(x, u) - \sum_{y\in\mathbb{P}_i} \sum_{u\in\mathbb{U}_y} T(x|y, u) q_{k-1}(y, u) \leq \alpha_k(x), \tag{3.2c}$$

$$q_N(x) - \sum_{y\in\mathbb{P}_i} \sum_{u\in\mathbb{U}_y} T(x|y, u) q_{N-1}(y, u) \leq \alpha_N(x), \tag{3.2d}$$

$$\forall k \in \mathbb{N}_{[0,N-1]} : q_k(x, u) \geq 0, \forall u \in \mathbb{U}_x, \tag{3.2e}$$

$$q_N(x) \geq 0. \tag{3.2f}$$

That is, when $q^*_k(x, u^*) > 0$, $\mu^*_{k,\mathbb{Q}}(x) = u^*$, $\forall x \in \mathbb{P}_i$ and $\forall k \in \mathbb{N}_{[0,N-1]}$.

Theorem 3.2. *When the state and control spaces are both finite sets, the N-step ϵ-PCIS within a given set $\mathbb{Q} \subset \mathbb{X}$ can be computed by Algorithm 3.1 in a finite number of steps, i.e., there exists a finite number $i \in \mathbb{N}$ such that $\mathbb{P}_{i+1} = \mathbb{P}_i$. Furthermore, at each iteration, $V^*_{0,\mathbb{P}_j}(x)$, the value of $\forall x \in \mathbb{P}_j$, $j \in \mathbb{N}_{[0,i]}$ can be computed via the LP (3.1) and the corresponding optimal Markov policy can be determined by the LP (3.2).*

Proof. From $\mathbb{P}_{i+1} = \mathbb{X}^*_{\epsilon,N}(\mathbb{P}_i)$, we have that $\mathbb{P}_{i+1} \subseteq \mathbb{P}_i$. Since the state and control spaces are finite, the maximum iteration number to achieve $\mathbb{P}_{i+1} = \mathbb{P}_i$ in Algorithm 3.1 is the cardinality of the set $\mathbb{Q}$. The remaining directly follows from Proposition 3.3 and Corollary 3.2. □

Continous state and control action spaces

If the state and control spaces are both continuous, the computation of $V^*_{0,\mathbb{P}_i}(x)$, $\forall x \in \mathbb{P}_i$ in Algorithm 3.1 is in general computationally intractable. We first discretize the continuous space to obtain an approximated stochastic control system with finite state and control spaces. Then, we can compute the approximation of $V^*_{0,\mathbb{P}_i}(x)$ via an LP of Proposition 3.3.

Assume that $\mathbb{X} \subseteq \mathbb{R}^{n_x}$, $\mathbb{U} \subset \mathbb{R}^{n_u}$, and the set $\mathbb{Q} \subset \mathbb{X}$ is compact. For each $x \in \mathbb{X}$, we denote by $\mathbb{U}_x$ the nonempty set of admissible control actions. We assume that the stochastic kernel $T(\cdot|x,u)$ admits a density $t(y|x,u)$, which represents the probability density of y when the current state is x and the control action taken is u. Given a set $\mathbb{Q} \subset \mathbb{X}$, we first define $\phi(\mathbb{Q}) = Leb(\mathbb{Q})$ where $Leb(\cdot)$ denotes the Lebesgue measure of sets. The compactness of $\mathbb{Q}$ ensures the finiteness of $\phi(\mathbb{Q})$.

Assumption 3.1. *For any $x, x', y, y' \in \mathbb{Q}$, and $u, u' \in \mathbb{U}$, there exists a constant L such that $|t(y|x,u) - t(y'|x',u')| \leq L(\|y - y'\| + \|x - x'\| + \|u - u'\|)$.*

Remark 3.3. *Assumption 3.1 is weaker than that in [78], which also discretizes both state and control spaces.*

The following lemma shows that the value functions in (2.1) and (2.2) are Lipschitz continuous.

Lemma 3.1. *Under Assumption 3.1, for any $x, x' \in \mathbb{Q}$, the value functions V^*_k in (2.1) and (2.2) satisfy*

$$|V^*_{k,\mathbb{Q}}(x) - V^*_{k,\mathbb{Q}}(x')| \leq \phi(\mathbb{Q})L\|x - x'\|, \forall k \in \mathbb{N}_{[0,N]}. \tag{3.3}$$

Proof. See Appendix. □

We discretize the compact set $\mathbb{Q} \subset \mathbb{X}$ into $\mathbb{Q} = \cup_{i=1}^{m_x}\mathbb{Q}_i$, where $\mathbb{Q}_i, \forall \mathbb{N}_{[1,m_x]}$ are pair-wise disjoint nonempty Borel sets, i.e., $\mathbb{Q}_i \in \mathcal{B}(\mathbb{X})$ and $\mathbb{Q}_i \cap \mathbb{Q}_j = \emptyset, \forall i \neq j$. For each $i \in \mathbb{N}_{[1,m_x]}$, we pick a representative state from the set $\mathbb{Q}_i$, denoted by q_i. The set of all discretized states in the $\mathbb{Q}$ is denoted by $\tilde{\mathbb{Q}} = \{q_i, i \in \mathbb{N}_{[1,m_x]}\}$. The diameter of $\mathbb{Q}_i$ is defined as $d_i = \sup_{x,y \in \mathbb{Q}_i} \|x - y\|$. Then, the grid size of the state space is $D_x = max_{i \in \mathbb{N}_{[1,m_x]}} d_i$.

Similarly, the compact control space $\mathbb{U}$ is devided into $\mathbb{U} = \cup_{i=1}^{m_u}\mathbb{C}_i$, where $\mathbb{C}_i, i \in \mathbb{N}_{[1,m_u]}$, are pair-wise disjoint nonempty Borel sets, i.e., $\mathbb{C}_i \in \mathcal{B}(\mathbb{U})$ and $\mathbb{C}_i \cap \mathbb{C}_j = \emptyset, \forall i \neq j$. For each $i \in \mathbb{N}_{[1,m_u]}$, we pick a representative control input from the set $\mathbb{C}_i$, denoted by $\tilde{u}_i$. The set of all discretized control actions is denoted by $\tilde{\mathbb{U}} = \{\tilde{u}_i, i \in \mathbb{N}_{[1,m_u]}\}$. The diameter of $\mathbb{C}_i$ is defined as $l_i = \sup_{x,y\in\mathbb{C}_i} \|x-y\|$. Then, the grid size of the control space is $D_u = max_{i\in\mathbb{N}_{[1,m_u]}} l_i$.

Assumption 3.2. *There exists a constant δ such that $D_x \leq \delta$ and $D_u \leq \delta$.*

For each $x \in \mathbb{Q}$, there exists only one $\mathbb{Q}_i$ such that $x \in \mathbb{Q}_i$. For notational convenience, we denote by s_x the representative state of $\mathbb{Q}_i$ to which x belongs, i.e., $s_x = q_i$ if $x \in \mathbb{Q}_i$. For each $x \in \mathbb{Q}$, the set of admissible discrete control actions is defined by

$$\tilde{\mathbb{U}}_x = \{\tilde{u} \in \tilde{\mathbb{U}} \mid \|u - \tilde{u}\| \leq \eta \text{ for some } u \in \mathbb{U}_{s_x}\}. \tag{3.4}$$

Lemma 3.2. *[78] If $D_u \leq \eta$, then the set $\tilde{\mathbb{U}}_{q_i}$ is nonempty for each $q_i \in \tilde{\mathbb{Q}}$. Furthermore, the set $\tilde{\mathbb{U}}_x$ is nonempty for each $x \in \mathbb{Q}$ and $\tilde{\mathbb{U}}_x = \tilde{\mathbb{U}}_y = \tilde{\mathbb{U}}_{q_i}$, $\forall x, y \in \mathbb{Q}_i$.*

Proof. For each $x \in \mathbb{Q}$, the admissible control set $\mathbb{U}_{s_x}$ is nonempty. For any $u \in \mathbb{U}_{s_x}$, if $D_u \leq \eta$, there exists $\tilde{u} \in \tilde{\mathbb{U}}$ such that $\|u - \tilde{u}\| \leq \eta$. Hence, by the definition of s_x, we have that the set $\tilde{\mathbb{U}}_{q_i}$ is nonempty for each $q_i \in \tilde{\mathbb{Q}}$. Furthermore, from (3.4), it is easy to obtain that the set $\tilde{\mathbb{U}}_x$ is nonempty for each $x \in \mathbb{Q}$ and $\tilde{\mathbb{U}}_x = \tilde{\mathbb{U}}_y = \tilde{\mathbb{U}}_{q_i}$, $\forall x, y \in \mathbb{Q}_i$. □

Remark 3.4. *From Lemma 3.2, we have that each discretized state space $\mathbb{Q}_i$ corresponds to one nonempty admissible discretized control space.*

As in [78], let us define the function $\tilde{t} : \mathbb{Q} \times \tilde{\mathbb{U}} \to \mathbb{R}$

$$\tilde{t}(y|s_x, \tilde{u}) = \begin{cases} \frac{t(s_y|s_x,\tilde{u})}{\int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz}, & \text{if } \int_{\mathbb{Q}} t(s_z|s_x, \tilde{u})dz \geq 1, \\ t(s_y|s_x, \tilde{u}), & \text{otherwise.} \end{cases} \tag{3.5}$$

From (3.5), we observe that all states $y \in \mathbb{Q}_i$ enjoy the same stochastic kernel. Since our focus is on solving the stochastic optimal control problem which involves the compact set $\mathbb{Q} \subset \mathbb{X}$, the approximation of the stochastic control system with continuous spaces is only taken with respect to $\mathbb{Q}$. The new stochastic control system is given by a triple $\tilde{\mathcal{S}}_{\mathbb{Q}} = (\tilde{\mathbb{Q}}, \tilde{\mathbb{U}}, \tilde{T})$. Here, $\tilde{\mathbb{Q}}$ and $\tilde{\mathbb{U}}$ are the set of all discretized states in $\mathbb{Q}$ and the set of all discretized control actions in $\mathbb{U}$, respectively. And the transition probability $\tilde{T}(q_j|q_i, \tilde{u})$ is defined by

$$\tilde{T}(q_j|q_i, \tilde{u}) = \int_{\mathbb{Q}_j} \tilde{t}(y|q_i, \tilde{u})dy,$$

where $q_i \in \mathbb{Q}_i$, $q_j \in \mathbb{Q}_j$, and $\tilde{u} \in \tilde{\mathbb{U}}$, $\forall i, j \in \mathbb{N}_{[1,m_x]}$.

Lemma 3.3. *[78] Under Assumptions 3.1 and 3.2, for all $y \in \mathbb{Q}$, $s_x \in \tilde{\mathbb{Q}}$ and $\tilde{u} \in \tilde{\mathbb{U}}$,*

$$\int_{\mathbb{Q}} |\tilde{t}(y|q_i, \tilde{u}) - t(y|q_i, \tilde{u})|dy \leq 2\phi(\mathbb{Q})L\delta.$$

Proof. See Appendix. □

The discretized version of the DP (2.1)–(2.2) is given by

$$\begin{cases} \hat{V}^*_{N,\mathbb{Q}}(q_i) = 1, & \text{if } q_i \in \tilde{\mathbb{Q}}, \\ \hat{V}^*_{N,\mathbb{Q}}(x) = \hat{V}^*_{N,\mathbb{Q}}(q_i), & \text{if } x \in \mathbb{Q}_i, \\ \forall k \in \mathbb{N}_{[0,N-1]}: & \\ \hat{V}^*_{k,\mathbb{Q}}(q_i) = \max_{\tilde{u}\in\tilde{\mathbb{U}}}(\sum_{j=1}^{m_x} \hat{V}^*_{k+1,\mathbb{Q}}(q_j)\tilde{T}(q_j|q_i,\tilde{u}), & \text{if } q_i \in \tilde{\mathbb{Q}}, \\ \hat{V}^*_{k,\mathbb{Q}}(x) = \hat{V}^*_{k,\mathbb{Q}}(q_i), & \text{if } x \in \mathbb{Q}_i. \end{cases} \tag{3.6}$$

We define the discretized optimal Markov policy $\hat{\mu}^* = (\hat{\mu}^*_0, \ldots, \hat{\mu}^*_{N-1})$, $\hat{\mu}^*_k : \mathbb{Q} \to \tilde{\mathbb{U}}$, $\forall k \in \mathbb{Q}_{[0,N-1]}$, as follows:

$$\begin{cases} \hat{\mu}^*_{k,\mathbb{Q}}(q_i) = \arg\max_{\tilde{u}\in\tilde{\mathbb{U}}} \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i,\tilde{u})dy, & \text{if } q_i \in \tilde{\mathbb{Q}}, \\ \hat{\mu}^*_{k,\mathbb{Q}}(x) = \hat{\mu}^*_{k,\mathbb{Q}}(q_i), & \text{if } x \in \mathbb{Q}_i, \end{cases}$$

and further define

$$\begin{cases} V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x) = \int_{\mathbb{Q}} V^{\hat{\mu}^*}_{k+1,\mathbb{Q}}(y)t(y|x,\hat{\mu}^*_k)dy, & \text{if } k \in \mathbb{N}_{[0,N-1]}, \\ V^{\hat{\mu}^*}_{N,\mathbb{Q}}(x) = 1, \text{if } x \in \mathbb{Q}. \end{cases}$$

Since the approximated system $\tilde{\mathcal{S}}_{\mathbb{Q}} = (\tilde{\mathbb{Q}}, \tilde{\mathbb{U}}, \tilde{T})$ is with finite state and control action spaces, the value of $\hat{V}^*_{k,\mathbb{Q}}$ can be computed via the LP (3.1) and the corresponding optimal policy can be determined by (3.2).

Theorem 3.3. *Under Assumptions 3.1 and 3.2, for any $x \in \mathbb{Q}$, the functions $V^*_{k,\mathbb{Q}}(x)$, $\hat{V}^*_{k,\mathbb{Q}}(x)$, and $V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)$ satisfy*

$$|V^*_{k,\mathbb{Q}}(x) - \hat{V}^*_{k,\mathbb{Q}}(x)| \le \tau_k\delta, \tag{3.7}$$

$$|V^*_{k,\mathbb{Q}}(x) - V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)| \le \rho_k\delta, \tag{3.8}$$

where $\tau_k = 4\phi(\mathbb{Q})L + \tau_{k+1}$, $k \in \mathbb{N}_{[0,N-1]}$, with initialization $\tau_N = 0$, and $\rho_k = \tau_k + \tau_{k+1} + 3\phi(\mathbb{Q})L + \rho_{k+1}$, $k \in \mathbb{N}_{[0,N-1]}$, with initialization $\rho_N = 0$.

Proof. See Appendix. □

Remark 3.5. *Theorem 3.3 guarantees the convergence as the grid size trends to* 0 *when computing the finite-horizon PCIS and generalizes the case in [60], which only discretizes the state space for a given finite control space.*

Corollary 3.3. *Consider a set $\mathbb{Q} \subseteq \mathbb{X}$ and a discretized set $\tilde{\mathbb{Q}}$ of $\mathbb{Q}$. If $\tilde{\mathbb{Q}}$ is a N-step ϵ-PCIS for the discretized stochastic system $\tilde{\mathcal{S}}_{\mathbb{Q}} = (\tilde{\mathbb{Q}}, \tilde{\mathbb{U}}, \tilde{T})$, and $\epsilon \ge \tau_0\delta$, the set $\mathbb{Q}$ is a N-step $\tilde{\epsilon}$-PCIS for system $\mathcal{S}$, where $\tilde{\epsilon} = \epsilon - \tau_0\delta$.*

Proof. According to the construction of the discretized system $\tilde{\mathcal{S}}_{\mathbb{Q}}$, we have that $\forall k \in \mathbb{N}_{[0,N]}$, $\forall i \in \mathbb{N}_{[1,m_x]}$ and $\forall x \in \mathbb{Q}_i$, $\hat{V}^*_{k,\mathbb{Q}}(x) = \hat{V}^*_{k,\mathbb{Q}}(q_i)$. Since $\tilde{\mathbb{Q}}$ is a N-step ϵ-PCIS, it follows that $\forall x \in \mathbb{Q}$, $\hat{V}^*_0(x) \geq \epsilon$. By (3.7) and triangular inequality, we have

$$V^*_{0,\mathbb{Q}}(x) \geq \hat{V}^*_{0,\mathbb{Q}}(x) - \tau_0\delta \geq \epsilon - \tau_0\delta, \forall x \in \mathbb{Q}.$$

Then, when $\epsilon \geq \tau_0\delta$, we conclude that the set $\mathbb{Q}$ is a N-step $\tilde{\epsilon}$-PCIS where $\tilde{\epsilon} = \epsilon - \tau_0\delta$. □

Theorem 3.4. *When the state and control spaces are both continuous sets, an approximate N-step ϵ-PCIS within a given set $\mathbb{Q} \subset \mathbb{X}$ can be computed by Algorithm 3.2 in a finite number of steps, i.e., there exists a finite number $i \in \mathbb{N}$ such that $\mathbb{P}_{i+1} = \mathbb{P}_i$. At each iteration j, the value of $\hat{V}^*_{0,\mathbb{P}_j}(q_k)$, $\forall j \in \mathbb{N}_{[0,i]}$, can be computed via the LP (3.1) and the corresponding optimal policy can be determined by (3.2).*

Proof. The proof is similar to that of Theorem 3.2. □

Algorithm 3.2 Approximate of N-step ϵ-PCIS

1: Discretize the sets $\mathbb{Q}$ and $\mathbb{U}$ to reformulate $\tilde{\mathcal{S}}_{\mathbb{Q}} = (\tilde{\mathbb{Q}}, \tilde{\mathbb{U}}, \tilde{T})$.
2: Initialize $i = 0$, $\mathbb{P}_i = \mathbb{Q}$, and $\tilde{\mathbb{P}}_i = \tilde{\mathbb{Q}}$.
3: Compute $\hat{V}^*_{0,\mathbb{P}_i}(q_j)$, $\forall q_j \in \tilde{\mathbb{P}}_i$.
4: Compute the set $\tilde{\mathbb{P}}_{i+1} = \mathbb{X}^*_{\epsilon,N}(\tilde{\mathbb{P}}_i)$ for $\tilde{\mathcal{S}}$ and $\mathbb{P}_i = \cup_{q_j \in \tilde{\mathbb{P}}_i} \mathbb{Q}_j$
5: If $\tilde{\mathbb{P}}_{i+1} = \tilde{\mathbb{P}}_i$, stop. Else, set $i = i + 1$ and go to step 3.

3.3 Extension to Infinite-horizon PCIS

Now let us extend the finite-horizon ϵ-PCIS to the infinite-horizon ϵ-PCIS. In this section, we define the infinite-horizon ϵ-PCIS and explore its structure. Furthermore, we provide algorithms to compute the infinite-horizon ϵ-PCIS within a given set.

Definition 3.2. *(Infinite-horizon PCIS) Given a confidence level $0 \leq \epsilon \leq 1$, a set $\mathbb{Q} \subseteq \mathbb{X}$ is an infinite-horizon ϵ-PCIS for system $\mathcal{S}$ if for any $x \in \mathbb{Q}$, there exists at least one stationary policy $\mu \in \mathcal{M}$ such that $p^{\mu}_{\infty,\mathbb{Q}}(x) \geq \epsilon$.*

Remark 3.6. *If $\epsilon = 1$, an infinite-horizon ϵ-PCIS becomes a RCIS or just a controlled invariant set, which has been widely studied.*

According to Theorem 2.2, an infinite-horizon ϵ-PCIS can be verified by checking if $p^*_{\infty,\mathbb{Q}}(x) \geq \epsilon$ for every $x \in \mathbb{Q}$, as stated by the following proposition.

Proposition 3.4. *A set $\mathbb{Q} \subset \mathbb{X}$ is an infinite-horizon ϵ-PCIS for system $\mathcal{S} = (\mathbb{X}, \mathbb{U}, T)$ if the following conditions are satisfied:*

(i) $G^*_{\infty,\mathbb{Q}}(x) \geq \epsilon$ for all $x \in \mathbb{Q}$, where $G^*_{\infty}(x)$ is calculated by (2.3)–(2.5);

(ii) there exists a $\bar{k} \geq 0$ *such that the set* $\mathbb{U}_k(x, \lambda) = \{u \in \mathbb{U} \mid \int_{\mathbb{X}} G^*_{k,\mathbb{Q}}(y)T(dy|x, u) \geq \lambda\}$ *is compact for all* $x \in \mathbb{Q}$*,* $\lambda \in \mathbb{R}$*, and* $k \in \mathbb{N}_{\geq \bar{k}}$*.*

An equivalent way to verify an infinite horizon ϵ-PCIS is based on the following stochastic backward reachable set:

$$\begin{aligned}
\mathbb{X}^*_{\epsilon,\infty}(\mathbb{Q}) &= \{x \in \mathbb{Q} \mid \exists \mu \in \mathcal{M}, p^{\mu}_{\infty,\mathbb{Q}}(x) \geq \epsilon\} \\
&= \{x \in \mathbb{Q} \mid \sup_{\mu \in \mathcal{M}} p^{\mu}_{\infty,\mathbb{Q}}(x) \geq \epsilon\} \\
&= \{x \in \mathbb{Q} \mid G^*_{\infty,\mathbb{Q}}(x) \geq \epsilon\}.
\end{aligned}$$

Proposition 3.5. *A set* $\mathbb{Q} \subseteq \mathbb{X}$ *is an infinite-horizon* ϵ*-PCIS for system* $\mathcal{S}$ *if and only if* $\mathbb{X}^*_{\epsilon,\infty}(\mathbb{Q}) = \mathbb{Q}$*.*

Proof. Follow from the definition of $\mathbb{X}^*_{\epsilon,\infty}(\mathbb{Q})$. □

Corollary 3.4. *The state space* $\mathbb{X}$ *is an infinite-horizon* ϵ*-PCIS for any* $0 \leq \epsilon \leq 1$*.*

Note that the infinite-horizon ϵ-PCISs also enjoy Properties 3.1–3.3 by replacing N with ∞. For more details, please refer to Properties 3.1–3.3.

Intuitively, the monotone decreasing of $G^*_{\infty,\mathbb{Q}}(x)$ may imply that the value of $G^*_{\infty,\mathbb{Q}}(x)$ is 1 or 0. However, it is possible to get $0 < G^*_{\infty,\mathbb{Q}}(x) < 1$ in some cases (see Examples 1 and 2 in Section 3.5). In the following, the underlying structure of an infinite-horizon ϵ-PCIS with positive probability is explored.

