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Decentralized Control of Networked Systems: Information Asymmetries and Limitations

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‘Indulge your passion for knowledge,’ says nature, ‘but seek knowledge of things that are human and directly relevant to action and society. As for abstruse thought and profound researches, I prohibit them, and if you engage in them I will severely punish you by the brooding melancholy they bring, by the endless uncertainty in which they involve you, and by the cold reception your announced discoveries will meet with when you publish them. Be a philosopher, but amidst all your philosophy be still a man.’

David Hume, *An Enquiry Concerning Human Understanding*, 1748.

Abstract

Designing local controllers for networked systems is challenging, because in these systems each local controller can often access only part of the overall information on system parameters and sensor measurements. Traditional control design cannot be easily applied due to the unconventional information patterns, communication network imperfections, and design procedure complexities. How to control large-scale systems is of immediate societal importance as they appear in many emerging applications, such as intelligent transportation systems, smart grids, and energy-efficient buildings. In this thesis, we make three contributions to the problem of designing networked controller under information asymmetries and limitations.

In the first contribution, we investigate how to design local controllers to optimize a cost function using only partial knowledge of the model governing the system. Specifically, we derive some fundamental limitations in the closed-loop performance when the design of each controller only relies on local plant model information. Results are characterized in the structure of the networked system as well as in the available model information. Both deterministic and stochastic formulations are considered for the closed-loop performance and the available information. In the second contribution of the thesis, we study decision making in transportation systems using heterogeneous routing and congestion games. It is shown that a desirable global behavior can emerge from simple local strategies used by the drivers to choose departure times and routes. Finally, the third contribution is a novel stochastic sensor scheduling policy for ad-hoc networked systems, where a varying number of control loops are active at any given time. It is shown that the policy provides stochastic guarantees for the network resources dynamically allocated to each loop.

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Farhad
Stockholm, February 2014

Contents

CHAPTER 1

Introduction

> "Thoughts without content are void; intuitions without conceptions, blind."
>
> Immanuel Kant, *The Critique of Pure Reason*[1], 1781.

RECENT developments in control engineering, embedded computing, and communication networks have enabled many complex systems, such as aircraft and satellite formations [1, 2], intelligent transportation infrastructures [3, 4], and flexible structures [5, 6]. A common feature of these large-scale control systems is that they are composed of several subsystems coupled through their dynamics, decision-making process, or performance objectives. When regulating these systems, it is often necessary to adopt a distributed architecture, in which the decision maker (e.g., controller, network manager, social planner) is composed of several interconnected units. Each local decision maker can only access a subset of the global information (e.g., sensor measurements, model parameters) and actuate on a subset of the inputs, perhaps in its vicinity. This distributed architecture is typically imposed because otherwise the central decision maker with full access to information might become very complex and not possible to implement, or because different subsystems may belong to competing entities that wish to retain a level of autonomy. Therefore, in this thesis, we try to mathematically formulate the effects of such information asymmetry and limitation in some control and estimation problems for complex networked systems.

The thesis consists of three parts. In the first part, we focus on decentralized control design under limited plant model information. We remove a common, but often implicit, assumption in the control literature, namely, that control design is performed in a centralized fashion with full knowledge of the plant model (even if

[1] Kritik der reinen Vernunft, translated by John M. D. Meiklejohn, 2011.

the controller is decentralized and has access only to a subset of the state). For these problems, we are interested in understanding how to optimize a social cost function using only partial knowledge of the model governing the system (in addition to the partial knowledge of the system state measurements). In the second part, we study decision making in road traffic using heterogeneous routing and congestion games. Specifically, we model the drivers' decision-making process for selecting departure time and route. Also in this case, the decision makers do not have access to the full information (e.g., the preferences of other players) when making their decisions. A desirable global behavior still can be achieved under certain conditions. Finally, in the third part, we propose a stochastic scheduling policy with the ability to balance the sensor sampling and transmission rates in ad-hoc networked control systems.

In the following chapters, we present motivating applications, review the existing literature, and discuss the contributions of the appended papers. Specifically, in the remainder of this chapter, we discuss the challenges that we face when controlling large-scale systems under asymmetric information regimes. In Section 1.1, we discuss power networks and transportation systems as two motivating applications. In Section 1.2, we present the main questions that we address in this thesis. In Section 1.3, we mathematically formulate several illustrating examples, which we use in Chapter 3 as well as in the attached papers, to demonstrate the developed results. Finally, in Section 1.4, we outline the thesis.

1.1 Motivating Applications

We start by presenting two motivating applications to illustrate the challenges we face in optimal control and estimation of shared infrastructures in power networks and transportation systems.

1.1.1 Power Networks

Consider the Baltic sea region electricity transmission grid portrayed in Figure 1.1. Most of the power is generated in a few large power generators and transmitted through the network to the consumers. The power network consists of tens-of-thousands of components (generators, transmission lines, converters, etc) connected together. These components have local interactions with each other through the grid and through a supporting communication network, which results in a structured networked control system.

For a power transmission grid, one of the design goals is to optimally regulate voltage, active and reactive power, and frequency in the face of variable demand, stochastic generation (mainly due to renewable energy sources), and faults. Sensors (e.g., phasor measurement units) measure voltages, phase angles, and frequencies among other variables and transmit these measurements over a communication network to the control stations. Due to the complexity of the grid (and because of the communication limitations), all sensor information is not used in every controller in the system. Therefore, the local controllers do not use full state measurements, but

Figure 1.1: Electricity transmission grid in the Baltic sea region. Picture provided courtesy of Nordregio http://www.nordregio.se/, Designer: P.G. Lindblom.

only a subset of the overall state. This constraint brings challenges in the design of stabilizing and optimal controllers.

Let us consider the overall problem of controlling power networks in a bit more detail. Power networks are highly complex time-varying dynamical systems, which are hard to model in detail for several reasons. First, these systems are social-technical systems meaning that they are composed of a technical layer (electrical and mechanical components and their interconnections) and a social layer working together [7]. The social layer consists of the end users, who put physical constraints on the technical layer, and the human operators, who change the structure of the technical layer and manage the production levels to control the power flow. At the control design, the behavior of the social layer is partially unknown (although to some extent predictable by the historical data and the regulations). Second, several companies produce varying levels of power based on the prices and the public demand. As a consequence, a varying set of generators (thermal, wind, hydro, etc) at each time instant provide the power needed across the network. These companies might be unwilling to share their information about their own production capacities and local network as it might compromise the company's financial benefits by giving tactical advantages to other companies in the energy generation market. Third, power networks consist of many nonlinear components, although it is common to design linear controllers with acceptable closed-loop performance based on linearized models. These controllers are functions of the linearized model and, in turn, functions of their operating points. Finally, safety constraints must be satisfied at all time instances to protect the electrical equipments and end users from harm due to faulty conditions or other hazardous situations. Therefore, safety switches automatically connect or disconnect electrical components or transmission lines (to meet these safety requirements). The switches change the topology of the network and the transmission lines impedances.

Due to the complexities mentioned above and because power networks are implemented over a vast geographical area (even across multiple countries), it is difficult, if not impossible, to gather all the model information (e.g., entire network topology, line impedances, and operating conditions) at one place. Even if one could gather all the information, the controller based on that information necessarily needs to be very complex. This motivates our interest in designing local controllers based on only local model information.

1.1.2 Transportation Systems

Consider the Swedish road network in Figure 1.2. According to Statistics Sweden[2], in December 31, 2006, there were 4 202 463 passenger cars registered in Sweden, which is 461 vehicles per 1000 inhabitants [9]. In addition, there were 479 794

[2] Statistics Sweden is an administrative agency aimed at supplying customers with statistics for decision making and research. For more information, visit their webpage http://www.scb.se/.

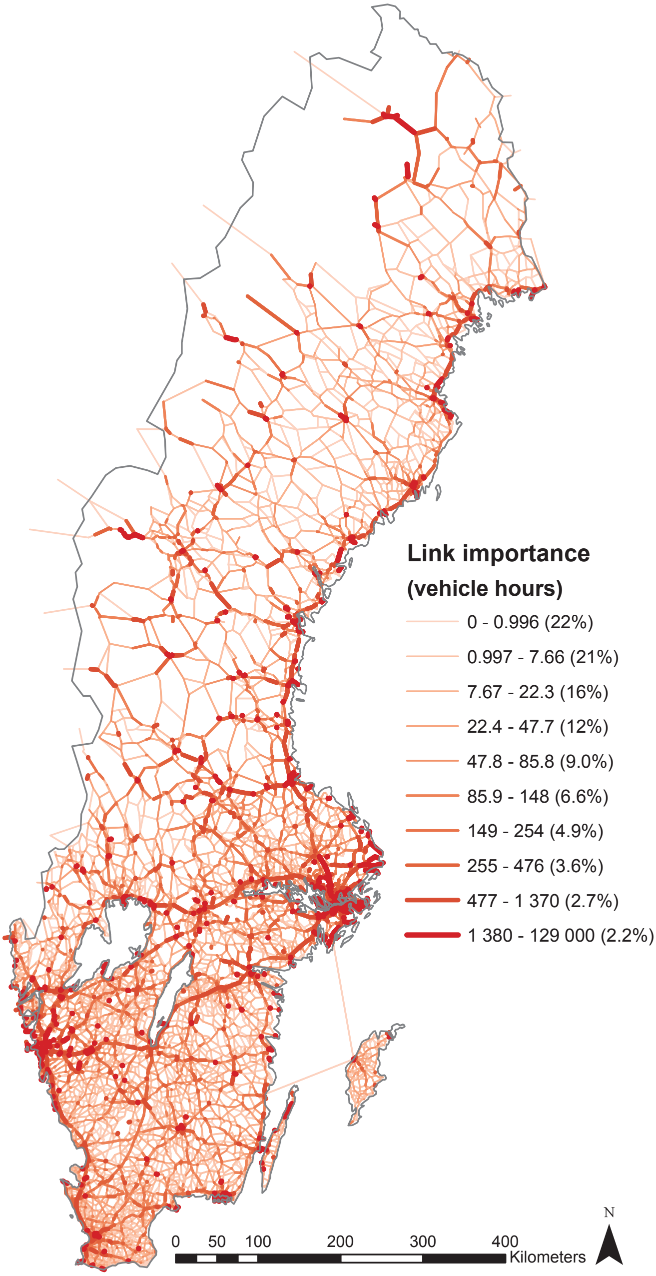

Figure 1.2: Road network in Sweden. Each road is color-coded according to its importance (a measure which is closely related to the number of vehicles using it). The figure is provided courtesy of Erik Jenelius; see [8] for more information.

Figure 1.3: Heavy-duty vehicles can form platoons to reduce the air drag coefficient and thereby improve their fuel efficiency. Picture provided courtesy of Scania http://www.scania.com/.

registered lorries[3] and 13 363 buses [9]. The Swedish road network together with mainly these millions of vehicles form a large complex dynamic system with severe resource constraints and almost no centralized control.

Traffic congestion creates many problems, such as increased transportation delays and fuel consumption, air pollution, and dampened economic growth in heavily congested areas [10–12]. A recent study [12] shows that the transportation has contributed to approximately 15% of the total man-made carbon-dioxide since preindustrial era and suggests that it will be responsible for roughly 16% of the carbon-emission over the next century. In addition to these environmental and economical issues, there are also a high number of injuries and deaths associated with the use of motor vehicles. Transport Analysis[4] details that during 2012, a total of 16 458 road traffic accidents involving personal injury (including fatal, severe, and slight injury) were reported by the Swedish police. They caused the death of 285 individuals [13].

To circumvent some of the problems with traffic congestion, the local governments in some urban areas introduced congestion taxes. For instance, Stockholm

[3]Petroleum tankers, trucks, vans, tractors, and other means of carrying goods (e.g., special tankers for transporting dairy products, water, and chemicals).

[4]Transport Analysis is a government agency in Sweden with the aim of providing decision makers with relevant policy advice and statistics in transportation. For more information, visit their webpage http://www.trafa.se/en/.

implemented a congestion taxing system in August, 2007 after a seven-month trial period in 2006. A survey of the influence of the congestion taxes over the trial period can be found in [14], which shows significant improvements in travel times as well as favorable economic and environmental effects. Behavioral aspects and other influences of the Stockholm congestion taxing system is discussed in [15–18].

Intelligent transportation solutions, such as vehicle-to-vehicle communication, dynamic toll administration, and commercial fleet management [10, 19], can be employed for reducing the fuel consumption in conjunction with improving the road safety. One way to improve the fuel efficiency is vehicle platooning (see Figure 1.3), as vehicles experience a reduced air drag when they travel in platoons [20–24]. Heavy-duty vehicles can significantly improve their fuel efficiency by platooning. In [20], the authors report 4.7%-7.7% reduction in the fuel consumption (depending on the distance between the vehicles among other factors) when two identical trucks platoon close together at 70 km/h. In addition to improving the fuel efficiency, platooning is suggested to reduce the road fatalities by around 10% [25, 26].

The problem of coordinating heavy-duty vehicle platoons can be decomposed into three main layers [27]. At the top layer, we have transport planning and route optimization to determine the vehicle routes and their timing along the route. At the middle layer, we have road planning and road segment optimization, which decides for instance about the platoon velocity. At the lowest layer, we have platoon coordination in which decisions are made on merging with other platoons, splitting platoons, and changing the order of the vehicles in a platoon. This layer also handles real-time inter-vehicle control and vehicle cruise control in which the vehicles communicate state measurements and other information to regulate the distance between vehicles. Optimizing these layers to achieve decreased fuel consumption and increased safety is a challenging task. Let us discuss this challenge in some detail for two specific platooning layers.

First, we focus on the transport planning and the route optimization. Consider a future scenario when all heavy-duty vehicles are equipped with platooning equipments. The number of vehicles that need to be coordinated is then enormous and, typically, they are geographically scattered across large areas. Therefore, gathering all the required information at one place is a time-consuming and complex task. Even assuming that this information can be gathered in a single place, a global decision-maker might become extremely complex to implement and execute. In addition, heavy-duty vehicles often belong to competing entities. These entities may wish not to share their private information with a central decision maker due to privacy constraints enforced by their clients or because of the fact that the released information might give a competitive advantage to other companies. Hence, it would be interesting to study if a desirable behavior, such as using the road at the same time or choosing the same path among alternative routes, can emerge from simple local strategies, such as appropriate monetary (e.g., taxing or subsidy) policies.

Second, let us consider the real-time inter-vehicle control of the layered platoon architecture. The control design might be constrained by that each vehicle should only rely on the parameters of its own vehicle due to several reasons. For instance, it might be the case that the controller of each vehicle should be fixed. Arguably, safety constraints might be a motive for this as time-varying controllers may result in behaviors harder to predict. Furthermore, the local controller of each vehicle cannot be designed based on the model information of all possible vehicles it may cooperate with in future traffic scenarios. Finally, the vehicle parameters (e.g., its mass) might not be available to other trucks because these vehicles may belong to other competing entities. In this case, it is interesting to see if the fleet owners still can guarantee a reasonable bound on the closed-loop performance of the platoon in terms of reduced fuel consumption.

1.2 Challenges

As illustrated by the motivating applications, it is often the case that when regulating a large-scale system composed of several interconnected subsystems, one needs to adopt a decentralized control architecture. In addition, when designing each local controller, we may not have access to the full model information. For instance, it might be desirable that each local controller is a function of only local parameters, so that it does not need to be modified if the model parameters of a particular system (that is not in its vicinity) change over time, or due to privacy constraints or other reasons, as discussed previously. This way, we can ensure simple control systems tuning and maintenance, if we are still able to guarantee good closed-loop performance. Hence, it is important to consider decentralized control design under limited model information. One question could be to study how far the best control design with limited model information is from the optimal control design with full model information in terms of the closed-loop performance. This can potentially shed some light on inherent limitations caused by the lack global model information. Another important question could be to study if it is possible to reduce the gap between the best control design with limited model information and the one with full model information through constructing more complex[5] control laws. For instance, when dealing with linear time-invariant systems, the optimal control design strategy with full model information and full state feedback is static; however, this observation may not extend when migrating to limited model information regime. In that case, we need to characterize the "simplest" control design strategy that one should construct to achieve a reasonable performance. We can also study whether it is possible to capture the value of information; i.e., the level of improvement in the closed-loop performance caused by moving from a given information regime to a richer one. Using this notion of value of information, we can understand what parts of the model information are more important when designing a local controller and, hence, we must acquire even at a high cost. This is highly relevant because

[5] We measure complexity in control laws in the sense that nonlinear control laws are more complex than their linear but dynamic counterparts which, in turn, are more complex than static controllers.

we typically have a finite budget (in terms of time, money, or computational resources) in the control design procedure. In all the above mentioned problems, we can consider two different approaches for handling the unknown parts of the model. We can either consider the worst-case possible combination of the parameters or use statistical data (where applicable and, more importantly, available) to remedy the average behavior of the closed-loop system.

Another challenge that we consider in this thesis is strategic decision making by drivers in transportation networks. In these systems, the drivers compete over a common resource (i.e., the road network). The choice of route, departure time, and speed of each driver affects some of the other drivers in the network. We model the drivers' decision making using game theory since they wish to optimize their own costs rather than contributing the social welfare (e.g., the total time wasted in traffic). In our model, we explicitly account for the heterogeneity of the drivers and their vehicles. An interesting questions is to understand whether desirable properties, such as the existence of an equilibrium in which no one can improve her cost by unilaterally changing her decision, can be guaranteed. We can also study how difficult it is to find such an equilibrium. For instance, we can investigate the convergence properties of various decentralized learning dynamics. Another interesting question could be to study if it is possible to encourage the drivers to take socially responsible decisions through appropriate monetary (i.e., taxing or subsidy) policies. Finally, we can also use these setups to better understand the incentives of cooperative driving scenarios, such as heavy-duty vehicle platooning, in transportation networks.

The third challenge we consider is optimal resource allocation for control and estimation of large-scale networked systems. When transmitting sensor measurements in a networked system, such as the power grid, we need to assign time intervals in which each sender transmits its measurement across the shared communication network (e.g., wireless communication network, Internet) to its designated estimation unit. Noting that there are potentially a huge number of sensors employed, we need to efficiently coordinate these sensors to avoid packet collisions and dropouts while maintaining an acceptable sampling rate. Furthermore, communication resources in large networks almost always are varying over time due to the needs from the individual users and physical communication constraints. In many practical networked systems, a varying number of control loops may be active at any given time. Therefore, a very interesting problem could be to design a scheduling policy for ad-hoc networked systems so that it adapts itself to the number of active control loops and their closed-loop performance requirements.

1.3 Illustrative Examples

In this section, we briefly introduce a few numerical examples to demonstrate the main problems considered in the thesis. We revisit these examples in the subsequent chapters and the attached papers to illustrate the developed results.

1.3.1 Power Grid Regulation

Let us consider the power network composed of two generators shown in Figure 1.4 from [28, pp. 64–65], see also [29]. We can model this power network as

$$\begin{aligned}
\dot{\delta}_1(t) &= \omega_1(t),\\
\dot{\omega}_1(t) &= \frac{1}{M_1}\big[(P_1(t)+w_1(t)) - \xi_{12}^{-1}\sin(\delta_1(t)-\delta_2(t)) - \xi_1^{-1}\sin(\delta_1(t)) - D_1\omega_1(t)\big],
\end{aligned}$$

and

$$\begin{aligned}
\dot{\delta}_2(t) &= \omega_2(t),\\
\dot{\omega}_2(t) &= \frac{1}{M_2}\big[(P_2(t)+w_2(t)) - \xi_{12}^{-1}\sin(\delta_2(t)-\delta_1(t)) - \xi_2^{-1}\sin(\delta_2(t)) - D_2\omega_2(t)\big],
\end{aligned}$$

where $\delta_i(t)$, $\omega_i(t)$, $P_i(t)$, and $w_i(t)$ are the phase angle of the terminal voltage, the rotation frequency, the input mechanical power, and the exogenous input of generator i, respectively. We assume that $P_1(t) = P_1^0 + M_1 v_1(t)$ and $P_2(t) = P_2^0 + M_2 v_2(t)$, where $v_1(t)$ and $v_2(t)$ are the continuous-time control inputs of this system, and P_1^0 and P_2^0 are constant references. Now, we can find the equilibrium point (δ_1^*, δ_2^*) of the system and linearize it around this equilibrium. Furthermore, let us discretize the linearized system by applying Euler's constant step scheme with sampling time ΔT, which results in

$$x(k+1) = Ax(k) + Bu(k) + Hw(k),$$

where

$$x(k) = \begin{bmatrix} \Delta\delta_1(k) \\ \Delta\omega_1(k) \\ \Delta\delta_2(k) \\ \Delta\omega_2(k) \end{bmatrix}, \qquad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \qquad w(k) = \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & \Delta T & 0 & 0 \\ \frac{-\Delta T(\xi_{12}^{-1}\cos(\delta_1^*-\delta_2^*)+\xi_1^{-1}\cos(\delta_1^*))}{M_1} & 1-\frac{\Delta T D_1}{M_1} & \frac{\Delta T\cos(\delta_1^*-\delta_2^*)}{\xi_{12}M_1} & 0 \\ 0 & 0 & 1 & \Delta T \\ \frac{\Delta T\cos(\delta_2^*-\delta_1^*)}{\xi_{12}M_2} & 0 & \frac{-\Delta T(\xi_{12}^{-1}\cos(\delta_2^*-\delta_1^*)+\xi_2^{-1}\cos(\delta_2^*))}{M_2} & 1-\frac{\Delta T D_2}{M_2} \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 \\ 1/M_1 & 0 \\ 0 & 0 \\ 0 & 1/M_2 \end{bmatrix}.$$

Here, $\Delta\delta_1(k)$, $\Delta\delta_2(k)$, $\Delta\omega_1(k)$, and $\Delta\omega_2(k)$ denote the deviation of $\delta_1(t)$, $\delta_2(t)$, $\omega_1(t)$, and $\omega_2(t)$ from their equilibrium points at time instances $t = k\Delta T$. Additionally, let the actuators be equipped with a zero order hold unit which corresponds

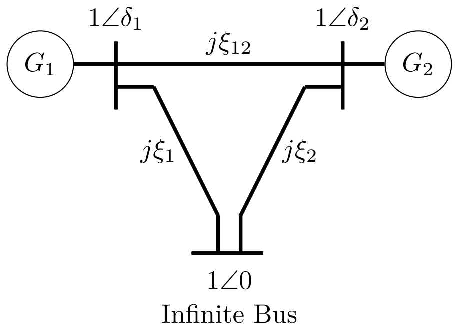

Figure 1.4: Schematic diagram of the power network.

to $v_i(t) = u_i(k)$ for all $k\Delta T \le t < (k+1)\Delta T$. Finally, we use the notation $w_i(k)$ to capture the equivalent influence of $w_i(t)$ over $k\Delta T \le t < (k+1)\Delta T$.

Alternatively, we can consider DC power generators such as solar farms and batteries. Suppose these sources are connected to AC transmission lines through DC/AC converters that are equipped with a droop-controller [30, 31]. Let us assume that both power generators in Figure 1.4 are DC power generators equipped with droop-controlled converters. We can then model this power network as

$$\dot{\delta}_1(t) = \frac{1}{D_1}\big[(P_1(t) + w_1(t)) - \xi_{12}^{-1}\sin(\delta_1(t) - \delta_2(t)) - \xi_1^{-1}\sin(\delta_1(t)) - D_1\omega_1(t)\big],$$
$$\dot{\delta}_2(t) = \frac{1}{D_2}\big[(P_2(t) + w_2(t)) - \xi_{12}^{-1}\sin(\delta_2(t) - \delta_1(t)) - \xi_2^{-1}\sin(\delta_2(t)) - D_2\omega_2(t)\big],$$

where $\delta_i(t)$, $1/D_i > 0$, and $P_i(t)$ are respectively the phase angle of the terminal voltage of converter i, its converter droop-slope, and its input power. Now, we can find the equilibrium point of this nonlinear system and linearize it around this equilibrium, which results in

$$x(k+1) = Ax(k) + Bu(k) + Hw(k),$$

where

$$x(k) = \begin{bmatrix} \Delta\delta_1(k) \\ \Delta\delta_2(k) \end{bmatrix}, \qquad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \qquad w(k) = \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix},$$

$$A = \begin{bmatrix} \frac{-\Delta T(\xi_{12}^{-1}\cos(\delta_1^* - \delta_2^*) + \xi_1^{-1}\cos(\delta_1^*))}{D_1} & \frac{\Delta T\cos(\delta_1^* - \delta_2^*)}{\xi_{12}D_1} \\ \frac{\Delta T\cos(\delta_2^* - \delta_1^*)}{\xi_{12}D_2} & \frac{-\Delta T(\xi_{12}^{-1}\cos(\delta_2^* + \delta_1^*) - \xi_2^{-1}\cos(\delta_2^*))}{D_2} \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1/D_1 & 0 \\ 0 & 1/D_2 \end{bmatrix}.$$

We are interested in the optimal control of this power network. Whenever we restrict our considerations to linear time-invariant controllers, the closed-loop performance measure is given by

$$J = \|T_{yw}(\mathfrak{z})\|_2^2,$$

where $T_{yw}(\mathfrak{z})$ denotes the closed-loop transfer function from the exogenous input $w(k)$ to output vector $y(k) = [x(k)^\top \, u(k)^\top]^\top$ in which $\mathfrak{z}$ is the symbol for the one time-step forward shift operator. Through minimizing such a cost function, we guarantee that the frequency of the generators stays close to its nominal value (e.g, 50 Hz in Sweden) without wasting too much energy. For the design of nonlinear controllers, we consider the cost function

$$J = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} x(k)^\top x(k) + u(k)^\top u(k).$$

This cost function is equal to the H_2-norm of the closed-loop transfer function for linear time-invariant systems excited by exogenous inputs that are elements of a sequence of independently and identically distributed Gaussian random variables with zero mean and unit covariance.