Theorem 3.5. *Given* $0 < \epsilon \leq 1$*, a nonempty set* $\mathbb{Q} \in \mathcal{B}(\mathbb{X})$ *is an infinite-horizon* ϵ*-PCIS for system* $\mathcal{S}$ *if and only if there exists a Borel set* $\mathbb{Q}_f \in \mathcal{B}(\mathbb{X})$ *with* $\mathbb{Q}_f \subseteq \mathbb{Q}$ *for which*

(i)$\forall x \in \mathbb{Q}_f$*, there exists* $u \in \mathbb{U}$ *such that* $T(\mathbb{X} \setminus \mathbb{Q}|x, u) = 0$*;*

(ii) $\forall x \in \mathbb{Q} \setminus \mathbb{Q}_f$*, there exists* $u \in \mathbb{U}$ *such that* $T(\mathbb{Q}_f|x, u) \geq \epsilon$*.*

Proof. The sufficiency is easy to check according the definition of infinite-horizon PCIS. Let us prove the necessity by first assuming that the nonexistence of such set $\mathbb{Q}_f$. That is, $\forall x \in \mathbb{Q}$ and $\forall u \in \mathbb{U}$, the transition probability from x to $\mathbb{X}\setminus\mathbb{Q}$ is positive, i.e., $T(\mathbb{X}\setminus\mathbb{Q}|x, u) > 0$. In this case, the value function defined in (2.3)–(2.4) is strictly decreasing, which implies that $G^*_{\infty,\mathbb{Q}}(x) = 0$, $\forall x \in \mathbb{Q}$. This contradicts the definition of infinite-horizon PCIS since $0 < \epsilon \leq 1$. The proof is completed. □

Remark 3.7. *The essence of the set* $\mathbb{Q}_f$ *in Theorem 3.5 is a RCIS despite the stochastic transition. One interpretation is that an infinite-horizon* ϵ*-PCIS with* $0 < \epsilon \leq 1$ *consists of two parts: one part is a RCIS and another part admits some transition to the RICS with expected positive probability.*

Direct way to compute infinite-horizon PCISs

As an adaption of Algorithm 3.1, Algorithm 3.3 is a direct way to compute the infinite-horizon ϵ-PCIS within $\mathbb{Q}$. The difference with Algorithm 3.1 is that the value of $G^*_{\infty,\mathbb{P}_i}(x)$, instead of $V^*_{0,\mathbb{P}_i}(x)$, $\forall x \in \mathbb{P}_i$, is computed by (2.3)–(2.5). Furthermore, the updated set $\mathbb{P}_{i+1} = \mathbb{X}^*_{\epsilon,\infty}(\mathbb{P}_i)$, which is a stochastic backward reachable set within $\mathbb{P}_i$ with respect to infinite horizon and a probability level ϵ. According to Propositions 3.4 and 3.5, the algorithm terminates when $\mathbb{P}_{i+1} = \mathbb{P}_i$.

Theorem 3.6. *Consider a set $\mathbb{Q} \subset \mathbb{X}$. If $\mathbb{Q}$ contains a nonempty RCIS, Algorithm 3.3 converges to a nonempty set. Moreover, this set is the maximal infinite-horizon PCIS within $\mathbb{Q}$.*

Proof. A RCIS is a special infinite-horizon PCIS with probability 1. If $\mathbb{Q}$ contains a RCIS, this RCIS is a fixed-point of $\mathbb{X}^*_{\epsilon,\infty}$. Thus, Algorithm 3.3 converges to a nonempty set. Furthermore, by the similar proof of Proposition 3.1, the convergent set via Algorithm 3.3 is the maximal infinite-horizon ϵ-PCIS within $\mathbb{Q}$. □

Algorithm 3.3 Infinite-horizon ϵ-PCIS

1: Initialize $i = 0$ and $\mathbb{P}_i = \mathbb{Q}$.
2: Compute $G^*_{\infty,\mathbb{P}_i}(x)$ for all $x \in \mathbb{P}_i$.
3: Compute the set $\mathbb{P}_{i+1} = \mathbb{X}^*_{\epsilon,\infty}(\mathbb{P}_i)$.
4: If $\mathbb{P}_{i+1} = \mathbb{P}_i$, stop. Else, set $i = i + 1$ and go to step 2.

The computational tractability of Algorithm 3 is dependent on the computation of $G^*_{\infty,\mathbb{P}_i}(x)$. In general, it is nontrivial to compute the exact value of $G^*_{\infty,\mathbb{P}_i}(x)$ even when state and control action spaces are finite. Next we focus on the computation of $G^*_{\infty,\mathbb{P}_i}(x)$ for finite spaces when implementing Algorithm 3.3.

When the state and control action spaces are finite, we adopt the same assumptions as in Section 3.2. In the second step of Algorithm 3.3, we need to compute $G^*_{\infty,\mathbb{P}_i}(x)$. As shown in (2.3)–(2.5) (replacing $\mathbb{Q}$ with $\mathbb{P}_i$), $G^*_{\infty,\mathbb{P}_i}(x)$ is the limitation of $G^*_{k,\mathbb{P}_i}$ as $k \to \infty$. For notational convenience, we use $G^*_{k,\mathbb{P}_i}$ to denote the vector form of $G^*_{k,\mathbb{P}_i}(x)$, $x \in \mathbb{P}_i$. And the optimization problems $\max_{u\in\mathbb{U}_x} \sum_{y\in\mathbb{P}_i} G^*_{k,\mathbb{P}_i}(y)T(y|x,u), x \in \mathbb{P}_i$ are rewritten as $\max_{\mu\in\mathcal{M}} T^\mu G^*_{k,\mathbb{P}_i}$. The following lemma provides the uniqueness of $G^*_{\infty,\mathbb{P}_i}$.

Lemma 3.4. *The sequence $(G^*_{0,\mathbb{P}_i}, G^*_{1,\mathbb{P}_i}, \ldots)$ converges to a unique fixed point satisfying (2.5).*

Proof. By contradiction, assume that the sequence $(G^*_{0,\mathbb{P}_i}, G^*_{1,\mathbb{P}_i}, \ldots)$ could converge to two different fixed points satisfying (2.5), denoted by $G^{1,*}_{\infty,\mathbb{P}_i}$ and $G^{2,*}_{\infty,\mathbb{P}_i}$. Then, we have

$$\begin{aligned} 0 < \|G^{1,*}_{\infty,\mathbb{P}_i} - G^{2,*}_{\infty,\mathbb{P}_i}\| &\le \|\max_{\mu\in\mathcal{M}} T^\mu G^{1,*}_{\infty,\mathbb{P}_i} - \max_{\mu\in\mathcal{M}} T^\mu G^{2,*}_{\infty,\mathbb{P}_i}\| \\ &\le \max_{\mu\in\mathcal{M}} \|T^\mu (G^{1,*}_{\infty,\mathbb{P}_i} - G^{2,*}_{\infty,\mathbb{P}_i})\| \end{aligned}$$

$$\leq \|G^{1,*}_{\infty,\mathbb{P}_i} - G^{2,*}_{\infty,\mathbb{P}_i}\|. \tag{3.9}$$

In (3.9), the equality holds if and only if for each $x \in \mathbb{P}_i$, there exists $u \in \mathbb{U}_x$ such that $\sum_{y\in\mathbb{P}_i} T(y|x,u) = 1$. In this case, it is easy to check that $G^*_{\infty,\mathbb{P}_i}(x) = G^*_{0,\mathbb{P}_i}(x) = 1$ for each $x \in \mathbb{P}_i$ so $G^*_{\infty,\mathbb{P}_i}$ is unique. For other cases, we have a contradiction. Hence, the sequence $(G^*_{0,\mathbb{P}_i}, G^*_{1,\mathbb{P}_i}, \ldots)$ converges to a unique fixed point satisfying (2.5). □

Corollary 3.5. *The convergence point $G^*_{\infty,\mathbb{P}_i}$ of the sequence $(G^*_{0,\mathbb{P}_i}, G^*_{1,\mathbb{P}_i}, \ldots)$ is the maximum fixed point satisfying (2.5).*

Proof. The monotone decrease of the sequence $(G^*_{0,\mathbb{P}_i}, G^*_{1,\mathbb{P}_i}, \ldots)$ and the unique convergence point imply that $G^*_{\infty,\mathbb{P}_i}$ is the maximum fixed point satisfying (2.5). □

In the following, we compute the maximum fixed point satisfying (2.5) by solving a computationally tractable MILP.

Proposition 3.6. *Given any set $\mathbb{P}_i \subseteq \mathbb{X}$ and any positive real number $\beta(x)$, $\forall x \in \mathbb{P}_i$, the function G^*_∞ in (2.5) can be obtained by solving the MILP:*

$$\max_{g(x),\kappa(x,u)} \sum_{x\in\mathbb{P}_i} \beta(x)g(x) \tag{3.10a}$$

subject to $\forall x \in \mathbb{P}_i$,

$$g(x) \geq \sum_{y\in\mathbb{P}_i} g(y)T(y|x,u), \forall u \in \mathbb{U}_x, \tag{3.10b}$$

$$g(x) \leq \sum_{y\in\mathbb{P}_i} g(y)T(y|x,u) + (1-\kappa(x,u))\Delta, \forall u \in \mathbb{U}_x, \tag{3.10c}$$

$$\sum_{u\in\mathbb{U}_x} \kappa(x,u) \geq 1, \tag{3.10d}$$

$$0 \leq g(x) \leq 1, \kappa(x,u) \in \{0,1\}, \forall u \in \mathbb{U}_x, \tag{3.10e}$$

where $\kappa(x,u)$ is a 0-1 *variable and Δ is a constant greater than* 1. *That is, $G^*_{\infty,\mathbb{P}_i}(x) = g^*(x)$, $\forall x \in \mathbb{P}_i$.*

Proof. From Lemma 3.4 and Corollary 3.5, $G^*_{\infty,\mathbb{P}_i}$ is the maximum fixed point satisfying (2.5). Hence, the equivalent form of $G^*_{\infty,\mathbb{P}_i}$ can be written as MILP (3.10), where the constraints (3.10b)–(3.10d) guarantee that there exists $u \in \mathbb{U}_x$ such that the equality in (2.5) holds. □

Corollary 3.6. *For each $x \in \mathbb{Q}$, the optimal stationary policy is $\bar{\mu}^*_{\mathbb{P}_i}(x) = u^*$ such that $\kappa^*(x,u^*) = 1$ and $u^* \in \mathbb{U}_x$, where κ^* is the optimal solution of the MILP (3.10).*

Remark 3.8. *Since* 0 *is a trivial solution of (2.5), we cannot directly reformulate (2.3)–(2.5) as an LP, which is the traditional way to deal with infinite-horizon stochastic optimal control problem [80].*

Theorem 3.7. *When the state and control spaces are both finite sets, the infinite-horizon ϵ-PCIS within a given set $\mathbb{Q} \subset \mathbb{X}$ can be computed by Algorithm 3.3 in a finite number of steps, i.e., there exists a finite number $i \in \mathbb{N}$ such that $\mathbb{P}_{i+1} = \mathbb{P}_i$. At each iteration j, the value of $G^*_{\infty,\mathbb{P}_j}(x)$, $\forall j \in \mathbb{N}_{[0,i]}$, can be computed via the MILP (3.10) and the corresponding optimal policy can be determined by Corollary 3.6.*

Proof. The proof is similar to that of Theorem 3.2. □

Indirect way to compute infinite-horizon PCIS

Based on Theorem 3.5, this subsection provides an indirect way to compute the infinite-horizon ϵ-PCIS within a given set $\mathbb{Q}$. Different from Algorithm 3.3, which at each iteration requires the computation of $G^*_{\infty,\mathbb{P}_i}(x)$, $\forall x \in \mathbb{P}_i$, Algorithm 3.4 generates an the infinite-horizon ϵ-PCIS by computing a backward stochastic reachable set from the RCIS $\mathbb{Q}_f$ contained in $\mathbb{Q}$.

The first step in Algorithm 3.4 is the computation of RCIS within a given set, which is a well-studied topic in the literature [11–13]. Then, based on RCIS $\mathbb{Q}_f$ within $\mathbb{Q}$, the stochastic backward reachable set

$$\mathbb{P} = \{x \in \mathbb{Q} \mid \exists u \in \mathbb{U}, \int_{\mathbb{Q}_f} T(dy|x,u) \geq \epsilon\} \tag{3.11}$$

is an infinite-horizon ϵ-PCIS within $\mathbb{Q}$. Furthermore, if the RCIS $\mathbb{Q}_f$ is the maximal RCIS within $\mathbb{Q}$, the resulting $\mathbb{P}$ is the maximal infinite-horizon ϵ-PCIS within $\mathbb{Q}$. We remark that Algorithm 3.4 is applicable with both discrete and continuous spaces (see Examples 1 and 2).

Algorithm 3.4 Infinite-horizon ϵ-PCIS

1: Compute the RCIS within $\mathbb{Q}$, denoted by $\mathbb{Q}_f$.
2: Compute the stochastic backward reachable set from $\mathbb{Q}_f$, i.e., $\mathbb{P} = \{x \in \mathbb{Q} \mid \exists u \in \mathbb{U}, \int_{\mathbb{Q}_f} T(dy|x,u) \geq \epsilon\}$.

3.4 Discussion

Computational complexity

The computational complexities of Algorithms 1-4 are discussed in the following.

- When implementing Algorithm 3.1 to a system with finite spaces, the maximal iteration number is $|\mathbb{Q}|$. At each iteration, an LP is solved to compute the value of $V^*_{0,\mathbb{P}_i}(x)$, $\forall x \in \mathbb{P}_i$. The number of the decision values in the LP is at most $|\mathbb{Q}|(N+1)$ and the number of the constraints is at most $|\mathbb{Q}|(N|\mathbb{U}|+1)$. It is well known that the interior-point methods can be used to solve the LP in polynomial time [81].

- When implementing Algorithm 3.2 to a system with continuous spaces, the maximal iteration number is m_x, i.e., the number of discretized subsets of $\mathbb{Q}$. Similar to Algorithm 1, an LP is solved at each iteration to compute the approximated value $\hat{V}^*_{0,\mathbb{P}_i}(q_j)$, $\forall q_j \in \tilde{\mathbb{P}}_i$. The number of the decision values in the LP is at most $m_x(N+1)$ and the number of the constraints is at most $m_x(Nm_u + 1)|$.

- When implementing Algorithm 3.3 to a system with finite spaces, the maximal iteration number is $|\mathbb{Q}|$. An MILP is used to compute the value of $G^*_{\infty,\mathbb{P}_i}(x)$, $\forall x \in \mathbb{P}_i$, at each iteration. The number of the continuous decision values is at most $|\mathbb{Q}|$, the number of the binary decision values is at most $|\mathbb{Q}||\mathbb{U}|$, and the number of the constraints is at most $|\mathbb{Q}|(2|\mathbb{U}| + 3)$. Some advanced softwares have been developed to solve large MILPs efficiently [82, 83].

- The complexity of Algorithm 3.4 depends on the computation of the RCIS, which is a classic topic in the literature [2, 11–13], and the computation of the backward stochastic reachable set, for which some results have been reported in [84]. Example 2 in Section 3.5 will show how to compute the backward stochastic reachable set for continuous spaces.

Comparison with [36]

By extending the definition of probabilistic invariant sets in [36], one can give another definition of infinite-horizon PCISs (Definition 3.3) than Definition 3.2.

Definition 3.3. *Given a confidence level* $0 \leq \epsilon \leq 1$*, a set* $\mathbb{Q} \subseteq \mathbb{X}$ *is an infinite-horizon* ϵ*-PCIS for system* $\mathcal{S}$ *if for any* $x_0 \in \mathbb{Q}$*, there exists at least one sequence of Markov policies* $\mu = (\mu_0, \mu_1, \ldots)$ *such that* $Pr\{x_k \in \mathbb{Q}\} \geq \epsilon$*,* $\forall k \in \mathbb{N}$*.*

Then, we can find that Definition 3.2 is sufficient for Definition 3.3 since $Pr\{\forall k \in \mathbb{N}, x_k \in \mathbb{Q}\} \geq \epsilon$ implies $Pr\{x_k \in \mathbb{Q}\} \geq \epsilon$, $\forall k \in \mathbb{N}$. One interpretation is that Definitions 3.2 and 3.1 capture the trajectory inclusion while Definition 3.3 captures the state inclusion at each time step.

Although Definition 3.2 seems stronger than Definition 3.3, it is easier to verify and compute the PCISs in Definitions 3.2 than that in Definition 3.3. This is because given $x_0 \in \mathbb{Q}$, it is very difficult to find a sequence of Markov policies $\mu = (\mu_0, \mu_1, \ldots)$ such that $Pr\{x_k \in \mathbb{Q}\}$ for $k \geq 1$. Moreover, for safety-critical control problem, the trajectory inclusion with probability provides much "safer" insight than the state inclusion with the same probability.

3.5 Examples

In this section, three examples are provided to illustrate the effectiveness of the proposed theoretical results. The first one involves the robot motion in a partitioned space with obstacles. The second one is concerned with comparison between PCIS and RCIS. The third one focuses on a temperature control system.

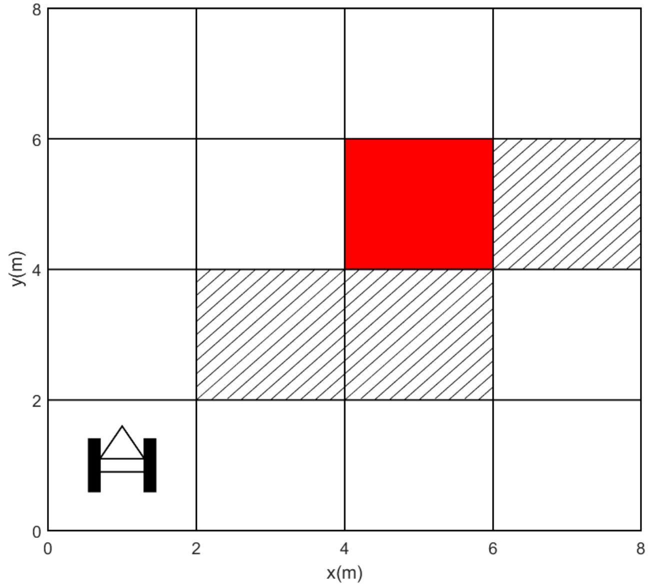


Figure 3.1: The workspace with obstacles (shadow) and 'absorbing' region (red).

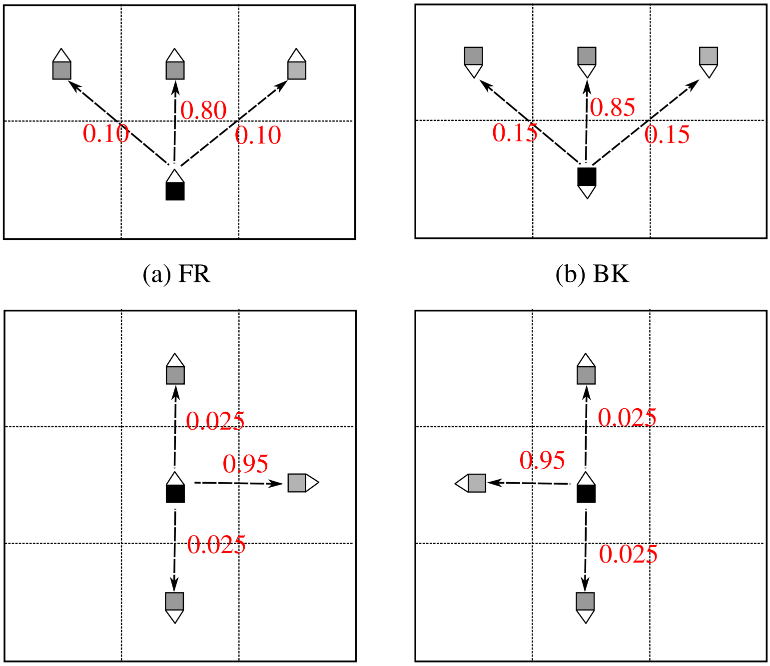
Example 1 (motion planning)

The motion planning example in [85] is adapted to seek an infinite-horizon PCIS within the workspace. Consider a partitioned workspace $8m \times 8m$ shown in Figure 3.1, where the shadowed cells are occupied by obstacles and the red cell is an 'absorbing' region, i.e., as long as the mobile robot enters in this region it will stay there forever.

Following [85], the dynamics of the mobile robot is modeled by

$$\begin{cases} \dot{x} = v\cos(\theta) \\ \dot{y} = v\sin(\theta) \\ \dot{\theta} = \omega. \end{cases}$$

The robot state is abstracted by the cell coordinate where it belongs to, i.e., $(x, y) \in \{1, 3, 5, 7\}^2$, and its four possible orientations $\{E, W, S, N\}$. Due to the actuation noise and drifting, the robot motion is stochastic. Here, we restrict the action space to be $\{FR, BK, TRFR, TLFR\}$, under which the possible transitions are shown in Figure 3.2. Specifically, action "FR" means driving forward for $2m$. The probability of reaching $2m$ forward is 0.80 and the probability of drifting to the left or the right by $2m$ is 0.1, respectively. Action "BW" can be defined similarly to "FR". Action "$TRFR$" means turning right for $\pi/2$ and driving forward for $2m$, of which the probability is 0.95. The probability of driving forward for $2m$ without turning right is 0.025 and the probability of



(a) FR (b) BK

(c) TRFR (d) TLFR

Figure 3.2: Transition probability under different actions.

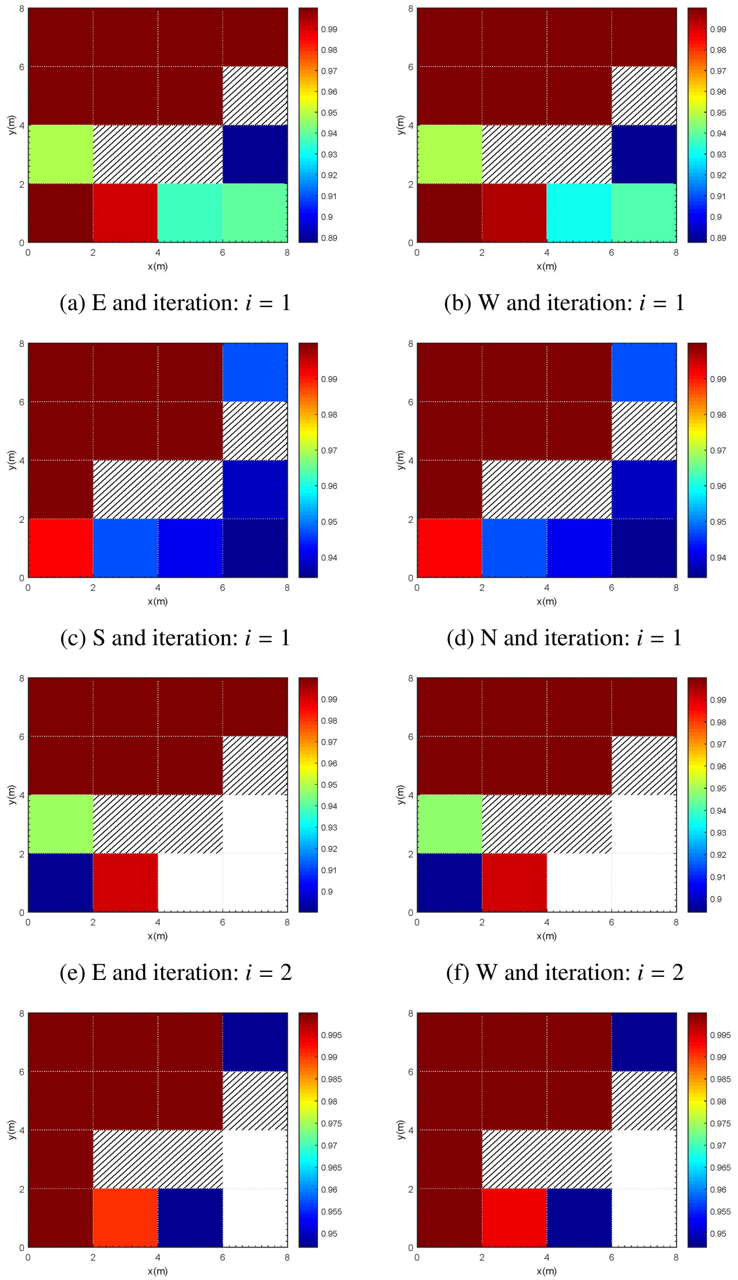
turning right for π and driving forward for $2m$ is 0.025. Similarly, we can define the action "$TLFR$".

Then, we construct an MDP with 64 states and 4 actions. The transition relation and probability can be defined based on the above description. We aim to compute the infinite-horizon 0.9-PCIS within the safe state space, i.e., the remaining of the state space by excluding the states associated with the obstacles. By implementing Algorithm 3.3, the iterative sets and the corresponding probability $p^*_{\infty,\mathbb{P}_i}(x)$ are shown in Figure 3.3, of which each subfigure corresponds to one orientation in $\{E, W, S, N\}$. The resulting infinite-horizon 0.9-PCIS provides a region where the admissible action can drive the robot moving without colliding with the obstacles with respect to probability 0.90.

Example 2 (MPC)

Consider the same model as in [25],

$$x_{k+1} = Ax_k + Bu_k + w_k$$



(a) E and iteration: $i = 1$

(b) W and iteration: $i = 1$

(c) S and iteration: $i = 1$

(d) N and iteration: $i = 1$

(e) E and iteration: $i = 2$

(f) W and iteration: $i = 2$

(g) S and iteration: $i = 2$

(h) N and iteration: $i = 2$

Figure 3.3: The iterative sets and the corresponding probability $p^*_{\infty,\mathbb{P}_i}(x)$ for different orientations.

where $A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The control input is constrained by $|u_k| \leq 0.25$. The region of interest is $\mathbb{Q} = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 0.5\}$. We will compare the maximal RCIS and our PCIS within $\mathbb{Q}$.

When computing a finite-horzion PCIS, assume that elements of w_k are i.i.d. Gaussian random variables with zero mean and variance $\sigma^2 = \frac{1}{30^2}$. This system can be represented as a triple $\mathcal{S} = \{\mathbb{X}, \mathbb{U}, T\}$:

$$\begin{cases} \mathbb{X} = \mathbb{R}^2, \\ \mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 0.1\}, \\ t(x_{k+1}|x_k, u_k) = \psi(\Lambda^{-1}(x_{k+1} - Ax_k - Bu_k)), \end{cases}$$

where $\psi(\cdot)$ is the density function of the standard normal distribution and $\Lambda = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$. We discretize the continuous spaces and implement Algorithm 3.2 to compute the 5-step 0.80-PCIS within $\mathbb{Q}$. The iterative sets and the corresponding probability $p^*_{5,\mathbb{P}_i}(x)$ are shown in Figure 3.4. The convergent set and the corresponding probability $p^*_{5,\mathbb{P}_i}(x)$ are shown in subfigure (h) of Figure 3.4.

To derive a RCIS for this system, we assume the disturbance belongs to the compact set $\mathbb{W} = \{w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.05\}$. By using the methods in [10, 13], we obtain the maximal RCIS, which is the blue region shown in Figure 3.5.

When computing an infinite-horizon PCIS, we choose the same bound on the disturbance as RCIS. Assume that elements of w_k are truncated i.i.d. Gaussian random variables with zero mean and variance $\sigma^2 = \frac{1}{30^2}$. This system can be represented as a triple in a similar way. Denote the obtained maximal RCIS by $\mathbb{Q}_f = \{x \in \mathbb{R}^2 \mid Hx \leq h\}$ where the matrix H and the vector h are with appropriate dimensions. As stated in Algorithm 3.4, the infinite-horizon 0.80-PCIS within $\mathbb{Q}$ is a stochastic backward reachable set from the RCIS associated with probability 0.80, i.e.,

$$\mathbb{P} = \{x \in \mathbb{Q} \mid \exists u \in \mathbb{U}, Pr\{H(Ax + Bu + w) \leq h\} \geq 0.80\}.$$

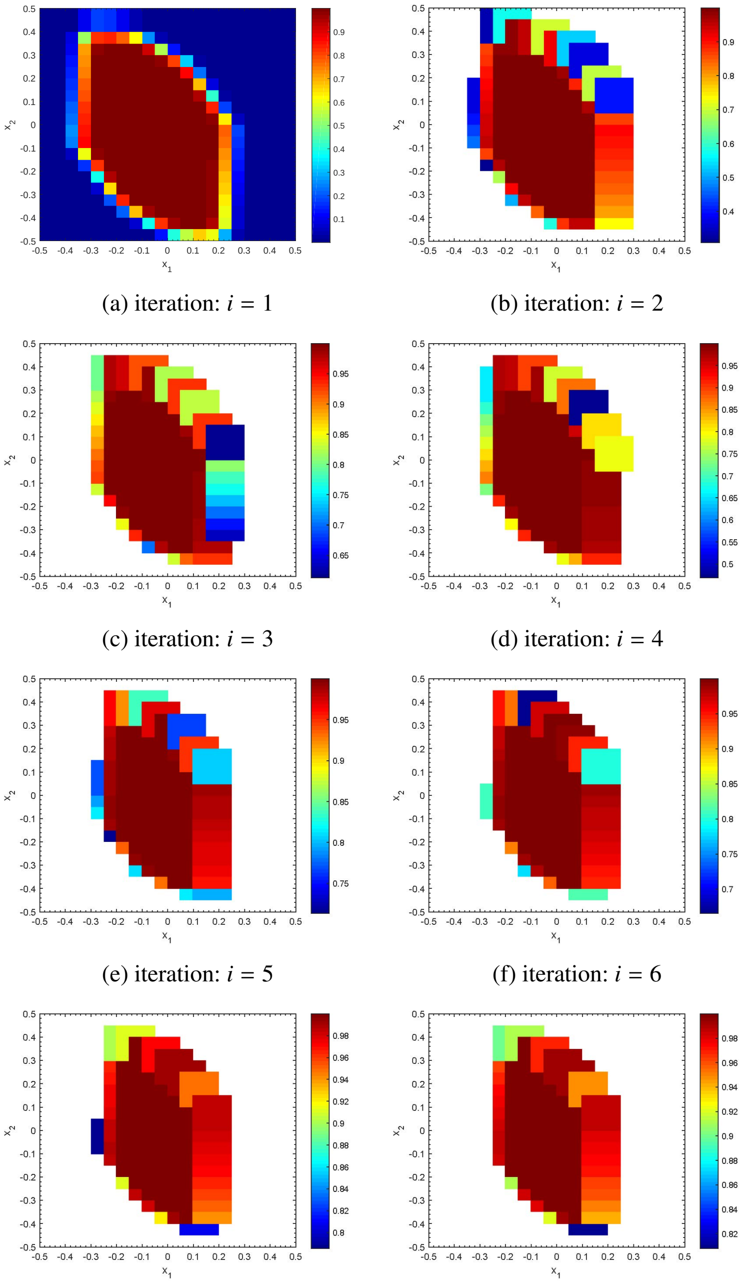
The set $\mathbb{P}$ can be further approximated by

$$\mathbb{P} = \{x \in \mathbb{Q} \mid \exists u \in \mathbb{U}, H(Ax + Bu) + h' \leq h\}$$

where h' is the solution of a chance constrained program

$$h' = \min \sum_j h'_j$$
$$\text{subject to } Pr\{Hw \leq h'\} = 0.8.$$

This chance program can be numerically solved by using the methods in [86, 87]. Then, the resulting infinite-horizon 0.80-PCIS within $\mathbb{Q}$ is the gray region shown in Figure 3.5. Note that the maximal RCIS and the infinite-horizon 0.80-PCIS are smaller than the 5-step 0.80-PCIS even though the disturbance is unbounded for finite-horizon case. One future work is to provide a convex approximation of finite-horizon PCIS.



(a) iteration: $i = 1$ (b) iteration: $i = 2$

(c) iteration: $i = 3$ (d) iteration: $i = 4$

(e) iteration: $i = 5$ (f) iteration: $i = 6$

(g) iteration: $i = 7$ (h) iteration: $i = 8$

Figure 3.4: The iterative sets and the corresponding probability $p^*_{5,\mathbb{P}_i}(x)$

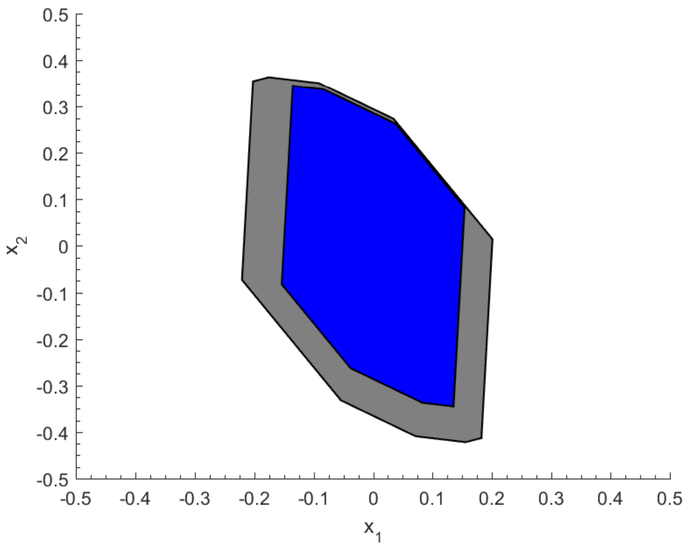


Figure 3.5: The maximal RCIS (blue) and the infinite-horizon 0.8-PCIS (gray).

Example 3 (temperature regulation)

This example involves the temperature regulation of a room. The discrete-time temperature dynamics can be modeled using a resistance-capacitance circuit analogy [88]:

$$x_{k+1} = \underbrace{(1 - \frac{\Delta t}{RC})}_{A} x_k + \underbrace{\frac{\Delta t}{C}}_{B} u_k + \underbrace{\frac{\Delta t}{RC}}_{C} y_k + w_k,$$

where x_k is the temperature of the room, u_k is the heating and cooling power input to the space, y_k is the temperature of outside air, and w_k is the external disturbance load generated by occupants, direct sunlight, and electrical devices. Here, Δt is the sampling time, R describes the thermal resistance of walls and windows isolating the room from the outside environment, and the parameter C represents the thermal capacitance of the room. The disturbance w_k is state-dependent and admits a density function $f : \mathbb{R} \to \mathbb{R}$:

$$f(w_k) = \begin{cases} \psi(\frac{w_k}{\sigma_1}), & \text{if } x_k \le \bar{x}, \\ \psi(\frac{w_k}{\sigma_2}), & \text{if } x_k \ge \bar{x}, \end{cases}$$

where $\bar{x}$ is a constant. The control input is constrained by $|u_k| \le \bar{u}$ where $\bar{u}$ is a positive constant. This system can be represented as a triple $\mathcal{S} = \{\mathbb{X}, \mathbb{U}, T\}$:

$$\begin{cases} \mathbb{X} = \mathbb{R}, \\ \mathbb{U} = \{u \in \mathbb{R} \mid |u| \le \bar{u}\}, \\ t(x_{k+1}|x_k, u_k) = f(x_{k+1} - Ax_k - Bu_k - Cy_k). \end{cases}$$

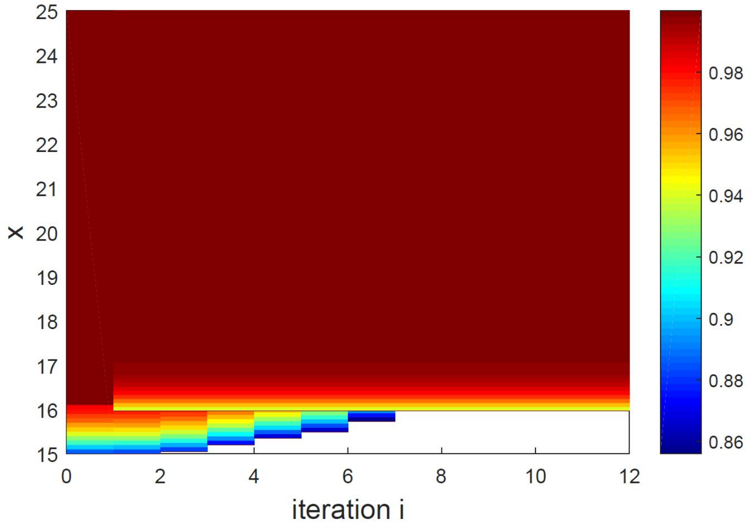


Figure 3.6: The iterative sets and the corresponding probability $p^*_{5,\mathbb{P}_i}(x)$ for temperature control system.

Choose the parameters $A = 0.8$, $B = 1$, $C = 0.2$, $y_k = 6$, $\sigma_1 = \sqrt{0.7}$, $\sigma_1 = \sqrt{0.3}$, $\bar{x} = 16$, and $\bar{u} = 3$. Provided a set $\mathbb{Q} = \{x \in \mathbb{R} \mid 15 \leq x \leq 25\}$, we discretize the continuous spaces and implement Algorithm 3.2 to compute the 5-step 0.90-PCIS within $\mathbb{Q}$. The iterative sets and the corresponding probability $p^*_{5,\mathbb{P}_i}$ are shown in Figure 3.6. After 7 iterations, the resulting set converges to $\mathbb{P}_7 = \{x \in \mathbb{R} \mid 16.01 \leq x \leq 25.00\}$, which is the approximated 5-step PCIS probability 0.90 within $\mathbb{Q}$. This implies that the temperature of the room can stay between $16°C$ and $25°C$ for 5 steps with probability 0.90. Below $16°C$, the temperature cannot retain with the desired probability due to the limitation of the control input.

3.6 Conclusion

We investigated the extension of set invariance in a stochastic sense for control systems. We proposed two definitions for PCISs: finite- and infinite-horizon PCISs, and provided their fundamental properties. We designed iterative algorithms to compute the PCIS within a given set. For systems with discrete state and control spaces, the finite- and infinite-horizon PCISs could be computed by solving an LP and an MILP at each iteration, respectively. We proved that the iterative algorithms were computationally tractable and could terminate in a finite number of steps. For systems with continuous state and control spaces, we established the approximation of stochastic control systems and proved its convergence when computing finite-horizon PCIS. In addition, thanks to the structure of the infinite-horizon PCIS, it could be also be computed by the stochastic backward reachable set from the RCIS contained in it. Numerical examples were given to illustrate the theoretical results.

Appendix

Proof of Lemma 3.1: Since $V^*_{N,\mathbb{Q}}(x) = 1$ for all $x \in \mathbb{Q}$, the inequality (3.3) holds for $k = N$. When $k \in \mathbb{N}_{[0,N-1]}$, for any $x, x' \in \mathbb{Q}$, we have

$$\begin{aligned}
&|V^*_{k,\mathbb{Q}}(x) - V^*_{k,\mathbb{Q}}(x')| \\
&= |\sup_{u\in\mathbb{U}} \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,u)dy - \sup_{u\in\mathbb{U}} \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x',u)dy| \\
&\leq \sup_{u\in\mathbb{U}} |\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)(t(y|x,u) - t(y|x',u))|dy \\
&\leq \sup_{u\in\mathbb{U}} \int_{\mathbb{Q}} |(t(y|x,u) - t(y|x',u))|dy \\
&\leq \phi(\mathbb{Q})L(\|x - x'\|),
\end{aligned}$$

which completes the proof.

Proof of Lemma 3.3: If $\int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz < 1$, it follows from Assumptions Assumptions 3.1 and 3.2 that

$$\int_{\mathbb{Q}} |\tilde{t}(y|q_i,\tilde{u}) - t(y|q_i,\tilde{u})|dy \leq \phi(\mathbb{Q})L\delta.$$

And if $\int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz \geq 1$, we first have

$$\begin{aligned}
0 &\leq \int_{\mathbb{Q}} t(s_y|q_i,\tilde{u})dy - 1 \\
&\leq \int_{\mathbb{Q}} t(s_y|q_i,\tilde{u})dy - \int_{\mathbb{Q}} t(y|q_i,\tilde{u})dy \\
&\leq \int_{\mathbb{Q}} |t(s_y|q_i,\tilde{u}) - t(y|q_i,\tilde{u})|dy \\
&\leq \phi(\mathbb{Q})L\delta.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\int_{\mathbb{Q}} |\tilde{t}(y|q_i,\tilde{u}) - t(y|q_i,\tilde{u})|dy \\
&= \int_{\mathbb{Q}} \frac{|t(s_y|q_i,\tilde{u}) - t(y|q_i,\tilde{u}) \int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz|}{\int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz} dy \\
&\leq \int_{\mathbb{Q}} |t(s_y|q_i,\tilde{u}) - t(y|q_i,\tilde{u}) \int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz|dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbb{Q}} |t(s_y|q_i,\tilde{u}) - t(y|q_i,\tilde{u})|dy + |\int_{\mathbb{Q}} t(s_z|s_x,\tilde{u})dz - 1| \int_{\mathbb{Q}} |t(y|q_i,\tilde{u})|dy \\
&\leq 2\phi(\mathbb{Q})L\delta.
\end{aligned}
$$

This completes the proof.

Proof of Theorem 3.3: First of all, let us prove the inequality (3.7). It is easy to check it for $k = N$ since $V^*_{N,\mathbb{Q}}(x) = \hat{V}^*_{k,\mathbb{Q}}(x) = 1, \forall x \in \mathbb{Q}$. By induction, we assume that

$$
|V^*_{k+1,\mathbb{Q}}(x) - \hat{V}^*_{k+1,\mathbb{Q}}(x)| \leq \tau_{k+1}\delta, x \in \mathbb{Q}.
$$

For any $q_i \in \mathbb{Q}_i$, $i \in \mathbb{N}_{[1,m_x]}$, we define

$$
\begin{aligned}
\mu^*_k &= \arg\sup_{u\in\mathbb{U}} \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,u)dy, \\
\hat{\mu}^*_k &= \arg\max_{\tilde{u}\in\tilde{\mathbb{U}}} \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i,\tilde{u})dy.
\end{aligned}
$$

According to the dicretization procedure of the control space and Assumption 3.2, we can choose some $\tilde{v}_k \in \tilde{\mathbb{U}}$ such that $||\mu^*_k - \tilde{v}_k|| \leq \delta$. Then, we have that

$$
\begin{aligned}
&V^*_{k,\mathbb{Q}}(q_i) - \hat{V}^*_{k,\mathbb{Q}}(q_i) \\
=& \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,\mu^*_k)dy - \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i,\hat{\mu}^*_k)dy \\
\leq& \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,\mu^*_k)dy - \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i,\tilde{v}_k)dy \\
\leq& |\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,\mu^*_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,\tilde{v}_k)dy| + \\
&|\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x,\tilde{v}_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|x,\tilde{v}_k)dy| + \\
&|\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|x,\tilde{v}_k)dy - \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|x,\tilde{v}_k)dy| \\
\leq& \phi(\mathbb{Q})L\delta + 2\phi(\mathbb{Q})L\delta + \tau_{k+1}\delta \\
=& (3\phi(\mathbb{Q})L + \tau_{k+1})\delta,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{V}^*_{k,\mathbb{Q}}(q_i) - V^*_{k,\mathbb{Q}}(q_i) \\
\leq& \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|x,\mu^*_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|q_i,\mu^*_k)dy \\
\leq& |\int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|x,\mu^*_k)dy - \int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)t(y|x,\mu^*_k)dy| + \\
&|\int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)t(y|x,\mu^*_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|q_i,\mu^*_k)dy|
\end{aligned}
$$

$$\leq (2\phi(\mathbb{Q})L + \tau_{k+1})\delta.$$

Thus, we have

$$|V^*_{k,\mathbb{Q}}(q_i) - \hat{V}^*_{k,\mathbb{Q}}(q_i)| \leq (3\phi(\mathbb{Q})L + \tau_{k+1})\delta.$$

For any $x \in \mathbb{Q}_i$, $i \in \mathbb{N}_{[1,m_x]}$, it follows that

$$\begin{aligned}
&|V^*_{k,\mathbb{Q}}(x) - \hat{V}^*_{k,\mathbb{Q}}(x)| \\
&= |V^*_{k,\mathbb{Q}}(x) - \hat{V}^*_{k,\mathbb{Q}}(q_i)| \\
&\leq |V^*_{k,\mathbb{Q}}(x) - V^*_{k,\mathbb{Q}}(q_i)| + |V^*_{k,\mathbb{Q}}(q_i) - \hat{V}^*_{k,\mathbb{Q}}(q_i)| \\
&\leq (4\phi(\mathbb{Q})L + \tau_{k+1})\delta = \tau_k\delta,
\end{aligned}$$

which completes the proof of the inequality (3.7).

Now let us move to prove the inequality (3.8). It is trivial to check it for $k = N$ since $V^*_{N,\mathbb{Q}}(x) = V^{\hat{\mu}^*}_{N,\mathbb{Q}}(x) = 1, \forall x \in \mathbb{Q}$. By induction, we assume that $|V^*_{k+1,\mathbb{Q}}(x) - V^{\hat{\mu}^*}_{k+1,\mathbb{Q}}(x)| \leq \rho_{k+1}\delta$, $x \in \mathbb{Q}$. For any $x \in \mathbb{Q}_i$, $i \in \mathbb{N}_{[1,m_x]}$,

$$\begin{aligned}
&|V^*_{k,\mathbb{Q}}(x) - V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)| \\
&\leq |V^*_{k,\mathbb{Q}}(x) - \hat{V}^*_{k,\mathbb{Q}}(x)| + |\hat{V}^*_{k,\mathbb{Q}}(x) - V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)| \\
&\leq \tau_k\delta + |\hat{V}^*_{k,\mathbb{Q}}(q_i) - V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)|.
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
&|\hat{V}^*_{k,\mathbb{Q}}(q_i) - V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)| \\
&\leq |\int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i, \hat{\mu}^*_k)dy - \int_{\mathbb{Q}} V^{\hat{\mu}^*}_{k+1,\mathbb{Q}}(y)t(y|x, \hat{\mu}^*_k)dy| \\
&\leq |\int_{\mathbb{Q}} \hat{V}^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i, \hat{\mu}^*_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i, \hat{\mu}^*_k)dy| + \\
&\quad |\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)\tilde{t}(y|q_i, \hat{\mu}^*_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|q_i, \hat{\mu}^*_k)dy| + \\
&\quad |\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|q_i, \hat{\mu}^*_k)dy - \int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x, \hat{\mu}^*_k)dy| + \\
&\quad |\int_{\mathbb{Q}} V^*_{k+1,\mathbb{Q}}(y)t(y|x, \hat{\mu}^*_k)dy - \int_{\mathbb{Q}} V^{\hat{\mu}^*}_{k+1,\mathbb{Q}}(y)t(y|x, \hat{\mu}^*_k)dy| \\
&\leq (\tau_{k+1} + 3\phi(\mathbb{Q})L + \rho_{k+1})\delta.
\end{aligned}$$

Then, it follows that

$$|V^*_{k,\mathbb{Q}}(x) - V^{\hat{\mu}^*}_{k,\mathbb{Q}}(x)| \leq (\tau_k + \tau_{k+1} + 3\phi(\mathbb{Q})L + \rho_{k+1})\delta = \rho_k\delta.$$

The proof is completed.