Let us assume that the impedance of the lines that connect each generator to the infinite bus in Figure 1.4 varies over time. We define α_i, $i = 1, 2$, as the deviation of the admittance ξ_i^{-1} from its nominal value. Notice that α_i only appears in the model of subsystem i. When designing the control laws, we assume that the information regarding the value of parameter α_i is only available in the design of the controller for subsystem i. One motivation for this can be that the generators are physically far apart from each other.

1.3.2 Heating, Ventilation, and Air Conditioning Systems

Let us consider the problem of regulating the temperature in N rooms on the 2nd floor of the Electrical Engineering building at KTH (see Figure 1.5). Let us, for the sake of simplicity, assume that each room can be heated by a single actuator. The corridors and stairways are supposed to have the ambient temperature $\bar{x}_a$ which may be assumed to be constant. Let us denote the average temperature of room i by $\bar{x}_i$. By applying Euler's constant step discretization scheme to the continuous-time model (both in time and space), we obtain the following difference equation

$$\bar{x}_i(k+1) = \sum_{j\neq i} \alpha_{ij}(\bar{x}_j(k) - \bar{x}_i(k)) + \beta_i(\bar{x}_a - \bar{x}_i(k)) + u_i(k), \tag{1.1}$$

where β_i and α_{ij} are constants representing the average heat loss rates of room i to the ambient and to room j, respectively. The goal is to regulate the temperature

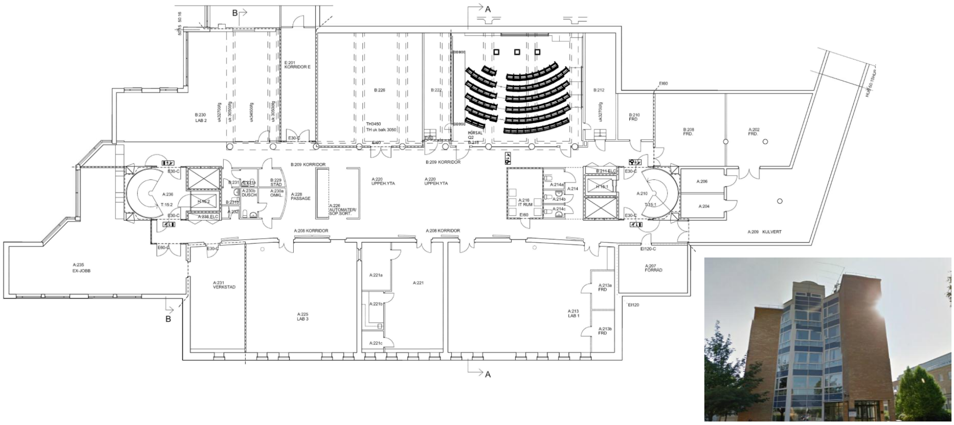

Figure 1.5: The architecture plan of the 2nd floor of Electrical Engineering building at KTH. Provided courtesy of Akademiska Hus http://www.akademiskahus.se/.

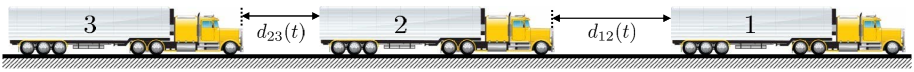

Figure 1.6: Regulating the distance between three trucks.

of each room at a prescribed value by minimizing the performance criterion

$$J = \sum_{k=0}^{\infty} \sum_{i=1}^{N} (\bar{x}_i(k) - r_i)^2 + (u_i(k) - u_i^*)^2, \tag{1.2}$$

where r_i, for each i, is the reference temperature of room i, and u_i^*, for each i, is the steady-state control signal of room i.

The characteristics of each room (such as opening doors and windows, place of furniture, etc) influence its model parameters $\{\beta_i\} \cup \{\alpha_{ij} \mid j \neq i\}$. Sometimes it could be desirable to let the controller of each room not depend on the parameters of other rooms. Another interesting problem here could be to propose a scheduling policy for the sensors in each room to communicate their measurements to the neighboring rooms as well as to a central estimation unit. Certainly, this scheduling policy should be able to adapt itself to the number of control loops that are active at any given time because not all the rooms are occupied with people at all times.

1.3.3 Heavy-Duty Vehicle Platooning

Consider a physical example where three trucks are following each other closely in a platoon (see Figure 1.6). Each truck can be modeled as a continuous-time linear system described by

$$\begin{bmatrix} \dot{x}_i(t) \\ \dot{v}_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\varrho_i/m_i \end{bmatrix} \begin{bmatrix} x_i(t) \\ v_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_i/m_i \end{bmatrix} u_i(t) + \begin{bmatrix} w_{i1}(t) \\ w_{i2}(t) \end{bmatrix},$$

where $v_i(t)$, $x_i(t)$, and $u_i(t)$ denote the velocity, the position, and the control input (i.e., the acceleration) of truck i, respectively. In addition, $w_{i1}(t)$ and $w_{i2}(t)$ are the exogenous inputs to truck i (i.e., the effect of wind, road quality, friction, etc). Finally, ϱ_i is the viscous drag coefficient of vehicle i and b_i is its power conversion quality coefficient. These parameters are all scaled by the maximum allowable mass of each vehicle. Let us define $d_{ij}(t)$ as the distance between vehicles i and j (see Figure 1.6). Now, we can model the whole platoon as

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) + w(t),$$

where

$$x(t) = \begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_{12}(t) \\ w_{11}(t) - w_{21}(t) \\ w_{22}(t) \\ w_{21}(t) - w_{31}(t) \\ w_{32}(t) \end{bmatrix},$$

and

$$A(\alpha)=\begin{bmatrix} -\varrho_1/m_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho_2/m_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\varrho_3/m_3 \end{bmatrix}, \quad B(\alpha)=\begin{bmatrix} b_1/m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2/m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_3/m_3 \end{bmatrix}.$$

In this example, we assume $\alpha = [m_1 \; m_2 \; m_3]^\top \in \mathbb{R}^3$ is the vector of parameters with m_i denoting the mass of vehicle i (scaled by its maximum allowable mass). We define the state of each subsystem as

$$x_1(t) = \begin{bmatrix} v_1(t) \\ d_{12}(t) \end{bmatrix}, \quad x_2(t) = v_2(t), \quad x_3(t) = \begin{bmatrix} d_{23}(t) \\ v_3(t) \end{bmatrix}.$$

For safety reasons, we want to ensure that the exogenous inputs do not significantly influence the distances between the vehicles. However, we would like to guarantee this fact using as little control action as possible. We capture this goal by minimizing the H_∞-norm of the closed-loop transfer function from the exogenous inputs $w(t)$ to

$$z(t) = \begin{bmatrix} d_{12}(t) & d_{23}(t) & u_1(t) & u_2(t) & u_3(t) \end{bmatrix}^\top.$$

Therefore,

$$J = \|T_{zw}(s)\|_\infty,$$

Figure 1.7: The dashed black curve shows the segment of northbound E4 highway between Lilla Essingen and Fredhällstunneln in Stockholm.

where $T_{zw}(s)$ denotes the closed-loop transfer function from $w(t)$ to $z(t)$ in which s is the symbol for the Laplace transform variable. For practical reasons, it could be desirable to let the controller of each vehicle not depend on the model parameters of the other vehicles. It is interesting to understand what limitations such privacy constraint put on the achievable closed-loop performance of the overall platoon.

1.3.4 Decision Making in Transportation Systems

In this subsection, we model the traffic flow at various time intervals of the day on the segment of northbound E4 highway between Lilla Essingen and Fredhällstunneln in Stockholm (see Figure 1.7) using an atomic congestion game. Let us divide the time window of interest into $R \in \mathbb{N}$ non-overlapping intervals and denote each interval by r_i for $1 \leq i \leq R$. The set of all these intervals is denoted by $\mathcal{R} = \{r_1, r_2, \dots, r_R\}$. Here, we assume there are two types of agents, namely, cars and trucks. Let $z = \{z_i\}_{i=1}^N$ and $x = \{x_i\}_{i=1}^M$ denote the actions of N cars and M trucks that are participating in the congestion game. Now, we describe the utilities of these players.

Car i, $1 \leq i \leq N$, maximizes its utility given by

$$U_i(z_i, z_{-i}, x) = \xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) + v_{z_i}(z, x) + p_i^{\mathrm{c}}(z, x),$$

where the mapping $\xi_i^{\mathrm{c}} : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ describes the penalty for deviating from the preferred time interval for using the road denoted by $T_i^{\mathrm{c}} \in \mathcal{R}$ (e.g., due to being late for work or delivering goods), $v_{z_i}(z, x)$ is the average velocity of the traffic flow at time interval z_i, and $p_i^{\mathrm{c}}(z, x)$ is a potential congestion tax for using the road on a specific time interval. The choice of the penalty mappings ξ_i^{c}, $1 \leq i \leq N$, can capture various models of cars. For instance, we can use $\xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) = \alpha_i^{\mathrm{c}}|z_i - T_i^{\mathrm{c}}|$, with flexibility parameter $\alpha_i^{\mathrm{c}} < 0$, to describe the case where the driver of car i is penalized symmetrically by deviating from its preferred time interval T_i^{c}. Following [32–34], we assume that the average velocity at time interval $r \in \mathcal{R}$ is an affine function of the total number of vehicles (both cars and trucks) that are using the road at that time interval

$$n_r(z, x) = \sum_{\ell=1}^{N} \mathbf{1}_{\{z_\ell = r\}} + \sum_{\ell=1}^{M} \mathbf{1}_{\{x_\ell = r\}}.$$

We use real traffic data from sensors on this stretch of highway to extract reasonable parameters for modeling the average velocity at any time interval as a function of the total number of vehicles that are using the road at that time interval. The measurements are extracted during October 1–15, 2012. Figure 1.8 shows the average velocity of the flow as a function of the number of vehicles. As we can see, for up to 1000 vehicles, a linear relationship $v_r(z, x) = a n_r(z, x) + b$ with $a = -0.0110$ and $b = 84.9696$ describes the data well. However, for higher numbers of vehicles, it fails to capture the behavior of around 20% of the data (shown by the red dots in Figure 1.8). Some of these outlier measurements can be caused by traffic accidents, sudden weather changes during the day, or temporary road constructions.

In the congestion game, truck j, $1 \leq j \leq M$, maximizes its utility given by

$$V_j(x_j, x_{-j}, z) = \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) + v_{x_j}(z, x) + p_i^{\mathrm{t}}(z, x) + \beta v_{x_j}(z, x) m_{x_j}(x),$$

where, similar to the utilities of the cars, $\xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}})$ is the penalty for deviating from its preferred time T_j^{t} for using the road, $v_{x_j}(z, x)$ is the average velocity of the traffic flow, and $p_i^{\mathrm{t}}(z, x)$ is a potential congestion tax for using the road at time interval x_j. Trucks have an extra term $\beta v_{x_j}(z, x) m_{x_j}(x)$ in their utility because of their benefit in using the road at the same time as the other trucks in which $m_{x_j}(x)$ denotes the number of trucks that are using the road at time interval $x_j \in \mathcal{R}$. The increased utility can be justified by the fact that whenever there are many trucks on the road at the same time interval, they can potentially collaborate to form platoons and thereby increase the fuel efficiency. Note that this extra utility is a function of the average velocity of the flow since trucks cannot save a significant amount of fuel through platooning at low velocities [20, 27].

It is interesting to find the equilibria of this strategic game. In particular, we study decentralized learning dynamics that do not use the knowledge of the utilities of other vehicles.

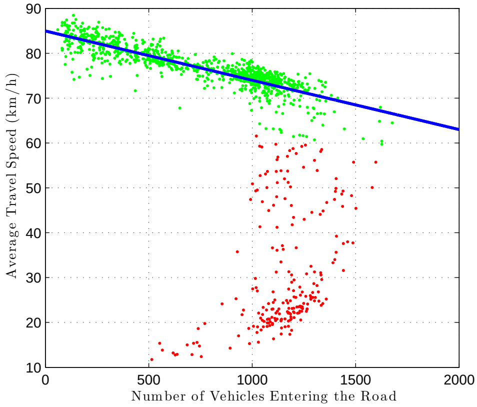

Figure 1.8: Average velocity of the traffic flow as a function of the number of vehicles that are entering the segment of northbound E4 highway between Lilla Essingen and Fredhällstunneln for 15 min time intervals.

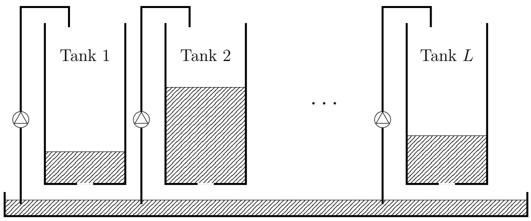

Figure 1.9: An example of a networked system with water tanks composed of decoupled scalar subsystems.

1.3.5 Water Tank Regulation

Consider a networked system composed of L decoupled water tanks illustrated in Figure 1.9, where each tank is linearized about its stationary water level h_ℓ as

$$\mathrm{d}z_\ell(t) = -\frac{a_\ell}{a'_\ell}\sqrt{\frac{g}{2h_\ell}}\, z_\ell(t)\mathrm{d}t + \mathrm{d}w_\ell(t);\ z_\ell(0) = z_\ell^0. \tag{1.3}$$

The exogenous inputs $\{w_\ell(t)\}_{t\in\mathbb{R}_{\geq 0}}$, $1 \leq \ell \leq L$, are statistically independent Wiener processes with zero mean. They represent input flow fluctuations and other disturbances. In this model, a'_ℓ is the cross-section of water tank ℓ, a_ℓ is the cross-section of its outlet hole, and g is the acceleration of gravity. Furthermore, $z_\ell(t) \in \mathbb{R}$ denotes the deviation of the tank's water level from its stationary point.

It is interesting to develop an optimal scheduling policy to sample the water levels in these tanks and transmit these measurements to their respective estimation units over a shared communication medium. Certainly, it is preferable to construct a scheduling policy to deal with ad-hoc networked systems since a varying number of water tanks may be utilized at any given time. In addition, since these water tanks may work around slowly-varying stationary levels, their linearized models change over time. Therefore, the designer may want to only rely on local model information for designing the controller to avoid redesigning the whole controller whenever a single parameter changes in the system.

1.4 Thesis Outline

This thesis is a compilation thesis. In the remainder of this chapter, we discuss the organization of the chapters and the papers.

First, we present the introductory material. Specifically, Chapter 2 gives a review of the pre-existing literature on cooperative and competitive decision making with limited information. We particularly focus on networked control systems and strategic decision making in transportation systems. In Chapter 3, we discuss the contributions of the thesis in control design with limited model information, strategic decision making in transportation networks, and stochastic sensor scheduling with application to networked control systems. We present the conclusions and possible directions for future research in Chapter 4.

Part 1: Control Design with Limited Model Information

The first part of the thesis consists of six papers on optimal control design with limited plant model information. In what follows, we briefly discuss these papers.

Paper 1: Optimal Structured Static State-Feedback Control Design with Limited Model Information for Fully-Actuated Systems

In this paper, we introduce the family of limited model information control design methods, which construct controllers by accessing the plant's model in a constrained way, according to a given design graph. We investigate the closed-loop performance achievable by such control design methods for fully-actuated discrete-time linear time-invariant systems, under a separable quadratic cost. We restrict our study to control design methods which produce structured static state feedback controllers, where each subcontroller can at least access the state measurements of those subsystems that affect its corresponding subsystem. We compute the optimal control design strategy (in terms of the competitive ratio and domination metrics) when the control designer has access to the local model information and the global interconnection structure of the plant-to-be-controlled. Finally, we study the trade-off between the amount of model information exploited by a control design method and the best closed-loop performance (in terms of the competitive ratio) of controllers it can produce. This paper is published as:

F. Farokhi, C. Langbort, K. H. Johansson, "Optimal Structured Static State-Feedback Control Design with Limited Model Information for Fully-Actuated Systems," *Automatica*, vol. 49, no. 2, pp. 326–337, 2013.

A preliminary version of the paper was presented as:

F. Farokhi, C. Langbort, K. H. Johansson, "Control Design with Limited Model Information," in *Proceedings of the American Control Conference*, pp. 4697–4704, 2011.

Paper 2: Dynamic Control Design Based on Limited Model Information

The design of optimal H_2 dynamic controllers for interconnected linear systems under limited plant model information is considered in this paper. An explicit minimizer of the competitive ratio is found. It is shown that this control design strategy is not dominated by any other strategy with the same amount of model information. The result applies to a wide class of system interconnections, controller structures, and design information. This paper was presented as:

F. Farokhi, K. H. Johansson, "Dynamic Control Design Based on Limited Model Information," in *Proceedings of the 49th Annual Allerton Conference on Communication, Control, and Computing*, pp. 1576–1583, 2011.

Paper 3: Decentralized Disturbance Accommodation with Limited Plant Model Information

The optimal control design for disturbance accommodation with limited model information is considered in this paper. As it is shown in Papers 1 and 2, when it comes to designing optimal centralized or partially structured decentralized state-feedback controllers with limited model information, the best control design strategy (in terms of competitive ratio and domination) is static. This is true even though the optimal partially structured decentralized state-feedback controller with full model information is dynamic. In this paper, we show that, in contrast, the best limited model information control design strategy for the disturbance accommodation problem gives a dynamic controller. We find an explicit minimizer of the competitive ratio and we show that it is undominated. This optimal controller can be separated into a static feedback law and a dynamic disturbance observer. This paper was published as:

F. Farokhi, C. Langbort, K. H. Johansson, "Decentralized Disturbance Accommodation with Limited Plant Model Information," *SIAM Journal on Control and Optimization*, vol. 51, no. 2, pp. 1543–1573, 2013.

An early version of this paper was presented as:

F. Farokhi, C. Langbort, K. H. Johansson, "Optimal Disturbance-Accommodation with Limited Model Information," in *Proceedings of the American Control Conference*, pp. 4757–4764, 2012.

Paper 4: Optimal Control Design under Structured Model Information Limitation Using Adaptive Algorithms

In this paper, we show that with an adaptive networked controller with limited plant model information, it is indeed possible to achieve a competitive ratio equal to one. Therefore, by migrating from the set of linear control laws studied in Papers 1-3 to the set of nonlinear control laws, we can drastically reduce the competitive ratio. The plant model considered in the paper belongs to a compact set of stochastic linear time-invariant systems and the closed loop performance measure is the ergodic mean of a quadratic function of the state and control input. This paper is under review for journal publication as:

F. Farokhi, K. H. Johansson, "Optimal Control Design under Structured Model Information Limitation Using Adaptive Algorithms," Submitted.

Paper 5: Optimal H_∞ Control Design under Model Information Limitations and State Measurement Constraints

In this paper, we present a numerical algorithm for constructing control design strategies that rely on limited model information and partial state measurements. The algorithm is based on successive local minimizations and maximizations (using the subgradients) of the H_∞–norm of the closed-loop transfer function with respect to the controller gains and the system parameters. This paper was recently presented as:

F. Farokhi, H. Sandberg, K. H. Johansson, "Optimal H_∞ Control Design under Model Information Limitations and State Measurement Constraints," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 6218–6225, 2013.

Paper 6: Optimal Control Design under Limited Model Information for Discrete-Time Linear Systems with Stochastically-Varying Parameters

Here, we design optimal state-feedback controllers for interconnected discrete-time linear systems with stochastically-varying parameters. The design of each controller relies only on exact local plant model information and statistical beliefs about the model of the rest of the system. Therefore, as opposed to Papers 1-4, in this formulation, we may use statistical beliefs about the model of other systems. For quadratic cost functions, we show that the optimal controller is a linear function of the state measurements. Furthermore, we study the value of model information in optimal control design using the performance degradation ratio (a concept related to that of competitive ratio) which is defined as the supremum (over all possible initial conditions) of the ratio of the cost of the optimal controller with limited model information scaled by the cost of the optimal controller with full model information. This paper is submitted for journal publication as:

> F. Farokhi, K. H. Johansson, "Optimal Control Design under Limited Model Information for Discrete-Time Linear Systems with Stochastically-Varying Parameters," Submitted.

A preliminary version of this paper was presented as:

> F. Farokhi, K. H. Johansson, "Limited Model Information Control Design for Linear Discrete-Time Systems with Stochastic Parameters," in *Proceedings of the 51st IEEE Conference on Decision and Control*, pp. 855–861, 2012.

Part 2: Strategic Decision Making in Transportation Systems

The second part of the thesis consists of two papers on strategic decision making in transportation systems. In what follows, we present a brief recapitulation of these papers.

Paper 7: When Do Potential Functions Exist in Heterogeneous Routing Games?

We study a heterogeneous routing game in which vehicles might belong to more than one type. The type determines the cost of traveling along an edge as a function of the flow of various types of vehicles over that edge. We extend the available results to present necessary and sufficient conditions for the existence of a potential function. We characterize a set of tolls that guarantee the existence of a potential function when only two types of users are participating in the game. We present an upper bound for the price of anarchy (i.e., the worst-case ratio of the social cost calculated for a Nash equilibrium over the social cost for a socially optimal flow) for the case in which only two types of players are participating in a game with affine edge cost functions. This paper is a technical report:

> F. Farokhi, W. Krichene, A. M. Bayen, and K. H. Johansson, "When Do Potential Functions Exist in Heterogeneous Routing Games?," Technical Report TRITA-EE 2014:009, 2014.

A preliminary version of this paper was presented as:

> F. Farokhi, W. Krichene, A. M. Bayen, and K. H. Johansson, "A Heterogeneous Routing Game," in *Proceedings of the Annual Allerton Conference on Communication, Control, and Computing*, pp. 448–455, 2013.

Paper 8: A Study of Truck Platooning Incentives Using a Congestion Game

Another aspect of transportation systems in the time at which vehicles decide to use the transportation network. In this paper, we introduce an atomic congestion game with two types of agents, cars and trucks, to model the traffic flow on a road over various time intervals of the day. Cars maximize their utility by finding a trade-off between the time they choose to use the road, the average velocity of the flow at that time, and the dynamic congestion tax that they pay for using the road. In addition to these terms, the trucks have an incentive for using the road at the same time as their peers because they have platooning capabilities, which allow them to save fuel. The dynamics and equilibria of this game-theoretic model for the interaction between car traffic and truck platooning incentives are investigated. We use traffic data from Stockholm to validate parts of the modeling assumptions and extract reasonable parameters for the simulations. We use joint strategy fictitious play and average strategy fictitious play to learn a pure strategy Nash equilibrium of this game. We perform a comprehensive simulation study to understand the influence of various factors, such as the drivers' value of time and the percentage of the trucks that are equipped with platooning devices, on the properties of the Nash equilibrium. This paper is submitted for journal publication as:

> F. Farokhi, K. H. Johansson, "A Study of Truck Platooning Incentives Using a Congestion Game," Submitted.

Preliminary versions of this result may be found in:

> F. Farokhi, K. H. Johansson, "A Game-Theoretic Framework for Studying Truck Platooning Incentives," in *Proceedings of the 16th International IEEE Annual Conference on Intelligent Transportation Systems*, pp. 1253–1260, 2013.

> F. Farokhi, K. H. Johansson, "Investigating the Interaction Between Traffic Flow and Vehicle Platooning Using a Congestion Game," Accepted for Presentation at the 19th World Congress of the International Federation of Automatic Control (IFAC), 2014.

The first conference paper focuses on motivating the modeling assumptions and to extract appropriate simulation parameters using real traffic data from Stockholm while the second conference paper studies the problem from a theoretical perspective to show the existence of a pure strategy Nash equilibrium and to prove the convergence of the learning algorithms.

Part 3: Stochastic Sensor Scheduling

The last part of the thesis consists of one paper on stochastic sensor scheduling with application to networked control and estimation.

Paper 9: Stochastic Sensor Scheduling for Networked Control Systems

Here, we model sensor measurement and transmission instances using jumps between states of a continuous-time Markov chain. We introduce a cost function for this Markov chain as the summation of terms depending on the average sampling frequencies of the subsystems and the effort needed for changing the parameters of the underlying Markov chain. By minimizing this cost function, we extract an optimal scheduling policy to fairly allocate the network resources among the control loops. We study the statistical properties of this scheduling policy in order to compute upper bounds for the closed-loop performance of the networked system, where several decoupled scalar subsystems are connected to their corresponding estimator or controller through a shared communication medium. This paper is recently published as:

> F. Farokhi, K. H. Johansson, "Stochastic Sensor Scheduling for Networked Control Systems," *IEEE Transactions on Automatic Control*, 2014. To Appear.

A preliminary version of this paper was presented in a conference as:

> F. Farokhi, K. H. Johansson, "Stochastic Sensor Scheduling with Application to Networked Control," in *Proceedings of the American Control Conference*, pp. 2325–2332, 2013.

Other Publications

The following articles were published during the course of my studies; however, they are not included in the thesis.

> F. Farokhi, A. M. H. Teixeira, C. Langbort, "Gaussian Cheap Talk Game with Quadratic Cost Functions: When Herding Between Strategic Senders is a Virtue," Accepted for Presentation at the American Control Conference, 2014.

> T. Tanaka, F. Farokhi and C. Langbort, "Faithful Implementations of Distributed Algorithms and Control Laws," Submitted.

> T. Tanaka, F. Farokhi and C. Langbort, "A Faithful Distributed Implementation of Dual Decomposition and Average Consensus Algorithms," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 2985–2990, 2013.

> F. Farokhi, H. Sandberg, K. H. Johansson, "Complexity reduction for parameter-dependent linear systems," in *Proceedings of the American Control Conference*, pp. 2618–2624, 2013.

F. Farokhi, A. Shirazinia, K. H. Johansson, “Networked Estimation Using Sparsifying Basis Prediction,” in *Proceedings of the 4th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, pp. 174–181, 2013.

F. Farokhi, I. Shames, and K. H. Johansson, “Distributed MPC Via Dual Decomposition and Alternating Direction Method of Multipliers,” in J. M. Maestre and R. R. Negenborn, (Eds.), *Distributed Distributed Model Predictive Control Made Easy*, Intelligent Systems, Control and Automation: Science and Engineering, 69, Springer, 2013.