Chapter 4

Stochastic Self-triggered Model Predictive Control

4.1 Introduction

This chapter considers a self-triggered implementation of stochastic MPC for linear systems with stochastic disturbances. As mentioned in Chapter 2, one main feature of stochastic MPC is the presence of probabilistic constraints, which require the constraints to be satisfied with given probability thresholds. Such constraints can mitigate the conservativeness introduced by hard constraints of robust MPC. Stochastic MPC has found applications in diverse fields, e.g., building climate control [9, 89] or chemical processes [90]. One remarkable challenge is how to characterize the 'propagation' of uncertainties during two sampling instants and formulate a computationally tractable optimization problem for determining sampling instants and control design.

The stochastic self-triggered MPC algorithm of this chapter extends considerably the literature, as detailed below. The idea employed in [54] is extended to the case of infinite prediction horizon, where the prediction horizon is divided into three parts and feedback is applied only after a designed inter-sampling time. To achieve a better trade-off between performance and communication, as in [53], the inter-sampling time is maximized at each instant while guaranteeing that the associated cost is not much higher than the cost when sampling at every time instant. Under this self-triggering mechanism, deterministic constraints on the nominal system are derived based on the knowledge of the probability distribution of the disturbances. Following the ideas of tube-based MPC [25], we construct stochastic tubes as tight as possible by explicitly using the distributions of the disturbances. Since a crucial assumption of feedback at every time step in [25] is not satisfied in the self-triggered setting (which allows open-loop operations between sampling instants), some appropriate and non-trivial modifications are needed: (i) by considering the multi-step open-loop operation between control updates, three predicted controllers are defined for different phases of the prediction horizon, making it more complex than [25] to evaluate the effect of the uncertainty on predictions and construct equivalent deterministic constraints; (ii) the inter-sampling time as an optimizing variable is included in the cost function and a tuning parameter is introduced to provide a trade-off between performance and communication; (iii) an improved terminal set, which is adapted to different inter-sampling times, is designed to make the constraints recursively feasible.

The main contributions are summarized in the following.

(1) Our joint design of the self-triggering mechanism and the stochastic MPC effectively reduces the amount of communication, while guaranteeing control performance with specific level of trade-off.

(2) The MPC optimization problem is transformed into a tractable quadratic programming problem by using information on the disturbance distribution.

(3) For the self-triggering mechanism, the probability of constraint violation can be tight to the specified limit.

(4) Both recursive feasibility and closed-loop stability are guaranteed. To illustrate the effectiveness of the algorithm, numerical experiments are carried out to compare the proposed stochastic self-triggered MPC with a periodically-triggered stochastic MPC, robust self-triggered MPC, and LQR.

The remainder of this chapter is structured as follows. Problem formulation is set up in Section 4.2. In Section 4.3, a multi-step open-loop MPC optimization problem is formulated incorporating probabilistic constraints and specific terminal sets. In Section 4.4, a stochastic self-triggered MPC algorithm is developed and main results are established. Section 4.5 presents numerical simulations and Section 4.6 concludes.

4.2 Problem Statement

Consider a discrete-time linear time-invariant system described by

$$x_{k+1} = Ax_k + Bu_k + w_k,\ k \in \mathbb{N}, \tag{4.1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w(k) \in \mathbb{R}^{n_w}$ is the stochastic disturbance, and (A, B) is a stabilizable pair. Note that $n_w = n_x$. We assume that $\{w_0, w_1, \ldots\}$ is independent and identically distributed (i.i.d.) for all $k \in \mathbb{N}$ and that the elements of w_k have zero mean. The distribution F_i of the ith element of w_k is assumed to be known and continuous with a bounded support $[-\sigma_i, \sigma_i]$, $\sigma_i > 0$, and correspondingly we have $w_k \in \mathbb{W} \triangleq \{w \mid |w| \le \sigma\}$, $\sigma = [\sigma_1\ \sigma_2 \ldots \sigma_{n_w}]^T$. Moreover, system (4.1) is subject to n_c probabilistic constraints given by

$$Pr\{g_\ell^T x_k \le h_\ell\} \ge p_\ell,\ \ell \in \mathbb{N}_{[1,n_c]},\ k \in \mathbb{N},$$

where $g_\ell \in \mathbb{R}^{n_x}$, $h_\ell \in \mathbb{R}$, and $p_\ell \in [0, 1]$. In the sequel, we will focus on one probabilistic constraint

$$Pr\{g^T x_k \le h\} \ge p,\ k \in \mathbb{N}, \tag{4.2}$$

as the other constraints can be treated in a similar way.

In a periodically-triggered MPC scheme, the predictive control input at time k can be designed as

$$u_{k+i|k} = Kx_{k+i|k} + c_{k+i|k},\ i \in \mathbb{N}, \tag{4.3}$$

where $K \in \mathbb{R}^{n_u \times n_x}$ is chosen offline such that the matrix $\Phi \triangleq A + BK$ is Schur stable and for a prediction horizon $N \in \mathbb{N}_{\geq 1}$, perturbations $c_{k+i|k} \in \mathbb{R}^{n_u}$ for $i \in \mathbb{N}_{\leq N-1}$ are optimization variables and $c_{k+i|k} = \mathbf{0}$ for $i \in \mathbb{N}_{\geq N}$. At each time instant k, $u_k = Kx_k + c_{k|k}$ is applied to the system.

To reduce the amount of communication, in the self-triggered scheme, the states x_k are only measured and transmitted to the controller at the sampling instants $k_j \in \mathbb{N}$, $j \in \mathbb{N}$, which evolve as $k_{j+1} = k_j + M_j$ with $k_0 = 0$. The inter-sampling time $M_j \in \mathbb{N}_{[1,N-1]}$ is determined by a self-triggering mechanism based on the state at the sampling instant k_j. In the self-triggered scheme, at the time instants between k_j and k_{j+1}, the shutdown of the sensor and the uncertain x_{k_j+i} resulting from the stochastic disturbance $w_{k_j}, w_{k_j+1}, \ldots, w_{k_j+i-1}$ make the system operate in an open-loop fashion. As a result, the predictive control sequence in (4.3) is not applicable and we redefine the predictive control sequence in the self-triggered setup as

$$u_{k_j+i|k_j} = Kz_{k_j+i|k_j} + c_{k_j+i|k_j}, i \in \mathbb{N}_{\leq M_j-1}, \tag{4.4}$$

$$u_{k_j+i|k_j} = Kx_{k_j+i|k_j} + c_{k_j+i|k_j}, i \in \mathbb{N}_{[M_j,N-1]}, \tag{4.5}$$

$$u_{k_j+i|k_j} = Kx_{k_j+i|k_j},\ i \in \mathbb{N}_{\geq N}, \tag{4.6}$$

where the nominal trajectory $z_{k_j+i|k_j} = \mathbb{E}[x_{k_j+i|k_j}]$ evolves as

$$z_{k_j+i+1|k_j} = \Phi z_{k_j+i|k_j} + Bc_{k_j+i|k_j},\ i \in \mathbb{N},$$

with the initial condition $z_{k_j|k_j} = x_{k_j}$. The predictive controller (4.4) is designed with respect to nominal state predictions for prediction time instants $k_j + i$, $i \in \mathbb{N}_{\leq M_j-1}$, and (4.5) is designed with respect to disturbed state predictions for $i \in \mathbb{N}_{[M_j,N-1]}$. After the Nth prediction time, the predictive controller (4.6) is given by the state feedback law without perturbations. Note that the number of decision variables is also a finite N. At each sampling instant k_j, after solving an MPC optimization problem parameterized by x_{k_j}, the first M_j control inputs, i.e., $u_{k_j|k_j}, u_{k_j+1|k_j}, \ldots, u_{k_j+M_j-1|k_j}$, are transmitted to the actuator and are applied until the next sampling instant k_{j+1}.

The goal is to design a perturbation sequence $\mathbf{c}_{k_j} = [c^T_{k_j|k_j}\ c^T_{k_j+1|k_j} \ldots c^T_{k_j+N-1|k_j}]^T \in \mathbb{R}^{n_u \times N}$ and to maximize the sampling interval M_j at each sampling instant k_j, such that a low frequency of control updates and communication is achieved, while stabilizing a bounded set containing the origin and guaranteeing constraint satisfaction with a specified probability.

4.3 Optimization Problem Formulation

In this section, we formulate the problem described in Section 4.2 to a computationally tractable MPC optimization problem with a fixed inter-sampling time $M \in \mathbb{N}_{[1,N-1]}$. To

$\mathcal{P}_o^M(\mathbf{c}_{k_j})$:

$$\min_{\mathbf{c}_{k_j}} J^M(\mathbf{c}_{k_j}) \triangleq \frac{1}{\alpha}\sum_{i=0}^{M-1}\mathbb{E}_{k_j}[||x_{k_j+i|k_j}||_Q^2 + ||u_{k_j+i|k_j}||_R^2 - \ell_{ss}] + \sum_{i=M}^{\infty}\mathbb{E}_{k_j}[||x_{k_j+i|k_j}||_Q^2 + ||u_{k_j+i|k_j}||_R^2 - \ell_{ss}] \tag{4.7}$$

subject to

$$z_{k_j|k_j} = x_{k_j} \tag{4.8a}$$
$$\forall i \in \mathbb{N}_{\leq M-2} : \ z_{k_j+i+1|k_j} = Az_{k_j+i|k_j} + Bu_{k_j+i|k_j} \tag{4.8b}$$
$$\forall i \in \mathbb{N} : \ x_{k_j+i+1|k_j} = Ax_{k_j+i|k_j} + Bu_{k_j+i|k_j} + w_{k_j+i} \tag{4.8c}$$
$$\forall i \in \mathbb{N}_{\leq M-1} : \ u_{k_j+i|k_j} = Kz_{k_j+i|k_j} + c_{k_j+i|k_j} \tag{4.8d}$$
$$\forall i \in \mathbb{N}_{[M,N-1]} : \ u_{k_j+i|k_j} = Kx_{k_j+i|k_j} + c_{k_j+i|k_j} \tag{4.8e}$$
$$\forall i \in \mathbb{N}_{\geq N} : \ u_{k_j+i|k_j} = Kx_{k_j+i|k_j} \tag{4.8f}$$
$$\forall i \in \mathbb{N}_{\geq 1} : \ Pr\{g^T x_{k_j+i|k_j} \leq h\} \geq p \tag{4.8g}$$

---

achieve this, following the idea of [25], we construct a stochastic tube as tight as possible by making explicit use of the distributions defining the disturbances. Since the control input (4.4)-(4.6) in this chapter is different from the control input (4.3) used in [25], some appropriate modifications are needed with respect to the definition of cost function, the handling of probabilistic constraints, and the construction of terminal sets.

Given a state x_{k_j} of system (4.1) and a fixed $M \in \mathbb{N}_{[1,N-1]}$, define the prototype MPC optimization problem $\mathcal{P}_o^M(\mathbf{c}_{k_j})$ on the decision variable $\mathbf{c}_{k_j}$. See (4.7) and (4.8). Therein, $Q = Q^T \succ 0$ and $R = R^T \succ 0$ are the weighting matrices and the scalar $\alpha \geq 1$ is a tuning parameter. The positive parameter $\ell_{ss} \triangleq \lim_{i\to\infty}\mathbb{E}_{k_j}[||x_{k_j+i|k_j}||_Q^2 + ||u_{k_j+i|k_j}||_R^2]$ can be determined offline, see for example [34].

Remark 4.1. *Based on the inter-sampling time M, the cost function in (4.7) is divided into two parts similar to [53] and [54]. The main differences are the choices of the predictive horizon and the definition of the stage cost. In [53] and [54], the cost function consists of finite-horizon costs and a terminal cost which are defined by the nominal system. Due to the presence of stochastic disturbances, we define the cost function in expectation, which indicates the average performance of the system.*

Remark 4.2. *Using the probabilistic distribution of w and extending the proof of Theorem 2 in [34], the infinite-horizon cost function in (4.7) can be written as a quadratic form of* *the decision variable* $\mathbf{c}_{k_j}$

$$J^M(\mathbf{c}_{k_j}) = \mathbf{c}_{k_j}^T P_{cc}\mathbf{c}_{k_j} + \mathbf{c}_{k_j}^T P_{cx} x_{k_j} + f_{k_j}(x) + f_{k_j}(w),$$

where $P_{cc} \succ 0$, P_{cc} *and* P_{cx} *are constant matrices, and* $f_{k_j}(x)$ *and* $f_{k_j}(w)$ *are determined, respectively, by the state* x_{k_j} *and the distribution of disturbances regardless of the choice of* $\mathbf{c}_{k_j}$.

Note that although the infinite-horizon cost function in (4.7) can be expressed as a quadratic function, solving optimization problem $\mathcal{P}_o^M(\mathbf{c}_{k_j})$ online is still unrealistic due to the presence of an infinite number of probabilistic constraints.

Probabilistic constraint handling strategy

To render the optimization problem $\mathcal{P}_o^M(\mathbf{c}_{k_j})$ computationally feasible, we will convert the probabilistic constraints (4.8g) to deterministic ones, such that the observed probability of constraint satisfaction is the same as the specified value and the derived constraints are recursively feasible for the closed-loop system.

Under the assumption that the first M inputs in the sequence are applied to the system in an open-loop fashion, Lemma 4.1 gives the equivalent form of the probabilistic constraints (4.8g).

Lemma 4.1. *For any* $M \in \mathbb{N}_{[1,N-1]}$ *and any sampling instant* k_j, $j \in \mathbb{N}$, *probabilistic constraints* $Pr\{g^T x_{k_j+i|k_j} \le h\} \ge p$ *for* $i \in \mathbb{N}_{\ge 1}$ *hold if and only if* $\mathbf{c}_{k_j}$ *satisfies*

$$g^T \Phi^i x_{k_j} + g^T H^i \mathbf{c}_{k_j} \le h - \gamma_i^M,\ i \in \mathbb{N}_{\ge 1}, \tag{4.9}$$

where $H^i \triangleq [\Phi^{i-1}B \dots B\, \mathbf{0} \dots \mathbf{0}]$ *and* γ_i^M *is defined as the minimum value such that*

$$\begin{cases} Pr\{g^T A^{i-1} w_{k_j|k_j} + \dots + g^T w_{k_j+i-1|k_j} \le \gamma_i^M\} = p,\ i \in \mathbb{N}_{[1,M]}, \\ Pr\Big\{g^T \Phi^{i-M} \sum_{\ell=0}^{M-1} A^\ell w_{k_j+\ell|k_j} + \sum_{\ell=0}^{i-M-1} g^T \Phi^\ell w_{k_j+i-1-\ell|k_j} \le \gamma_i^M\Big\} = p,\ \ i \in \mathbb{N}_{\ge M+1}. \end{cases} \tag{4.10}$$

Proof. From (4.1) and (4.4), it follows that for $i \in \mathbb{N}_{[1,M]}$

$$x_{k_j+i|k_j} = \Phi^i x_{k_j} + H^i \mathbf{c}_{k_j} + A^{i-1} w_{k_j|k_j} + \dots + w_{k_j+i-1|k_j}.$$

Further, from (4.1) and (4.5) for all $i \in \mathbb{N}_{\ge M+1}$, it holds that

$$x_{k_j+i|k_j} = \Phi^i x_{k_j} + H^i \mathbf{c}_{k_j} + \Phi^{i-M} \sum_{\ell=0}^{M-1} A^\ell w_{k_j+\ell|k_j} + \sum_{\ell=0}^{i-M-1} \Phi^\ell w_{k_j+i-1-\ell|k_j}.$$

Hence, it follows directly by the definition of γ_i^M in (4.10) that $Pr\{g^T x_{k_j+i|k_j} \le h\} \ge p$ for $i \in \mathbb{N}_{\ge 1}$ is equivalent to the deterministic constraints (4.9). □

$$\prod = \begin{bmatrix} \gamma_1^M & \cdots & \gamma_M^M & \gamma_{M+1}^M & \gamma_{M+2}^M & \gamma_{M+3}^M & \cdots \\ 0 & \cdots & 0 & b_{M+1}^M + \xi_{M+1}^M & b_{M+2}^M + \xi_{M+2}^M & b_{M+3}^M + \xi_{M+3}^M & \cdots \\ 0 & \cdots & 0 & 0 & b_{M+2}^M + d_{M+2}^M + \xi_{M+1}^M & b_{M+3}^M + d_{M+3}^M + \xi_{M+2}^M & \cdots \\ 0 & \cdots & 0 & 0 & 0 & b_{M+3}^M + d_{M+3}^M + d_{M+2}^M + \xi_{M+1}^M & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (4.13)$$

The values of γ_i^M can be calculated using multivariate convolution integrals, or alternatively approximated using univariate convolutions with arbitrarily small approximation error (see [91], Remark 3.2). The computation can be performed offline. By modifying Lemma 4.1, recursively feasible constraints are derived in Theorem 4.1 ensuring the satisfaction of the probabilistic constraints (4.2) along closed-loop trajectories.

Theorem 4.1. *Given any $M \in \mathbb{N}_{[1,N-1]}$ and system state $x(k_j)$ with $j \in \mathbb{N}$, let for all $k \in \mathbb{N}_{[k_j,k_{j+1}-1]}$*

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k, \\ u_k = Kz_{k|k_j} + c_{k|k_j}. \end{cases} \quad (4.11)$$

Then the probabilistic constraints (4.2) are satisfied for all $k \in \mathbb{N}_{[k_j+1,k_{j+1}]}$, $j \in \mathbb{N}$, for the closed-loop system (4.11), if $\mathbf{c}_{k_j}$ satisfies

$$g^T \Phi^i x_{k_j} + g^T H^i \mathbf{c}_{k_j} \leq h - \beta_i^M, \ i \in \mathbb{N}_{\geq 1}, \quad (4.12)$$

where β_i^M is defined as the maximum element of ith column of $\prod$ (which is defined in (4.13)) with

$$b_i^M \triangleq \max_{w \in \mathbb{W}} g^T \Phi^{i-M} \sum_{\ell=0}^{M-1} A^\ell w,$$
$$d_i^M \triangleq \max_{w \in \mathbb{W}} g^T \Phi^{i-M-1} w,$$

and ξ_i^M the minimum value such that $Pr\{\sum_{\ell=0}^{i-M-1} g^T \Phi^\ell w \leq \xi_i^M\} = p$. Furthermore, if the constraints in (4.12) are feasible at time k_j, it can be guaranteed that there exists at least one solution $\mathbf{c}_{k_{j+i}}$ satisfying (4.12) for $M = 1$ at all future sampling instants k_{j+i}, $i \in \mathbb{N}_{\geq 1}$.

Proof. By Lemma 4.1 and the definition of β_i^M, constraints (4.12) ensure that $Pr\{g^T x_{k_j+i|k_j} \leq h\} \geq p$, $i \in \mathbb{N}_{[1,M]}$, are satisfied for any possible realization of the disturbances $w_{k_j}, w_{k_j+1}), \ldots, w_{k_j+M-1}$, which implies that (4.2) is satisfied for all $k \in \mathbb{N}_{[k_j+1,k_{j+1}]}$.

To show recursive feasibility of (4.12), we first consider the prediction of the state at time $k_{j+1} + i$, $i \in \mathbb{N}_{\geq 1}$, based on the information obtained at the next sampling instant k_{j+1}. By defining T as the shift matrix with ones on the superdiagonal and zeros elsewhere and constructing the perturbation sequence as $\tilde{\mathbf{c}}_{k_{j+1}} = T^M \mathbf{c}_{k_j}$, it yields that for $i \in \mathbb{N}_{\geq 1}$,

$$x_{k_{j+1}+i|k_{j+1}} = \Phi^{i+M} x_{k_j} + H^{i+M} \mathbf{c}_{k_j} + \Phi^i e_{k_j+M|k_j+M} + \Phi^{i-1} w_{k_j+M|k_j+M} + \ldots + w_{k_j+M+i-1|k_j+M},$$

where

$$e_{k_j+M|k_j+M} = A^{M-1}w_{k_j|k_j} + \ldots + w_{k_j+M-1|k_j}$$

has already been realized at time k_{j+1}. To ensure that $\tilde{\mathbf{c}}_{k_{j+1}}$ is a feasible solution at time k_{j+1}, the worst-case bound on $e_{k_j+M|k_j+M}$ and the probabilistic bound on $\Phi^{i-1}w_{k_j+M|k_j+M} + \ldots + w_{k_j+M+i-1|k_j+M}$ need to be considered explicitly at time k_j. Hence, to ensure feasibility at time k_{j+1} for $M = 1$, we must require at time k_j

$$g^T\Phi^i x_{k_j} + g^T H^i \mathbf{c}_{k_j} \leq h - (b_i^M + \xi_i^M),\ i \in \mathbb{N}_{\geq M+1}.$$

By the same arguments, it follows that the feasibility of (4.12) at time $k_{j+\ell}$, $\ell \in \mathbb{N}_{\geq 1}$, can be ensured, if it holds that

$$g^T\Phi^i x_{k_j} + g^T H^i \mathbf{c}_{k_j} \leq h - (b_i^M + d_i^M + d_{i-1}^M + \ldots + d_{i-\ell+2}^M + \xi_{i-\ell+1}^M),\ i \in \mathbb{N}_{\geq M+\ell}. \quad (4.14)$$

To ensure feasibility at all sampling instants $k_j, k_{j+1}, \ldots$, taking the intersection of (4.9) and (4.14) for all $\ell \in \mathbb{N}_{\geq 1}$ and using the fact that $\gamma_1^M = \xi_{M+1}^M$, it yields that

$$g^T\Phi^i x_{k_j} + g^T H^i \mathbf{c}_{k_j} \leq h - \beta_i^M,\ i \in \mathbb{N}_{\geq 1},$$

where β_i^M is taken as the maximum element of ith column of (4.13). □

Note that the constraint parameters γ_i^M in (4.9) and β_i^M in (4.12) are decided by not only the length i of the predicted time steps but also the length M of the open-loop steps. By setting $M = 1$, the results in Lemma 4.1 and Theorem 4.1 are reduced to the corresponding results in [25], in which a periodically-triggered scheme is considered.

In the following, two properties of the sequence β_i^M, $i \in \mathbb{N}_{\geq 1}$, will be established.

Lemma 4.2. *For all $M \in \mathbb{N}_{[1,N-1]}$, it holds that*

$$\beta_i^M = \begin{cases} \gamma_i^M,\ i \in \mathbb{N}_{[1,M]}, \\ b_i^M + \sum_{\ell=M+2}^{i} d_\ell^M + \gamma_1^M,\ \ i \in \mathbb{N}_{\geq M+1}. \end{cases} \quad (4.15)$$

Proof. For $i \in \mathbb{N}_{[1,M]}$, (4.15) holds directly. For $i \in \mathbb{N}_{\geq M+1}$, by the definition of γ_1^M, we have

$$Pr\{b_i^M + \sum_{j=M+2}^{i} d_j^M + g^T w \leq b_i^M + \sum_{j=M+2}^{i} d_j^M + \gamma_1^M\} = p.$$

Further, from the fact

$$g^T\Phi^{i-M}\sum_{j=0}^{M-1} A^j w + \sum_{j=0}^{i-M-1} g^T\Phi^j w \leq b_i^M + \sum_{j=M+2}^{i} d_j^M + g^T w,$$

it follows that

$$Pr\{g^T\Phi^{i-M}\sum_{j=0}^{M-1}A^jw+\sum_{j=0}^{i-M-1}g^T\Phi^jw\le b_i^M+\sum_{j=M+2}^{i}d_j^M+\gamma_1^M\}\ge p.$$

Hence, according to the definition of γ_i^M, we have

$$\gamma_i^M\le b_i^M+\sum_{j=M+2}^{i}d_j^M+\gamma_1^M. \tag{4.16}$$

Then, it can be concluded that β_i^M is equal to the last non-zero element in the ith column of (4.13), which gives the second row of (4.15). □

Lemma 4.3. *For all $M\in\mathbb{N}_{[1,N-1]}$ and all $i\in\mathbb{N}_{\ge1}$, it holds that*

$$\beta_{i+M}^M=b_{i+M}^M+\beta_i^1. \tag{4.17}$$

Proof. From Lemma 4.2, β_i^1 can be rewritten as

$$\beta_i^1=\sum_{\ell=1}^{i-1}\max_{w\in\mathbb{W}}g^T\Phi^\ell w+\gamma_1^1,\ i\in\mathbb{N}_{\ge M+1}.$$

Then, it follows from (4.15) and the fact that $\gamma_1^1=\gamma_1^M$ that

$$\beta_{i+M}^M=b_{i+M}^M+\sum_{\ell=1}^{i-1}\max_{w\in\mathbb{W}}g^T\Phi^\ell w+\gamma_1^M=b_{i+M}^M+\beta_i^1,\ i\in\mathbb{N}_{\ge1}.$$

We complete the proof. □

Terminal set

To ensure that constraints (4.12) are satisfied over an infinite prediction horizon, a terminal set is used. First, due to $c_{k_j+N+i|k_j}=\mathbf{0}$ for all $i\in\mathbb{N}$, the terminal dynamics of the nominal system can be rewritten as

$$z_{k_j+N+i+1|k_j}=\Phi z_{k_j+N+i|k_j},\ i\in\mathbb{N}.$$

Define the constraint set for $z_{k_j+N|k_j}$ as

$$\{z\mid g^T\Phi^iz\le h-\beta_{N+i}^M,\ i\in\mathbb{N}\}. \tag{4.18}$$

Then given some $\hat{N}\in\mathbb{N}$, we split the infinite prediction horizon in (4.18) into two stages $i\in\mathbb{N}_{\le M+\hat{N}}$ and $i\in\mathbb{N}_{\ge M+\hat{N}+1}$. In the second stage, an upper bound of the sequence β_i^M will be used, which is introduced through the following lemma.

Lemma 4.4. *For all $M \in \mathbb{N}_{[1,N-1]}$, the sequence β_i^M for $i \in \mathbb{N}_{\geq M+1}$ is upper bounded by*

$$\beta_i^M \leq \bar{\beta}^M \triangleq \bar{b}^M + \sum_{\ell=M+2}^{v-1} d_\ell^M + \frac{\rho^v}{1-\rho}\|g\|_S + \gamma_1^M, \tag{4.19}$$

with any given $v \in \mathbb{N}_{\geq M+3}$ and

$$\bar{b}^M \triangleq \max_{i \in \mathbb{N}_{\geq M+1}, w \in \mathbb{W}} g^T \Phi^{i-M} \sum_{\ell=0}^{M-1} A^\ell w.$$

The scalar ρ and matrix S can be obtained by solving a semidefinite program as in [92].

Proof. The strict stability of Φ and the bounded support of w ensure the existence of the upper bound on $g^T \Phi^{i-M} \sum_{\ell=0}^{M-1} A^\ell w$ and the conclusion in (4.19) is then obtained by using similar treatments as in [92]. □

Replace β_{N+i}^M in (4.18) by the bound $\bar{\beta}^M$ over the horizon $i \in \mathbb{N}_{\geq M+\hat{N}+1}$ and define an inner approximation of (4.18) as

$$\{z \mid g^T \Phi^i z \leq h - \beta_{N+i}^M,\ i \in \mathbb{N}_{\leq M+\hat{N}},\ g^T \Phi^i z \leq h - \bar{\beta}^M,\ i \in \mathbb{N}_{\geq M+\hat{N}+1}\}. \tag{4.20}$$

Although the revised set (4.20) only contains a finite number of parameters including β_{N+i}^M, $i \in \mathbb{N}_{\leq M+\hat{N}}$, and $\bar{\beta}^M$, the number of constraints in (4.20) remains infinite. By utilizing the results obtained in [13], there exists $n^* \in \mathbb{N}_{\geq 1}$ such that the infinite number of constraints in (4.20) can be ensured through the first $M + \hat{N} + n^*$ constraints. Therefore, the terminal set for $z(k_j + N|k_j)$ is constructed as follows:

$$\mathbb{X}_f^M \triangleq \{z \mid g^T \Phi^i z \leq h - \beta_{N+i}^M,\ i \in \mathbb{N}_{\leq M+\hat{N}},\ g^T \Phi^i z \leq h - \bar{\beta}^M,\ i \in \mathbb{N}_{[M+\hat{N}+1, M+\hat{N}+n^*]}\}, \tag{4.21}$$

where the smallest allowable value of n^* can be computed offline by solving a finite number of linear programs, see [13] for more details.

Optimization problem

Given a state x_{k_j} of system (4.1) and any $M \in \mathbb{N}_{[1,N-1]}$, with the constraints defined in (4.12) and the terminal set constructed in (4.21), the constraints imposed on the decision variable $\mathbf{c}_{k_j}$ are summarized as follows:

$$z_{k_j|k_j} = x_{k_j}, \tag{4.22a}$$
$$\forall i \in \mathbb{N}_{\leq N-1} : z_{k_j+i+1|k_j} = \Phi z_{k_j+i|k_j} + B c_{k_j+i|k_j}, \tag{4.22b}$$
$$\forall i \in \mathbb{N}_{\leq M-1} : u_{k_j+i|k_j} = K z_{k_j+i|k_j} + c_{k_j+i|k_j}, \tag{4.22c}$$
$$\forall i \in \mathbb{N}_{[1,N-1]} : g^T \Phi^i x_{k_j} + g^T H^i \mathbf{c}_{k_j} \leq h - \beta_i^M, \tag{4.22d}$$
$$z_{k_j+N|k_j} \in \mathbb{X}_f^M. \tag{4.22e}$$

Note that there is only a finite number of deterministic constraints in (4.22), which can be computed for the predictions of the nominal model.

Define the set of all decision variables $\mathbf{c}_{k_j}$ satisfying (4.22) as

$$\mathbb{F}^M(x_{k_j}) \triangleq \{\mathbf{c}_{k_j} \mid (4.22a) - (4.22e) \text{ hold}\},$$

and the state x_{k_j} feasible if $\mathbb{F}^M(x_{k_j}) \neq \emptyset$.

At the sampling instant k_j, for any state x_{k_j} and any $M \in \mathbb{N}_{[1,N-1]}$, an MPC optimization problem $\mathcal{P}^M(\mathbf{c}_{k_j})$, which is the deterministic version of $\mathcal{P}_o^M(\mathbf{c}_{k_j})$, can now be formulated as

$$V^M(k_j) \triangleq \min_{\mathbf{c}_{k_j} \in \mathbb{F}^M(x_{k_j})} J^M(\mathbf{c}_{k_j}),$$

$$\mathbf{c}^*_{k_j} \triangleq \arg \min_{\mathbf{c}_{k_j} \in \mathbb{F}^M(x_{k_j})} J^M(\mathbf{c}_{k_j}),$$

where $V^M(k_j)$ denotes the optimal value function and $\mathbf{c}^*_{k_j}$ the corresponding optimal solution.

Remark 4.3. *Clearly constraints (4.22) are all affine functions in the decision variable* $\mathbf{c}_{k_j}$ *and* $J^M(\mathbf{c}_{k_j})$ *is a quadratic cost function, see Remark 4.2. It can be shown that the optimization problem* $\mathcal{P}^M(\mathbf{c}_{k_j})$ *is thus a quadratic programming problem. More importantly, the computational complexity of the optimization problem is not increased compared with the periodically-triggered MPC scheme in [25] regarding the number of constraints and decision variables.*

4.4 Stochastic Self-triggered MPC

Using the above MPC optimization problem as a basis, a stochastic self-triggered MPC algorithm is designed in this section. In the self-triggered setup, the goal at every sampling instant k_j is to decide not only the perturbation sequence $\mathbf{c}_{k_j}$ but also the next sampling instant k_{j+1}. To reduce the computation and communication cost, we need to find the largest $M_j \in \mathbb{N}_{[1,N-1]}$ such that $\mathcal{P}^{M_j}(\mathbf{c}_{k_j})$ is feasible for some $\mathbf{c}_{k_j} \in \mathbb{F}^{M_j}(x_{k_j})$ while still maintaining certain performance of the closed-loop system. Following the ideas in [54], for a system state x_{k_j} and any $k_j \in \mathbb{N}$ with $j \in \mathbb{N}$, we define the self-triggered MPC problem $\mathcal{S}(x_{k_j})$ as

$$M^*_j \triangleq \max\{M \in \mathbb{N}_{[1,M_{\max}]} \mid \mathbb{F}^M(x_{k_j}) \neq \emptyset, V^M(k_j) \leq V^1(k_j)\}, \tag{4.23}$$

$$\mathbf{c}^*_{k_j} \triangleq \arg \min_{\mathbf{c}_{k_j} \in \mathbb{F}^{M^*_j}(x_{k_j})} J^{M^*_j}(\mathbf{c}_{k_j}), \tag{4.24}$$

where $M_{\max} \in \mathbb{N}_{[1,N-1]}$ is an *a priori* maximum of the inter-sampling time and $V^1(k_j)$ is the optimal value function at sampling instant k_j corresponding to the MPC scheme in which control updates take place at every time instant.

Remark 4.4. *The idea adopted in (4.23) to determine the length of the inter-sampling time M is similar to [45, 53, 54], in which, by introducing a tuning parameter* α *in the* *cost function as in (4.7), the optimal value function of the M-step open-loop MPC scheme is required to be not worse than that of a periodically-triggered MPC scheme. The main difference is that in [45] the system considered is undisturbed, and in [53, 54] the system is subject to bounded disturbances and hard constraints. Furthermore, for the robust self-triggered MPC schemes in [53, 54], the choice of the cost function and the design of the tightened constraint sets are significantly different from that in (4.23).*

Remark 4.5. *By employing the triggering mechanism in (4.23), parameter α may be used to trade off the control performance and the usage of network resources. If $\alpha = 1$, the cost function is transformed into the standard infinite-horizon cost function [25]. In this case, since the first M inputs in the prediction horizon are applied in an open-loop way, the optimal cost function $V^M(k_j)$ for $M \in \mathbb{N}_{>1}$ is in general larger than the optimal cost function $V^1(k_j)$ corresponding to the periodically-triggered MPC scheme. Therefore, from (4.23), it may lead to a small inter-sampling time M. To reduce the frequency of control updates, we can increase the value of α to counter the effect of the open-loop control making it possible to obtain a larger inter-sampling time M, while possibly sacrificing slightly the control performance.*

The resulting stochastic self-triggered MPC algorithm is summarized in Algorithm 4.1.

Algorithm 4.1 Stochastic self-triggered MPC

Offline:

Determine α, Q, R, K, N, $\hat{N}$, $M_{\max}$, and n^*. For any $M \in [1, M_{\max}]$, compute parameters γ_i^M, $i \in \mathbb{N}_{[1,M]}$, b_{M+i}^M, $i \in \mathbb{N}_{[1,N+\hat{N}]}$, d_{M+i}^M, $i \in \mathbb{N}_{[2,N+\hat{N}]}$, ξ_{M+1}^M, and $\bar{\beta}^M$ in (4.22d) and (4.22e).

Online:

1: Initialize $k = 0$, measure the initial state x_k and obtain M^* and $\mathbf{c}_k^*$ by solving the optimization problem $\mathcal{S}(x_k)$.
2: For all $i \in \mathbb{N}_{\leq M^*-1}$, apply the input $u_{k+i} = Kz_{k+i|k} + c_{k+i|k}^*$ to the system.
3: Take the next sampling instant as $k + M^*$ and set $k = k + M^*$.
4: Measure the current state x_k of system (4.1).
5: Solve the optimization problem $\mathcal{S}(x_k)$ to obtain M^* and $\mathbf{c}_k^*$.
6: Go to step 2.

Compared with the periodically-triggered MPC scheme, in the proposed self-trigged algorithm, it is necessary to solve at most $M_{\max}$ quadratic programs at each sampling instant to obtain the maximum M^* such that (4.23) holds. However, it is worth noting that computation of the control inputs and the next update time and communication from sensors to controller and from controller to actuators only happen at the sampling times k_j, $j \in \mathbb{N}$, while the periodically-triggered MPC needs to perform these computations for all times. Though there is no computation and communication required at the time instants between two sampling instants, the self-triggered MPC scheme still preserves recursive feasibility and stability (see Theorem 4.2 and 4.3 below).

By applying Algorithm 4.1, the resulting closed-loop system is

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + w_k, \\ u_k = Kz_{k|k_j} + c^*_{k|k_j}, \ k \in \mathbb{N}_{[k_j, k_{j+1}-1]}, \end{cases} \tag{4.25a}$$

$$k_{j+1} = k_j + M_j^*, \tag{4.25b}$$

for all $j \in \mathbb{N}$ and $k_0 = 0$, which has the following properties.

Theorem 4.2. *(Recursive feasibility and constraint satisfaction). If optimization problem $\mathcal{S}(x_{k_0})$ is feasible at sampling instant k_0, then the feasibility of $\mathcal{S}(x_{k_j})$ can be ensured at every sampling instant k_j, $j \in \mathbb{N}$, for the closed-loop system (4.25a)–(4.25b) under Algorithm 4.1. Furthermore, for all $k \in \mathbb{N}$, it holds that $Pr\{g^T x_k \le h\} \ge p$.*

Proof. Consider any two successive sampling instants k_j and k_{j+1}. Let M_j and $\mathbf{c}_{k_j}$ be a solution of $\mathcal{S}(x_{k_j})$ at sampling instant k_j. Define $\tilde{\mathbf{c}}_{k_{j+1}} = T^{M_j}\mathbf{c}_{k_j}$. We will show that $M_{j+1} = 1$ and $\tilde{\mathbf{c}}_{k_{j+1}}$ are the feasible solution of $\mathcal{S}(x_{k_{j+1}})$ at sampling instant k_{j+1}, i.e., $\mathbf{c}_{k_{j+1}} \in \mathbb{F}^1(x_{k_{j+1}})$.

The constraints (4.22a)–(4.22c) in $\mathbb{F}^1(x_{k_{j+1}})$ are trivially satisfied. The satisfaction of (4.22d) in $\mathbb{F}^1(x_{k_{j+1}})$ by $\tilde{\mathbf{c}}_{k_{j+1}}$ is obtained from the proof of Theorem 4.1.

By assumption that $\mathbf{c}_{k_j} \in \mathbb{F}^{M_j}(x_{k_j})$, it follows that $z_{k_j+N|k_j} \in \mathbb{X}_f^{M_j}$. Further, for all $i \in \mathbb{N}$, we have

$$\begin{aligned} g^T\Phi^i z_{k_j+M_j+N|k_j+M_j} &= g^T\Phi^{M_j+N+i}z_{k_j|k_j} + g^T\Phi^i H^{M_j+N}\mathbf{c}_{k_j} + g^T\Phi^{N+i}\sum_{\ell=0}^{M_j-1}A^\ell w \\ &= g^T\Phi^{M_j+i}z_{k_j+N|k_j} + g^T\Phi^{N+i}\sum_{\ell=0}^{M_j-1}A^\ell w \\ &\le h - \beta^{M_j}_{N+i+M_j} + b^{M_j}_{N+i+M_j} \\ &\le h - \beta^1_{N+i}, \end{aligned}$$

where the last inequality follows from (4.17). Therefore, we immediately obtain that $z_{k_{j+1}+N|k_{j+1}} \in \mathbb{X}_f^1$, i.e., constraint (4.22e) in $\mathbb{F}^1(x_{k_{j+1}})$ is satisfied.

From all of the above, it can be concluded that at sampling instant k_{j+1}, optimization problem $\mathcal{S}(x_{k_{j+1}})$ is feasible and further by induction optimization problems $\mathcal{S}(x_{k_j})$ are feasible at all sampling instants k_j, $j \in \mathbb{N}$.

For the closed-loop system (4.25a)–(4.25b) and all $k \in \mathbb{N}$, the satisfaction of the probabilistic constraints (4.2) is guaranteed directly from Theorem 4.1. □

The stability result of the closed-loop system is established in the following theorem.

Theorem 4.3. *(Stability). For any realization of the disturbances w_k, $k \in \mathbb{N}$, the resulting closed-loop system (4.25a)–(4.25b) under Algorithm 4.1 satisfies the stability condition*

$$\lim_{k_r \to \infty} \frac{1}{k_r} \sum_{k=0}^{k_r-1} \mathbb{E}[\|x_k\|_Q^2 + \|u_k\|_R^2] \le \ell_{ss}.$$

Proof. For any $j \in \mathbb{N}$, let M_j and $\mathbf{c}_{k_j}$ be the optimal solution at sampling instant k_j and the corresponding optimal cost function $V^{M_j}(k_j)$ be a Lyapunov function candidate. Further, as in the proof of Theorem 4.2, $\tilde{\mathbf{c}}_{k_{j+1}} = T^{M_j}\mathbf{c}_{k_j}$ together with $M_{j+1} = 1$ is a feasible solution at sampling instant k_{j+1}. Define $\tilde{V}^1(k_{j+1})$ as the value function associated with this feasible solution.

Using the fact that $\alpha \geq 1$, it holds for the closed-loop system that

$$\begin{aligned}\mathbb{E}_{k_j}[\tilde{V}^1(k_{j+1})] &\leq V^{M_j}(k_j) - \frac{1}{\alpha}\sum_{i=0}^{M_j-1}\mathbb{E}_{k_j}[\|x_{k_j+i|k_j}\|_Q^2 + \|u_{k_j+i|k_j}\|_R^2 - \ell_{ss}] \\ &\leq V^1(k_j) - \frac{1}{\alpha}\sum_{i=0}^{M_j-1}\mathbb{E}_{k_j}[\|x_{k_j+i|k_j}\|_Q^2 + \|u_{k_j+i|k_j}\|_R^2 - \ell_{ss}],\end{aligned}$$

where the second inequality follows from the definition of M_j in (4.23).

The optimality of the solution leads to

$$\mathbb{E}_{k_j}[V^1(k_{j+1})] \leq V^1(k_j) - \frac{1}{\alpha}\sum_{i=0}^{M_j-1}\mathbb{E}_{k_j}[\|x_{k_j+i|k_j}\|_Q^2 + \|u_{k_j+i|k_j}\|_R^2 - \ell_{ss}].$$

Summing the inequality for $j \in \mathbb{N}_{[0,r-1]}$ and taking expectation on both sides,

$$\sum_{j=0}^{r-1}\frac{1}{\alpha}\sum_{i=0}^{M_j-1}\mathbb{E}[\|x_{k_j+i|k_j}\|_Q^2 + \|u_{k_j+i|k_j}\|_R^2 - \ell_{ss}] \leq \mathbb{E}[V^1(k_0)] - \mathbb{E}[V^1(k_r)].$$

Since $\mathbb{E}[V^1(k_0)]$ is finite by assumption and $\mathbb{E}[V^1(k_r)]$ is lower bounded for every $r \in \mathbb{N}$ due to Remark 4.2, it holds that

$$\lim_{r\to\infty}\frac{1}{k_r}\sum_{j=0}^{r-1}\sum_{i=0}^{M_j-1}\mathbb{E}[\|x_{k_j+i|k_j}\|_Q^2 + \|u_{k_j+i|k_j}\|_R^2] \leq \ell_{ss},$$

which implies

$$\lim_{k_r\to\infty}\frac{1}{k_r}\sum_{k=0}^{k_r-1}\mathbb{E}[\|x_k\|_Q^2 + \|u_k\|_R^2] \leq \ell_{ss},$$

thereby completing the proof. □

4.5 Example

In this section, we will provide some simulation studies to show the effectiveness and the advantages of the proposed stochastic self-triggered MPC in comparison with periodically-triggered stochastic MPC (by setting $M_j = 1$), robust self-triggered MPC (by setting $p = 1$), and the unconstrained LQR control.

To this end, consider a system as in [23, 86]:

$$x_{k+1} = \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix} x_k + \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix} u_k + w_k,\ k \in \mathbb{N}$$

subject to the probabilistic constraint

$$Pr\{\begin{bmatrix} 1 & 0 \end{bmatrix} x_k \leq 2\} \geq 0.8.$$

Elements of the disturbance w_k are assumed to be i.i.d. truncated Gaussian random variables with zero mean, variance 0.04^2, and bounded by $|w_{i,k}| \leq 0.1$ for $i = 1, 2$. In the cost function (4.7), the parameters are given by $Q = \text{diag}\{1, 3.5\}$, $R = 0.1$, and $\alpha = 1.2$. The limit value $\ell_{ss} = 0.37$ is precomputed using the method in [34]. The feedback matrix K in (4.22c) is chosen as the unconstrained LQR gain $K = [0.263\ -0.329]$. The prediction horizon and horizons in terminal set (4.22e) are $N = 8$, $\hat{N} = 12$, and $n^* = 1$. The maximal open-loop length is $M_{\max} = 8$. Furthermore, to obtain β_i^M in (4.22d), parameters γ_i^M, $i \in \mathbb{N}_{[1,M]}$, and ξ_{M+1}^M are calculated according to the approximation method in Remark 3.2 of [91]. For the computation of $\bar{\beta}^M$ in (4.22e), parameters ρ and S in $\bar{\beta}^M$ are calculated by the method proposed in [92] and the approximation parameter v is chosen to $v = 13$.