M. Larsson, J. Lindberg, J. Lycke, K. Hansson, A. Khakulov, E. Ringh, F. Svensson, I. Tjernberg, A. Alam, J. Araujo, F. Farokhi, E. Ghadimi, A. Teixeira, D. V. Dimarogonas, and K. H. Johansson, “Towards an Indoor Testbed for Mobile Networked Control Systems,” in *Proceedings of the 1st Workshop on Research, Development, and Education on Unmanned Aerial Systems*, pp. 51–60, 2011.

F. Farokhi, H. Sandberg, “A Robust Control-Design Method Using Bode’s Ideal Transfer Function,” in *Proceedings of the Mediterranean Conference on Control and Automation*, pp. 712–717, 2011.

Contribution by the Author

The order of the authors’ names reflect the work load of the paper where the first author has the most important contributions. In all the listed publications, all the authors were actively involved in developing the results and in writing the final paper.

CHAPTER 2

Background

> "Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information on it."
>
> Samuel Johnson[1] (1709–1784)

In this thesis, we focus on the effects of information asymmetries and limitations in decision making. To discuss decision making with limited information, one must first define and quantify information[2]. The theory of knowledge dates back to Greek philosophers, such as Plato and Aristotle [35, 36]. However, it was not until much later that Descartes, Berkeley, and Hume, among many other prominent philosophers, started to formally study the problem [37–39]. Later, the term epistemology was coined by Ferrier to describe a branch of philosophy concerned with knowledge [40]. There are also many mathematical approaches for modeling information or knowledge [41]. The idea of quantifying information has attracted much attention to the point of creating the field of "information theory" in mathematics, communication theory, and signal processing [42, 43].

A common approach to model the information available to an agent (e.g., human beings in a market) is to partition the space of all the possible scenarios (i.e., state space) and, upon realization of one state, the agent can only know that an element of the partition, to which the realized state belongs, has occurred [44–47]. In this framework, an agent has more (detailed) information if her partition becomes finer. The agents can also form beliefs (i.e., probability distributions on the state space) and update these beliefs based on their observations [41, 48]. In information theory, the uncertainty in capturing a random variable can be quantified using the concept of entropy [43]. Another approach is to use semantics to model the information available to each agent [49, 50]. Using these definitions, several studies

[1] Quoted in "Life of Samuel Johnson" by James Boswell, 1791.

[2] Different academic disciplines use different terminologies to refer to what we call information in this thesis. For instance, economists and philosophers use the term knowledge instead. In this chapter, we will use these two terms interchangeably when reviewing results from the literature.

have developed results in decision making with limited information. In the rest of this chapter, we discuss some of them.

Specifically, in Section 2.1, we review decision making with limited information in both cooperative and competitive settings. In Section 2.2, we focus on networked control and estimation as an application of cooperative decision making with limited information. In Section 2.3, we review parts of the literature on routing and congestion games as applications of competitive decision making with limited information.

2.1 Decision Making with Limited Information

The problem of decision making with incomplete information and the value of information is a well-studied problem in economics[3] and computer science [53–57]. For instance, in [54], Arrow studied the degradation in economic decisions caused by the lack of information and communication between both competing and cooperating agents. He also gave an estimate of the value of information in a network using this degradation factor. In [55, 56], the value of information in distributed algorithmic decision making was studied. The value of information was captured using the competitive ratio [58, 59], which was defined based on the so-called regret ratio in economics [53]. We can study distributed decision making in both cooperative or competitive settings, that is, the agents that are involved in determining a decision may have aligned performance criteria or conflicting ones. We review these two settings in the remainder of this section.

2.1.1 Cooperative Decision Making

Cooperative decision making with limited information arises naturally in many scenarios where agents must make decisions in a system in which they influence each others through dynamics or performance criteria; however, due to several reasons, such as communication constraints or memory limitations, they cannot share all their information or process all the available information. Examples of this problem can be found in optimal coordination of multi-agent systems and decentralized control of interconnected systems [60–66]. A key feature of cooperative decision making is that although the agents have different information regarding the system, the cost function that they are trying to optimize is the same.

First, it might be case that the agents prefer to calculate the outcome in a distributed fashion to improve the robustness of the overall system because if they

[3]Arguably, economic decision making with limited information is historically as old as the classic economics literature itself. In 1776, Adam Smith, a prominent philosopher and the pioneer of political economy, discussed competitive markets as means to dynamically solve large-scale resource allocation problems with only local information about demand and production levels [51]. There is much controversy regarding effectiveness, practicality, and stability of the considered economic system [52]; however, one certainly cannot argue with Smith's genius in recognizing the issues present because of the lack of global information in the system.

designate a node *a priori* to make such decisions and transmit all their information to that node, there is a chance that the designated node might fail (e.g., due to a hardware malfunction) or get hijacked (by malicious agents to change the collective decision of the group). Another reason could also be that each agent does not have the necessary computation capabilities to solve the problem on its own and, hence, they require to collaborate in order to solve the problem. These reasons motivate a large body of studies in distributed consensus-seeking, synchronization, and formation acquisition [60, 61, 67, 68]. These studies started with [69] in which Reynolds examined bird flocks to understand how local decision rules may result in complex global behaviors. Later, in [70], a dynamical model was introduced to study the coordination of particles and it was observed that if the agents update their heading to be equal to the average of their neighbors, all of them eventually move in the same direction. A theoretical proof for this observation was later presented in [66]. Following these studies, several authors proposed synchronization and consensus-seeking algorithms using both discrete-time and continuous-time models [60, 71, 72]. An extension of these results to time-varying communication links as well as asynchronous communications was presented in [61, 73–76]. These results had applications in vehicle formation [77, 78], attitude alignment [79, 80], rendezvous operations [81], synchronization of oscillators [82, 83], and flocking [71, 84], among many other problems. For a comprehensive survey of the applications of consensus algorithms, see [85]. As mentioned earlier, the lack of necessary computational resources can also be a motivation for decision making with limited information. The lack of computational resources in decision making has inspired the research in distributed computation and optimization [86–89]. For instance, decomposition algorithms for large-scale programming were introduced in [90–92]. Later, dual decomposition was introduced and explored as a powerful method for solving large-scale optimization problems [93, 94]. Recently, the idea of alternating direction method of multipliers (based on earlier studies in [95, 96]) was revived and applied in distributed optimization [88]. A survey of various decomposition methods can be found in [88, 89, 97, 98].

Another reason for decision making with limited information might be privacy constraints. This reason has motivated studies in distributed optimization with privacy constraints [99–102]. For instance, the authors in [101] studied the standard linear programming problem when each agent just knows a subset of the coefficients that appear in the constraints. They motivated this problem using distributed decision making in network management (see also [103, 104]), distributed task assignment problem, and organization theory. The problem was generalized to dynamic cases (i.e., multistage optimization problems) in [105, 106]. Since these studies are the origin of the definitions of competitive ratio and domination for decentralized control design with limited plant model information in this thesis, we review them in detail in Subsection 2.2.4.

In addition to the above mentioned reasons, constraints in communication infrastructure may also force the agents to only rely on information that they can measure directly. Decentralized control design is mainly motivated by such communication restrictions [62–64, 107]. We review the available results in distributed and decentralized control in Subsection 2.2.3.

2.1.2 Competitive Decision Making

Competitive decision making, opposed to the cooperative one studied in the previous subsection, appears when agents belong to competing entities. Examples can be found in economic and financial systems, such as inventory management, supply chains, and competitive markets [108–111]. An important ingredient of these results is game theory. Game theory has a rich history dating back to 1713 when de Montmort introduced an example of mixed strategy using Waldegrave's problem[4] in probability [112]. Much later, in 1838, Cournot considered a duopoly problem and presented a solution concept for it [113]. In this study, an equilibrium concept was introduced that is closely related to the Nash equilibrium (i.e., a set of decisions for which no agent can improve her cost/utility by unilaterally changing her decision). However, game theory, as we know it today, did not exist until it was formalized in 1928 with the celebrated result of von Neumann [114] and his subsequent book with Morgenstern in 1944 [115]. The concept of mixed strategy Nash equilibrium for zero-sum games was introduced in [115]. This notion was generalized to arbitrary games with a finite number of players by Nash himself [116]. Many studies also considered the existence of an equilibrium [116, 117]. In setups with many decision makers, the idea of common knowledge (an event that everyone knows about, everyone knows that everyone knows about it, and so on) was introduced and formally characterized (see [48, 50, 118, 119]) to better understand decision making in a distributed setup[5]. The introduced studies typically focused on games with complete information, that is, all the players' payoff functions and parameters were common knowledge. In 1967, Harsanyi introduced the notion of games with incomplete information (i.e., Bayesian games) and formulated the Bayesian Nash equilibrium in a series of papers [120–122]. For more information about games with complete and incomplete information, see [123].

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[4]Waldegrave's problem was originally discussed in a letter from Pierre Rémond de Montmort to Nicolas Bernoulli on April 10, 1711 (see [112, p. 318-320] with its translation provided courtesy of `http://cerebro.xu.edu/math/Sources/`) and was called "Probléme de la Poule" (or Problem of Pool) as

> "... This same Geometer [Mr. Waldegrave] who is a Gentleman of much intellect, has proposed to me lately & has resolved a quite pleasing Problem which is here. Pierre, Paul & Jacques play a pool at Trictrac or at Piquet. After one has deduced whom will play it is found that Pierre and Paul begin. We demand, 1°. what is the advantage of Jacques. 2°. How great are the odds that Pierre or Paul will win rather than Jacques. 3°. How many games must the pool naturally endure. [sic]"

[5]For instance, the knowledge that green traffic light means that the drivers may pass through the intersection should be common knowledge between the drivers in a society otherwise they cannot make any decision in a distributed manner regarding how to pass an intersection [50]. Common knowledge is a prerequisite for achieving agreement in any setup.

It is known that Nash equilibria in strategic[6] games are typically not socially optimal [124]. To quantify this inefficiency, the notion of price of anarchy (i.e., the worst-case ratio of the social cost function calculated for a Nash equilibrium over the social cost function calculated for a socially optimal decision) was first introduced in [125, 126]. Later, this criterion was utilized in various games [127–131]. In parallel, the theory of competitive analysis of distributed algorithms was used to compare the cost of a distributed on-line algorithm to the cost of an optimal distributed algorithm [132]. The authors of [133, 134] discussed a setting in which several agents jointly solve a coordination game and studied the value of information in these games. Many related measures, such as price of stability, information, cooperation, and fairness, were later introduced and studied [127, 135, 136].

Motivated by the above mentioned results, we know that agents (that are employed to make a decision in a distributed manner) do not blindly follow instructions to optimize a social cost function. Therefore, we need to incentivize them using appropriate monetary schemes to follow the intended algorithms. Mechanism design theory[7] is concerned with how to create and enforce a socially preferable decision in the presence of strategic agents [137–140]. A direct revelation mechanism design problem considers how the leader can design a decision-making process, possibly with a side payment mechanism, so that followers are incentivised to tell the truth. A central positive result about this problem is the celebrated Vickrey–Clarke–Groves (VCG) mechanisms (in which the tax imposed on an agent is equal to its marginal contribution to the rest of the society) to encourage truthful reports [140–143]. However, there are also several negative results in mechanism design theory. For instance, a leader can only implement trivial decision rules (in an appropriate sense) without introducing monetary policies [140]. Furthermore, generally, there does not exist any mechanism except the VCG mechanisms in order to make truth-telling a dominant strategy [144]. VCG mechanisms require the computation of optimal social decisions in the central node (based on the reported information by the agents). A recent study in [145] proposed a framework for distributed implementation of VCG mechanisms; however, these results may not be easily used in any context since, in general, VCG mechanisms combined with an approximated solution can destroy incentive compatibility (i.e., cannot guarantee that truth-telling is a dominant strategy) [146]. Generalization of mechanism design to dynamic situations may be found in [147, 148].

A natural question in game theory is whether it is possible to propose dynamics to actually determine an equilibrium (e.g., Nash equilibrium, correlated equilibrium, etc). For instance, in 1951, Brown introduced the idea of fictitious play for learning Nash equilibria in which each player plays its best response to the empirical frequency of the observed actions [149]. In [150], it was shown that if fictitious

[6]Sometimes, the term strategic is employed to emphasize the fact that the agents are strategically optimizing their own cost or utility (which are conflicting with each other).

[7]Mechanism design is sometimes referred to, albeit informally, as reverse game theory since in mechanism design, one typically wants to design a game (based on reverse engineering) that has an equilibrium with appropriate properties.

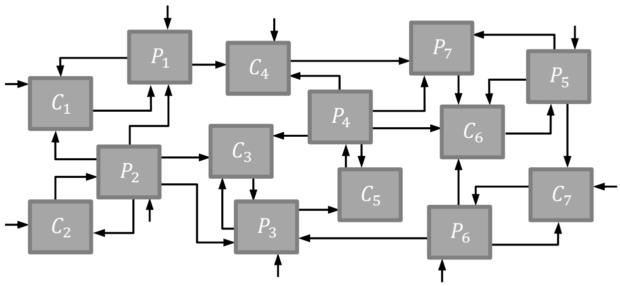

Figure 2.1: Illustrative example of a networked control system.

play converges to any distribution, those probabilities correspond to an equilibrium of the game; however, the fictitious play may not converge in general [151]. Stronger convergence results for fictitious play exits in the case of zero-sum two-player games [152] and potential games (i.e, the games for which there exists a potential function, such that its variation when only one agent changes her action is equal to the variation of the utility of the corresponding agent) [153]. Other variants of fictitious play were subsequently proposed and studied [32, 154]. There exist many other learning algorithms in game theory such as Nash dynamics [155, 156], reinforcement learning [157], no regret learning [158, 159], and trial and error learning [160].

2.2 Networked Control and Estimation

According to [65], a networked control system is "a spatially distributed system in which sensors, actuators, and controllers are connected to each other through a band-limited digital communication network". Figure 2.1 illustrates an example of a networked control system which is composed of several controllers C_i and systems P_i connected to each other through a communication network. The network topology defines how sensors communicate with controllers and how controllers send their commands to the corresponding actuators.

Networked control systems have several characteristics. First, these systems are typically distributed geographically over a vast area (as the power grid application in Subsection 1.1.1). It is natural to assume that a given subsystem can only influence a subset of neighboring subsystems (due to the geographical constraints). Therefore, the geographical profile of the system and its underlying physical characteristics dictate the interconnection pattern between subsystems. In many situations, the interconnections of the subsystems are fixed (and given) in advance. This property of large-scale control systems has attracted a lot of attention and many have studied the generic properties of structured systems. We take a deeper look into structured systems in Subsection 2.2.1.

Second, any communication medium brings limitations such as band-limited channels, sampling, quantization, variable delays, and packet drop-outs. A realistic communication network has band-limited channels, that is, it can only relay a limited amount of data per unit of time. Therefore, it might not make sense to assume that each subcontroller has access to the full state measurements of the plant. Note that even if each channel has high bandwidth, the point-to-point capacity of a large multi-hop network can still be very limited [161]. We first review some results in scheduling algorithms for communication networks in Subsection 2.2.2. The absence of full state information gives rise to several challenges in designing stabilizing and optimal controllers which we discuss in Subsection 2.2.3.

Finally, in large-scale dynamical systems, it may be extremely difficult to identify all system parameters and update them globally. One can only hope that the designer knows the local parameter variations and update the corresponding subcontroller based on them. This fact motivates optimal control design with limited model information. We briefly review the related literature in Subsection 2.2.4.

2.2.1 Generic Properties of Structured Systems

The study of structured systems dates back almost four decades [162–166]. In [162], the author first introduced the definition that a pair of matrices (A, B) is structurally controllable if there exists a controllable pair of matrices (A', B') with the same structure as (A, B). A structurally controllable system can be shown to be controllable for almost all parameter combinations, except for a measure-zero set that might occur when the system parameters satisfy certain equality constraints [162–164]. Thus, structural controllability helps the designer to overcome the inherently incomplete knowledge of the system parameters. There exist graph-theoretic conditions for verifying structured controllability [162]. A set of algebraic conditions has been presented in [163, 165] to check structured controllability. It is interesting to note that, as structured controllability gives controllability of a continuum of linearized systems, the aforementioned results may also provide a sufficient condition for controllability of many nonlinear systems [167–169].

Many classical control results have been generalized to structured systems. For instance, the problem of input–output decoupling of structured systems was discussed in [170–172]. The problem of disturbance rejection and disturbance decoupling was addressed initially in [173–175]. Decentralized control of structured systems was considered in [176–179]. For instance, the authors of [176] presented necessary and sufficient conditions for controllability under a decentralized information structure. In [177], the authors studied geometric properties of structured systems using graph-theoretic tools. They also obtained graph-theoretic conditions to determine stabilizability of structured interconnected systems via decentralized feedback control. Decentralized stabilization and pole placement of structured system were discussed in [180]. Parts of these results were generalized to descriptor systems in [181]. More related studies can be found in a recent survey of structured systems and their generic properties [182]. There has also been some work on fault detection and isolation for structured systems. For instance, in [183], the authors provided necessary and sufficient graph-theoretic conditions under which the fault detection and isolation problem has a solution. Later, the sensor location problem for fault diagnosis in structured systems was discussed in [184]. Recently, a necessary and sufficient graph-theoretic condition for the existence of vulnerabilities that are inherent to the power network interconnection structures was developed in [185].

2.2.2 Scheduling Algorithms

In many practical scenarios, several agents are competing or collaborating on a shared medium to provide or to receive services, such as on power grids, economic markets, or communication networks. This problem has attracted much attention from various research communities, such as computer sciences, economics, and industrial and communication engineering [186–189]. For instance, in networked control systems, communication resources need to be efficiently shared between multiple control loops in order to guarantee a good closed-loop performance under communication constraints; e.g., bit-rate constraints [190–193] and packet loss [194–197]. The authors of [189] proposed a scheduler to allocate time slots between several users over a long horizon. In that scheduler, the designer must first manually assign shares (of a communication medium or processing unit) that an individual user should receive. Then, each user achieves its pre-assigned share by means of probabilistic or deterministic algorithms [189, 198]. The authors in [186, 199] proved that implementing the task with the earliest deadline achieves the optimum latency in case of both synchronous and asynchronous job arrivals. In [200], a scheduling policy based on static priority assignment to the tasks was introduced. Many studies in the communication literature have also considered the problem of developing protocols in order to avoid the interference between several information sources when using a common communication medium. Examples of such protocols are both time-division and frequency-division multiple access [201, 202]. As a continuation of these studies, several authors have focused on proposing distributed algorithms for solving resource allocation problems (i.e., determining time shares for communication in time-division multiple access protocols or bandwidth assignment in frequency-division multiple access) [203–207]. There have been studies in stochastic sensor scheduling algorithms. For instance, in [208], the authors developed a stochastic sensor scheduling policy using discrete-time Markov chains.

2.2.3 Distributed and Decentralized Control Design

Band-limited channels in a networked control system force us to design distributed and decentralized controllers as subcontrollers in the overall system that may access only a strict subset of the state measurements. Distributed and decentralized control and estimation in large-scale and networked systems is a well-studied problem [64, 209–211].

There is a huge body of literature on stabilizing decentralized systems. For instance, the authors of [63, 212–214] showed that the absence of so-called fixed modes is a necessary and sufficient condition for stabilizability of a linear time-invariant dynamical system with a time-invariant decentralized controller. Later, this result was extended to show that a time-varying controller might be able to eliminate the fixed modes that are not structurally fixed modes and, as a result, a linear time-invariant dynamical system could be stabilized with a decentralized controller even when fixed modes are present [215, 216]. Fixed modes may also be eliminated with vibrational control or sampling techniques [217–219]. It was also shown that if a fixed mode cannot be eliminated by a decentralized periodically time-varying controller, then it cannot be eliminated by any decentralized controller [220, 221].

There are contributions in multi-agent systems related to distributed control, such as a Nyquist-like condition for stability of a formation using the individual plant transfer function and the Laplacian of the graph describing the network topology [77]. This work was generalized to the stability of multi-input multi-output dynamical systems with arbitrary dynamical interconnection between the subsystems with fixed interaction topology [222]. The coordination of a group of autonomous agents when the graph topology changes over time was considered in [61, 66]. These works were generalized to a framework for stability analysis of interconnected systems where the topology can potentially be time-varying [78]. The authors of [223] presented an algorithm for designing controllers that preserve the stability of the closed-loop system under any interconnection and communication typology.

There has been a great effort in designing optimal distributed and decentralized controllers. Witsenhausen showed that, in general, a linear controller is not optimal for a quadratic performance criterion with a linear time-invariant system subject to Gaussian noise under the distributed information constraint and that the cost function is not necessarily convex in the controller variables [224]. The authors of [225, 226] established that the discrete-time version of the Witsenhausen counter-example is NP-complete. There have been some efforts also to identify the cases in which a linear solution is optimal. For instance, Witsenhausen identified some cases where the resulting optimal controllers are linear [227]. The authors of [228] showed that under a partially nested information pattern the optimal controller is a linear controller. It was shown in [229] that the optimal controller is linear if each subcontroller has access to all the previously implemented control values and observations made by any other subsystem in the system before the current time and its own observations including the current time. There were some studies under the spatial invariance assumption [62, 230]. Some other control structures were shown to result in optimal linear controllers [231, 232]. In [233], the author presented a solution to the optimal decentralized state-feedback control design problem for partially nested information structure. Recently, it was shown that under quadratic invariance and internal quadratic invariance information patterns, one can formulate structured H_∞- and H_2-optimal control design as convex optimization problems [234–237]. This formulation resulted in an explicit solution for the problem of designing decentralized H_2-optimal controllers for a special class of systems [238–242]. Also using partially ordered sets, the authors of [243–245] introduced an explicit solution to the decentralized state-feedback H_2-optimal control design problem for some classes of plant interconnection and information structure. The problem of designing optimal distributed controllers was recently approached using team decision theory in [246, 247]. That work was further generalized to solve the stochastic linear quadratic control problem under power constraints [248]. In that work, the output-feedback problem was also considered. Later, the team decision theory was used to develop optimal distributed H_∞-optimal controllers when each subsystems has access to the state measurements and control signals of those subsystems that can affect it [249].

There have been studies on designing optimal controllers for positive systems with more general structures. For instance, the authors of [250, 251] gave a necessary and sufficient condition for existence of a diagonal Lyapunov function for positive systems. They also showed that, in this case, the H_∞-optimal control design problem can be written as a convex optimization problem (and, therefore, it is computationally tractable). Later, the author of [252] proved that H_∞- and ℓ_1-norms of transfer functions are equal for single-input single-output positive discrete-time linear time-invariant systems. It was also shown that the problem of designing an optimal controller for these systems can be written as a convex optimization problem under some conditions on the controller structure.

There have been studies on sub-optimal distributed and decentralized control design because, as it was mentioned earlier, the problem of synthesizing optimal controllers for arbitrary information patterns is computationally expensive. The authors of [253] considered the problem of designing sub-optimal static and fixed-order dynamic structured compensators. Some approaches were based on gradient descent, Newton, and quasi-Newton algorithms [254–259]. A set of sufficient linear matrix inequalities for finding distributed controllers was presented in [260]. In [261], the authors presented an algorithm for designing a near-optimal decentralized controller that replicates the behavior of the optimal centralized controller. The problem of near-optimal decentralized output regulation of hierarchical systems subject to disturbances was studied in [262]. In [263, 264], the problem of designing an optimal decentralized state-feedback controller was solved on a finite-horizon using dynamic programming. In those papers, the authors provided both a computationally intensive optimal solution and a sub-optimal solution that is more computationally tractable. A receding horizon approach to develop a sub-optimal controller was considered in [265, 266]. A recent result was introduced in [267] using decomposition methods in distributed optimization accompanied with a special stopping criteria to synthesize a sub-optimal controller with closed-loop performance guarantees.

2.2.4 Limited Model Information Control Design

The problem of designing controllers using uncertain plant model information is a classical topic in control theory [268–273]. In robust control design, the goal is to design a controller such that some level of performance of the controlled system is guaranteed irrespective of changes in the plant dynamics within a predefined bound around a given nominal global model. This is different from designing an optimal controller without a global model since in optimal control design with limited model information, subsystems do not have any prior information about the other subsystems' model (i.e., there are no nominal models for the subsystems in the design procedure) and there are no, *a priori* known, bounds on the model uncertainties. In addition, in control design with limited model information, the uncertainty sets are different from the perspective of each designer since the design of each local controller is done based on different parts of the model information.

There have been some interesting approaches for tackling the limited model information control design problem, although they are not specifically tailored for it. For instance, references [274–277] introduced methods for designing sub-optimal decentralized controllers without a global dynamical model of the system. In these papers, the authors assume that the plant consists of an interconnection of weakly coupled subsystems. They designed an optimal controller for each subsystem using only the corresponding local model and connect the obtained subcontrollers to construct a global controller. They showed that, when the coupling is negligible, this latter controller is satisfactory in terms of closed-loop stability and performance. However, as coupling strength increases, even closed-loop stability guarantees are lost. The motivation behind their studies was to design fully-decentralized near-optimal controllers for large-scale dynamical systems and to avoid numerical complications, stemming from the high dimension of the system, by splitting the original problem into several smaller ones. The idea of "decentralized design" for systems that may be decomposed into several weakly coupled subsystems was further investigated in [278, 279]. Other approaches, such as [4, 266], are based on receding horizon control and use decomposition methods to solve each step's optimization problem in a decentralized manner with only limited information exchange between subsystems.