Simulation for the four control schemes are performed with 1000 realizations of the uncertainty sequence, initial condition $[2.5\ 2.8]^T$, and a simulation length of $T_{\text{run}} = 18$ steps. The simulations are implemented in Matlab R2012b with Yalmip and SeDuMi solver.

Stability and constraint violation: The state trajectories $\{x_k,\ k = 0, 1, \ldots\}$ for 100 realizations of the uncertainty sequence are depicted in Figures 4.1-4.2 with the black dotted lines being the constraint bounds. The right plots of subfigure (a)–(b) in Figures 4.1-4.2 enlarge the region of constraint bound to show the constraint violation. As it turns out, with the proposed stochastic self-triggered MPC, the observed probabilities of constraint violation in the first 5 steps are 19.7%, 20.4%, 19.8%, 20.2%, and 16.3%, while by periodically-triggered stochastic MPC, the violation rates are 19.8%, 20.1%, 19.9%, 16.8%, and 9% for the same 1000 realizations. Furthermore, as expected, the robust self-triggered MPC achieves no constraint violations, whereas violation rate is 100% in the first 3 steps under the unconstrained LQR control. The simulation results indicate that by the proposed stochastic self-triggered MPC, the closed-loop state converges to a neighborhood of the origin and the constraint violation is tight to the specified violation value 20%.

Average inter-sampling time and performance: To illustrate the decreased communication achieved by stochastic self-triggered MPC, Figures 4.3-4.4 show, respectively, the state trajectory and input trajectory under stochastic self-triggered MPC and periodically-triggered stochastic MPC for 1 realization of the uncertainty sequence. The sampling instants are highlighted by red solid circles. It can be observed that the number of the sampling instants is significantly reduced. The associated average number of steps between sampling instants is compared. For each scheme, the same uncertainty sequences are used and the average is taken over 1000 realizations and 18 steps. The average inter-sampling time is $\overline{M} = 2.9$ under the self-triggered scheme, which leads to an average reduction in communication by 65.5% compared to the scheme with updates at every time instant.

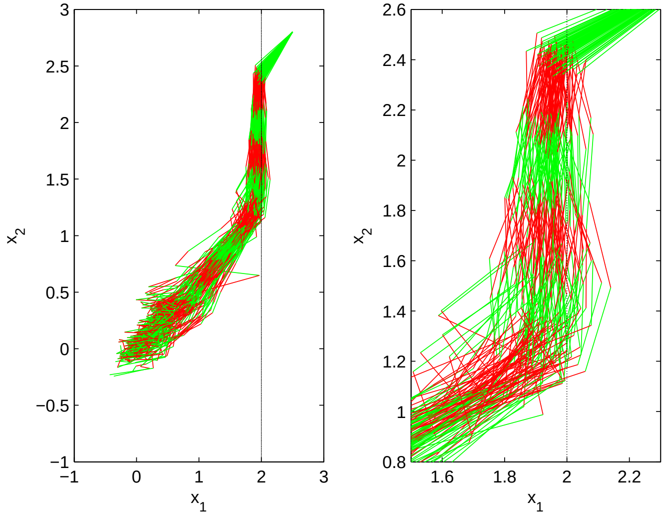
Let us compare the performance measure

$$J_{\text{perf}} = \frac{1}{T_{\text{run}}} \sum_{k=0}^{T_{\text{run}}-1} \{\|x_k\|_Q^2 + \|u_k\|_Q^2 - \ell_{ss}\}.$$

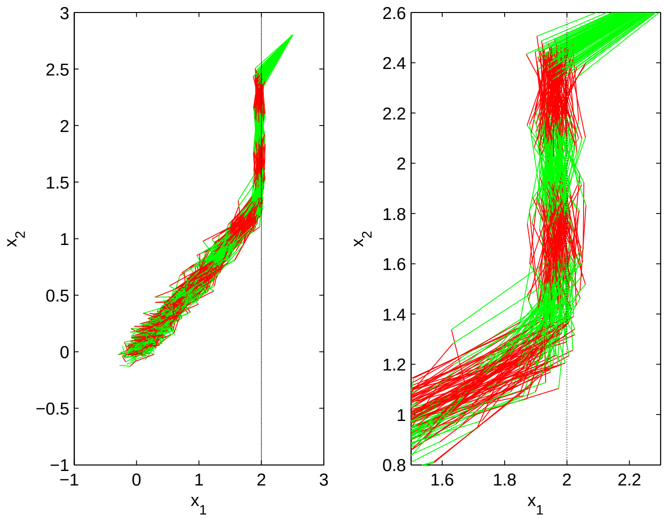
It is 6.82 for stochastic self-triggered MPC, as compared with 6.75 for periodically-triggered stochastic MPC. It can be concluded that by the proposed stochastic self-triggered MPC, communication is decreased without much loss in performance, which can also be illustrated by Figures 4.3-4.4.

4.6 Conclusion

We proposed a stochastic and self-triggered MPC strategy denoted by stochastic self-triggered MPC for the stabilization of systems with additive disturbances and probabilistic constraints. It was shown that the required amount of communication was reduced while simultaneously guaranteeing a specific performance loss when compared with a periodically-triggered scheme. By taking the disturbances occurring during the inter-sampling period into account and making use of their probability distribution, a set of deterministic constraints and the terminal sets were constructed to formulate a computationally tractable MPC optimization problem. The probabilistic constraints were guaranteed at each time instant despite the open-loop operation between any two sampling instants. Moreover, recursive feasibility and stability were proved for the closed-loop system. The results were compared in simulations with other MPC methods from the literature.

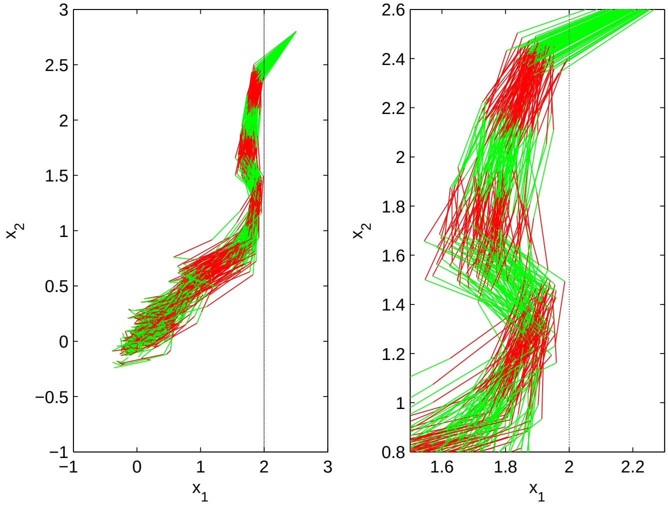


(a) Stochastic self-triggered MPC

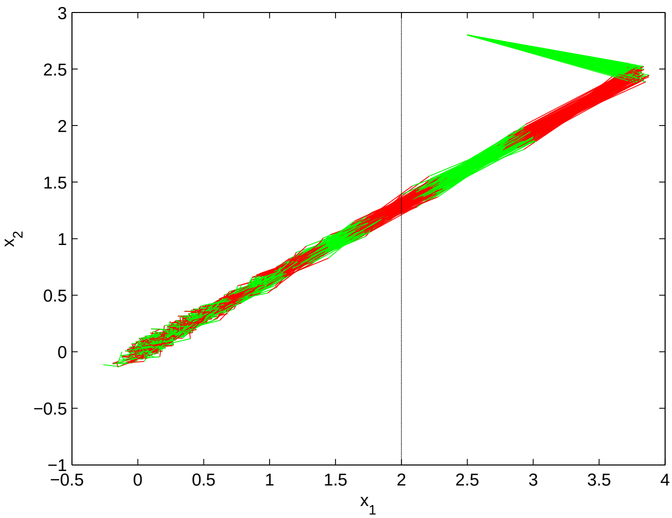


(b) Periodically-triggered stochastic MPC

Figure 4.1: Closed-loop trajectories under stochastic self-triggered MPC and periodically-triggered stochastic MPC for 100 realizations of the uncertainty sequence.

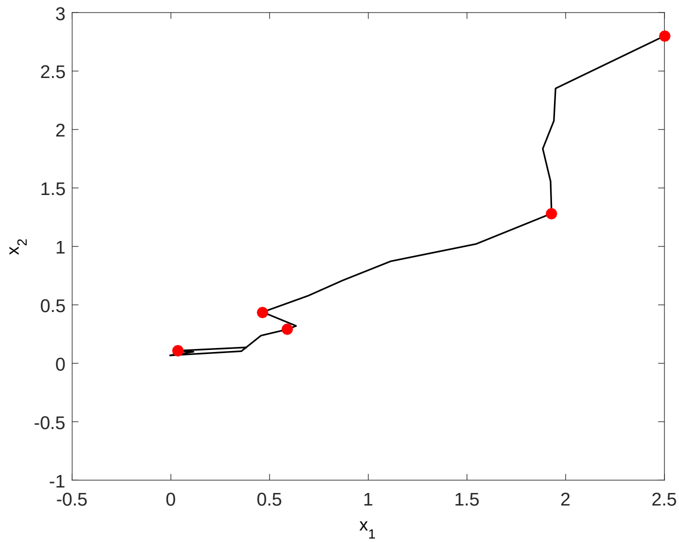


(a) Robust self-triggered MPC

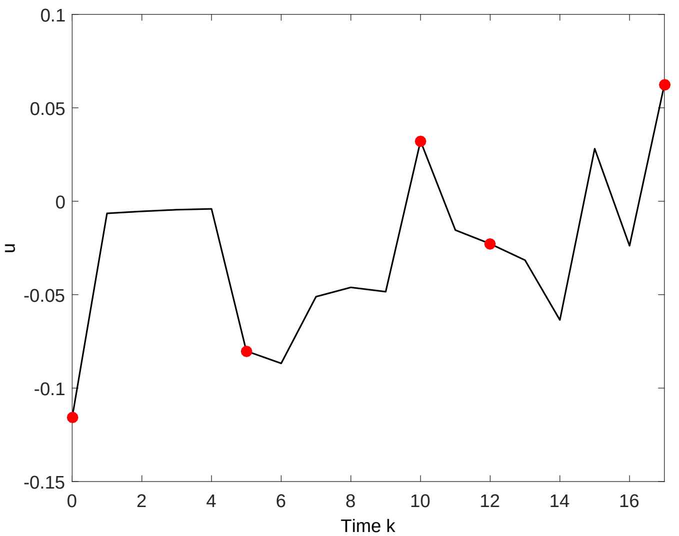


(b) LQR

Figure 4.2: Closed-loop trajectories under robust self-triggered MPC and LQR for 100 realizations of the uncertainty sequence.

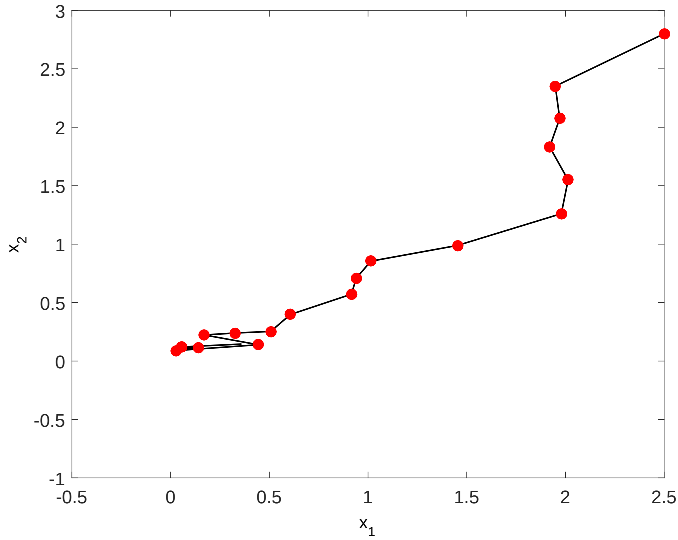


(a) State trajectories

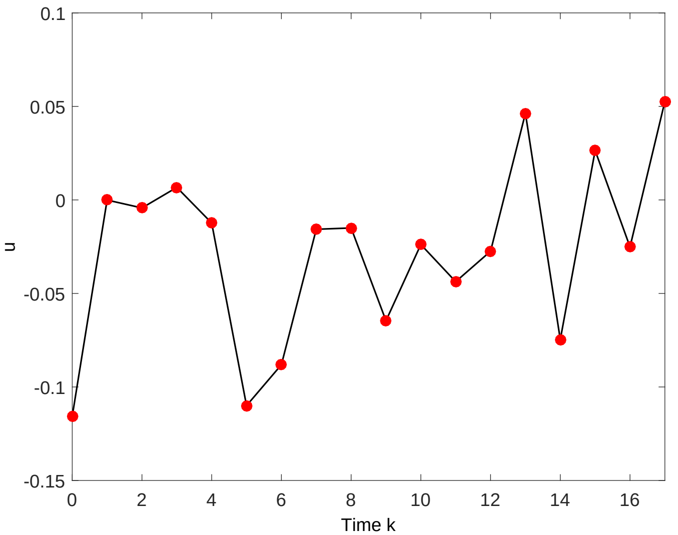


(b) Input trajectories

Figure 4.3: State trajectories and input trajectories under stochastic self-triggered MPC for 1 realization of the uncertainty sequence.



(a) State trajectories



(b) Input trajectories

Figure 4.4: State trajectories and input trajectories under periodically-triggered stochastic MPC for 1 realization of the uncertainty sequence.

Chapter 5

Robust Self-triggered Control via Reachability Analysis

5.1 Introduction

As shown in Chapter 4, the incorporation of the self-triggered scheme into MPC does not immediately preserve the conventional recursive feasibility and closed-loop stability of MPC. In order to guarantee these two properties, the price of more computational effort at the sampling instants is paid to satisfy the constraints on the cost function when maximizing the inter-sampling time.

Different from the above results, this paper aims at proposing a robust self-triggered control framework for time-varying and uncertain systems with constraints. To the best of our knowledge, this topic has not been explored up to now and cannot be handled by the previous mentioned methods, such as self-triggered MPC. The main challenges are: (1) how to guarantee recursive feasibility in a time-varying setup by self-triggered control; (2) how to ensure constraint satisfaction for any disturbance realization. In this work, we make full use of reachability analysis to handle these issues. Although reachability has been widely studied [69–71], the incorporation of reachability into self-triggered control is novel. One recent work [93] uses reachability-based self-triggered control to design the variable sampling period for sampled-data linear systems. However, neither constraints nor uncertainties are considered in [93].

The use of reachability analysis in this paper provides a geometric interpretation for the self-triggered control from a set theoretical point of view. Available geometry software tools facilitate the implementation of our algorithms. Some practical applications of our algorithm include control of hybrid systems and robot motion planning (see Example 2). The main contributions are summarized below.

- We propose a novel robust self-triggered control algorithm (Algorithm 1) for time-varying and uncertain systems with constraints. Constraint satisfactions and recursive feasibility are shown to be guaranteed based on reachability analysis. We calculate the maximum inter-sampling time by solving the corresponding optimization problem ($\mathcal{P}^1_{[k_i,N]}(x_{k_i})$) only once at each sampling instant, which avoids the repetitive computation required in the self-triggered MPC. In the particular case when there is no uncertainty, we develop a control method with minimum number of samplings over a finite time horizon. This is achieved by solving the optimization problem ($\mathcal{P}^2_{[0,N]}(x_0)$) only once.

- When the plant is linear and the constraints are polyhedral, we prove that all the optimization problems ($\mathcal{P}^1_{[k_i,N]}(x_{k_i})$ and $\mathcal{P}^2_{[0,N]}(x_0)$) can be reformulated as mixed integer linear programming (MILP) problems, which are, in our cases, computationally tractable (Theorems 5.1 and 5.2). The numerical comparisons (Example 1) show that our algorithm achieves a better communication reduction and faster online computation than the robust self-triggered MPC in [50] without much loss in performance.

The remainder of the chapter is organized as follows. The problem statement is given in Section 5.2. Section 5.3 presents the robust self-triggered control algorithm. The specialization to linear plants with polyhedral constraints is provided in Section 5.4. Two examples in Section 5.5 illustrate the effectiveness of our approach. Finally, Section 5.6 concludes this chapter.

5.2 Problem Statement

Consider a discrete-time dynamic control system

$$x_{k+1} = f_k(x_k, u_k) + w_k, \tag{5.1}$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$, $w_k \in \mathbb{R}^{n_x}$, and $f_k : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. The control input u_k at time k is constrained by a set $\mathbb{U}_k \subset \mathbb{R}^{n_u}$. The additive disturbance w_k at time instant k belongs to a compact set $\mathbb{W}_k \subset \mathbb{R}^{n_x}$. The initial state x_0 is contained in a given set $\mathbb{X}_0 \subset \mathbb{R}^{n_x}$. In addition, given a finite time horizon $N \in \mathbb{N}$, the system (5.1) is subject to a target tube, denoted by $\{(\mathbb{X}_k, k), k \in \mathbb{N}_{[1,N]}\}$, where $\mathbb{X}_k \subseteq \mathbb{R}^{n_x}$, $\forall k \in \mathbb{N}_{[1,N]}$. It is assumed that the function f_k and the disturbance set $\mathbb{W}_k$ are known for all $k \in \mathbb{N}_{[0,N-1]}$.

Assumption 5.1. *The function $f_k(x, u)$, $\forall k \in \mathbb{N}_{[0,N-1]}$, is continuous in x and u, respectively.*

Assumption 5.2. *The sets $\mathbb{U}_k$, $\forall k \in \mathbb{N}_{[0,N-1]}$, and $\mathbb{X}_k$, $\forall k \in \mathbb{N}_{[0,N]}$, are compact.*

The objective of this chapter is to develop a self-triggered control algorithm for the system (5.1), thereby yielding a sequence of piecewise constant control inputs. More specifically, we aim to determine a sequence of sampling instants $\{k_0, k_1, \ldots, k_T\}$ with $k_0 = 0$, $k_{i+1} = k_i + M_i$, and $k_T = N$ such that $u_j = u_l$, $\forall j, l \in \mathbb{N}_{[k_i,k_{i+1}-1]}$ and all the constraints are satisfied at each time instant $k \in \mathbb{N}_{[0,N]}$. Here, $T + 1$ is the total number of samplings within N time instants, which quantifies the communication consumption, and M_i denotes the inter-sampling time between k_i and k_{i+1}.

5.3 Self-triggered Control via Reachability Analysis

In this section, we will provide a reachability-based self-triggered control algorithm for the uncertain constrained system (5.1). Furthermore, a control method with minimum number of samplings will be designed when the system (5.1) is reduced to be deterministic, i.e., $\mathbb{W}_k = \{0\}$.

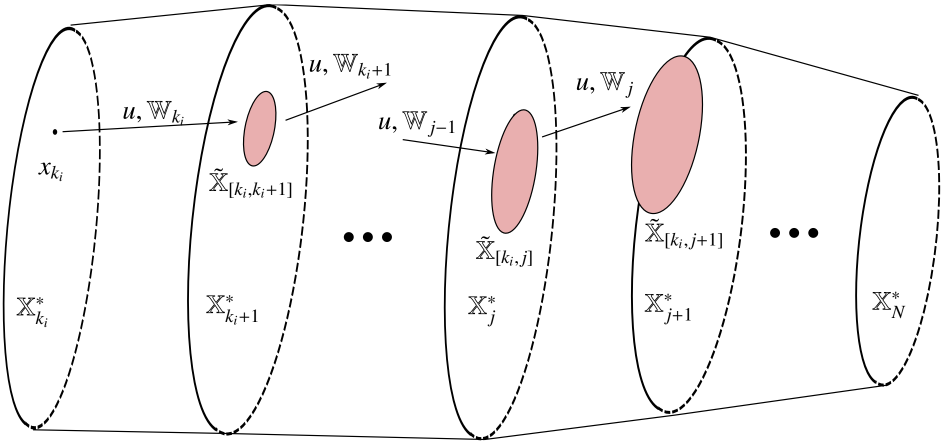


Figure 5.1: Geometrical interpretation for $\mathcal{P}^3_{[k_i,N]}(x_{k_i})$

Robust self-triggered control

Computation of reachable sets

Let $\mathbb{X}^*_N = \mathbb{X}_N$. For $k \in \mathbb{N}_{[0,N-1]}$, the backward reachable set $\mathbb{X}^*_k$ for the system (5.1) is recursively computed by:

$$\mathbb{P}_k = \{z \in \mathbb{R}^{n_x} \mid \exists u_k \in \mathbb{U}_k, f_k(z, u_k) \oplus \mathbb{W}_k \subseteq \mathbb{X}^*_{k+1}\}, \tag{5.2a}$$

$$\mathbb{X}^*_k = \mathbb{P}_k \cap \mathbb{X}_k. \tag{5.2b}$$

According to Proposition 2.1,the target tube $\{(\mathbb{X}_k, k), k \in \mathbb{N}_{[1,N]}\}$ is reachable from the initial state $x_0 \in \mathbb{X}_0$ if and only if $x_0 \in \mathbb{X}^*_0$. Furthermore, according to [71], Assumptions 5.1 and 5.2 make the resulting reachable sets compact. Proposition 2.1 indicates that if the state $x_j \in \mathbb{X}^*_j$, the recursive feasibility and the constraint satisfactions can be guaranteed.

Remark 5.1. *There exist some methods to compute the reachable sets for a nonlinear system (5.1), e.g., [71, 75]. In addition, there are results on the inner approximations of the reachable sets $\mathbb{X}^*_k$, e.g., [94, 95]. Note that these inner approximations are applicable also for the following algorithms, since they provide constraint satisfaction and recursive feasibility guarantees.*

Remark 5.2. *Given the initial state x_0, one can choose the minimal horizon N such that $\{(\mathbb{X}_j, j), j \in \mathbb{N}_{[0,N]}\}$ is reachable from x_0.*

Algorithm

Define the self-triggered condition for the system (5.1) as

$$\begin{aligned} k_{i+1} = \max\{k \mid k_i < k \leq N \text{ such that } u_j = u \in \mathbb{U}_j,\\ j \in \mathbb{N}_{[k_i,k-1]}, \text{ and the target tube}\\ \{(\mathbb{X}_j, j), j \in \mathbb{N}_{[k_i,N]}\} \text{ of (5.1) is reachable}\}. \end{aligned} \tag{5.3}$$

Recall that $k_{i+1} = k_i + M_i$. The following lemma provides the formulation to compute M_i.