The problem of designing an optimal controller with limited model information, in the setup that we are considering in this thesis, was first approached in [105, 106]. In these papers, the authors introduced control design strategies as mappings from the set of plants of interest to the set of eligible controllers. They investigated the quality of the controllers that these control design strategies construct. This quality was measured by a quadratic closed-loop performance criterion. They introduced the competitive ratio as a performance metric and the domination as a partial order on the set of limited model information control design strategies to study the intrinsic limitations of limited model information control design strategies. Previously, there were no other metrics specifically proposed for control design strategies. The authors defined the competitive ratio as the worst case ratio of the cost of a control design strategy to the cost of the optimal control design with full model information. They worked with communication-less control design strategies as an extreme family of limited model information control design strategies that only rely on each subsystem model for designing the corresponding subcontroller. They used the term communication-less to illustrate the fact that different parts of these control design strategies do not exchange model information (and, equivalently, do not communicate) with each other. The subsystems were assumed to be scalar. Under these assumptions, it was proven that, when dealing with continuous-time linear time-invariant dynamical systems, the competitive ratio of any control design strategy is always unbounded. Thus, they focused on discrete-time linear time-invariant systems and found an explicit minimizer of the competitive ratio over the set of limited model information control design strategies. Since this minimizer might not be unique, they also proved that it is undominated, that is, there is no other control design method that acts always better while having the same worst-case ratio. This undominated minimizer of the competitive ratio was shown to be the deadbeat control design strategy. Towards the end, they briefly studied the amount of information needed to find a control design strategy with a lower competitive ratio than the deadbeat control design strategy or to dominate it.

2.3 Congestion and Routing Games

A problem that has attracted much attention is modeling the traffic flow in transportation systems and communication networks using congestion games or routing games [280–286]. Rosenthal [283] presented a noncooperative game in which a finite number of players compete for the use of a finite set of resources with application to transport networks. He showed that a class of these games admit at least one pure strategy Nash equilibrium. However, later, in [287], Rosenthal showed that this result may not be generalized to arbitrary weighted multicommodity congestion games. When cost functions (e.g., latency) of each road are affine in the number of vehicles that use it, an equilibrium certainly exists [288]. Later, the authors of [153] showed that atomic congestion games are indeed potential games under some conditions and, hence, one can find a Nash equilibrium by minimizing the potential function. A class of congestion games that are not in general potential games were studied in [289]. For a survey of these (and related) results, see [290]. Most of these studies modeled the route selection using an atomic congestion game; however, recently, the authors of [32] utilized a congestion game for modeling the time when drivers use a road.

In the context of transportation networks, routing games were originally studied in [286]. This study also formulated the definition of an equilibrium in routing games. Researchers in different academic communities use different names for the equilibrium such as user-optimizing flow [291, 292], Wardrop equilibrium [292–294], Wardrop first principle [293], and Nash equilibrium [131, 295]. The term Wardrop equilibrium is common in transportation literature due to the pioneering work of [286] as well as the fact that the term pure strategy Nash equilibrium is primarily utilized in the context of games with finitely many players [294]. It is vital to note that the definition of Nash equilibrium in [131, 295] is indeed different from that of [294], which shows that by increasing the number of users (in a game with finitely many players), the Nash equilibrium converges to the Wardrop equilibrium under appropriate assumptions. Later, in [296], it was shown that under some mild conditions, the routing game admits a potential function and the minimizers of this potential function are the equilibria of the routing game which guarantees the existence of an equilibrium for these games.

2.3.1 Inefficiency of the Equilibrium

Considering that the Nash equilibria are not efficient in general, we can use the price of anarchy as a measure to determine the degradation caused in the social cost function due to the selfish nature of the agents. The price of anarchy of atomic congestion games with linear latency functions was studied in [297]. This problem was also studied for routing games. For instance, in [130] and [298], an upper bound and a lower bound for the price of anarchy was presented, respectively. The lower bound was called the Pigou bound due to an example presented by Pigou, an influential scientists in welfare economics, in [299]. Later, in [298, 300], it was shown that the Pigou bound is tight, i.e., it can be achieved by a special class of routing games. Due to this inherent inefficiency of the Nash equilibrium, there have been several studies in reducing the inefficiency by imposing tolls on the roads in the transportation network [296, 299, 301, 302] and rerouting a fixed percentage of the flow [303–305]. For instance, in [299], Pigou suggested marginal congestion taxes (i.e., taxes corresponding to the increase in cost of the flow on a road caused by adding one user to that road) in order to guarantee that the socially optimal solution becomes a Nash equilibrium and, hence, eliminating the inefficiency of the equilibrium. This result is very useful when we can impose tolls on all the edges of the network rather than only a subset of them. Later, in [302], it was shown that, when we can only impose tolls on a strict subset of the roads in the network, the problem of computing optimal tolls is NP-hard even for only two commodities and linear latency functions. However, in the same study, the authors presented a polynomial-time algorithm for finding optimal tolls in a single-commodity routing game with linear latency functions over a parallel link network. As mentioned earlier, another approach to reduce the inefficiency is to reroute a fixed percentage of the flow. This approach is known as Stackelberg routing since the problem can be formulated as a Stackelberg game (i.e., a game in which the leader announces her strategy as a function of the actions of the followers in advance and the followers react to it). For instance, in [305], it was proven that computing the optimal Stackelberg strategy is NP-hard for a class of routing games; however, a simple algorithm for achieving a reasonable suboptimal strategy was proposed. For parallel networks and a special category of cost functions, the optimal Stackelberg strategy can be computed efficiently [306].

2.3.2 Heterogeneous Routing Games

An important category of routing and congestion games are heterogeneous games, as many factors may result in different cost functions for different classes of drivers.

For instance, in a transportation network, if we include the fuel consumption of the vehicles in the cost functions, two vehicles (of different types) may experience different costs for using a road even if their travel times are equal. For example, this phenomenon can be caused by the fact that heavy-duty vehicles experience an increased efficiency when a higher number of heavy-duty vehicles are present on the same road (because of a higher possibility of platooning and, therefore, a higher fuel efficiency [20]), while such an increased efficiency may not be true for cars. Drivers may have different sensitivities to the latency under different circumstances or depending on their personality and background. In addition, due to economic advantages, heavy-duty vehicles might be more sensitive to latency in comparison to cars (because they need to deliver their goods at specific times). Finally, the drivers generally react differently to road tolls, e.g., based on the reason of the trip or their socioeconomic background [307].

Heterogeneous routing games have been studied extensively over the past starting with the pioneering works in [291, 308]. In these studies, a routing game with multi-class users was introduced and the definition of equilibrium was given. Furthermore, in [291], the author introduced a sufficient condition for transforming the problem of finding an equilibrium to that of an optimization problem (equivalent to the existence of a potential function [153, 287]). The sufficient condition holds if over each edge, the users of any two types influence each other equally, i.e., the increased cost of a user of the first type due to addition of one more user of the second type is equal to the increased cost of a user of the second type due to addition of one more user of the first type [291]. This condition was considered later in [307] in which it was also noted that satisfaction of this symmetry condition may depend on the units (e.g., time or money) adopted for representing the cost functions when the users' types are determined by their value of time (i.e., a scalar factor that balances the relationship between the latency and the imposed tolls). This result is of special interest since the equilibrium does not change by using different units for the cost functions (if the latency only depends on the sum of the flows of various types over the edge, not the individual flows, and the value of time appears linearly in the cost functions) [309]. Necessary and sufficient conditions for the existence of potential functions in games with finite number of players were recently investigated in [310]; however, these results were not generalized to games with a continuum of players as in heterogeneous routing games. The authors of [292] studied the existence of an equilibrium in heterogeneous routing games even if such a symmetry condition does not hold. In contrast to these articles that assumed a finite set of types to which the users may belong, a wealth of studies also considered the case in which the users may belong to a continuum of types [311, 312]. The problem of finding tolls for general heterogeneous routing games as well as the case in which the types of users is determined by their value of time have been considered extensively [313–318]. For instance, in [313], the problem of determining tolls on each edge or path for heterogeneous routing games was studied. Guarantees were provided for the socially optimal solution (also referred to as system-optimizing flow [291]) to be an equilibrium of the game. However, in that article, the users were assumed to be equally sensitive to the imposed tolls. The problem of finding optimal tolls for routing game in which the users' value of time belong to a continuum was studied in [314].

CHAPTER 3

Contributions

"Science is facts, just as houses are made of stones ... But a pile of stones is not a house and a collection of facts is not necessarily science."

Jules Henri Poincaré, *Science and Hypothesis*[1], 1903

CONTRIBUTIONS of the thesis can be categorized in three parts: Decentralized control design with limited model information, strategic decision making in heterogeneous transportation networks, and stochastic sensor scheduling with application to networked systems. In Section 3.1, a brief recapitulation of our results in designing decentralized control laws using limited model information is presented. In Section 3.2, we review our contributions in strategic decision making in transportation systems. Finally, in Section 3.3, we discuss our results in stochastic sensor scheduling using continuous-time Markov chains.

3.1 Control Design with Limited Model Information

In this section, we discuss our contributions in decentralized control design with limited model information. First, in Subsections 3.1.1 and 3.1.2, we revisit two numerical examples introduced in Chapter 1. Then, in Subsection 3.1.3, we outline the contributions.

3.1.1 Numerical Example: Power Grid Regulation

In this subsection, we revisit the example presented in Subsection 1.3.1, specifically, when controlling DC power generators. At first, we are interested in static control

[1] La science et l'hypothése. Translation of the quote provided in "What constitutes a theoretical contribution?" by David A. Whetten.

laws of the form

$$u(k) = Kx(k),$$

where $K \in \mathcal{K}$ with $\mathcal{K}$ denoting the set of admissible controllers. Let us construct full state feedback controllers and, hence, set $\mathcal{K} = \mathbb{R}^{2\times 2}$. Now, define control design strategies as mappings $\Gamma : \mathcal{A} \to \mathcal{K}$ where $\mathcal{A}$ denotes the set of all plausible parameters[2]. Let us define the deadbeat control design strategy $\Gamma^{\Delta}(\alpha) = -A$, where the dependency to parameters α appears in matrix A. Clearly, Γ^{Δ} is a control design with limited model information, because design of control i only relies on α_i (due to the structure of matrix A). The results of Papers 1 and 2 show that

$$J(\Gamma^{\Delta}(\alpha)) \leq 2J(K^*(\alpha)), \ \forall \alpha \in \mathcal{A},$$

where $K^*(\alpha) \in \mathcal{K}$ is the optimal control design with full model information, that is, for any given $\alpha \in \mathcal{A}$, $J(K^*(\alpha)) \leq J(K)$ for all $K \in \mathcal{K}$. Recalling from Subsection 1.3.1, the cost function $J(\cdot)$ is the H_2-norm of the closed-loop transfer function. Therefore, the deadbeat control design strategy, which uses only local model information, is never worse than twice the optimal controller. However, if we relax the set of admissible controllers $\mathcal{K}$ to also contain adaptive control laws, the results of Paper 4 shows that there exists Γ such that

$$J(\Gamma(\alpha)) = J(K^*(\alpha)), \ \forall \alpha \in \mathcal{A},$$

for all $\alpha \in \mathcal{A}$ except a measure-zero set. Again recalling from Subsection 1.3.1, it is easy to see that the closed-loop system is no longer linear when using a nonlinear (adaptive) control law and, hence, we utilize the ergodic mean of a quadratic function of the state and control input as the cost function $J(\cdot)$ (which coincides with the H_2-norm of the closed-loop transfer function when using linear control laws). This control design strategy follows naturally from generalization of [319] to a setting in which each controller separately uses a maximum-likelihood estimator with regularization term to estimate the global model of the system and, then, applies the optimal control law designed using that estimate.

3.1.2 Numerical Example: Heavy-Duty Vehicle Platooning

In this subsection, we revisit the example presented in Subsection 1.3.3. Let us fix $\varrho_i = 0.1$ and $b_i = 1$ for all $i = 1, 2, 3$. We assume that

$$\mathcal{A} = \{\alpha \in \mathbb{R}^3 \,|\, \alpha_i \in [0.5, 1.0] \text{ for all } i = 1, 2, 3\}.$$

We are interested in static control laws of the form

$$u_i(k) = K_{ii} y_i(k), \quad \forall i = 1, 2, 3,$$

[2]Notice that this definition is slightly different from those of Papers 1 and 2 since in this example only some of the parameters can vary and the rest are fixed. However, irrespective of this difference, the following portion of the results holds.

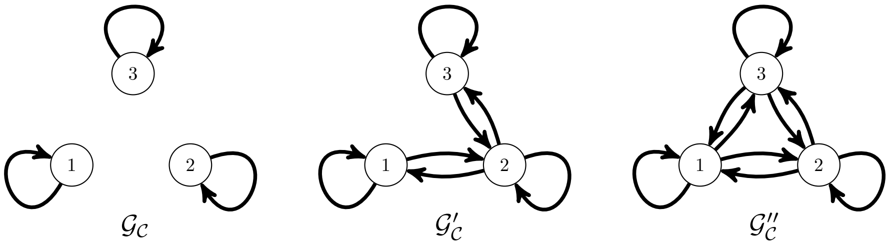

Figure 3.1: The design graphs utilized in the vehicle platooning.

where

$$y_1(t)=\begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \end{bmatrix}, \quad y_2(t)=\begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix}, \quad y_3(t)=\begin{bmatrix} v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix}.$$

Notice that the choice of these particular observation vectors is convenient as the vehicles can measure them directly (using velocity and distance sensors mounted on the front and the back of the vehicles) and they do not need to relay these measurements to each other through a communication medium. Let us again denote the set of admissible controllers by $\mathcal{K}$.

Let a directed graph $\mathcal{G}_\mathcal{C} = (\mathcal{V}_\mathcal{C}, \mathcal{E}_\mathcal{C})$ with vertex set $\mathcal{V}_\mathcal{C} = \{1, 2, 3\}$ and edge set $\mathcal{E}_\mathcal{C} \subseteq \mathcal{V}_\mathcal{C} \times \mathcal{V}_\mathcal{C}$ be given, which we refer to as the design graph. An edge $(i, j) \in \mathcal{E}_\mathcal{C}$ indicates that the parameter of truck i, i.e., its mass in this example, is available in the design of the controller for truck j. We can define a control design strategy as a mapping $\Gamma : \mathcal{A} \to \mathcal{K}$ and use the the notation $\mathcal{C}$ to denote the set of all such mappings that satisfy the pattern of information availability by $\mathcal{G}_\mathcal{C}$. Fix basis functions $\eta_1(\alpha) = 1$, $\eta_2(\alpha) = m_1$, $\eta_3(\alpha) = m_1^2$, $\eta_4(\alpha) = m_2$, $\eta_5(\alpha) = m_2^2$, $\eta_6(\alpha) = m_3$, and $\eta_7(\alpha) = m_3^2$. Now, define the set $\mathcal{C}((\eta_\ell)_{\ell=1}^7)$ as the intersection of the set $\mathcal{C}$ and the set of all mappings constructed by linear combinations of basis functions $(\eta_\ell)_{\ell=1}^7$. Our objective is to find the optimal control design strategy Γ through solving

$$\min_{\Gamma \in \mathcal{C}((\eta_\ell)_{\ell=1}^7)} \max_{\alpha \in \mathcal{A}} \|T_{zw}(s; \Gamma, \alpha)\|_\infty ,$$

where $T_{zw}(s; \Gamma, \alpha)$ denotes the closed-loop transfer function from the exogenous input $w(t)$ to the performance measurement vector $z(t)$ for $\alpha \in \mathcal{A}$. Since solving this problem is difficult in general, we settle for a local solution. We discuss the definition of a local solution and the method for constructing one at length in Paper 5. For now, let us demonstrate the achievable closed-loop performance under various plant model information availability regimes.

We start with the case where each local controller only relies on the mass of its own vehicle. This model information availability corresponds to the design graph $\mathcal{G}_{\mathcal{C}}$ in Figure 3.1. For this case, we get the performance

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^{\text{local}},\alpha\right)\right\|_{\infty} = 4.7905,$$

where Γ^{local} is the suboptimal control design strategy under this model information availability. In the next step, we let the neighboring vehicles communicate their mass to each other. This model information availability corresponds to the design graph $\mathcal{G}'_{\mathcal{C}}$ in Figure 3.1. For this information regime, we get

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^{\text{limited}},\alpha\right)\right\|_{\infty} = 3.5533,$$

where Γ^{limited} is the suboptimal control design strategy under this model information availability. Clearly, we get 25% improvement in comparison to Γ^{local}. Finally, we consider the case where each local controller has access to all the model parameters (i.e., the mass of all other vehicles). This model information availability corresponds to the design graph $\mathcal{G}''_{\mathcal{C}}$ in Figure 3.1. We get

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^{\text{full}},\alpha\right)\right\|_{\infty} = 3.3596,$$

where Γ^{full} the suboptimal control design strategy under this model information availability. It is interesting to note that with access to full model information, we only improve the closed-loop performance by another 5% in comparison to Γ^{limited}. This might be caused by the fact that the first and the third vehicles are not directly interacting.

3.1.3 Contributions

The results presented in Papers 1-6 of this thesis considerably extend the contributions of [105, 106]. Extensions are made by considering several different performance measures and notions of model information. The detailed descriptions of these results are as follows.

In Paper 1, we consider limited model information control design for interconnections of fully-actuated discrete-time linear time-invariant subsystems (of arbitrary order) with a separable quadratic cost function. We investigate the best closed-loop performance achievable by structured static state-feedback controllers constructed using limited model information design strategies. We show that the result depends crucially on the subsystems interconnection pattern and state measurement availability (i.e., the plant graph and the control graph). We extend the fact proven in [105] that the deadbeat strategy is the best limited model information control design method when there is no sink in the plant graph (i.e., a subsystem that cannot affect any other subsystem) and each subcontroller has access to at least the state measurements of those subsystems that affect it. However, the deadbeat control design strategy is dominated when there is a subsystem that could not affect any other subsystem. We find a better, undominated, limited model information control design method, which, although having the same competitive ratio as the deadbeat control design strategy, can achieve a better closed-loop performance in average. We also characterize the amount of model information needed to achieve a better competitive ratio than the deadbeat control design strategy.

In Paper 2, we generalize these results to structured dynamic state-feedback controllers when the closed-loop performance criterion is the H_2-norm of the closed-loop transfer function. Surprisingly, the optimal control design strategy (in the sense of competitive ratio) with limited model information is a static one. This is the case even though the optimal decentralized state-feedback controller with full model information is dynamic itself [239, 240]. We also partially remove the assumption that all the subsystems are fully-actuated and generalize the result for a class of under-actuated systems where the sinks in the plant graph are not required to be fully-actuated.

Later, in Paper 3, we also discuss the design of dynamic controllers for disturbance accommodation. This problem is of special interest because of the fact that the best limited model information control design is a dynamic control design strategy contrary to all previous results in Papers 1-2 where the best limited model information control design strategy was a static one. Interestingly, this dynamic control design strategy can be divided into two parts; a static part which was previously introduced in Papers 1-2 and an observer for canceling the disturbances. For constant disturbances, it is shown that this structure corresponds to proportional-integral control.

Following the observation that we cannot decrease the competitive ratio by migrating from the set of static control laws into dynamic ones, we embark upon designing a special class of nonlinear control laws using limited model information in Paper 4. Specifically, we show that with an adaptive networked controller with limited plant model information, it is indeed possible to achieve a competitive ratio equal to one. To do so, we prove that an adaptive controller exists (constructed by extending the control law introduced in [319]) that asymptotically achieves the closed-loop performance of the optimal centralized controller with full model information. However, in this paper, we consider plants that belong to a compact set of stochastic linear time-invariant systems and the closed loop performance measure is defined to be the ergodic mean of a quadratic function of the state and the control input.

Noticing that not much has been done in optimal control design under limited model information for continuous-time systems and the fact that we have not also presented a systematic approach for constructing optimal control design strategies for general plant model availability in our earlier results, we propose a numerical method for characterizing control design strategies under arbitrary model information limitations and state measurement constraints for parameter-dependent continuous-time systems in Paper 5. The algorithm is based on successive local minimizations and maximizations using the subgradients of H_∞-norm[3] of the closed-loop transfer function with respect to the controller gains and the system parameters.

Up to now, the model information of other subsystems were assumed to be completely unknown which typically results in conservative controllers because it forces the designer to study the worst-case behavior of the control design methods. However, one can sometimes use historical data to construct a probabilistic model for the rest of the subsystems. Therefore, in Paper 6, we design optimal state-feedback controllers for interconnected discrete-time linear systems with stochastically-varying parameters and assume that a statistical model is available for the parameters of the other subsystems. Specifically, we assume that the design of each controller relies only on exact local plant model information and statistical beliefs about the model of the rest of the system. Interestingly, for both finite- and infinite-horizon quadratic cost functions, the optimal controller is shown to be linear in the state. In order to study the value of model information, we also introduce performance degradation ratio (a concept closely related to that of the competitive ratio), which is defined as the supremum (over all possible initial conditions) of the ratio of the cost of the optimal controller with limited model information to the cost of the optimal controller with full model information. Moreover, we calculate an upper bound for it when all the subsystems are fully-actuated.

3.2 Strategic Decision Making in Transportation Systems

In this section, we discuss the problem of decision making in transportation systems. Specifically, in Subsection 3.2.1, we revisit a numerical example from Chapter 1. The contributions are subsequently discussed in Subsection 3.2.2.

3.2.1 Numerical Example: Decision Making in Transportation Systems

Let us revisit the example presented in Subsection 1.3.4. Consider the segment of the highway illustrated in Figure 1.7 from 7:00am to 9:00am on a daily basis. Assume that $N = 10000$ cars and $M = 100$ trucks are using the northbound lanes. We divide the time horizon into eight equal non-overlapping intervals. Hence, we fix the action set as $\mathcal{R} = \{1, \dots, 8\}$, where each number represents an interval of 15 min. Let T_i^{c}, $1 \leq i \leq N$, be randomly chosen from the set $\mathcal{R}$ using the discrete distribution

$$\mathbb{P}\{T_i^{\mathrm{c}} = n\} = \begin{cases} 1/6, & n = 2, 4, \\ 1/4, & n = 3, \\ 1/12, & \text{otherwise.} \end{cases}$$

Let us also use a similar probability distribution to extract T_j^{t}, $1 \leq j \leq M$. Hence, we consider the case where the drivers statistically prefer to use the road at time

[3] The choice of H_∞-norm is not crucial and one can construct a similar algorithm when using H_2-norm of the closed-loop transfer function as the performance measure. In such case, we can use the results of [255, 320] for constructing the gradient of the performance measure.

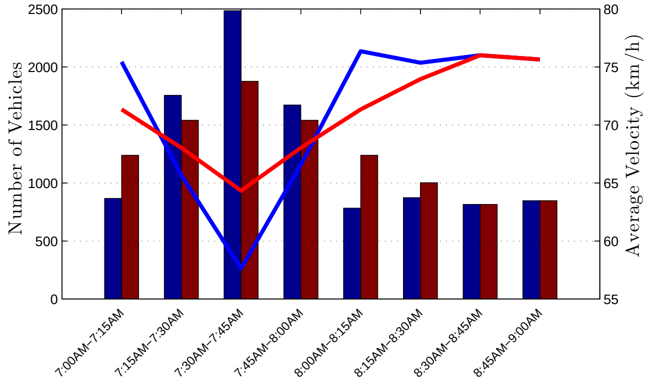

Figure 3.2: Number of the vehicles and the average velocity of the traffic flow in each time interval for the case where the drivers neglect the congestion in their decision making (blue) and for the learned pure strategy Nash equilibrium (red).

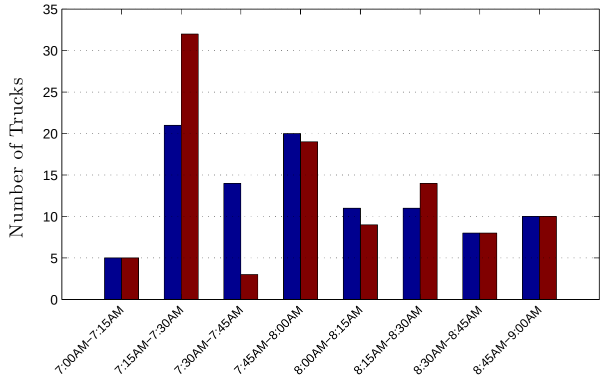

Figure 3.3: Number of the trucks in each time interval for the case where the drivers neglect the congestion in their decision making (blue) and for the learned pure strategy Nash equilibrium (red).

interval $r = 3$ which corresponds to 7:30am to 7:45am. Let α_i^{c}, $1 \leq i \leq N$, and α_j^{t}, $1 \leq j \leq M$, be randomly generated following a uniform distribution within the interval $[-7.5, -2.5]$. Now, to ensure the congestion game admits a potential function, we ask car $1 \leq i \leq N$ and truck $1 \leq j \leq M$ to pay congestion taxes

$$p_i^{\mathrm{c}}(z, x) = a\beta m_{z_i}(x)(m_{z_i}(x) + 1)/2, \tag{3.1a}$$

$$p_j^{\mathrm{c}}(z, x) = 0, \tag{3.1b}$$

for using the road at time interval $z_i \in \mathcal{R}$ and $x_j \in \mathcal{R}$, respectively. Recalling from Subsection 1.3.4, $m_r(x)$ denotes number of trucks that are using time interval $r \in \mathcal{R}$.

Figure 3.2 shows number of the vehicles in each time interval and the corresponding average velocity in that time interval. The blue color denotes the case where the drivers do not consider the congestion in their decision making; i.e., they commute at their best convenience, $z_i = T_i^{\mathrm{c}}$ for all $1 \leq i \leq N$ and $x_j = T_j^{\mathrm{t}}$ for all $1 \leq j \leq M$. The red color denotes the case where the drivers implement a pure strategy Nash equilibrium. As we can see in this figure, the proposed congestion game reduces the average commuting time (increases the average velocity). Figure 3.3 illustrates number of the trucks that are using the road on various time intervals. In contrast to the case where the drivers do not consider the congestion in their decision making, at the Nash equilibrium, thirty two trucks use the time interval 7:15am to 7:30am while most of them avoid using 7:30am to 7:45am because it is highly congested (and they would not save much fuel if they commuted at this time).

3.2.2 Contributions

The contributions of this part are presented in two papers. The detailed descriptions of these results are as follows.

In Paper 7, we study the routing problem in heterogeneous transportation systems. Specifically, we formulate a general heterogeneous routing game in which the vehicles might belong to more than one type. The type determines the cost of traveling along an edge as a function of the flow of all types of vehicles over that edge. This setup has applications in studying the platooning incentives in route selection by heavy-duty vehicles. We extend available results by presenting necessary and sufficient conditions for the existence of a potential function for these games when only two types of vehicles are participating. Under these conditions, we can pose the problem of finding a Nash equilibrium for the heterogeneous routing game as an optimization problem. We characterize a set of tolls that guarantee the existence of a potential function. We present an upper bound for the price of anarchy for the case in which only two types of players are participating in a heterogeneous routing game with affine edge cost functions.

Later, in Paper 8, we study the time at which the drivers decide to the use the road to study heavy-duty vehicle platooning incentives. We introduce an atomic congestion game with two types of agents, cars and trucks, to model the traffic flow on a road over various time intervals of the day. Cars maximize their utility by finding a trade-off between the time they choose to use the road, the average velocity of the flow at that time, and the dynamic congestion tax that they pay for using the road. In addition to these terms, the trucks have an incentive for using the road at the same time as their peers because they have platooning capabilities, which allow them to save fuel. We investigate if a desirable behavior can emerge from simple local strategies such as congestion taxes or subsidies. Specifically, we

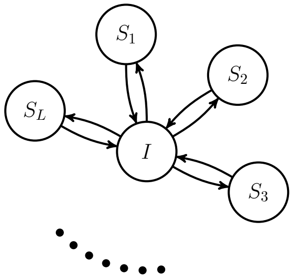

Figure 3.4: Flow diagram of the continuous-time Markov chain used for modeling the proposed stochastic scheduling policy.

propose taxing mechanisms for which the underlying congestion game is a potential game and, then, use joint strategy fictitious play and average strategy fictitious play to learn a pure strategy Nash equilibrium.

3.3 Stochastic Sensor Scheduling

Finally, we discuss our contributions in stochastic sensor scheduling with application to networked control and estimation. We revisit the numerical example on water tank regulation and discuss our contributions in stochastic sensor scheduling in Subsections 3.3.1 and 3.3.2, respectively.

3.3.1 Numerical Example: Water Tank Regulation

Here, we revisit the numerical example introduced in Subsection 1.3.5. Let us consider a networked system that can admit up to $L = 70$ identical subsystems described by (1.3) with $\gamma_\ell = 0.3$ and $\sigma_\ell = 1.0$ for $1 \leq \ell \leq 70$. Let us assume that for $t \in [0, 5)$, only 30 subsystems are active, for $t \in [5, 10)$, all 70 subsystems are active, and finally, for $t \in [10, 15]$, only 10 subsystems are active. Estimator ℓ (of active subsystems) receives state measurements $\{y_i^\ell\}_{i=0}^\infty$ at time instances $\{T_i^\ell\}_{i=0}^\infty$, such that

$$y_i^\ell = z_\ell(T_i^\ell) + n_i^\ell; \qquad \forall i \in \mathbb{Z}_{\geq 0}, \tag{3.2}$$

where $\{n_i^\ell\}_{i=0}^\infty$ denotes the measurement noise sequence, which is composed of independently and identically distributed Gaussian random variables with zero mean and standard deviation $\eta_\ell = 0.3$. Let each subsystem adopt a simple estimator of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_\ell(t) = -\gamma_\ell \hat{z}_\ell(t); \qquad \hat{z}_\ell(T_i^\ell) = y_i^\ell, \tag{3.3}$$

for $t \in [T_i^\ell, T_{i+1}^\ell)$. Define the estimation error $e_\ell(t) = z_\ell(t) - \hat{z}_\ell(t)$.

We use time instances of the jumps between states of this continuous-time Markov chain to model the sampling instances, i.e., whenever there is a jump

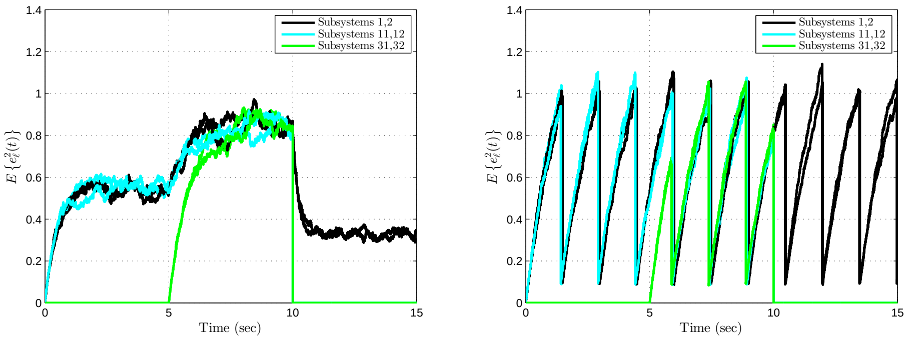

Figure 3.5: Estimation error $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations over an ad-hoc networked system with the stochastic scheduling policy (left) and the periodic scheduling policy (right).

from an idle state in the Markov chain to a state that represents a subsystem in the networked system, we sample that particular subsystem and transmit its state measurement across the shared communication network to the corresponding subcontroller. Figure 3.4 illustrates the flow diagram of the proposed Markov chain. Every time that a jump from the idle node I to node S_ℓ, $1 \leq \ell \leq L$, occurs in this continuous-time Markov chain, we sample subsystem ℓ and send its state measurement to estimator ℓ. The idle state I helps to tune the sampling rates of the subsystems independently. We introduce a cost function that is a combination of the average sampling frequencies of the subsystems (i.e., the average frequency of the jumps between the idle state and the rest of the states in the Markov chain) and the effort needed for changing the scheduling policy (i.e., changing the underlying Markov chain parameters). In Paper 9, we find an explicit minimizer of the cost function and develop the optimal scheduling policy accordingly. To capture the changes in the networked system, when some of the subsystems are inactive, we simply remove their corresponding nodes from the Markov chain flow diagram in Figure 3.4 and set their corresponding terms in the cost function to be equal to zero.

Figures 3.5 (left) and (right) illustrate the estimation error variance $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations when using the optimal scheduling policy and the periodic scheduling policy, respectively. Since we have to fix the sampling instances in advance for the periodic scheduling policy, we must determine the sampling periods according to the worst-case scenario (i.e., when the networked system is composed of 70 subsystems). Therefore, when using the periodic sampling, the networked system is not using its true potential for $t \in [0, 5)$ and $t \in [10, 15]$. The proposed stochastic scheduling policy adapts to the demand of the system. For instance, as shown in Figure 3.5 (left), when subsystems 31 and 32 become active for $t \in [5, 10)$, the overall sampling frequencies of the subsystems decreases (and, in turn, the estimation error variance increases), but when they become inactive again for $t \in [10, 15]$, the average sampling frequencies increase (and, in turn, the estimation error variance decreases). Hence, this example illustrates the dynamic benefits of our proposed stochastic scheduling approach.

3.3.2 Contributions

The contributions of this part are presented in Paper 9. In this study, we introduce a stochastic sensor scheduling policy with application to networked control and estimation. In the presented scheduling algorithm, we model sensor measurement and transmission instances using jumps between states of a continuous-time Markov chain. We introduce a cost function for this Markov chain which is the summation of terms depending on the average sampling frequencies of the subsystems and the effort needed for changing the parameters of the underlying Markov chain. By minimizing this cost function (through extending the results of [321]), we extract an optimal scheduling policy to fairly allocate the communication network resources among the control loops. We study the statistical properties of this scheduling policy in order to compute upper bounds for the closed-loop performance of the networked system, where several decoupled subsystems are connected to their corresponding estimator or controller through a shared communication medium. The proposed optimal scheduling policy works particularly well for ad-hoc sensor networks.

CHAPTER 4

Conclusions and Future Work

"Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning."

Sir Winston Churchill, Speech[1] in November 1942

IN THIS chapter, we present a brief conclusion and some directions for future research.

4.1 Summary

As described in the previous chapter, Papers 1-6 of this thesis focus on the design of networked control systems under limited plant model information. Specifically, in Paper 1, we presented a framework to study static control design under limited model information, and investigated the connection between the quality of controllers produced by a design method and the amount of plant model information available to it. This was done for a set of discrete-time linear time-invariant plants under a separable quadratic performance measure with structured static state-feedback controllers. We showed that the best performance achievable by a limited model information control design method crucially depends on the structure of the plant graph and thereby giving the designer access to this graph, even without a detailed model of all plant subsystems, results in superior design, in the sense of domination. In Paper 2, we considered optimal H_2 dynamic control design for interconnected linear systems under limited plant model information. We found an explicit undominated minimizer of the competitive ratio for a large class of system interconnections, controller structure, and design information. It was also shown

[1]Quoted in page 149 of "Oxford Dictionary of Quotations by Subject" by Susan Ratcliffe, Oxford University Press, 2010.

when it comes to designing optimal centralized or partially structured decentralized state-feedback controllers with limited model information, the best control design strategy (in terms of competitive ratio) is a static one. This is true even though the optimal structured decentralized state-feedback controller with full model information is dynamic. We were also able to relax the assumption that all the subsystems are fully-actuated for sinks in the plant graph. In Paper 3, we studied the design of optimal disturbance accommodation controllers with limited model information. We adapted the notion of limited model information control design strategies to handle disturbance accommodation to study the cases where the best limited model information control design is a dynamic control design strategy. We found an explicit minimizer of the competitive ratio and we showed that it is undominated. We split this optimal control design strategy into a static part for regulating the state of the systems and a deadbeat observer for canceling the disturbance effect. As a generalization of earlier mentioned results, in Paper 4, we searched over the set of control design strategies that construct adaptive controllers. We found a minimizer of the competitive ratio both in average and supremum senses. We used an adaptive control law to achieve a competitive ratio equal to one, that is, this adaptive controller asymptotically achieves closed-loop performance equal to the optimal centralized controller with full model information. We presented a numerical approach for calculating optimal decentralized control design strategies under various model information availability regimes in Paper 5. Specifically, we focused on continuous-time linear parameter-dependent systems and defined the control design strategies as mappings from the set of parameters to the set of control laws. Then, we expanded these mappings using some basis functions and proposed a numerical optimization method based on consecutive local minimizations and maximizations of the H_∞–norm of the closed-loop transfer function with respect to the control design strategy gains and the system parameters. Finally, in Paper 6, we presented a statistical framework for the study of control design under limited model information. We found the best performance achievable by a limited model information control design method and studied the value of information in control design using the performance degradation ratio for discrete-time systems with stochastically-varying parameters.

In Papers 7 and 8, we studied heterogeneous transportation networks. Specifically, in Paper 7, we considered a heterogeneous routing game in which the players may belong to more than one type. The type of each player determines the cost of using an edge as a function of the flow of all types over that edge. We proved that this heterogeneous routing game admits at least one Nash equilibrium. Additionally, for the case where only two types of vehicles are participating in the heterogeneous routing game, we gave a necessary and sufficient condition for the existence of a potential function for the introduced routing game, which indeed implies that we can transform the problem of finding a Nash equilibrium into an optimization problem. We also developed tolls to guarantee the existence of a potential function and studied the price of anarchy. Later, in Paper 8, we introduced a model for traffic flow on a specific road at various time intervals per day using an atomic congestion game with two types of agents (namely, cars and trucks). Cars only optimize their trade-off between using the road at the time they prefer, the average velocity of the traffic flow, and the congestion tax they are paying. However, trucks benefit from using the road at the same time as the other trucks. We motivated this extra utility using an increased possibility of platooning with the other trucks and, as a result, saving fuel. We used congestion data from Stockholm to validate the affine relationship between the average velocity of commuting and the number of the vehicles that are using the road at that time. We devised appropriate tax or subsidy policies to create a potential game. Then, we used the joint strategy fictitious play and the average strategy fictitious play to learn a pure strategy Nash equilibrium of this game.

Finally, in Paper 9, we used a continuous-time Markov chain to schedule measurement and transmission time instances in a sensor network. As applications of this stochastic scheduling policy, we studied networked estimation and control of large-scale system that are composed of several decoupled scalar stochastic subsystems. We studied the statistical properties of this scheduling policy to compute bounds on the closed-loop performance of the networked system.

4.2 Future Work

There are several directions to further expand the work presented in the thesis. We list some of these directions below.

In the adaptive control law presented in Paper 4, all the subcontrollers required having access to the full state measurement and the implemented control inputs of other subsystems. An interesting direction for future research could be to use a decentralized adaptive controller in which each subcontroller may only rely on local state measurements (perhaps not even the implemented control inputs of other subsystems). In this scenario, one might be able to use the adaptive control law presented in [322] to construct input-output models for the rest of the subsystems and then using that model to calculate a control law with a reasonable closed-loop performance.

There is also much more to be done in numerical methods for finding an optimal control design strategy. For instance, the results of Paper 5 only guarantee recovering a local solution if the algorithm converges (which might not happen because saddle point solutions may not exists). It would be interesting to propose numerical algorithms that give guarantees on the global performance of the control design strategy. In addition, in Paper 5, we used a finite basis for expanding the control design strategies to convert the underlying infinite-dimensional optimization problem to a finite-dimensional one. As a viable direction for future work, we can focus on finding the best basis functions for expanding the control design strategies. We can also study the rate at which the closed-loop performance improves with increasing number of basis functions. Furthermore, because of the general pattern of the state-measurement availability, we could only search for a suboptimal control law which introduces conservativeness in the results (unless we restrict ourselves to information patters that result in a convex optimization problem, and assume that the order of the optimal controller is finite and, more importantly, known). Hence, it is interesting to find a bound on the suboptimality of the calculated control laws.

A general extension in the presented framework for control design using limited plant model information could be to let designers communicate with each other and to explore the question of characterizing an appropriate way of signaling between the designers without revealing all the parameters, for instance by finding the minimum amount of information needed for designing the optimal controller (e.g., see [323–325] for such bounds on distributed computation). However, the communication opens a door to many malicious behaviors by the designers (if they only wish to strategically optimize their own cost and not the social welfare function). In this case, it would be interesting to study the possibility of imposing taxes or subsidies to incentivize the agents to communicating truthfully. We may also use the idea of contract-based design (see [326] for an application of the idea in control of hybrid systems) to let the designers of various controllers negotiate acceptable input-output behaviors under which they can bound the closed-loop performance of the system.

In Paper 7, we only considered static routing games. Due to the time-varying nature of the traffic in transportation systems, it would be of great interest to extend the model to account for the time in which the drivers decide to use the road in addition to the path that they select. This can be done by combining the model in Paper 7 with the one presented in Paper 8. Another interesting extension of the results in both Papers 7 and 8 could be to consider games with more than two types or even a continuum of types.

Finally, in Paper 9, we presented upper bounds for the estimation error and the closed-loop performance of decoupled systems. Future research can focus on modeling the whole systems in conjunction with the optimal scheduling algorithm as a Markov jump linear system. Then, hopefully, we can present upper bounds for quality of the estimation or the control when dealing with coupled subsystems. Another direction for future research could be to focus on combining the estimation and control results for achieving a reasonable closed-loop performance when dealing with observable and controllable subsystems of arbitrary dimension.

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Papers

Part 1:

Control Design with Limited Model Information

PAPER 1

Optimal Structured Static State-Feedback Control Design with Limited Model Information for Fully-Actuated Systems

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Abstract–We introduce the family of limited model information control design methods, which construct controllers by accessing the plant's model in a constrained way, according to a given design graph. We investigate the closed-loop performance achievable by such control design methods for fully-actuated discrete-time linear time-invariant systems, under a separable quadratic cost. We restrict our study to control design methods which produce structured static state feedback controllers, where each subcontroller can at least access the state measurements of those subsystems that affect its corresponding subsystem. We compute the optimal control design strategy (in terms of the competitive ratio and domination metrics) when the control designer has access to the local model information and the global interconnection structure of the plant-to-be-controlled. Finally, we study the trade-off between the amount of model information exploited by a control design method and the best closed-loop performance (in terms of the competitive ratio) of controllers it can produce.

1 Introduction

Many modern control systems, such as aircraft and satellite formation [1, 2], automated highways and other shared infrastructure [3, 4], flexible structures [5], and supply chains [6, 7], consist of a large number of subsystems coupled through their performance goals or system dynamics. When regulating this kind of plant, it is often advantageous to adopt a distributed control architecture, in which the controller itself is composed of interconnected subcontrollers, each of which accesses a strict subset of the plant's output. Several control synthesis methods have been proposed over the past decades that result in distributed controllers of this form, with various types of closed-loop stability and performance guarantees (e.g., [8–16]). Most recently, the tools presented in [17] and [18] revealed how to exploit the specific interconnection of classes of plants (the so-called quadratically invariant systems) to formulate convex optimization problems for the design of structured H_∞- and H_2- optimal controllers. A common thread in this part of the literature is the assumption that, even though the controller is structured, its design can be performed in a centralized fashion, with full knowledge of the plant model. However, in some applications (described in more detail in the next paragraph), this assumption is not always warranted, as the design of each subcontroller may need to be carried out by a different control designer, with no access to the global model of the plant, although its interconnection structure and the common closed-loop cost function to be minimized are public knowledge. This class of problems, which we refer to as "limited model information control design problems", is the main object of interest in the present paper.

Limited model information control design occurs naturally in contexts where the subsystems belong to different entities, which may consider their model information private and may thus be reluctant to share it with others. In this case, the designers may have to resort to "communication-less" strategies in which subcontroller K_i depends solely on the description of subsystem i's model. This case is well illustrated by supply chains, where the economic incentives of competing companies might limit the exchange of model information (such as, inventory volume, transportation efficiency, raw material sources, and decision process) inside a layer of the chain (see [7, 19–21] for a detailed review of modeling and control of supply chains). Another reason for using communication-less strategies in more general design situations, even when the circulation of plant information is not restricted a priori, is that the resulting subcontroller K_i does not need to be modified if the characteristics of a particular subsystem, which is not directly connected to subsystem i, vary. For instance, consider a chemical plant in the process industry, with thousands of local controllers. In such a large-scale system, the tuning of each local controller should not require model parameters from other parts of the system so as to simplify maintenance and limit controller complexity. Note that engineers often implement these large-scale systems as a whole using commercially available pre-designed modules. These modules are designed, in advance, with no prior knowledge of their possible use or future operating condition. This lack of availability of the complete model of the plant, at the time of the design, constrains the designer to only use its own model parameters in each module's control design.

Control design based on uncertain plant model information is a classic topic in the robust control literature [22–25]. However, designing an optimal controller without a global model is different from a robust control problem. In optimal control design with limited model information, subsystems do not have any prior information about the other subsystems' model; i.e., there is no nominal model for the design procedure and there is no bound on the model uncertainties. There have been some interesting approaches for tackling this problem. For instance, references [26–29] introduced methods for designing sub-optimal decentralized controllers without a global dynamical model of the system. In these papers, the authors assume that the large-scale system to be controlled consists of an interconnection of weakly coupled subsystems. They design an optimal controller for each subsystem using only the corresponding local model, and connect the obtained subcontrollers to construct a global controller. They show that, when coupling is negligible, this latter controller is satisfactory in terms of closed-loop stability and performance. However, as coupling strength increases, even closed-loop stability guarantees are lost. Other approaches such as [4, 6] are based on receding horizon control and use decomposition methods to solve each step's optimization problem in a decentralized manner with only limited information exchange between subsystems. What is missing from the literature, however, is a rigorous characterization of the best closed-loop performance that can be attained through limited model information design and, a study of the trade off between the closed-loop performance and the amount of exchanged information. We tackle this question in the present paper for a particular class of systems (namely, the set of fully-actuated discrete-time linear time-invariant dynamical systems) and a particular class of control laws (namely, the set of structured linear static state feedback controllers where each subcontroller can at least access the state measurements of those subsystems that affect its corresponding subsystem).

In this paper, we study the properties of limited model information control design methods. We investigate the relationship between the amount of plant information available to the designers, the nature of the plant interconnection graph, and the quality (measured by the closed-loop control goal) of controllers that can be constructed using their knowledge. To do so, we look at limited model information and communication-less control design methods as belonging to a special class of maps between the plant and controller sets, and make use of the competitive ratio and domination metrics introduced in [30] to characterize their intrinsic limitations. To the best of our knowledge, there are no other metrics specifically tuned to control design methods. We address much more general classes of subsystems and of limitations on the model information available to the designer than is done in [30]. Specifically, we consider limited model information structured static state-feedback control design for interconnections of fully-actuated (i.e., with invertible B-matrix) discrete-time linear time-invariant subsystems with quadratic separable (i.e., with block diagonal Q- and R-matrices) cost function. Our choice of such a cost function is motivated by our interest in applications such as power grids [31–34] and [4, Chs. 5,10], supply chains [6, 7], and water level control [4, Ch. 18], which have been shown to be well-modeled by dynamically-coupled but cost-decoupled interconnected systems. We show in the last section of the paper that the assumption on the B-matrix can be partially removed for the sinks (i.e., subsystems that cannot affect any other subsystem) in the plant graph.

We investigate the best closed-loop performance achievable by structured static state feedback controllers constructed by limited model information design strategies. We show that the result depends crucially on the plant graph and the control graph. In the case where the plant graph contains no sink and the control graph is a supergraph of the plant graph, we extend the fact proven in [30] that the deadbeat strategy is the best communication-less control design method. However, the deadbeat control design strategy is dominated when the plant graph has sinks, and we exhibit a better, undominated, communication-less control design method, which, although having the same competitive ratio as the deadbeat control design strategy, takes advantage of the knowledge of the sinks' location to achieve a better closed-loop performance in average. We characterize the amount of model information needed to achieve better competitive ratio than the deadbeat control design strategy. This amount of information is expressed in terms of properties of the design graph; a directed graph which indicates the dependency of each subsystem's controller on different parts of the global dynamical model.

This paper is organized as follows. After formulating the problem of interest and defining the performance metrics in Section 2, we characterize the best communication-less control design method according to both competitive ratio and domination metrics in Section 3. In Section 4, we show that achieving a strictly better competitive ratio than these control design methods requires a complete design graph when the plant graph is itself complete. Finally, we end with a discussion on extensions in Section 5 and the conclusions in Section 6.

1.1 Notation

Sets will be denoted by calligraphic letters, such as $\mathcal{P}$ and $\mathcal{A}$. If $\mathcal{A}$ is a subset of $\mathcal{M}$ then $\mathcal{A}^c$ is the complement of $\mathcal{A}$ in $\mathcal{M}$, i.e., $\mathcal{M} \setminus \mathcal{A}$.

Matrices are denoted by capital roman letters such as A. A_j will denote the j^{th} row of A. A_{ij} denotes a sub-matrix of matrix A, the dimension and the position of which will be defined in the text. The entry in the i^{th} row and the j^{th} column of the matrix A is a_{ij}.

Let S^n_{++} (S^n_{+}) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$. $A > (\geq)0$ means that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq)B$ means that $A - B > (\geq)0$.

$\underline{\lambda}(Y)$ and $\bar{\lambda}(Y)$ denote the smallest and the largest eigenvalues of the matrix Y, respectively. Similarly, $\underline{\sigma}(Y)$ and $\bar{\sigma}(Y)$ denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i denotes the column-vector with all entries zero except the i^{th} entry, which is equal to one.

All graphs considered in this paper are directed, possibly with self-loops, with vertex set $\{1, ..., q\}$ for some positive integer q. If $G = (\{1, ..., q\}, E)$ is a directed graph, we say that i is a sink if there does not exist $j \neq i$ such that $(i, j) \in E$. A loop of length t in G is a set of distinct vertices $\{i_1, ..., i_t\}$ such that $(i_t, i_1) \in E$ and $(i_p, i_{p+1}) \in E$ for all $1 \leq p \leq t-1$. We will sometimes refer to this loop as $(i_1 \to i_2 \to \cdots \to i_t \to i_1)$. The adjacency matrix S of graph G is the $q \times q$ matrix whose entries satisfy

$$s_{ij} = \begin{cases} 1 & \text{if } (j, i) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since the set of vertices is fixed here, a subgraph of G is a graph whose edge set is a subset of the edge set of G and a supergraph of G is a graph of which G is a subgraph. We use the notation $G' \supseteq G$ to indicate that G' is a supergraph of G.

2 Control Design with Limited Model Information

In this section, we introduce the system model and the problem under consideration, but first, we present a simple illustrative example.

2.1 Illustrative Example

Consider a discrete-time linear time-invariant dynamical system composed of three subsystems represented in state-space form as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_{11}u_1(k) \\ b_{22}u_2(k) \\ b_{33}u_3(k) \end{bmatrix},$$

where, for each subsystem i, $x_i(k) \in \mathbb{R}$ is the state and $u_i(k) \in \mathbb{R}$ is the control signal. This system, which is illustrated in Figure 1, is a simple networked control system. Networked control systems have several important characteristics. First, they are often distributed geographically. Therefore, it is natural to assume that a given subsystem can only influence its neighboring subsystems. We capture this fact using a directed graph called the plant graph like the one presented in Figure 2(a) for this example. This star graph corresponds to applications like unmanned aerial vehicles formation, platoon of vehicles, and composite formations of power systems [35, 36].

Second, any communication medium that we use to transmit the sensor measurements and actuation signals in networked control systems brings some limitations. For instance, every communication network has band-limited channels. Therefore, when designing subcontrollers, it might not make sense to assume that it can instantaneously access full state measurements of the plant. The state measurement availability in this example is

$$\begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & k_{23} \\ 0 & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}.$$

We use a control graph to characterize the controller structure. Control graph $G_{\mathcal{K}}$ in Figure 2(b) represents the state-measurement availability in this example. It corresponds to the case where neighboring subsystems transmit their state-measurements to each other, which is common for unmanned aerial vehicles formation, autonomous ground vehicles platoons, and biological system of particles [1, 2, 37, 38].