Proposition 5.1. *Given the state $x_{k_i} \in \mathbb{X}^*_{k_i}$, $k_i \in \mathbb{N}_{[0,N-1]}$, the inter-sampling time M_i is obtained by solving the following optimization problem, denoted by $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$:*

$$\max_u \sum_{j=k_i+1}^{N} r_j \tag{5.4a}$$

subject to

$$\tilde{\mathbb{X}}_{[k_i,k_i]} = \{x_{k_i}\}, \tag{5.4b}$$

$$\forall j \in \mathbb{N}_{[k_i,N-1]} : \tilde{\mathbb{X}}_{[k_i,j+1]} = f_j(\tilde{\mathbb{X}}_{[k_i,j]}, u) \oplus \mathbb{W}_j, \tag{5.4c}$$

$$\forall j \in \mathbb{N}_{[k_i+1,N]} :$$

$$r_j = \begin{cases} \mathbb{1}_{\mathbb{X}^*_{k_i+1}}(\tilde{\mathbb{X}}_{[k_i,k_i+1]})\mathbb{1}_{\mathbb{U}_{k_i}}(u), & j = k_i + 1, \\ r_{j-1}\mathbb{1}_{\mathbb{X}^*_j}(\tilde{\mathbb{X}}_{[k_i,j]})\mathbb{1}_{\mathbb{U}_{j-1}}(u), & j > k_i + 1, \end{cases} \tag{5.4d}$$

where $f_j(\mathbb{X}, u) = \{z \in \mathbb{R}^{n_x} \mid z = f(x, u), \forall x \in \mathbb{X}\}$. That is, $M_i = \sum_{j=k_i+1}^{N} r^*_j$, where r^*_j corresponds to the optimal solution of $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$.

Proof. The definition of r_j characterizes the successive constraint satisfactions from time k_i for all possible disturbances $w_l \in \mathbb{W}_l, \forall l \in \mathbb{N}_{[k_i,j-1]}$. Then, the proof directly follows from Proposition 2.1 and the objective function of $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$. □

The geometric interpretation of the optimization problem $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$, as shown in Figure 5.1, is to seek a fixed control input u such that starting from time k_i, the time length, during which the state constraints and the control input constraints are satisfied for all possible disturbances, is maximized.

We denote by u^* the optimal solution to the optimization problem $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$. The following Algorithm 5.1 presents the robust self-triggered control for the uncertain constrained system (5.1).

Algorithm 5.1 Robust self-triggered control

Offline:

Determine a sequence of backward reachable sets $\{(\mathbb{X}_j^*, j), j \in \mathbb{N}_{[0,N]}\}$ by (5.2).

Online:

1: Initialize $i = 0$. If $x_0 \in \mathbb{X}_0^*$, continue. Else, stop.
2: Sample the state x_{k_i}, solve $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$ to obtain u^* and M_i.
3: Set $k_{i+1} = k_i + M_i$. Implement $u_j = u^*$, $\forall j \in \mathbb{N}_{[k_i,k_{i+1}-1]}$ to system (5.1) for some realizations $w_j \in \mathbb{W}_j$, $j \in \mathbb{N}_{[k_i,k_{i+1}-1]}$.
4: Set $i = i + 1$.
5: If $k_i < N$, go to step 2. Else, stop.

Control with minimum number of samplings

This subsection will provide a control method with minimum number of samplings, denoted by $T^* + 1$, over a given finite horizon for the system (5.1). Without loss of generality, we assume that $N \geq 2$.

Proposition 5.2. *The minimum number of samplings $T^* + 1$ is obtained by solving the following optimization problem, denoted by $\mathcal{P}^2_{[0,N]}(x_0)$,*

$$\min_{u_0,\Delta_j} \sum_{j=0}^{N-2} (1 - \mathbb{1}_{\{0\}}(\Delta_j)) \tag{5.5a}$$

subject to

$$\forall j \in \mathbb{N}_{[0,N-1]} : x_{j+1} = f_j(x_j, u_j), \tag{5.5b}$$

$$\forall j \in \mathbb{N}_{[0,N-1]} : u_j = \begin{cases} u_0, & j = 0 \\ u_{j-1} + \Delta_{j-1}, & j \geq 1, \end{cases} \tag{5.5c}$$

$$\forall j \in \mathbb{N}_{[1,N]} : x_j \in \mathbb{X}_j^*, \tag{5.5d}$$

$$\forall j \in \mathbb{N}_{[0,N-1]} : u_j \in \mathbb{U}_j. \tag{5.5e}$$

That is, $T^* = \sum_{j=0}^{N-2}(1-\mathbb{1}_{\{0\}}(\Delta_j^*))$, where Δ_j^* corresponds to the optimal solution of $\mathcal{P}^2_{[0,N]}(x_0)$.

Proof. In (5.5c), Δ_{j-1} denotes the difference between u_j and u_{j-1}. The objective function of $\mathcal{P}^2_{[0,N]}(x_0)$ aims at maximizing the number of zero (i.e., $\Delta_j = 0$) over the time interval $\mathbb{N}_{[0,N-1]}$. Thus, the optimal solution generates a sequence of piecewise constant control inputs with minimum number of switching times. □

Note that Algorithm 5.1 is applicable for deterministic systems. In this case, the difference is that Algorithm 5.1 cannot guarantee the achievement of minimum number of samplings for deterministic systems.

Remark 5.3. *For uncertain constrained systems, it is difficult to design the control with minimum number of samplings since at each sampling instant k_i, the exact executions of the future disturbance w_j, $\forall j \in \mathbb{N}_{[k_i,N-1]}$, are unknown.*

5.4 Self-triggered Control for Linear Systems with Polyhedral Constraints

The development of geometry software allows us to compute the sets $\mathbb{X}_k^*$ exactly and efficiently if the system is linear and the constraint sets are polyhedral [96]. This section will specialize the systems (5.1) to be linear and the constraint sets to be polyhedral. We can reformulate the optimization problems $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$ and $\mathcal{P}^2_{[0,N]}(x_0)$ to be computationally tractable MILP problems.

If the model f_k is linear, the system (5.1) becomes

$$x_{k+1} = A_k x_k + B_k u_k + w_k. \tag{5.6}$$

Here A_k and B_k are deterministic real matrices with appropriate dimensions at each time $k \in \mathbb{N}_{[0,N-1]}$. The control input sets $\mathbb{U}_k$, $k \in \mathbb{N}_{[0,N-1]}$, are compact polyhedra. Each set $\mathbb{X}_k$ of the target tube $\{(\mathbb{X}_k, k), k \in \mathbb{N}_{[1,N]}\}$ is a compact polyhedron. The disturbance sets $\mathbb{W}_k$, $k \in \mathbb{N}_{[0,N-1]}$, are compact polyhedra.

Now, the computation of the sets $\mathbb{X}_k^*$ in (5.2) is given as follows. Note that the following equations involve only set operations and the corresponding sets can be well-defined even if the matrix A_k is not invertible. Hence, we do not impose any assumption on A_k.

Lemma 5.1. *[69] For the uncertain linear system (5.6) with polyhedral constraints, the set $\mathbb{X}_k^*$ in (5.2) evolves as*

$$\mathbb{Q}_k = \mathbb{X}_k^* \ominus \mathbb{W}_k, \tag{5.7a}$$

$$\mathbb{P}_k = A_k^{-1}(\mathbb{Q}_{k+1} \oplus (-B_k \mathbb{U}_k)), \tag{5.7b}$$

$$\mathbb{X}_k^* = \mathbb{P}_k \cap \mathbb{X}_k,\ \mathbb{X}_N^* = \mathbb{X}_N. \tag{5.7c}$$

Remark 5.4. *Since the sets $\mathbb{X}_k$, $k \in \mathbb{N}_{[0,N]}$, are compact, the sets $\mathbb{X}_k^*$, $k \in \mathbb{N}_{[0,N]}$, are also compact even when the matrices A_k are not invertible.*

The polyhedral sets $\mathbb{U}_k$ and $\mathbb{X}_k^*$ are written as

$$\mathbb{U}_k = \{z \in \mathbb{R}^{n_u} \mid E_k z \le e_k\}, \tag{5.8}$$

$$\mathbb{X}_k^* = \{z \in \mathbb{R}^{n_x} \mid F_k z \le f_k\}, \tag{5.9}$$

where E_k and F_k are matrices with appropriate dimensions, and e_k and f_k are vectors with appropriate dimensions.

Robust self-triggered control

Before providing the main result, we need some preliminary lemmas.

Lemma 5.2. *The set $\tilde{\mathbb{X}}_{[k_i,j]}$, $j \in \mathbb{N}_{[k_i,N]}$, in (5.4c) can be written as*

$$\tilde{\mathbb{X}}_{[k_i,j]} = (G_{[k_i,j]} x_k + H_{[k_i,j]} u) \oplus \mathbb{Z}_{[k_i,j]}, \tag{5.10}$$

where $G_{[k_i,j]} = \prod_{l=k_i}^{j-1} A_l$, $H_{[k_i,j]} = \sum_{m=k_i}^{j-1} \prod_{l=m+1}^{j-1} A_l B_m$, $\mathbb{Z}_{[k_i,j]} = \bigoplus_{m=k_i}^{j-1} \prod_{l=m+1}^{j-1} A_l \mathbb{W}_m$. Furthermore, the set $\mathbb{Z}_{[k_i,j]}$ is a closed polyhedron.

Proof. When $j = k_i$, (5.10) implies that $\tilde{\mathbb{X}}_{[k_i,k_i]} = \{x_{k_i}\}$. According to the definition of $\tilde{\mathbb{X}}_{[k_i,j]}$, $j \in \mathbb{N}_{[k_i+1,N]}$, in (5.4c), by induction, it follows

$$
\begin{aligned}
\tilde{\mathbb{X}}_{[k_i,j+1]} &= A_j\tilde{\mathbb{X}}_{[k_i,j]} \oplus B_j u \oplus \mathbb{W}_j \\
&= A_j(G_{[k_i,j]}x_{k_i} + H_{[k_i,j]}u) \oplus \mathbb{Z}_{[k_i,j]}) \oplus B_j u \oplus \mathbb{W}_j \\
&= (A_j \prod_{l=k_i}^{j-1} A_l x_{k_i} + (A_j \sum_{m=k_i}^{j-1} \prod_{l=m+1}^{j-1} A_l B_m + B_j)u) \oplus (A_j \bigoplus_{m=k_i}^{j-1} \prod_{l=m+1}^{j-1} A_l \mathbb{W}_m \oplus \mathbb{W}_j) \\
&= (\prod_{l=k_i}^{j} A_l x_{k_i} + \sum_{m=k_i}^{j} \prod_{l=m+1}^{j} A_l B_m) \oplus (\bigoplus_{m=k_i}^{j} \prod_{l=m+1}^{j} A_l \mathbb{W}_m) \\
&= (G_{[k_i,j+1]}x_{k_i} + H_{[k_i,j+1]}u) \oplus \mathbb{Z}_{[k_i,j+1]}.
\end{aligned}
$$

Since the sets $\mathbb{W}_m$ are compact polyhedra, we have that the sets $\mathbb{Z}_{[k_i,j]}$ are closed polyhedra. □

Lemma 5.3. *[2] Given two polyhedra $\mathbb{P} = \{z \in \mathbb{R}^n \mid Pz \leq p\}$ and $\mathbb{Q} = \{z \in \mathbb{R}^n \mid Qz \leq q\}$, $\mathbb{P} \subseteq \mathbb{Q}$ holds if and only if there exists a non-negative matrix S such that*

$$SP = Q, \tag{5.11}$$

$$Sp \leq q. \tag{5.12}$$

Assume now that $\mathbb{Z}_{[k_i,j]} = \{z \in \mathbb{R}^n \mid V_{[k_i,j]}z \leq v_{[k_i,j]}\}$, $j \in \mathbb{N}_{[k_i+1,N]}$, where the matrix $V_{[k_i,j]}$ and vector $v_{[k_i,j]}$ can be computed offline according to $\mathbb{Z}_{[k_i,j]} = \bigoplus_{m=k_i}^{j-1} \prod_{l=m+1}^{j-1} A_l \mathbb{W}_m$. By Lemma 5.3, we derive the following result.

Lemma 5.4. *For the sets $\tilde{\mathbb{X}}_{[k_i,j]}$ and $\mathbb{X}_j^*$, $\tilde{\mathbb{X}}_{[k_i,j]} \subseteq \mathbb{X}_j^*$ holds if and only if there exists a non-negative matrix $S_{[k_i,j]}$ such that*

$$S_{[k_i,j]}V_{[k_i,j]} = F_j, \tag{5.13}$$

$$S_{[k_i,j]}(v_{[k_i,j]} + V_{[k_i,j]}(G_{[k_i,j]}x_{k_i} + H_{[k_i,j]}u)) \leq f_j. \tag{5.14}$$

Proof. This directly follows from Lemma 5.2 and Lemma 5.3. □

Since u is the decision variable, the constraints (5.14) are nonlinear. To remedy this, we can calculate the nonnegative matrices $S_{[k_i,j]}$ offline to satisfy (5.13) by solving an LP [97]:

$$(S_{[k_i,j]})_l = \underset{a^T}{\operatorname{argmin}}\{\mathbf{1}^T a \mid a^T V_{[k_i,j]} = (F_j)_l, a \geq \mathbf{0}\}, \tag{5.15}$$

where a is a vector with appropriate dimension and $(S)_l$ denotes the lth row of the matrix S.

Remark 5.5. *The LP in (5.15) admits a nonnegative matrix solution with minimum infinity norm, which could lead to a larger feasible region for the optimization problem $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$ than other nonnegative solutions to (5.13).*

The next theorem shows that the robust self-triggered control for the system (5.6) with polyhedral constraints can be designed by solving a tractable MILP.

Theorem 5.1. *For the uncertain linear system (5.6) with polyhedral constraints, the optimization problem $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$ can be reformulated as an MILP, denoted by $\mathcal{P}^3_{[k_i,N]}(x_{k_i})$,*

$$\max_{u,\delta_j} \sum_{j=k_i+1}^{N} (1-\delta_j) \tag{5.16a}$$

subject to

$$\forall j \in \mathbb{N}_{[k_i+1,N]}: \quad S_{[k_i,j]}(\tilde{G}_{[k_i,j]}x_{k_i} + \tilde{H}_{[k_i,j]}u) \le \tilde{f}_{[k_i,j]} + \delta_j\Gamma\mathbf{1}, \tag{5.16b}$$

$$\forall j \in \mathbb{N}_{[k_i,N-1]}: E_j u \le e_j + \delta_j\Gamma\mathbf{1}, \tag{5.16c}$$

$$\forall j \in \mathbb{N}_{[k_i+1,N-1]}: \delta_j \le \delta_{j+1}, \tag{5.16d}$$

$$\forall j \in \mathbb{N}_{[k_i+1,N]}: \delta_j \in \{0,1\}, \tag{5.16e}$$

where $\tilde{G}_{[k_i,j]} = V_{[k_i,j]}G_{[k_i,j]}$, $\tilde{H}_{[k_i,j]} = V_{[k_i,j]}H_{[k_i,j]}$, $\tilde{f}_{[k_i,j]} = f_j - S_{[k_i,j]}v_{[k_i,j]}$, and Γ is a positive constant satisfying

$$\Gamma > \max\{\max_{j\in\mathbb{N}_{[k_i,N-1]}} \|\mathbb{U}_j\|_\infty, \max_{j\in\mathbb{N}_{[k_i+1,N]}} \max_{u\in\mathbb{U}_j} \|S_{[k_i,j]}(\tilde{G}_{[k_i,j]}x_{k_i} + \tilde{H}_{[k_i,j]}u) - \tilde{f}_{[k_i,j]}\|_\infty\}. \tag{5.17}$$

Proof. Recall that $\forall j \in \mathbb{N}_{[k_i+1,N]}$,

$$r_j = \begin{cases} \mathbb{1}_{\mathbb{X}^*_{k_i+1}}(\tilde{\mathbb{X}}_{[k_i,k_i+1]})\mathbb{1}_{\mathbb{U}_{k_i}}(u), & j = k_i+1, \\ r_{j-1}\mathbb{1}_{\mathbb{X}^*_j}(\tilde{\mathbb{X}}_{[k_i,j]})\mathbb{1}_{\mathbb{U}_{j-1}}(u), & j > k_i+1. \end{cases}$$

Let us introduce a sequence of 0-1 variables δ_j, $j \in \mathbb{N}_{[k_i+1,N]}$. Obviously, it follows from $r_j = 1-\delta_j$ that

$$\begin{cases} \forall l \in \mathbb{N}_{[k_i+1,j]}: \\ \quad S_{[k_i,l]}(\tilde{G}_{[k_i,l]}x_{k_i} + \tilde{H}_{[k_i,l]}u) \le \tilde{f}_{[k_i,l]} + \delta_l\Gamma\mathbf{1}, \\ \forall l \in \mathbb{N}_{[k_i,j-1]}: E_l u \le e_l + \delta_l\Gamma\mathbf{1}, \end{cases}$$

where Γ is a positive number satisfying (5.17). Furthermore, $\forall j \in \mathbb{N}_{[k_i+1,N-1]}$, $r_j \ge r_{j+1}$ can be rewritten as $\delta_j \le \delta_{j+1}$. Then, we get the optimization problem $\mathcal{P}^2_{[k_i,N]}(x_{k_i})$. □

Remark 5.6. *Let us discuss the complexity of the MILP (5.16). The only continuous decision variable is u with dimension n_u and the number of binary variables δ_j is linear in the horizon N. The number of constraints at each time instant is determined by the reachable sets. Considering the special constraint (5.16d) on δ_j, it is easy to construct an enumeration tree and recursively solve a linear program until the problem is infeasible. Hence, we conclude that the problem (5.16) can be solved in polynomial time. Note that several software tools have been developed to solve large MILPs in the past few years, e.g., [82], allowing us to solve our problem online efficiently.*

Remark 5.7. *Following similar operations as for the linear case, the previous optimization problem $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$ can be reformulated as an integer program with a constraint like (5.16d). The resulting integer program can be iteratively cast as a classic constrained robust nonlinear control problem.*

Remark 5.8. *In general, Γ can be arbitrarily chosen to be sufficiently a large positive constant. The lower bound on Γ defined in (5.17) aims to quantify how large Γ should be, which can be calculated by solving a linear program (LP) since the sets $\mathbb{U}_j$ are compact polyhedra and the norm is ℓ_∞−norm.*

Control with minimum number of samplings

When the disturbance set $\mathbb{W}_k = \{0\}$, $\forall k \in \mathbb{N}_{0,N-1}$, we can also reformulate the optimization problem $\mathcal{P}^2_{[0,N]}(x_0)$ as a tractable MILP.

Theorem 5.2. *For the deterministic linear system (5.6) with polyhedral constraints, the optimization problem $\mathcal{P}^2_{[0,N]}(x_0)$ can be reformulated as an MILP, denoted by $\mathcal{P}^4_{[0,N]}(x_0)$,*

$$\max_{u_0,\Delta_j,\delta_j,c_j} J_{[0,N]}(x_0) = \sum_{j=0}^{N-2}(1-\delta_j) \tag{5.18a}$$

subject to

$$\forall j \in \mathbb{N}_{[0,N-1]} : x_{j+1} = A_j x_j + B_j u_j, \tag{5.18b}$$

$$\forall j \in \mathbb{N}_{[0,N-1]} : u_j = \begin{cases} u_0, & j = 0 \\ u_{j-1} + \Delta_{j-1}, & j \geq 1, \end{cases} \tag{5.18c}$$

$$\forall j \in \mathbb{N}_{[1,N]} : F_j x_j \leq f_j, \tag{5.18d}$$

$$\forall j \in \mathbb{N}_{[0,N-1]} : E_j u_j \leq e_j, \tag{5.18e}$$

$$\forall j \in \mathbb{N}_{[0,N-2]} : \begin{cases} \Delta_j \leq \Gamma\delta_j \mathbf{1}, \\ \Delta_j \geq -\gamma\delta_j \mathbf{1}, \\ \Delta_j \leq c_j + \gamma(1-\delta_j)\mathbf{1}, \\ \Delta_j \geq c_j - \Gamma(1-\delta_j)\mathbf{1}, \end{cases} \tag{5.18f}$$

$$\forall j \in \mathbb{N}_{[0,N-2]} : -\gamma\mathbf{1} \leq c_j \leq \Gamma\mathbf{1}, \tag{5.18g}$$

$$\forall j \in \mathbb{N}_{[0,N-2]} : \delta_j \in \{0,1\}, \tag{5.18h}$$

where γ and Γ are two large positive constants satisfying

$$\gamma, \Gamma > \max_{j \in \mathbb{N}_{[0,N-2]}} \max_{u \in \mathbb{U}_j, v \in \mathbb{U}_{j+1}} \|u - v\|_\infty. \tag{5.19}$$

Proof. By introducing new variables $c_j \in \mathbb{R}^{n_u}$ and 0-1 variables δ_j, $j \in \mathbb{N}_{[k+2,N]}$, we define $\Delta_j = \delta_j c_j$, i.e., $\Delta_j = 0$ if $\delta_j = 0$ and $\Delta_j = c_j$ if $\delta_j = 1$. And it follows that

$$\forall j \in \mathbb{N}_{[k+1,N]} : \Delta_j = \delta_j c_j \tag{5.20}$$

$$\Leftrightarrow \begin{cases} \Delta_j \leq \delta_j \Gamma \mathbf{1}, \\ \Delta_j \geq \delta_j \gamma \mathbf{1}, \\ \Delta_j \leq c_j + (1-\delta_j)\gamma \mathbf{1}, \\ \Delta_j \geq c_j - (1-\delta_j)\Gamma \mathbf{1}, \\ -\gamma \mathbf{1} \leq c_j \leq \Gamma \mathbf{1}, \end{cases} \tag{5.21}$$

where Γ and γ are two positive numbers satisfying (5.19). Then, we get the optimization problem $\mathcal{P}^4_{[0,N]}(x_0)$. □

Remark 5.9. *The statements in Remark 5.8 also apply with γ and Γ in (5.19). In addition, the optimization problem $\mathcal{P}^4_{[0,N]}(x_0)$ is equivalent to $\mathcal{P}^2_{[0,N]}(x_0)$, while the optimal solution to $\mathcal{P}^3_{[k_i,N]}(x_{k_i})$ is a suboptimal solution to $\mathcal{P}^1_{[k_i,N]}(x_{k_i})$.*

5.5 Examples

This section provides two examples to illustrate the effectiveness of our proposed algorithms. The following numerical experiments were run in Matlab R2016a with MPT toolbox [96] on a Dell laptop with Window 7, Intel i7-6600U CPU 2.80GHz and 16.0 GB RAM.