Finally, in large-scale dynamical systems, it might be extremely difficult (if not impossible) to identify all system parameters and update them globally. One can only hope that the designer has access to the local parameter variations and update the corresponding subcontroller based on them. Therefore, it makes sense to assume that each local controller only has access to model information from its corresponding subsystem; i.e., designer of subcontroller i uses only $\{a_{i1}, a_{i2}, a_{i3}\}$ in the design procedure

$$[k_{i1} \;\; k_{i2} \;\; k_{i3}] = \Gamma_i \left([a_{i1} \;\; a_{i2} \;\; a_{i3}], b_{ii}\right),$$

where $\Gamma_i : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the control design map. Note that assuming subcontroller i has access to the state-measurements of those subsystems that can affect subsystem i, it can identify parameters $\{a_{i1}, a_{i2}, a_{i3}\}$. However, identifying parameters $\{a_{1i}, a_{2i}, a_{3i}\}$ might not be possible since subcontroller i may not have access to the state-measurements of all the subsystems that it can influence. The block-diagram in Figure 1 does not specify Γ. We will use a directed graph called the design graph to capture structural properties of Γ. Figure 2(c') represents the plant model information availability in this example. This totally disconnected graph corresponds to applications such as supply chain management [7, 21] or vehicle platooning [39, 40], where subsystems potentially belong to different entities and privacy concerns might restrict plant model information circulation. In the rest of this section, we formalize the above notions for more general design problems.

2.2 Plant Model

Let a graph $G_{\mathcal{P}} = (\{1, ..., q\}, E_{\mathcal{P}})$ be given, with adjacency matrix $S_{\mathcal{P}} \in \{0,1\}^{q\times q}$. We define the following set of matrices associated with $S_{\mathcal{P}}$:

$$\begin{aligned}\mathcal{A}(S_{\mathcal{P}}) = \{A \in \mathbb{R}^{n\times n} \mid A_{ij} = 0 \in \mathbb{R}^{n_i\times n_j} \text{ for all} \\ 1 \le i, j \le q \text{ such that } (s_{\mathcal{P}})_{ij} = 0\},\end{aligned} \tag{1}$$

where for each $1 \le i \le q$, integer number n_i is the dimension of subsystem i. Implicit in these definitions is the fact that $\sum_{i=1}^{q} n_i = n$. Also, for a given scalar $\epsilon > 0$, we let

$$\mathcal{B}(\epsilon) = \{B \in \mathbb{R}^{n\times n} \mid \underline{\sigma}(B) \ge \epsilon, B_{ij} = 0 \in \mathbb{R}^{n_i\times n_j} \text{ for all } 1 \le i \ne j \le q\}. \tag{2}$$

The set $\mathcal{B}(\epsilon)$ defined in (2) is made of invertible block-diagonal square matrices since $\underline{\sigma}(B) \ge \epsilon > 0$ for each matrix $B \in \mathcal{B}(\epsilon) \subseteq \mathbb{R}^{n\times n}$. With these definitions, we can introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant dynamical systems of the form

$$x(k+1) = Ax(k) + Bu(k) \; ; \; x(0) = x_0, \tag{3}$$

with $A \in \mathcal{A}(S_\mathcal{P})$, $B \in \mathcal{B}(\epsilon)$, and $x_0 \in \mathbb{R}^n$. Clearly $\mathcal{P}$ is isomorph to $\mathcal{A}(S_\mathcal{P}) \times \mathcal{B}(\epsilon) \times \mathbb{R}^n$ and, slightly abusing notation, we will thus identify a plant $P \in \mathcal{P}$ with the corresponding triple (A, B, x_0).

A plant $P \in \mathcal{P}$ can be thought of as the interconnection of q subsystems, with the structure of the interconnection specified by the graph $G_\mathcal{P}$ (i.e., subsystem j's output feeds into subsystem i only if $(j, i) \in E_\mathcal{P}$). As a consequence, we refer to $G_\mathcal{P}$ as the "plant graph". We will denote the ordered set of state indices pertaining to subsystem i as $\mathcal{I}_i$; i.e., $\mathcal{I}_i := (1 + \sum_{j=1}^{i-1} n_j, \dots, n_i + \sum_{j=1}^{i-1} n_j)$. For subsystem i, state vector and input vector are defined as

$$\underline{x}_i = \begin{bmatrix} x_{\ell_1} & \cdots & x_{\ell_{n_i}} \end{bmatrix}^T, \qquad \underline{u}_i = \begin{bmatrix} u_{\ell_1} & \cdots & u_{\ell_{n_i}} \end{bmatrix}^T$$

where the ordered set of indices $(\ell_1, \dots, \ell_{n_i}) \equiv \mathcal{I}_i$, and its dynamics is specified by

$$\underline{x}_i(k+1) = \sum_{j=1}^{q} A_{ij} \underline{x}_j(k) + B_{ii} \underline{u}_i(k).$$

According to the specific structure of $\mathcal{B}(\epsilon)$ given in (2), each subsystem is fully-actuated, with as many input as states, and controllable in one time-step. Possible generalization of the results to a (restricted) family of under-actuated systems is discussed in Section 5.

Figure 2(a) shows an example of a plant graph $G_\mathcal{P}$. Each node represents a subsystem of the system. For instance, the second subsystem in this example may affect the first subsystem and the third subsystem; i.e., sub-matrices A_{12} and A_{32} can be nonzero. The self-loop for the second subsystem shows that A_{22} may be non-zero. The plant graph $G_\mathcal{P}$ in Figure 2(a) does not contain any sink. In contrast, the first subsystem of the plant graph $G'_\mathcal{P}$ in Figure 2(a') is a sink. The control graph $G_\mathcal{K}$ is introduced in the next subsection.

2.3 Controller Model

Let a control graph $G_\mathcal{K}$ be given, with adjacency matrix $S_\mathcal{K}$. The control laws of interest in this paper are linear static state-feedback control laws of the form

$$u(k) = Kx(k),$$

where

$$K \in \mathcal{K}(S_\mathcal{K}) = \{K \in \mathbb{R}^{n \times n} | K_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_\mathcal{K})_{ij} = 0\}. \tag{4}$$

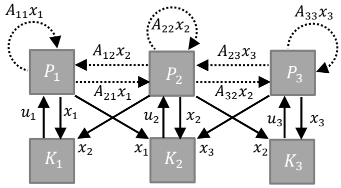

Figure 1: Physical interconnection between different subsystems and controllers corresponding to $G_{\mathcal{P}}$ and $G_{\mathcal{K}}$ in Figures 2(a) and 2(b), respectively.

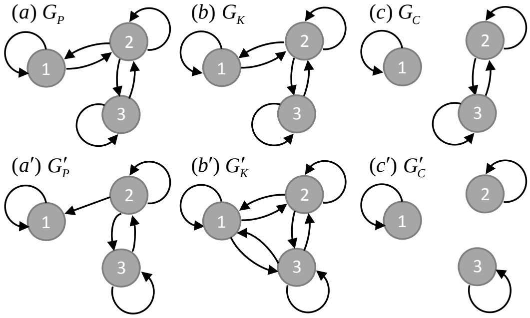

Figure 2: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs, $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs, and $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

In particular, when $G_{\mathcal{K}}$ is a complete graph, $\mathcal{K}(S_{\mathcal{K}}) = \mathbb{R}^{n\times n}$, while, if $G_{\mathcal{K}}$ is totally disconnected with self-loops, $\mathcal{K}(S_{\mathcal{K}})$ represents the set of fully-decentralized controllers. When adjacency matrix $S_{\mathcal{K}}$ is not relevant or can be deduced from context, we refer to the set of controllers as $\mathcal{K}$.

An example of a control graph $G_{\mathcal{K}}$ is given in Figure 2(b). Each node represents a subsystem-controller pair of the overall system. For instance, Figure 2(b) shows that the first subsystem's controller can use state measurements of the second subsystem besides its own state measurements. Figure 2(b') shows a complete graph, which indicates that each subsystem has access to full state measurements of all other subsystems; i.e., $\mathcal{K}(S_{\mathcal{K}}) = \mathbb{R}^{n\times n}$.

2.4 Linear State Feedback Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}$. Just like plants and controllers, a control design method can exhibit structure which, in turn, can be captured by a design graph. Let a control design method Γ

be partitioned according to subsystems dimensions as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{bmatrix} \tag{5}$$

and a graph $G_{\mathcal{C}} = (\{1, ..., q\}, E_{\mathcal{C}})$ be given, with adjacency matrix $S_{\mathcal{C}}$. Each block Γ_{ij} represents a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathbb{R}^{n_i \times n_j}$. Control design method Γ can be further partitioned in the form

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix},$$

where each γ_{ij} is a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathbb{R}$. We say that Γ has structure $G_{\mathcal{C}}$ if, for all i, the map $[\Gamma_{i1} \ \cdots \ \Gamma_{iq}]$ is only a function of

$$\{[A_{j1} \ \cdots \ A_{jq}], B_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}. \tag{6}$$

In words, a control design method has structure $G_{\mathcal{C}}$ if and only if, for all i, the subcontroller of subsystem i is constructed with knowledge of the plant model of only those subsystems j such that $(j, i) \in E_{\mathcal{C}}$. The set of all control design methods with structure $G_{\mathcal{C}}$ will be denoted by $\mathcal{C}$. In the particular case where $G_{\mathcal{C}}$ is the totally disconnected graph with self-loops (meaning that every node in the graph has a self-loop; i.e, $S_{\mathcal{C}} = I_q$), we say that a control design method in $\mathcal{C}$ is "communication-less", so as to capture the fact that subsystem i's subcontroller is constructed with no information coming from (and, hence, no communication with) any other subsystem j, $j \neq i$. Therefore, the design graph indicates knowledge (or lack thereof) of entire block rows in the aggregate system matrix. When $G_{\mathcal{C}}$ is not a complete graph, we refer to $\Gamma \in \mathcal{C}$ as being "a limited model information control design method".

Note that $\mathcal{C}$ can be considered as a subset of the set of functions from $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon)$ to $\mathcal{K}(S_{\mathcal{K}})$, since a design method with structure $G_{\mathcal{C}}$ is not a function of initial state x_0. Hence, when $\Gamma \in \mathcal{C}$ we will write $\Gamma(A, B)$ instead of $\Gamma(P)$ for plant $P = (A, B, x_0) \in \mathcal{P}$.

An example of a design graph $G_{\mathcal{C}}$ is given in Figure 2(c). Each node represents a subsystem-controller pair of the overall system. For instance, $G_{\mathcal{C}}$ shows that the third subsystem's model is available to the designer of the second subsystem's controller but not the first subsystem's model. Figure 2(c') shows a fully disconnected design graph with self-loops $G'_{\mathcal{C}}$. A local designer in this case can only rely on the model of its corresponding subsystem; i.e., the design strategy is communication-less.

2.5 Performance Metrics

The goal of this paper is to investigate the influence of the plant and design graph on the properties of controllers constructed by limited model information control design methods. To this end, we will use two performance metrics for control design methods. These performance metrics are adapted from the notions of competitive ratio and domination introduced in [30], so as to take plant, controller, and control design structures into account. Following the approach in [30], we start by associating a closed-loop performance criterion to each plant $P = (A, B, x_0) \in \mathcal{P}$ and controller $K \in \mathcal{K}$. As explained in the introduction, we are particularly interested in dynamically-coupled but cost-decoupled systems in this paper, hence, we use a cost of the form

$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k), \tag{7}$$

where $Q \in S_{++}^n$ and $R \in S_{++}^n$ are block diagonal matrices, with each diagonal block entry belonging to $S_{++}^{n_i}$. Note that the summation in the first term on the right-hand side of (7) starts from $k = 1$. This is without loss of generality as the removed term $x(0)^T Q x(0)$ is not a function of the controller. We make the following two standing assumptions:

Assumption 1.1 $Q = R = I$.

This is without loss of generality because the change of variables $(\bar{x}, \bar{u}) = (Q^{1/2}x, R^{1/2}u)$ transforms the performance criterion and state space representation into

$$J_P(K) = \sum_{k=1}^{\infty} \bar{x}(k)^T \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T \bar{u}(k), \tag{8}$$

and

$$\begin{aligned} \bar{x}(k+1) &= Q^{1/2} A Q^{-1/2} \bar{x}(k) + Q^{1/2} B R^{-1/2} \bar{u}(k) \\ &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \end{aligned}$$

respectively, without affecting the plant, control, or design graph (due to the block diagonal structure of Q and R).

Assumption 1.2 *The set of matrices $\mathcal{B}(\epsilon)$ is replaced with the set of diagonal matrices with diagonal entries greater than or equal to ϵ.*

This assumption is without loss of generality. Indeed, consider a plant $P = (A, B, x_0) \in \mathcal{P}$. Every sub-system's B_{ii} matrix has a singular value decomposition $B_{ii} = U_{ii} \Sigma_{ii} V_{ii}^T$ with $\Sigma_{ii} \geq \epsilon I_{n_i \times n_i}$. Combining these singular value decompositions together results in a singular value decomposition for matrix $B =$

$U\Sigma V^T$ where $U = \text{diag}(U_{11}, U_{22}, \cdots, U_{qq})$, $\Sigma = \text{diag}(\Sigma_{11}, \Sigma_{22}, \cdots, \Sigma_{qq})$, and $V = \text{diag}(V_{11}, V_{22}, \cdots, V_{qq})$. Defining $\bar{x}(k) = U^T x(k)$ and $\bar{u}(k) = V^T u(k)$ results in

$$\bar{x}(k+1) = U^T AU\bar{x}(k) + U^T BV\bar{u}(k),$$

where $U^T BV$ is diagonal. Because of the block diagonal structure of matrices U and V, the change of variables $(A, B, x_0) \mapsto (U^T AU, U^T BV, U^T x_0)$ does not affect the plant, control, or design graph. In addition, the cost function becomes

$$\begin{aligned} J_P(K) &= \sum_{k=1}^{\infty} \bar{x}(k)^T U^T U \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T V^T V \bar{u}(k) \\ &= \sum_{k=1}^{\infty} \bar{x}(k)^T \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T \bar{u}(k), \end{aligned}$$

which is of the form (8), because both U and V are unitary matrices. We are now ready to define the performance metrics of interest in this paper.

Definition 1.1 *(Competitive Ratio) Let a plant graph $G_{\mathcal{P}}$, control graph $G_{\mathcal{K}}$ and constant $\epsilon > 0$ be given. Assume that, for every plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that*

$$J_P(K^*(P)) \leq J_P(K), \ \forall K \in \mathcal{K}.$$

The competitive ratio of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,x_0)\in\mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$" equals one.

Note that the mapping $K^* : P \to K^*(P)$ is not itself required to lie in the set $\mathcal{C}$, as every component of the optimal controller may depend on all entries of the model matrices A and B.

Definition 1.2 *(Domination) A control design method Γ is said to dominate another control design method Γ' if*

$$J_P(\Gamma(A,B)) \leq J_P(\Gamma'(A,B)), \quad \forall \ P = (A, B, x_0) \in \mathcal{P}, \tag{9}$$

with strict inequality holding for at least one plant in $\mathcal{P}$. When $\Gamma' \in \mathcal{C}$ and no control design method $\Gamma \in \mathcal{C}$ exists that satisfies (9), we say that Γ' is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$.

2.6 Problem Formulation

With the definitions of the previous subsections in hand, we can reformulate the main question of this paper regarding the connection between closed-loop performance, plant structure, and limited model information control design as follows. For a given plant graph, control graph, and design graph, we would like to determine

$$\arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma). \tag{10}$$

Since several design methods may achieve this minimum, we are interested in determining which ones of these strategies are *undominated*.

In [30], this problem was solved in the case when $G_{\mathcal{P}}$ and $G_{\mathcal{K}}$ are complete graphs, $G_{\mathcal{C}}$ is a totally disconnected graph with self-loops (i.e., $S_{\mathcal{C}} = I_q$), and $\mathcal{B}(\epsilon)$ is replaced with singleton $\{I_n\}$. In this paper, we investigate the role of more general plant and design graphs. We also extend the results in [30] for scalar subsystems to subsystems of arbitrary order $n_i \geq 1$, $1 \leq i \leq q$.

3 Plant Graph Influence on Achievable Performance

In this section, we study the relationship between the plant graph and the achievable closed-loop performance in terms of the competitive ratio and domination.

Definition 1.3 *The deadbeat control design method* $\Gamma^{\Delta} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathcal{K}$ *is defined as*

$$\Gamma^{\Delta}(A,B) = -B^{-1}A, \text{ for all } P = (A,B,x_0) \in \mathcal{P}.$$

This control design method is communication-less; i.e., the control design for the subsystem i is a function of the model of subsystem i only, because subsystem i's controller gain $\left[\Gamma^{\Delta}_{i1}(A,B) \ \cdots \ \Gamma^{\Delta}_{iq}(A,B)\right]$ equals to $B_{ii}^{-1}\left[A_{i1} \ \cdots \ A_{iq}\right]$. The name "deadbeat" comes from the fact that the closed-loop system obtained by applying controller $\Gamma^{\Delta}(A,B)$ to plant $P = (A,B,x_0)$ reaches the origin in just one time-step [41].

Remark 1.1 *Note that for the case where the control graph $G_{\mathcal{K}}$ is a complete graph; i.e., $\mathcal{K} = \mathbb{R}^{n\times n}$, there exists a controller $K^*(P)$ satisfying the assumptions of Definition 1.1 for all $P \in \mathcal{P}$, namely, the optimal linear quadratic regulator which is independent of the initial condition of the plant. For incomplete control graphs, the optimal control design strategy $K^*(P)$ (if exists) might become a function of the initial condition [42]. Hence, we will use $K^*(A,B)$ instead of $K^*(P)$ when the control graph $G_{\mathcal{K}}$ is a complete graph for each plant $P = (A,B,x_0) \in \mathcal{P}$ to emphasize this fact.*

From Definition 1.1, the notation $K^*(P)$ is reserved for the optimal control design strategy for any given control graph $G_{\mathcal{K}}$. In contrast, when $G_{\mathcal{K}}$ is not the complete graph, we will refer to the optimal *unstructured* controller as $K^*_C(A,B)$.

Lemma 1.1 *Let the control graph $G_{\mathcal{K}}$ be a complete graph. The cost of the optimal control design strategy K^* is lower-bounded by*

$$J_P(K^*(A,B)) \geq \left(\frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B)+1}\right) J_P(\Gamma^{\Delta}(A,B)),$$

for all plants $P = (A, B, x_0) \in \mathcal{P}$.

Proof: See [43, p.73–74]. ∎

Theorem 1.2 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then the competitive ratio of the deadbeat control design method Γ^{Δ} is*

$$r_{\mathcal{P}}(\Gamma^{\Delta}) = 1 + 1/\epsilon^2.$$

Proof: Irrespective of the control graph $G_{\mathcal{K}}$ and for all plants $P \in \mathcal{P}$, it is true that $J_P(K_C^*(A,B)) \leq J_P(K^*(P))$. Therefore, we get

$$\frac{J_P(\Gamma^{\Delta}(A,B))}{J_P(K^*(P))} \leq \frac{J_P(\Gamma^{\Delta}(A,B))}{J_P(K_C^*(A,B))}. \tag{11}$$

Now, using Lemma 1.1, we know that

$$\frac{J_P(\Gamma^{\Delta}(A,B))}{J_P(K_C^*(A,B))} \leq 1 + \frac{1}{\underline{\sigma}^2(B)}, \tag{12}$$

for all $P = (A, B, x_0) \in \mathcal{P}$. Combining (12) and (11) results in

$$r_{\mathcal{P}}(\Gamma^{\Delta}) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma^{\Delta}(A,B))}{J_P(K^*(P))} \leq 1 + \frac{1}{\epsilon^2}.$$

To show that this upper bound is attained, let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ where $1 \leq i \neq j \leq q$ and $(s_{\mathcal{P}})_{ij} \neq 0$ (such indices i and j exist because plant graph $G_{\mathcal{P}}$ has no isolated node by assumption). Consider the system $A = e_{i_1} e_{j_1}^T$ and $B = \epsilon I$. The unique positive definite solution of the discrete algebraic Riccati equation

$$A^T X A - A^T X B (I + B^T X B)^{-1} B^T X A = X - I, \tag{13}$$

is $X = I + [1/(1+\epsilon^2)] e_{j_1} e_{j_1}^T$. Consequently, the centralized controller $K_C^*(A,B) = -\epsilon/(1+\epsilon^2) e_{i_1} e_{j_1}^T$ belongs to the set $\mathcal{K}(S_{\mathcal{K}})$ because $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Thus, we get

$$J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) \leq J_{(A,B,e_{j_1})}(K_C^*(A,B)) \tag{14}$$

since $K^*(P)$ has a lower cost than any other controller in $\mathcal{K}(S_{\mathcal{K}})$. On the other hand, it is evident that

$$J_{(A,B,e_{j_1})}(K_C^*(A,B)) \leq J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) \tag{15}$$

because the centralized controller has access to more state measurements. Using (14) and (15) simultaneously results in

$$\begin{aligned} J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) &= J_{(A,B,e_{j_1})}(K_C^*(A,B)) \\ &= 1/(1+\epsilon^2). \end{aligned}$$

On the other hand $\Gamma^\Delta(A,B) = -[1/\epsilon]e_{i_1}e_{j_1}^T$ and $J_{(A,B,e_{j_1})}(\Gamma^\Delta(A,B)) = 1/\epsilon^2$. Therefore, $r_\mathcal{P}(\Gamma^\Delta) = 1 + 1/\epsilon^2$. ■

Remark 1.2 *Consider the limited model information design problem given by the plant graph $G_\mathcal{P}$ in Figure 2(a) and the control graph $G'_\mathcal{K}$ in Figure 2(b′). Theorem 1.2 shows that, if we apply the deadbeat control design strategy to this particular problem, the performance of the deadbeat control design strategy, at most, can be $1 + 1/\epsilon^2$ times the cost of the optimal control design strategy K^*. For instance, when $\mathcal{B} = \{I\}$ as in [30], we have $1 + 1/\epsilon^2 = 2$ since in this case $\epsilon = 1$. Therefore, the deadbeat control design strategy is never worse than twice the optimal controller in this case.*

Remark 1.3 *There is no loss of generality in assuming that there is no isolated node in the plant graph $G_\mathcal{P}$, since it is always possible to design a controller for an isolated subsystem without any model information about the other subsystems and without impacting cost (7). In particular, this implies that there are $q \geq 2$ vertices in the graph because for $q = 1$ the only subsystem that exists is an isolated node in the plant graph.*

Remark 1.4 *For implementation of the deadbeat control design strategy in each node, we only need the state measurements of the neighbors of that node. For the implementation of the optimal control design strategy K^* when the control graph has many more links than the plant graph, the controller gain $K^*(P)$ is not necessarily a sparse matrix.*

With this characterization of Γ^Δ in hand, we are now ready to tackle problem (10).

3.1 First case: plant graph $G_\mathcal{P}$ with no sink

In this subsection, we show that the deadbeat control method Γ^Δ is undominated by communication-less control design methods for plants in $\mathcal{P}$, when $G_\mathcal{P}$ contains no sink. We also show that Γ^Δ exhibits the smallest possible competitive ratio among such control design methods. First, we state the following two lemmas.

Lemma 1.3 *Let the plant graph $G_\mathcal{P}$ contain no isolated node, the design graph $G_\mathcal{C}$ be a totally disconnected graph with self-loops, and $G_\mathcal{K} \supseteq G_\mathcal{P}$. A control design* *method* $\Gamma \in \mathcal{C}$ *has bounded competitive ratio only if the following implication holds for all* $1 \leq i \leq q$ *and all* j*:*

$$a_{\ell j} = 0 \text{ for all } \ell \in \mathcal{I}_i \Rightarrow \gamma_{\ell j}(A, B) = 0 \text{ for all } \ell \in \mathcal{I}_i,$$

where $\mathcal{I}_i$ *is the set of indices related to subsystem* i*; i.e.,* $\mathcal{I}_i = (1+\sum_{z=1}^{i-1} n_z, \ldots, n_i + \sum_{z=1}^{i-1} n_z)$*.*

Proof: See [43, p.75] or [44]. ■

Lemma 1.4 *Let the plant graph* $G_{\mathcal{P}}$ *contain no isolated node, the design graph* $G_{\mathcal{C}}$ *be a totally disconnected graph with self-loops, and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$*. Assume the plant graph* $G_{\mathcal{P}}$ *has at least one loop. Then,*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2 \tag{16}$$

for all limited model information control design method Γ *in* $\mathcal{C}$*.*

Proof: See [43, p.75–77]. ■

Using these two lemmas, we are ready to state and prove one of the main theorems in this paper and, as a result, find the solution to problem (10) when the plant graph $G_{\mathcal{P}}$ contains no sink.

Theorem 1.5 *Let the plant graph* $G_{\mathcal{P}}$ *contain no isolated node and no sink, the design graph* $G_{\mathcal{C}}$ *be a totally disconnected graph with self-loops, and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$*. Then the competitive ratio of any control design strategy* $\Gamma \in \mathcal{C}$ *satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2.$$

Proof: From Lemma 1.4.23 in [45], we know that a directed graph with no sink must have at least one loop. Hence $G_{\mathcal{P}}$ must contain a loop. The result then follows from Lemma 1.4. ■

Remark 1.5 *Theorem 1.5 shows that* $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Delta})$ *for any control design strategy* $\Gamma \in \mathcal{C}$*, and as a result the deadbeat control design method* Γ^{Δ} *becomes a minimizer of the competitive ratio function* $r_{\mathcal{P}}$ *over the set of communication-less design methods.*

We now turn our attention to domination properties of the deadbeat control design strategy.

Lemma 1.6 *Let the plant graph* $G_{\mathcal{P}}$ *contain no isolated node, the design graph* $G_{\mathcal{C}}$ *be a totally disconnected graph with self-loops, and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$*. The deadbeat control design strategy* Γ^{Δ} *is undominated, if there is no sink in the plant graph* $G_{\mathcal{P}}$*.*

Proof: See [43, p.77–79]. ■

The following theorem shows that the deadbeat control design strategy is undominated by communication-less design methods if and only if the plant graph $G_{\mathcal{P}}$ has no sink. It thus provides a good trade-off between worst-case and average performance.

Theorem 1.7 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then the deadbeat control design method Γ^{Δ} is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$ if and only if the plant graph $G_{\mathcal{P}}$ has no sink.*

Proof: Proof of the "if" part of the theorem, is given by Lemma 1.6.

For ease of notation in this proof, we use $[\Gamma]_i = [\Gamma_{i1} \cdots \Gamma_{iq}]$ and $[A]_i = [A_{i1} \cdots A_{iq}]$.

In order to prove the "only if" part of the theorem, we need to show that if the plant graph has a sink (i.e., if there exists j such that $(s_{\mathcal{P}})_{ij} = 0$ for every $i \neq j$), then there exists a control design method Γ which dominates the deadbeat control design method. We exhibit such a strategy.

Without loss of generality, we can assume that $(s_{\mathcal{P}})_{iq} = 0$ for all $i \neq q$, in which case every matrix A in $\mathcal{A}(S_{\mathcal{P}})$ has the structure

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1,q-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{q-1,1} & \cdots & A_{q-1,q-1} & 0 \\ A_{q1} & \cdots & A_{q,q-1} & A_{qq} \end{bmatrix}.$$

Define $\bar{x}_0 = [\ x_1(0) \quad \cdots \quad x_{q-1}(0)\ ]^T$, and let control design strategy Γ be defined by

$$\begin{bmatrix} -B_{11}^{-1}A_{11} & \cdots & -B_{11}^{-1}A_{1,q-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -B_{q-1,q-1}^{-1}A_{q-1,1} & \cdots & -B_{q-1,q-1}^{-1}A_{q-1,q-1} & 0 \\ K_{q1}(A,B) & \cdots & K_{q,q-1}(A,B) & K_{qq}(A,B) \end{bmatrix}$$

for all $P = (A, B, x_0) \in \mathcal{P}$, with

$$\begin{aligned} \bar{K}(A,B) :&= [\ K_{q1}(A,B) \quad \cdots \quad K_{q,q-1}(A,B) \quad K_{qq}(A,B)\ ] \\ &= -(I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq} [A]_q, \end{aligned}$$

where X_{qq} is the unique positive definite solution to the discrete algebraic Riccati equation

$$\begin{aligned} A_{qq}^T X_{qq} B_{qq} (I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq} A_{qq} \\ - A_{qq}^T X_{qq} A_{qq} + X_{qq} - I = 0. \end{aligned} \tag{17}$$

In words, control design strategy Γ applies the deadbeat strategy to subsystems 1 to $q-1$ while, on subsystem q, it uses the same subcontroller as in the optimal controller for the plant

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k), \tag{18}$$

with cost function

$$J^{(2)}_{(A,B,x_0)}(\bar{K}) = \sum_{k=1}^{\infty} \hat{x}(k)^T Q \hat{x}(k) + \sum_{k=0}^{\infty} \hat{u}(k)^T \hat{u}(k),$$

where $Q = \mathrm{diag}(0, \dots, 0, I_{n_q \times n_q})$, the matrix $\hat{A}$ is defined as $[\hat{A}]_q = [A]_q$ and $[\hat{A}]_z = 0$ for all $z \neq q$, and furthermore, the matrix $\hat{B}$ is defined as $\hat{B} = \mathrm{diag}(0, \dots, 0, B_{qq})$. Note that Γ is indeed communication-less since $\bar{K}(A, B)$ defined above can be computed with the sole knowledge of the q^{th} lower block of A and B. Because of the structure of matrices in $\mathcal{A}(S_{\mathcal{P}})$ and this characterization of Γ, we have

$$J_{(A,B,x_0)}(\Gamma(A,B)) = J^{(1)}_{(A,B,x_0)} + J^{(2)}_{(A,B,x_0)}(\bar{K}(A,B)),$$

where $J^{(1)}_{(A,B,x_0)} = \bar{x}_0^T \bar{A}^T \bar{B}^{-T} \bar{B}^{-1} \bar{A} \bar{x}_0$, with

$$\bar{A} = \begin{bmatrix} A_{11} & \cdots & A_{1,q-1} \\ \vdots & \ddots & \vdots \\ A_{q-1,1} & \cdots & A_{q-1,q-1} \end{bmatrix},$$

and $\bar{B} = \mathrm{diag}(B_{11}, \dots, B_{q-1,q-1})$ and $J^{(2)}_{(A,B,x_0)}(\bar{K}(A,B))$ is the closed-loop cost for system (18). Since $\bar{K}(A,B)$ is the optimal controller for this cost, $J^{(2)}_{(A,B,x_0)}(\bar{K}(A,B))$ $= x_0^T \hat{A}^T W \hat{A} x_0$, where

$$W = \mathrm{diag}(0, \dots, 0, X_{qq} - X_{qq} B_{qq}(I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq}).$$

Using part 2 of Subsection 3.5.2 in [46], we have the matrix inversion identity

$$X - XY(I + ZXY)^{-1} ZX = (X^{-1} + YZ)^{-1},$$

which results in

$$\begin{aligned} W_{qq} &= X_{qq} - X_{qq} B_{qq}(I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq} \\ &= (X_{qq}^{-1} + B_{qq} B_{qq}^T)^{-1} \\ &< B_{qq}^{-T} B_{qq}^{-1}. \end{aligned}$$

Note that X_{qq}^{-1} exists because $X_{qq} \geq I$ which follows from the discrete algebraic Riccati equation in (17). This inequality implies that

$$\hat{A}^T W \hat{A} < \hat{A}^T (\hat{B}^\dagger)^T \hat{B}^\dagger \hat{A}$$

where $\hat{B}^{\dagger} = \mathrm{diag}(0, \dots, 0, B_{qq}^{-1})$. Thus

$$\begin{aligned} J_{(A,B,x_0)}(\Gamma(A,B)) &= J^{(1)}_{(A,B,x_0)} + J^{(2)}_{(A,B,x_0)}(\bar{K}(A,B)) \\ &< J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B)), \end{aligned}$$

for all $P = (A, B, x_0) \in \mathcal{P}$ such that the q^{th} lower block of A is not zero, otherwise $J_{(A,B,x_0)}(\Gamma(A,B)) = J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B))$. Thus, control design method Γ dominates the deadbeat control design method Γ^{Δ}. ■

Remark 1.6 *Consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 2(a), the control graph $G'_{\mathcal{K}}$ in Figure 2(b′), and the design graph $G'_{\mathcal{C}}$ in Figure 2(c′). Theorems 1.5 and 1.7 show that the deadbeat control design strategy Γ^{Δ} is the best control design strategy that one can propose based on the local model of subsystems and the plant graph, because the deadbeat control design strategy is the minimizer of the competitive ratio and it is undominated.*

Remark 1.7 *It should be noted that, the proof of the "only if" part of the Theorem 1.7 is constructive. We use this construction to build a control design strategy for the plant graphs with sinks in next subsection.*

3.2 Second case: plant graph $G_{\mathcal{P}}$ with at least one sink

In this section, we consider the case where plant graph $G_{\mathcal{P}}$ has $c \geq 1$ sinks. Accordingly, its adjacency matrix $S_{\mathcal{P}}$ is of the form

$$S_{\mathcal{P}} = \left[\begin{array}{c|c} (S_{\mathcal{P}})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_{\mathcal{P}})_{21} & (S_{\mathcal{P}})_{22} \end{array}\right], \tag{19}$$

where

$$(S_{\mathcal{P}})_{11} = \begin{bmatrix} (s_{\mathcal{P}})_{11} & \cdots & (s_{\mathcal{P}})_{1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q-c,1} & \cdots & (s_{\mathcal{P}})_{q-c,q-c} \end{bmatrix},$$

$$(S_{\mathcal{P}})_{21} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,1} & \cdots & (s_{\mathcal{P}})_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q,1} & \cdots & (s_{\mathcal{P}})_{q,q-c} \end{bmatrix},$$

and

$$(S_{\mathcal{P}})_{22} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,q-c+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (s_{\mathcal{P}})_{qq} \end{bmatrix},$$

where we assume, without loss of generality, that the vertices are numbered such that the sinks are labeled $q-c+1,\ldots,q$. With this notation, let us now introduce the control design method Γ^{Θ} defined by

$$\Gamma^{\Theta}(A,B) = -\mathrm{diag}(B_{11}^{-1},\ldots,B_{q-c,q-c}^{-1},W_{q-c+1}(A,B),\ldots,W_q(A,B))A \tag{20}$$

for all $(A,B) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon)$, where

$$W_i(A,B) = (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} \tag{21}$$

for all $q-c+1 \leq i \leq q$ and X_{ii} is the unique positive definite solution of the discrete algebraic Riccati equation

$$A_{ii}^T X_{ii} B_{ii}(I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} A_{ii} - A_{ii}^T X_{ii} A_{ii} + X_{ii} - I = 0. \tag{22}$$

The control design method Γ^{Θ} applies the deadbeat strategy to every subsystem that is not a sink and, for every sink, applies the same optimal control law as if the node were decoupled from the rest of the graph. We will show that when the plant graph contains sinks, Γ^{Θ} has, in worst case, the same competitive ratio as the deadbeat strategy. Unlike the deadbeat strategy, it has the additional property of being undominated by communication-less methods for plants in $\mathcal{P}$ when the plant graph $G_{\mathcal{P}}$ has sinks.

Lemma 1.8 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Let Γ be a control design strategy in $\mathcal{C}$. Suppose that there exist i and $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$ and that node i is not a sink. The competitive ratio of Γ is bounded only if*

$$A_{ij} + B_{ii}\Gamma_{ij}(A,B) = 0, \quad \textit{for all } P = (A,B,x_0) \in \mathcal{P}.$$

Proof: See [43, p.79–80]. ∎

Remark 1.8 *Lemma 1.8 shows that a necessary condition for a bounded competitive ratio is to decouple the nodes that are not sinks from the rest of the network.*

Now, we are ready to compute the competitive ratio of the newly defined control design strategy Γ^{Θ}. This is done at first for the case where the control graph $G_{\mathcal{K}}$ is a complete graph.

Theorem 1.9 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink, and the control graph $G_{\mathcal{K}}$ be a complete graph. Then the competitive ratio of the communication-less design method Γ^{Θ} introduced in (20) is*

$$r_{\mathcal{P}}(\Gamma^{\Theta}) = \begin{cases} 1, & \textit{if } (S_{\mathcal{P}})_{11} = 0 \textit{ and } (S_{\mathcal{P}})_{22} = 0, \\ 1 + 1/\epsilon^2, & \textit{otherwise.} \end{cases}$$

Proof: Based on Theorem 1.2 we know that, for every plant $P = (A, B, x_0) \in \mathcal{P}$

$$J_{(A,B,x_0)}(K^*(A,B)) \geq \frac{\epsilon^2}{1+\epsilon^2} x_0^T A^T B^{-T} B^{-1} A x_0, \tag{23}$$

In addition, proceeding as in the proof of the "only if" part of the Theorem 1.7, we know that

$$J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B)) \geq J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B)). \tag{24}$$

Plugging equation (24) into equation (23) results in

$$\frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} \leq 1 + \frac{1}{\epsilon^2},\ \forall P = (A,B,x_0) \in \mathcal{P}.$$

As a result, $r_{\mathcal{P}}(\Gamma^{\Theta}) \leq 1 + 1/\epsilon^2$. To show that this upper-bound is tight, we now exhibit plants for which it is attained. We use a different construction depending on matrices $(S_{\mathcal{P}})_{11}$ and $(S_{\mathcal{P}})_{22}$. If $(S_{\mathcal{P}})_{11} \neq 0$, two situations can occur.

Case 1: $(S_{\mathcal{P}})_{11} \neq 0$ *and it is not diagonal.* There exist $1 \leq i \neq j \leq q - c$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. In this case, choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A = e_{i_1} e_{j_1}^T$ and $B = \epsilon I$. Then, for $x_0 = e_{j_1}$, we find that

$$\frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} = \frac{1/\epsilon^2}{1/(1+\epsilon^2)} = 1 + \frac{1}{\epsilon^2}$$

because the control design Γ^{Θ} acts like the deadbeat control design method on this plant.

Case 2: $(S_{\mathcal{P}})_{11} \neq 0$ *and it is diagonal.* There exists $1 \leq i \leq q - c$ such that $(s_{\mathcal{P}})_{ii} \neq 0$. Pick an index $i_1 \in \mathcal{I}_i$. In that case, consider $A(r) = r e_{i_1} e_{i_1}^T$ and $B = \epsilon I$. For $x_0 = e_{i_1}$, the optimal cost is

$$J_{(A(r),B,x_0)}(K^*(A(r),B)) = \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon^2},$$

which results in

$$\lim_{r \to 0} \frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} = 1 + \frac{1}{\epsilon^2}.$$

Now suppose that $(S_{\mathcal{P}})_{11} = 0$. Again, two different situations can occur.

Case 3: $(S_{\mathcal{P}})_{11} = 0$ *and* $(S_{\mathcal{P}})_{22} \neq 0$. There exists $q - c + 1 \leq i \leq q$ such that $(s_{\mathcal{P}})_{ii} \neq 0$. From the assumption that the plant graph contains no isolated node, we know that there must exist $1 \leq j \leq q - c$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Accordingly, let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and consider the 2-parameter family of matrices $A(r,s)$ in $\mathcal{A}(S_{\mathcal{P}})$ with all entries equal to zero except $a_{i_1 i_1}$, which is equal to r, and $a_{i_1 j_1}$, which is equal to s. Let $B = \epsilon I$. For any initial condition x_0, the corresponding closed-loop performance is

$$J_{(A(r,s),B,x_0)}(\Gamma^{\Theta}(A(r,s),B)) = \beta_{\Theta} x_0^T a(r,s) a(r,s)^T x_0,$$

where we have let $a(r,s) = A(r,s)_{i_1}^T$ and β_Θ is

$$\beta_\Theta = \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon^2 r^2}.$$

Besides, the optimal closed-loop performance can be computed as

$$J_{(A(r,s),B,x_0)}(K^*(A(r,s),B)) = \beta_{K^*} x_0^T a(r,s) a(r,s)^T x_0,$$

where β_{K^*} is

$$\beta_{K^*} = \frac{\epsilon^2 s^2 + r^2(1+\epsilon^2) - (\epsilon^2+1)^2 + \sqrt{c_+ c_-}}{2\epsilon^2(\epsilon^2+1)(s^2+r^2)},$$
$$c_\pm = (\epsilon^2 s^2 + (r^2 \pm 2r)(\epsilon^2+1) + (\epsilon^2+1)^2).$$

Then,

$$\begin{aligned} r_{\mathcal{P}}(\Gamma^\Theta) &\geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_{(A(r,s),B,x_0)}(\Gamma^\Theta(A(r,s),B))}{J_{(A(r,s),B,x_0)}(K^*(A(r,s),B))} \\ &= 1 + \frac{1}{\epsilon^2} \end{aligned}$$

Case 4: $(S_{\mathcal{P}})_{11} = 0$ *and* $(S_{\mathcal{P}})_{22} = 0$. Then, every matrix $A \in \mathcal{A}(S_{\mathcal{P}})$ has the form $\left[\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right]$ and, in particular, is nilpotent of degree 2; i.e., $A^2 = 0$. In this case, the Riccati equation yielding the optimal control gain $K^*(A,B)$ can be readily solved, and we find that $K^*(A,B) = -(I + B^T B)^{-1} B^T A$ for all (A,B). As a result, $K^*(A,B) = \Gamma^\Theta(A,B)$ for all plant $P = (A,B,x_0) \in \mathcal{P}$ (since $W_i(A,B) = (I + B_{ii}^T B_{ii})^{-1} B_{ii}^T$ for all $q - c + 1 \leq i \leq q$), which implies that the competitive ratio of Γ^Θ against plants in $\mathcal{P}$ is equal to one. ■

In Theorem 1.9, the control graph $G_{\mathcal{K}}$ is assumed to be a complete graph. We needed this assumption to calculate the cost of the optimal control design strategy $K^*(P)$ when $(S_{\mathcal{P}})_{11} = 0$ and $(S_{\mathcal{P}})_{22} \neq 0$ which is not an easy task when the control graph $G_{\mathcal{K}}$ is incomplete. However, more can be said if $(S_{\mathcal{P}})_{11} \neq 0$.

Corollary 1.10 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then*

$$r_{\mathcal{P}}(\Gamma^\Theta) = \begin{cases} 1, & \text{if } (S_{\mathcal{P}})_{11} = 0 \text{ and } (S_{\mathcal{P}})_{22} = 0, \\ 1 + 1/\epsilon^2, & \text{if } (S_{\mathcal{P}})_{11} \neq 0. \end{cases}$$

Proof: According to Theorem 1.9, for $(S_{\mathcal{P}})_{11} \neq 0$, we get

$$\begin{aligned} r_{\mathcal{P}}(\Gamma^\Theta) &= \sup_{P\in\mathcal{P}} \frac{J_{(A,B,x_0)}(\Gamma^\Theta(A,B))}{J_{(A,B,x_0)}(K^*(P))} \\ &\leq \sup_{P\in\mathcal{P}} \frac{J_{(A,B,x_0)}(\Gamma^\Theta(A,B))}{J_{(A,B,x_0)}(K_C^*(A,B))} = 1 + \frac{1}{\epsilon^2}. \end{aligned}$$

Case 1: $(S_{\mathcal{P}})_{11} \neq 0$ *and it is not diagonal.* For the special plant introduced in Case 1 in the proof of Theorem 1.9, we have $J_{(A,B,e_{j_1})}(K_C^*(A,B)) = J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1}))$ since $A = e_{i_1}e_{j_1}^T$ is a nilpotent matrix. The rest of the proof is similar to Case 1 in the proof of Theorem 1.9.
Case 2: $(S_{\mathcal{P}})_{11} \neq 0$ *and it is diagonal.* Note that, for the special plant introduced Case 2 in the proof of Theorem 1.9, we have

$$K_C^*(A,B) = -\frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon r^2} A$$

which shows $K_C^*(A,B) \in \mathcal{K}(S_{\mathcal{K}})$ and similar to the proof of Theorem 1.2, we get $J_{(A,B,e_{i_1})}(K_C^*(A,B)) = J_{(A,B,e_{i_1})}(K^*(A,B,e_{i_1}))$. The rest of the proof is similar to Case 2 in the proof of Theorem 1.9.
Case 3: $(S_{\mathcal{P}})_{11} = 0$ *and* $(S_{\mathcal{P}})_{22} = 0$. Then, every $A \in \mathcal{A}(S_{\mathcal{P}})$ is nilpotent matrix which results in $J_P(K^*(P)) = J_P(K_C^*(A,B))$. The rest of the proof is similar to Case 4 in the proof of Theorem 1.9. ∎

Now that we have computed the competitive ratio of the control design strategy Γ^Θ in the presence of sinks, we present a theorem to show that the competitive ratio of any other communication-less control design strategy is lower-bounded by the competitive ratio of Γ^Θ when the control graph $G_{\mathcal{K}}$ is a complete graph. Therefore, the control design strategy Γ^Θ is a minimizer of the competitive ratio over the set of limited model information control design strategies.

Theorem 1.11 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink, the control graph $G_{\mathcal{K}}$ be a complete graph, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops. Then the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2,$$

if either $(S_{\mathcal{P}})_{11}$ is not diagonal or $(S_{\mathcal{P}})_{22} \neq 0$.

Proof: *Case 1:* $(S_{\mathcal{P}})_{11} \neq 0$ *and it is not diagonal.* Then, there exist $1 \leq i, j \leq q-c$ and $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and consider the matrix A defined by $A = e_{i_1}e_{j_1}^T$ and $B = \epsilon I$. From Lemma 1.8, we know that a communication-less method Γ has a bounded competitive ratio only if $\Gamma(A,B) = -B^{-1}A$ (because node i is a part of $(S_{\mathcal{P}})_{11}$ and it is not a sink). Therefore

$$r_{\mathcal{P}}(\Gamma) \geq \frac{J_{(A,B,e_{j_1})}(\Gamma(A,B))}{J_{(A,B,e_{j_1})}(K^*(A,B))} = 1 + \frac{1}{\epsilon^2}$$

for any such method.
Case 2: $(S_{\mathcal{P}})_{22} \neq 0$. There thus exists $q - c + 1 \leq i \leq q$ such that $(s_{\mathcal{P}})_{ii} \neq 0$. Note that, there exists $1 \leq j \leq q - c$ such that $(s_{\mathcal{P}})_{ij} \neq 0$, since there is no isolated node in the plant graph. Choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Consider A defined as $A = re_{i_1}e_{j_1}^T + se_{i_1}e_{i_1}^T$ and $B = \epsilon I$. As indicated in the proof of Theorem 1.9,

control design strategy Γ^Θ yields the globally optimal controller with limited model information for plants in this family. Hence, we know that $r_\mathcal{P}(\Gamma) \geq 1 + 1/\epsilon^2$ for every communication-less strategy Γ. ■

In Theorem 1.11, we assume the control graph $G_\mathcal{K}$ is a complete graph. In the next corollary, we generalize this result to the case where $G_\mathcal{K}$ is a supergraph of $G_\mathcal{P}$ when $(S_\mathcal{P})_{11}$ is not diagonal.

Corollary 1.12 *Let the plant graph $G_\mathcal{P}$ contain no isolated node and at least one sink, the design graph $G_\mathcal{C}$ be a totally disconnected graph with self-loops, and $G_\mathcal{K} \supseteq G_\mathcal{P}$. Then the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_\mathcal{P}(\Gamma) \geq 1 + 1/\epsilon^2,$$

if $(S_\mathcal{P})_{11}$ is not diagonal.

Proof: Considering that for the nilpotent matrix $A = e_{i_1} e_{j_1}^T$, we get $J_{(A,B,e_{j_1})}(K^*(A, B, e_{j_1})) = J_{(A,B,e_{j_1})}(K_C^*(A, B))$, the rest of the proof is similar to Case 1 in the proof of Theorem 1.11. ■

Remark 1.9 *Combining Theorems 1.9 and 1.11 implies that if either $(S_\mathcal{P})_{11}$ is not diagonal or $(S_\mathcal{P})_{22} \neq 0$, control design method Γ^Θ exhibits the same competitive ratio as the deadbeat control strategy, which is the smallest ratio achievable by a communication-less control method. Therefore, it is a solution to problem (10). Furthermore, if $(S_\mathcal{P})_{11}$ and $(S_\mathcal{P})_{22}$ are both zero, then Γ^Θ is equal to K^*, which shows that Γ^Θ is a solution to problem (10), in this case too.*

Remark 1.10 *The case where $(S_\mathcal{P})_{11}$ is diagonal and $(S_\mathcal{P})_{22} = 0$ is still open.*

The next theorem shows that Γ^Θ is a more desirable control design method than the deadbeat control design strategy when the plant graph $G_\mathcal{P}$ has sinks, since it is then undominated by communication-less design methods.

Theorem 1.13 *Let the plant graph $G_\mathcal{P}$ contain no isolated node and at least one sink, the design graph $G_\mathcal{C}$ be a totally disconnected graph with self-loops, and $G_\mathcal{K} \supseteq G_\mathcal{P}$. The control design method Γ^Θ is undominated by any control design method $\Gamma \in \mathcal{C}$.*

Proof: See [43, p.80–82]. ■

Remark 1.11 *Consider the limited model information design problem given by the plant graph $G'_\mathcal{P}$ in Figure 2(a′), the control graph $G'_\mathcal{K}$ in Figure 2(b′), and the design graph $G'_\mathcal{C}$ in Figure 2(c′). Theorems 1.9, 1.11, and 1.13 together show that, the control design strategy Γ^Θ is the best control design strategy that one can propose based on the local model information and the plant graph, because the control design strategy Γ^Θ is a minimizer of the competitive ratio and it is undominated.*

Remark 1.12 *For general weight matrices Q and R appearing in the performance cost, the competitive ratio of both the deadbeat control design strategy Γ^Δ and the control design strategy Γ^Θ is $1 + \bar{\sigma}(R)/(\underline{\sigma}(Q)\epsilon^2)$. In particular, the competitive ratio has a limit equal to one as $\bar{\sigma}(R)/\underline{\sigma}(Q)$ goes to zero. We thus recover the well-known observation (e.g., [47]) that, for discrete-time linear time-invariant systems, the optimal linear quadratic regulator approaches the deadbeat controller in the limit of "cheap control".*

4 Design Graph Influence on Achievable Performance

In the previous section, we have shown that communicat-ion-less control design methods (i.e., $G_\mathcal{C}$ is totally disconnected with self-loops) have intrinsic performance limitations, and we have characterized minimal elements for both the competitive ratio and domination metrics. A natural question is "given plant graph $G_\mathcal{P}$, which design graph $G_\mathcal{C}$ is necessary to ensure the existence of $\Gamma \in \mathcal{C}$ with better competitive ratio than Γ^Δ and Γ^Θ ?". We tackle this question in this section.

Theorem 1.14 *Let the plant graph $G_\mathcal{P}$ and the design graph $G_\mathcal{C}$ be given and $G_\mathcal{K} \supseteq G_\mathcal{P}$. If one of the following conditions is satisfied then $r_\mathcal{P}(\Gamma) \geq 1 + 1/\epsilon^2$ for all $\Gamma \in \mathcal{C}$:*

(a) $G_\mathcal{P}$ contains the path $k \to i \to j$ with distinct nodes i, j, and k while $(j,i) \notin E_\mathcal{C}$.

(b) There exist $i \neq j$ such that $n_i \geq 2$ and $(i,j) \in E_\mathcal{P}$ while $(j,i) \notin E_\mathcal{C}$.

Proof: We prove the case when condition (a) holds. The proof for condition (b) is similar.