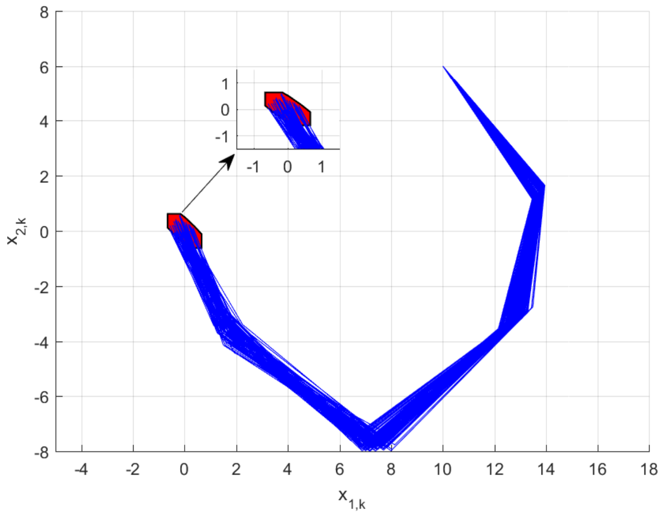
Example 1

Compare the proposed robust self-triggered algorithm with the robust self-triggered MPC in [50]. Consider a same linear time-invariant system in [50], where

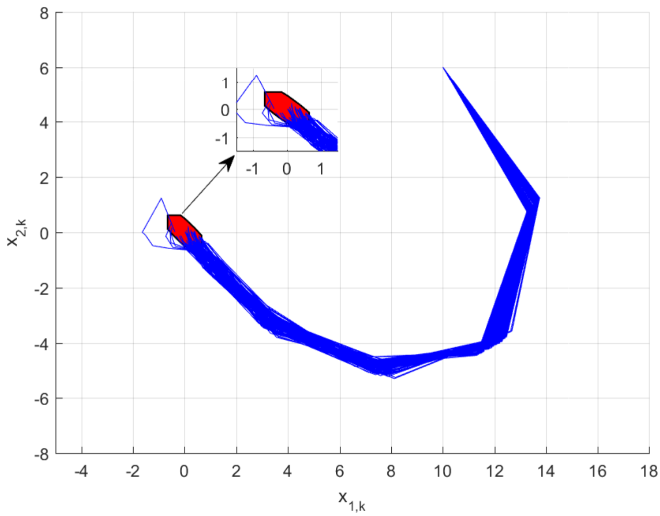
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

The constraint sets are $\mathbb{X} = \{z \in \mathbb{R}^2 \mid \begin{bmatrix} -20 \\ -8 \end{bmatrix} \leq z \leq \begin{bmatrix} 20 \\ 8 \end{bmatrix}\}$, $\mathbb{U} = \{z \in \mathbb{R} \mid -5 \leq z \leq 5\}$, $\mathbb{W} = \{z \in \mathbb{R}^2 \mid \begin{bmatrix} -0.25 \\ -0.25 \end{bmatrix} \leq z \leq \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}\}$. The initial state is $[10\ 6]^T$.

Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 0.1$ be the weight matrix in the objective function. By solving the discrete-time algebraic Riccati equation, we obtain the matrix $P = \begin{bmatrix} 2.0599 & 0.5916 \\ 0.5916 & 1.4228 \end{bmatrix}$ and the corresponding optimal feedback gain $K = [-0.6167\ -1.2703]$. The control objective is to steer the state to the robust invariant set, denoted by Ω (the red region in Figure 5.2), which is computed by the method in [51]. The implementation will stop if the state enters the robust invariant set.

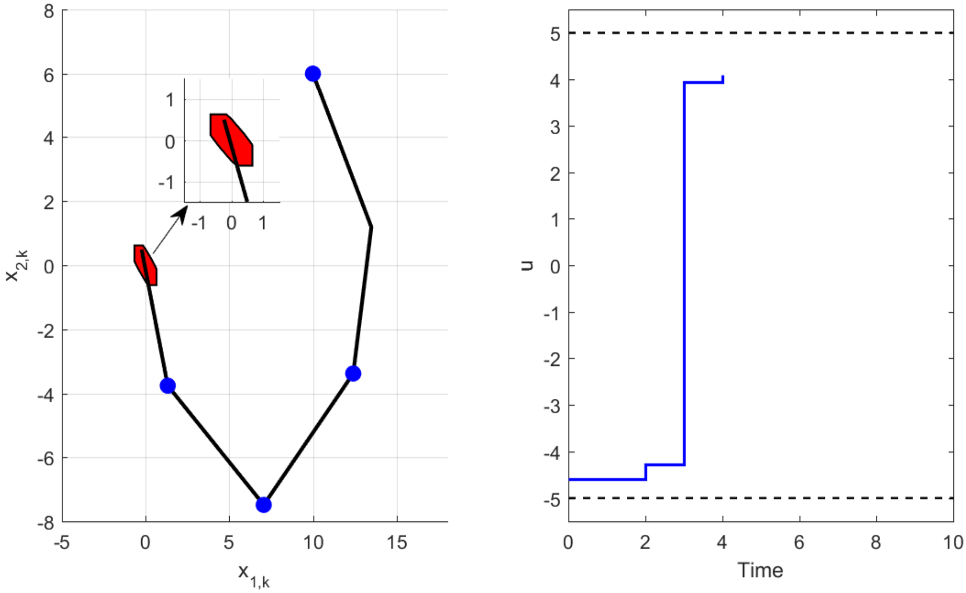


(a)

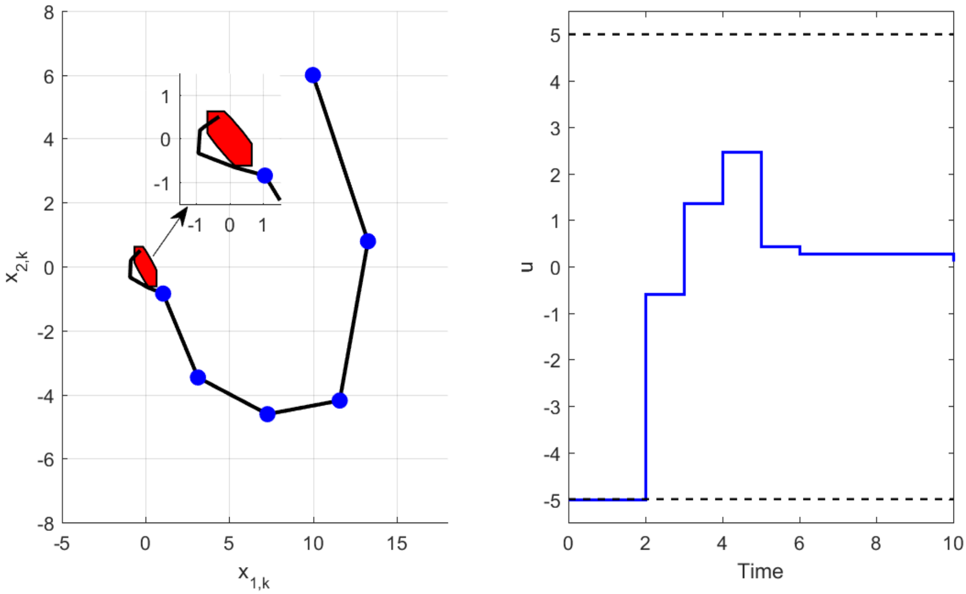


(b)

Figure 5.2: State trajectories under Algorithm 5.1 and robust self-triggered MPC [50] for 100 realizations of the uncertainty sequence.



(a) Algorithm 5.1



(b) Robust self-triggered MPC

Figure 5.3: State and control trajectories under Algorithm 5.1 and robust self-triggered MPC [50] for 1 realization of the uncertainty sequence. The algorithms terminate when the state enters the robust invariant set (the red region).

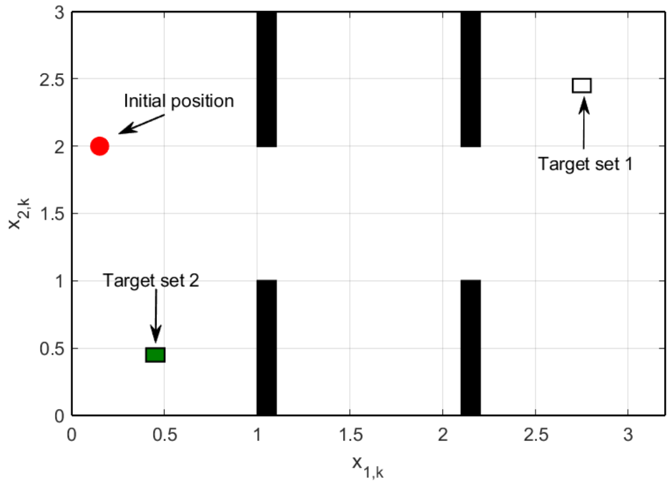


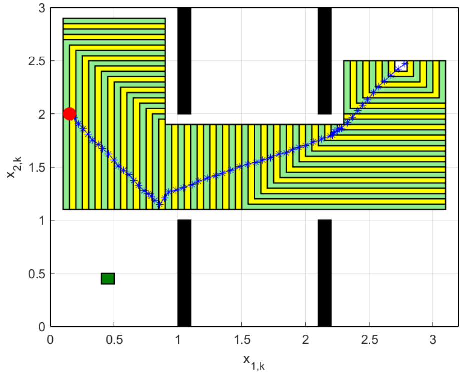
Figure 5.4: Scenario

For the robust self-triggered control in this paper, the terminal set of the target tube is Ω. For the robust self-triggered MPC in [50], we choose the maximal inter-sampling time $M_{max} = 4$. Figure 5.2 depicts the state trajectories under our algorithm and robust self-triggered MPC for 100 realizations of the uncertainty sequence. And Figure 5.3 shows the state and control trajectories for 1 realization of the uncertainty sequence. We highlight the sampling instants by blue solid circles. We compare the two different methods for several indexes, of which the average is taken over 500 realizations.

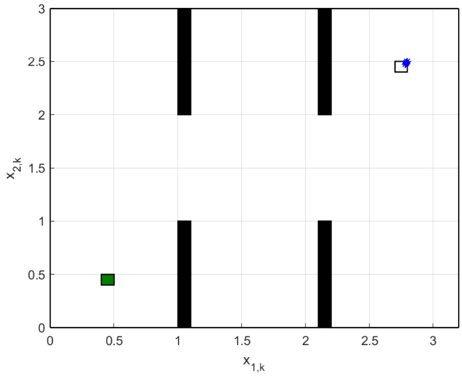
- *Average inter-sampling time*: The average inter-sampling time is $\bar{M} = 1.2045$ under the robust self-triggered MPC of [50] while it is $\bar{M} = 1.3333$ under the self-triggered scheme of this paper. Thus, our control scheme achieves an average communication reduction by 9.66% more than that of the self-triggered MPC.

- *Average online computation time*: The average online computation time at each sampling instant is $0.5758s$ for the robust self-triggered MPC while it is $0.1964s$ for our control method despite the presence of integer variables.

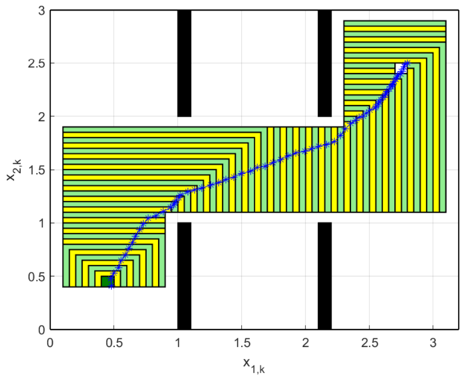
- *Average performance*: The performance measure is defined by $J = \sum_{k=0}^{T_{run}}(\|x_k\|_Q^2 + \|u_k\|_R^2)$ where T_{run} is the time when the state enters the robust invariant set. The robust self-triggered MPC achieves a slightly better average performance than our scheme. The performance measure is 591.9191 for MPC while it is 621.4552 for our scheme.



(a)



(b)



(c)

Figure 5.5: State trajectories under Algorithm 5.1.

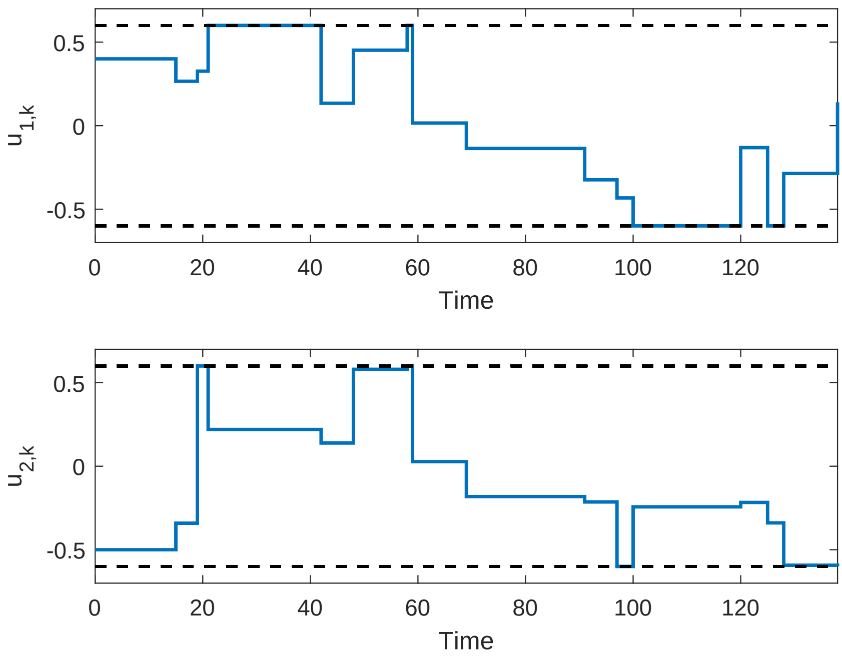


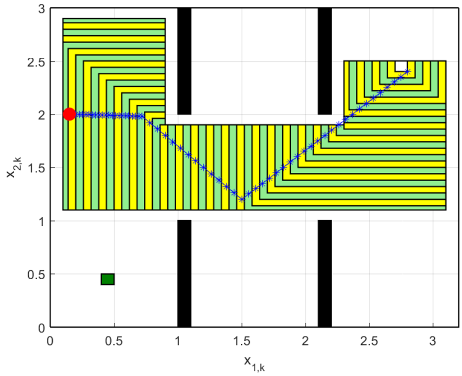
Figure 5.6: Example 2: control input trajectories under Algorithm 5.1

Example 2

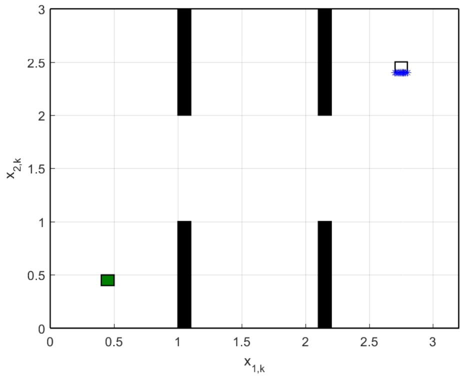
Consider a mobile robot with dynamics (5.6), where

$$A_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_k = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, k \in \mathbb{N}.$$

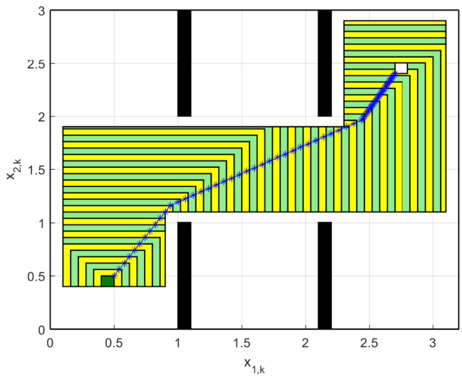
The input constraint set is $\mathbb{U}_k = \{z \in \mathbb{R}^2 \mid \begin{bmatrix} -0.6 \\ -0.6 \end{bmatrix} \leq z \leq \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}\}$, $k \in \mathbb{N}$. The robot moves in a closed workspace, as shown in Figure 5.4, in which there are some obstacles (the black rectangles). The robot should achieve collision avoidance with the obstacles and the boundaries of the workspace. We set the safe distance as 0.1. In addition, the robot can exchange the information (the position and the control input) with the control center via a bandwidth-limited communication network. At each time, the robot can only receive one control input from the control center. The initial position is $[0.15\ 2]^T$. The target set 1 is $\{z \in \mathbb{R}^2 \mid \begin{bmatrix} 2.7 \\ 2.4 \end{bmatrix} \leq z \leq \begin{bmatrix} 2.8 \\ 2.5 \end{bmatrix}\}$ and the target set 2 is $\{z \in \mathbb{R}^2 \mid \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \leq z \leq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\}$. A sequence of temporal constrained tasks for the mobile robot are



(a)



(b)



(c)

Figure 5.7: State trajectories by using control with minimum number of samplings.

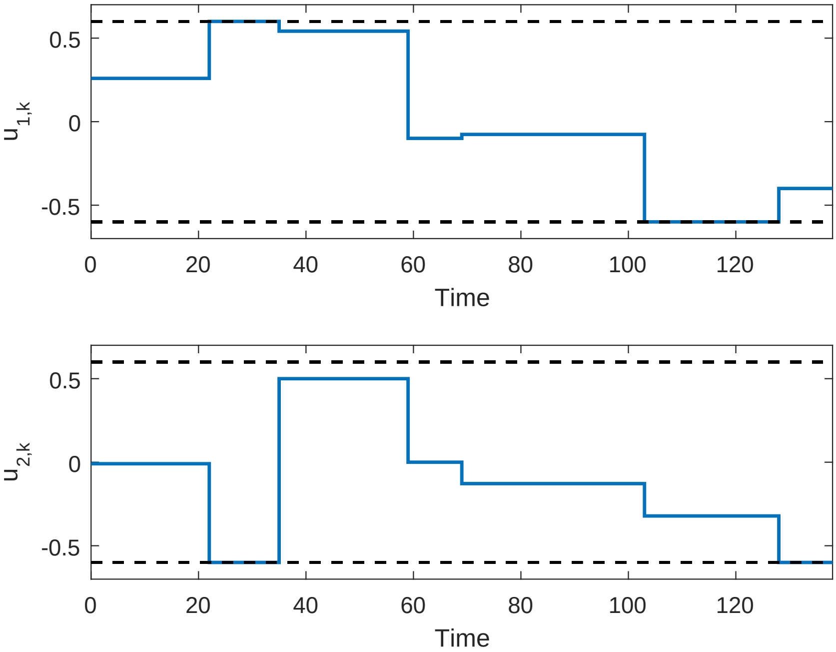


Figure 5.8: Example 2: control input trajectories by using control with minimum number of samplings.

- stage 1: the robot should arrive at the target set 1 before $k = 60$;
- stage 2: the robot should stay in target set 1 for at least 10 time steps after arrival.
- stage 3: the robot should arrive at the target set 2 before $k = 140$.

To save the communication recourses, our self-triggered control strategies are implemented. We choose the convex inner approximations of the backward reachable sets (which are the intersection between the predecessor sets and the safe regions). As mentioned in Remark 5.1, these approximations still respect the feasibility and the constraint satisfactions.

In the first case, assume that the disturbance set is $\mathbb{W}_k = \{z \in \mathbb{R}^2 \mid \begin{bmatrix} -0.01 \\ -0.01 \end{bmatrix} \leq z \leq \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}\}$. Subfigures (a)-(b) of Figure 5.5 depict the state trajectories for stage 1 and 3 under Algorithm 5.1. The yellow and lightgreen regions are the sets $\mathbb{X}_k^*$. The control input trajectory is shown in subfigure (c) of Figure 5.5 with the times of control update being 13.

In the second case, assume that $\mathbb{W}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\forall k \in \mathbb{N}$. Subfigures (a)-(b) of Figure 5.7 depict the state trajectories for stage 1 and 3 by using the control with minimum number of samplings. The yellow and lightgreen regions are the convex approximations of the sets $\mathbb{X}_k^*$. The control input trajectory is shown in subfigure (c) of Figure 5.7 with the times of control update being 6.

5.6 Conclusion

In this chapter, we proposed a robust self-triggered control algorithm for time-varying uncertain systems with constraints. By using reachability analysis, the constraint satisfactions and recursive feasibility were guaranteed. The proposed algorithm provided us a geometric interpretation for self-triggered control. The problem of control with minimum number of samplings was investigated for deterministic constrained systems. For linear systems with polyhedral constraints, the proposed methods were reformulated as computationally tractable MILP problems. In simulations, the results were compared with robust self-triggered MPC and applied to the robot motion planning.

Chapter 6

Conclusions and Future Research

In this chapter, we summarize the main results of this thesis and outline possible directions for future research.

6.1 Conclusions

This thesis studied stochastic invariance and aperiodic control for uncertain constrained systems. More precisely, we designed the algorithms to compute the PCIS within a given set for control systems, integrated self-triggered control and stochastic MPC for linear system with stochastic disturbances, and proposed a robust self-triggered control scheme for time-varying and uncertain constrained systems.

In Chapter 3, we extended controlled invariant set to stochastic systems. We proposed two definitions: finite- and infinite-horizon PCISs, and explored their relation to RCISs. We designed iterative algorithms to compute the PCIS within a given set by taking into account the issues of computation tractability and iteration convergence. For systems with discrete state and control spaces, the computations of the finite- and infinite-horizon PCISs at each iteration are based on linear programming and mixed integer linear programming, respectively. The algorithms are computationally tractable and terminate in a finite number of steps. For systems with continuous state and control spaces, we provided the discretizition procedure and proved the convergence of the approximation when computing the finite-horizon PCISs. In addition, it was shown that the infinite-horizon PCIS can be alternatively computed by the stochastic backward reachable set from the robust control invariant set contained in it.

In Chapter 4, we proposed a stochastic self-triggered MPC algorithm for linear systems subject to exogenous disturbances and probabilistic constraints. At each sampling instant, an optimization problem is solved to jointly determine both the next sampling instant and the control inputs to be applied between the two sampling instants. Although the self-triggered implementation achieves communication reduction, the control commands are necessarily applied in open-loop until the next sampling instant. To guarantee probabilistic constraint satisfaction, we derived a necessary and sufficient condition on the nominal systems by using the information on the distribution of the disturbances explicitly.

Moreover, based on a tailored terminal set, we transformed a multi-step open-loop MPC optimization problem with infinite prediction horizon into a tractable quadratic programming problem. It was further shown that the recursive feasibility and closed-loop stability are guaranteed.

In Chapter 5, we developed a robust self-triggered control algorithm for time-varying and uncertain systems with constraints. The algorithm is based on geometric necessary and sufficient conditions for reachability. The resulting piecewise constant control inputs achieve communication reduction and guarantee the constraint satisfactions. Particularly, when the system is deterministic, we proposed a control design with minimum number of samplings over a finite time horizon. Furthermore, when the plant is linear and the constraints are polyhedral, the proposed algorithms can be reformulated as computationally tractable mixed integer linear programming problems.

6.2 Future Research Directions

There are several interesting research directions based on the work of this thesis.

- First, an interesting problem to investigate further is how to incorporate PCISs into stochastic MPC as terminal sets and how to quantify the resulting conservatism reduction compared with existing stochastic MPC. When applying PCISs, one underlying challenge is the guarantee of the conventional recursive feasibility in MPC.

- A second problem for future study is how to serve as the safety measure for uncertain safety-critical systems through PCISs. For example, let us consider a human-robot interaction system. A robust control design by considering all possible human behaviors may lead to a conservative or even empty workspace for robot. It would be interesting to use PCISs to trade off the safe level for human and the flexibility for robot.

- Another problem of interest is how to design reachability-based self-triggered controller for continuous-time uncertain constrained systems. As shown in this thesis, reachability analysis is a good tool to guarantee the recursive feasibility and the constraint satisfaction. Different from the discrete-time case, one challenge is the exclusion of the Zeno behaviour.

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