Let i, j, and k be three distinct nodes such that $(s_\mathcal{P})_{ik} \neq 0$ and $(s_\mathcal{P})_{ji} \neq 0$ (i.e., the path $k \to i \to j$ is contained in the plant graph $G_\mathcal{P}$). Let us pick $i_1 \in \mathcal{I}_i$, $j_1 \in \mathcal{I}_j$ and $k_1 \in \mathcal{I}_k$ and consider the 2-parameter family of matrices $A(r,s)$ in $\mathcal{A}(S_\mathcal{P})$ with all entries equal to zero except $a_{i_1k_1}$, which is equal to r, and $a_{j_1i_1}$, which is equal to s. Let $B = \epsilon I$ and let $\Gamma \in \mathcal{C}$ be a limited model information with design graph $G_\mathcal{C}$. For $x_0 = e_{k_1}$, we have

$$J_{(A(r,s),B,e_{k_1})}(\Gamma(A(r,s),B)) \geq (r + \epsilon\gamma_{i_1k_1}(A,B))^2[\gamma_{j_1i_1}^2(A,B) + (s + \epsilon\gamma_{j_1i_1}(A,B))^2]$$

where $\gamma_{i_1k_1}$ cannot be a function of s because $(j,i) \notin E_\mathcal{C}$. Note that, irrespective of the choice of $\gamma_{j_1i_1}(A,B)$, we have

$$J_{(A(r,s),B,e_{k_1})}(\Gamma(A(r,s),B)) \geq \frac{(r + \epsilon\gamma_{i_1k_1}(A,B))^2 s^2}{1+\epsilon^2}.$$

The cost of the deadbeat control design on this plant satisfies

$$J_{(A(r,s),B,e_{k_1})}(\Gamma^\Delta(A(r,s),B)) = r^2/\epsilon^2,$$

and thus

$$\begin{aligned}
r_{\mathcal{P}}(\Gamma) &= \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))} \\
&= \sup_{P\in\mathcal{P}} \left[\frac{J_P(\Gamma(A,B))}{J_P(\Gamma^\Delta(A,B))}\frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(P))}\right] \\
&\geq \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(\Gamma^\Delta(A,B))}, \\
&\geq \lim_{s\to\infty} \frac{\epsilon^2(r+\epsilon\gamma_{i_1k_1}(A,B))^2 s^2}{(1+\epsilon^2)r^2}.
\end{aligned} \tag{25}$$

This shows that $r_{\mathcal{P}}(\Gamma)$ is unbounded unless $r + \epsilon\gamma_{i_1k_1}(A(r,s),B) = 0$ for all r, s. Now consider the 1-parameter family of matrices $\bar{A}(r)$ with all entries equal to zero except $a_{i_1k_1}$, which is equal to r. Because of $(j,i) \notin E_{\mathcal{C}}$, we know that $\Gamma_z(\bar{A}(r),B) = \Gamma_z(A(r,s),B)$ for all $z \in \mathcal{I}_i$. Thus

$$J_{(\bar{A}(r),B,e_{k_1})}(\Gamma(\bar{A}(r),B)) \geq r^2/\epsilon^2.$$

On the other hand, similar to the proof of Theorem 1.2, we can compute the optimal controller for systems in this 1−parameter family and find

$$\begin{aligned}
J_{(\bar{A}(r),B,e_{k_1})}(K^*(\bar{A}(r),B,e_{k_1})) &= J_{(\bar{A}(r),B,e_{k_1})}(K^*_C(\bar{A}(r),B)) \\
&= r^2/(1+\epsilon^2),
\end{aligned}$$

As a result, we get

$$r_{\mathcal{P}}(\Gamma) \geq \frac{r^2/\epsilon^2}{r^2/(1+\epsilon^2)} = 1 + \frac{1}{\epsilon^2},$$

which concludes the proof for this case. ■

Remark 1.13 *Consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 2(a), the control graph $G'_{\mathcal{K}}$ in Figure 2(b′), and the design graph $G_{\mathcal{C}}$ in Figure 2(c). Theorem 1.14 shows that, because the plant graph $G_{\mathcal{P}}$ contains the path $3 \to 2 \to 1$ but the design graph $G_{\mathcal{C}}$ does not contain $1 \to 2$, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ would be greater than or equal to $1 + 1/\epsilon^2$.*

Corollary 1.15 *Let both the plant graph $G_{\mathcal{P}}$ and the control graph $G_{\mathcal{K}}$ be complete graphs. If the design graph $G_{\mathcal{C}}$ is not equal to $G_{\mathcal{P}}$, then $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$ for all $\Gamma \in \mathcal{C}$.*

Proof: The proof is a direct application of Theorem 1.14 with condition (a) fulfilled. ■

Remark 1.14 *Corollary 1.15 shows that, when $G_{\mathcal{P}}$ is a complete graph, achieving a better competitive ratio than the deadbeat design strategy requires each subsystem to have full knowledge of the plant model when constructing each subcontroller.*

5 Extensions to Under-Actuated Sinks

In the previous sections, we gave an explicit solution to the problem in (10) under the assumption that all the subsystems are fully-actuated; i.e., all the matrices $B \in \mathcal{B}(\epsilon)$ are square invertible matrices. Note that this assumption stems from the fact that the subsystems that are not sinks in the plant graph are required to decouple themselves from the rest of the plant to avoid influencing highly sensitive (and potentially hard to control) subsystems in order to keep the competitive ratio finite (see Lemma 1.8). Therefore, we assume these subsystems are fully-actuated to easily decouple them from the rest of the system. As a future direction for improvement, one can try to replace this assumption with other conditions (e.g., geometric conditions) to ensure that the subsystems can decouple themselves. From the same argument, it should be expected that the assumption of a square invertible B-matrix is dispensable for sink nodes. In this section, we briefly discuss an extension of our results to the slightly more general, but still restricted, class of plants whose sinks are under-actuated.

Consider the limited model information control design problem given with the plant graph $G_{\mathcal{P}}$, the control graph $G_{\mathcal{K}}$, and the design graph $G_{\mathcal{C}}$ given in Figure 3. The state space representation of the system is given as

$$\begin{bmatrix} \underline{x}_1(k+1) \\ \underline{x}_2(k+1) \end{bmatrix} = A \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \end{bmatrix} + B \begin{bmatrix} \underline{u}_1(k) \\ \underline{u}_2(k) \end{bmatrix},$$

where

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix},$$

with $\underline{x}_1(k) \in \mathbb{R}^{n_1}$, $\underline{x}_2(k) \in \mathbb{R}^{n_2}$, $\underline{u}_1(k) \in \mathbb{R}^{n_1}$, and $\underline{u}_2(k) \in \mathbb{R}^{m_2}$ for some given integers $n_1 \geq 1$, $n_2 > m_2 \geq 1$. Thus, for the second subsystem the matrix $B_{22} \in \mathbb{R}^{n_2 \times m_2}$ is a non-square matrix, and as a result the second subsystem is an under-actuated subsystem. Let us assume that the matrices A_{21}, A_{22}, B_{22} satisfy the "matching condition"; i.e., the pair (A_{22}, B_{22}) is controllable and $\text{span}(A_{21}) \subseteq \text{span}(B_{22})$ [48]. Besides, assume that for all matrices B, we have $\underline{\sigma}(B) \geq \epsilon$ for some $\epsilon > 0$. For this case, we have

$$\Gamma^{\Theta}(A, B) = -\,\text{diag}(B_{11}^{-1}, W_2(A_{22}, B_{22}))A,$$

where $W_2(A_{22}, B_{22})$ is defined in (21). Note that we do not require the matrix B_{22} to be square invertible. Under some additional conditions and following a similar approach as above, it can be shown that the control design strategy Γ^{Θ} becomes an undominated minimizer of the competitive ratio over the set of limited model information control design strategies. This result can be generalized to cases with higher number of subsystems as long as the sinks in the plant graph $G_{\mathcal{P}}$ are the only under-actuated subsystems [49].

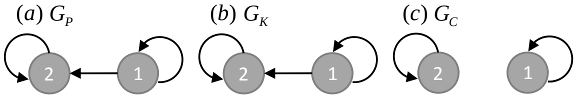

Figure 3: Plant graph $G_{\mathcal{P}}$, control graph $G_{\mathcal{K}}$, and design graph $G_{\mathcal{C}}$ used to illustrate an extension to under-actuated systems.

6 Conclusion

We presented a framework for the study of control design under limited model information, and investigated the connection between the quality of controllers produced by a design method and the amount of plant model information available to it. We showed that the best performance achievable by a limited model information control design method crucially depends on the structure of the plant graph and, thus, that giving the designer access to this graph, even without a detailed model of all plant subsystems, results in superior design, in the sense of domination. Possible future work will focus on extending the present framework to dynamic controllers and/or where disturbances are present.

7 Bibliography

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PAPER 2

Dynamic Control Design Based on Limited Model Information

Farhad Farokhi and Karl H. Johansson

Abstract–The design of optimal H_2 dynamic controllers for interconnected linear systems using limited plant model information is considered. Control design strategies based on various degrees of model information are compared using the competitive ratio as a performance metric, that is, the worst case control performance for a given design strategy normalized with the optimal control performance based on full model information. An explicit minimizer of the competitive ratio is found. It is shown that this control design strategy is not dominated by any other strategy with the same amount of model information. The result applies to a class of system interconnections and design information characterized through given plant, control, and design graphs.

1 Introduction

Many large-scale physical systems are composed of several smaller interconnected units. For these interconnected systems, it seems natural to employ local controllers which observe local states and control local inputs. The problem of designing such subcontrollers is usually addressed in the decentralized and distributed control literature [1–3]. Lately, there has been some efforts in formulating the problem of designing optimal decentralized controllers as a convex optimization problem for some specific classes of subsystem interconnection [4–8]. At the heart of all these decentralized and distributed control problems is the assumption that the control design is done with complete knowledge of the plant model. This is however not always possible in large-scale systems. It might be the case that (a) different subsystems belong to different individuals and they might be unwilling to share their model information since they may consider these information private, (b) the design of each subcontroller is done by a different designer with no access to the global plant model since in the time of design the complete model information is not available, or (c) the designer is interested in designing each subcontroller using only local model information, so that the resulting subcontrollers do not need to be modified if the model parameters of a particular subsystem change over time. We call this special class of control design problems limited model information control design problems [9, 10]. In these problems, we assume that only some part of the plant model information is available to each subcontroller designer, but that the system interconnection structure and the common closed-loop cost function to be minimized are global knowledge.

The main contribution of this paper is to study the influence of the subsystem interconnection, the controller structure, and the amount of model information available to each subdesign on the closed-loop performance that a limited model information control design method can produce. We compare the control design methods using a performance metric called the competitive ratio, that is, the worst case control performance for a given design strategy normalized with the optimal control performance based on full model information. We find an explicit minimizer of the competitive ratio for a wide range of problems. Since this minimizer might not be unique, we show that it is also undominated, that is, there is no other control design method that acts always better while having the same worst-case ratio.

This paper is organized as follows. We formulate the problem of interest in Section 2. We define a control design strategy and find its competitive ratio in Section 3. In Section 4, we study the influence of interconnection pattern between different subsystems on the best limited model information control design method. We further study the achievable performance of limited model information design strategies when the controllers that they can produce are structured in Section 5. The trade-off between the amount of plant information available to different parts of a control design strategy and the quality of controllers it can produce is considered in Section 6. Finally, we give the discussions on extensions in Section 7 and end with the conclusions in Section 8.

1.1 Notation

The sets of integer numbers, natural numbers, real numbers, and complex numbers are denoted respectively by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$. The boundary of the unit circle in $\mathbb{C}$ is shown by $\mathbb{T}$. The space of Lebesgue measurable functions that are bounded on $\mathbb{T}$ is presented by $\mathcal{L}_\infty$ and $\mathcal{RL}_\infty$ is the set of real proper rational transfer functions in $\mathcal{L}_\infty$. Additionally, all other sets are denoted by calligraphic letters such as $\mathcal{P}$ and $\mathcal{A}$.

Matrices are denoted by capital roman letters such as A. The entry in the i^{th} row and the j^{th} column of matrix A is a_{ij}. A_j will denote the j^{th} row of A. A_{ij} denotes a submatrix of matrix A, the dimension and the position of which will be defined in the text.

$A > (\geq) 0$ means that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq) B$ means $A - B > (\geq) 0$. Let $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$.

All graphs considered in this paper are directed with vertex set $\{1, \ldots, q\}$ for a given $q \in \mathbb{N}$. All self-loops are present in the graphs that we consider in this paper, that is, $(i,i) \in E$ for all $1 \leq i \leq q$. We say that a vertex i is a sink if there does not exist $j \neq i$ such that $(i,j) \in E$. The adjacency matrix $S \in \{0,1\}^{q\times q}$ of graph G is a matrix whose entry $s_{ij} = 1$ if $(j,i) \in E$ and $s_{ij} = 0$ otherwise for all $1 \leq i,j \leq q$. In this paper, since the set of vertices is fixed for all the graphs, a subgraph of a graph G is a graph whose edge set is a subset of the edge set of G and a supergraph of a graph G is a graph of which G is a subgraph. We use the notation $G' \supseteq G$ to indicate that G' is a supergraph of G.

$\underline{\sigma}(Y)$ and $\bar{\sigma}(Y)$ denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i denotes the column vector with all entries zero except the i^{th} entry which is equal to one. The function $\delta : \mathbb{Z} \to \{0,1\}$ is the unit-impulse function which is equal to one at origin and zero anywhere else.

2 Problem Formulation

2.1 Plant Model

Let a plant graph $G_\mathcal{P}$ with adjacency matrix $S_\mathcal{P}$ be given. Based on the adjacency matrix $S_\mathcal{P}$, we define the following set of matrices

$$\mathcal{A}(S_\mathcal{P}) = \{\bar{A} \in \mathbb{R}^{n\times n} \mid \bar{A}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i,j \leq q \text{ such that } (s_\mathcal{P})_{ij} = 0\},$$

where for each $1 \leq i \leq q$, $n_i \in \mathbb{N}$ is the order of subsystem i and consequently $\sum_{i=1}^q n_i = n$. Besides, we define

$$\mathcal{B}(\epsilon) = \{\bar{B} \in \mathbb{R}^{n\times n} \mid \underline{\sigma}(\bar{B}) \geq \epsilon,\ \bar{B}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i \neq j \leq q\},$$

for some given scalar $\epsilon > 0$ and

$$\mathcal{H} = \{\bar{H} \in \mathbb{R}^{n\times n} \mid \det(\bar{H}) \neq 0,\ \bar{H}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i \neq j \leq q\}.$$

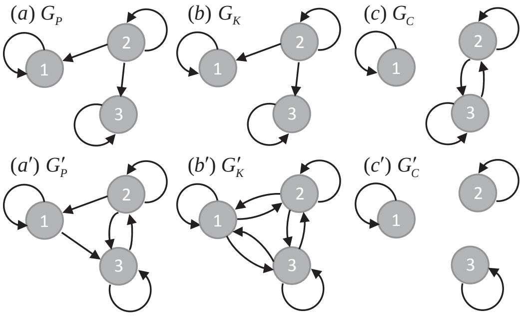

Figure 1: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs, $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs, and $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

Now we can introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant systems

$$x(k+1) = Ax(k) + Bu(k) + Hw(k) \ ; \ x(0) = 0, \tag{1}$$

with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $H \in \mathcal{H}$. With slightly abusing notation, we show a plant $P \in \mathcal{P}$ with triple (A, B, H) since the set $\mathcal{P}$ is clearly isomorph to $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{H}$. We will denote the ordered set of state indices related to subsystem i with $\mathcal{I}_i$, that is, $\mathcal{I}_i := (1 + \sum_{j=1}^{i-1} n_j, \ldots, n_i + \sum_{j=1}^{i-1} n_j)$. For subsystem i, state $\underline{\mathrm{x}}_i \in \mathbb{R}^{n_i}$, control input $\underline{\mathrm{u}}_i \in \mathbb{R}^{n_i}$, and exogenous input $\underline{\mathrm{w}}_i \in \mathbb{R}^{n_i}$ are defined as

$$\underline{\mathrm{x}}_i = \begin{bmatrix} x_{\ell_1} \\ \vdots \\ x_{\ell_{n_i}} \end{bmatrix}, \ \underline{\mathrm{u}}_i = \begin{bmatrix} u_{\ell_1} \\ \vdots \\ u_{\ell_{n_i}} \end{bmatrix}, \ \underline{\mathrm{w}}_i = \begin{bmatrix} w_{\ell_1} \\ \vdots \\ w_{\ell_{n_i}} \end{bmatrix}$$

where the ordered set of indices $(\ell_1, \ldots, \ell_{n_i}) \equiv \mathcal{I}_i$, and its dynamic is specified by

$$\underline{\mathrm{x}}_i(k+1) = \sum_{j=1}^{q} A_{ij}\underline{\mathrm{x}}_j(k) + B_{ii}\underline{\mathrm{u}}_i(k) + H_{ii}\underline{\mathrm{w}}_i(k).$$

An example of a plant graph $G_{\mathcal{P}}$ is given in Figure 1(a). For instance, the plant graph $G_{\mathcal{P}}$ shows that the second subsystem can affect the first and the third subsystems, that is, A_{12} and A_{32} can be nonzero. The first system is also a sink in the plant graph $G_{\mathcal{P}}$. An example of a plant graph $G'_{\mathcal{P}}$ without sink is given in Figure 1(a').

2.2 Controller

Let a control graph $G_{\mathcal{K}}$ with adjacency matrix $S_{\mathcal{K}}$ be given. In this paper, we are interested in dynamic discrete-time linear time-invariant state feedback control laws of the form

$$x_K(k+1) = A_K x_K(k) + B_K x(k)\ ;\ x_K(0) = 0,$$

$$u(k) = C_K x_K(k) + D_K x(k),$$

which can also be represented as the transfer function

$$K \triangleq \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = C_K(zI - A_K)^{-1}B_K + D_K,$$

where z is the symbol for one time-step forward shift operator. The controller K must belong to

$$\mathcal{K}(S_\mathcal{K}) = \{\bar{K} \in\ (\mathcal{RL}_\infty)^{n\times n} | \bar{K}_{ij} = 0 \in (\mathcal{RL}_\infty)^{n_i \times n_j} \\ \text{for all } 1 \le i, j \le q \text{ such that } (s_\mathcal{K})_{ij} = 0\}.$$

We refer to the set of controllers as $\mathcal{K}$ when adjacency matrix $S_\mathcal{K}$ can be deduced from the context or it is not relevant.

Figure 1(b) shows an example of an incomplete control graph $G_\mathcal{K}$ that characterizes a set of structured controllers. For instance, using control graph $G_\mathcal{K}$, we know that the third subsystem only has access to state measurements of the second subsystem beside its own state measurements, that is, $K_{31} = 0$ while K_{32} and K_{33} can be nonzero.

2.3 Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}$. Let a control design method Γ be partitioned according to subsystems dimensions like

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{bmatrix} \tag{2}$$

and a design graph $G_\mathcal{C}$ with adjacency matrix $S_\mathcal{C}$ be given. Each element Γ_{ij} is a mapping $\mathcal{A}(S_\mathcal{P}) \times \mathcal{B}(\epsilon) \times \mathcal{B} \rightarrow (\mathcal{RL}_\infty)^{n_i \times n_j}$. We say that Γ has structure $G_\mathcal{C}$ if, for all $1 \le i \le q$, the subsystem i subcontroller is constructed with the knowledge of those subsystems $1 \le j \le q$ plant model such that $(j, i) \in E_\mathcal{C}$, that is, the mapping $[\ \Gamma_{i1}\ \cdots\ \Gamma_{iq}\ ]$ is only a function of $\{[\ A_{j1}\ \cdots\ A_{jq}\ ], B_{jj}, H_{jj} \mid (s_\mathcal{C})_{ij} \neq 0\}$. The set of all these limited model information control design methods with structure $G_\mathcal{C}$ is denoted by $\mathcal{C}$.

Figure 1(c) shows an example of a design graph $G_\mathcal{C}$. For instance, using this design graph $G_\mathcal{C}$, we realize that the third subsystem model is available to the designer of the second subsystem controller but not the first subsystem model. Figure 1(c') illustrates an example of a fully disconnected design graph $G'_\mathcal{C}$ with self-loops only which shows that the controller of all subsystems are constructed using only their own model information.

2.4 Performance Metric

The considered performance metrics is a modified version of the performance metrics originally defined in [9, 10]. Let us start with introducing the closed-loop performance measure.

To each plant $P = (A, B, H) \in \mathcal{P}$ and controller $K \in \mathcal{K}$, we associate a performance measure which is the H_2 norm of the transfer function between the exogenous input $w(k)$ and the output

$$y(k) = \begin{bmatrix} C^T \ 0 \end{bmatrix}^T x(k) + \begin{bmatrix} 0 \ D^T \end{bmatrix}^T u(k),$$

where the matrices $C \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times n}$ are block diagonal *full-rank* matrices with each diagonal block entry belonging to $\mathbb{R}^{n_i \times n_i}$. Figure 2 illustrates the feedback system with the given controller K and the overall-plant

$$\hat{P} = \left[\begin{array}{c|cc} A & H & B \\ \hline \hat{C} & 0 & \hat{D} \\ I & 0 & 0 \end{array}\right]$$

where $\hat{C} = \begin{bmatrix} C^T \ 0 \end{bmatrix}^T$ and $\hat{D} = \begin{bmatrix} 0 \ D^T \end{bmatrix}^T$. Using the notation $\mathcal{F}(\hat{P}, K)$ for the closed-loop transfer function from $w(k)$ to $y(k)$, the performance measure can be written as

$$J_P(K) = \|\mathcal{F}(\hat{P}, K)\|_2. \tag{3}$$

We make the following standing assumption:

Assumption 2.1 $C = D = I$.

This is without loss of generality because the change of variables $(\bar{x}, \bar{u}) = (Cx, Du)$ transforms the output of the system and its state space representation into

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix}^T \bar{x}(k) + \begin{bmatrix} 0 & I \end{bmatrix}^T \bar{u}(k),$$

and

$$\bar{x}(k+1) = CAC^{-1}\bar{x}(k) + CBD^{-1}\bar{u}(k).$$

This is done without changing the plant, control, or design graphs because of the block diagonal structure of matrices C and D.

Definition 2.1 (Competitive Ratio) *Let a plant graph $G_{\mathcal{P}}$, a control graph $G_{\mathcal{K}}$, and a constant $\epsilon > 0$ be given. Let us assume that, for each plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that*

$$J_P(K^*(P)) \le J_P(K), \ \forall K \in \mathcal{K}.$$

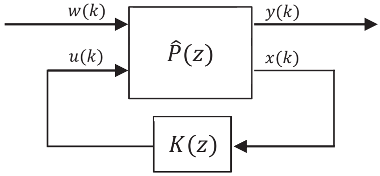

Figure 2: The feedback system with the given controller K and the overall-plant $\hat{P}$.

The competitive ratio of a control design method Γ *is defined as*

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,H)\in\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$*" equals one.*

Definition 2.2 (Domination) *A control design method* Γ' *is said to dominate another control design method* Γ *if*

$$J_P(\Gamma'(P)) \leq J_P(\Gamma(P)),\ \forall\ P = (A, B, H) \in \mathcal{P}, \tag{4}$$

with strict inequality holding for at least one plant in $\mathcal{P}$*. When* $\Gamma \in \mathcal{C}$ *and no control design method* $\Gamma' \in \mathcal{C}$ *exists that satisfies (4), we say that* Γ *is undominated in* $\mathcal{C}$*.*

2.5 Mathematical Problem Formulation

Now we can formulate the primary question concerning the connection between closed-loop performance and limited model information control design strategies. For a given plant graph $G_{\mathcal{P}}$, control graph $G_{\mathcal{K}}$, and design graph $G_{\mathcal{C}}$, we want to solve

$$\arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma). \tag{5}$$

Since the solution to this problem might not be unique, we are interested in finding a minimizer that is also undominated. These solutions are the best worst-case designs with limited model information.

3 Preliminary Results

In order to give the main results of the paper, we need to define a control design strategy and find its competitive ratio.

Definition 2.3 *Let a plant graph* $G_{\mathcal{P}}$ *and a constant* $\epsilon > 0$ *be given. The control design method* Γ^{Θ} *is defined as*

$$\Gamma^{\Theta}(P) = -\operatorname{diag}(W_1(P), \ldots, W_q(P))A, \tag{6}$$

for all plants $P = (A, B, H) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{H}$*, where*

$$W_i(P) = \begin{cases} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii}, & \text{if } i \text{ is a sink,} \\ B_{ii}^{-1}, & \text{otherwise,} \end{cases}$$

and for each sink i *the matrix* X_{ii} *is the unique positive definite solution of the discrete algebraic Riccati equation*

$$A_{ii}^T X_{ii} A_{ii} - A_{ii}^T X_{ii} B_{ii} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} A_{ii} - X_{ii} + I = 0.$$

The control design method Γ^{Θ} applies the so-called deadbeat strategy [10] to every subsystem that is not a sink (thus those closed-loop subsystems reach origin in just one time-step [11]) and, for every sink, applies the same optimal control law as if the node were decoupled from the rest of the graph.

Lemma 2.1 *The competitive ratio of the control design method* Γ^{Θ} *defined in (6) is* $r_{\mathcal{P}}(\Gamma^{\Theta}) = \sqrt{1 + 1/\epsilon^2}$ *if one of the following conditions is satisfied:*

(*a*) *the plant graph* $G_{\mathcal{P}}$ *contains no isolated node and the control graph* $G_{\mathcal{K}}$ *is a complete graph;*

(*b*) *the acyclic plant graph* $G_{\mathcal{P}}$ *contains no isolated node and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$*.*

Proof: Let $K_C^*(P)$ denotes the optimal static full-state feedback (centralized) controller for each plant $P \in \mathcal{P}$. According to the proof of the "only if" part of Theorem 3.6 in [10], we have

$$Z \leq A^T B^{-T} B^{-1} A + I, \tag{7}$$

for all plants $P = (A, B, H) \in \mathcal{P}$, where Z is the unique positive definite solution of discrete algebraic Lyapunov equation

$$(A + B\Gamma^{\Theta}(P))^T Z (A + B\Gamma^{\Theta}(P)) - Z + I + \Gamma^{\Theta}(P)^T \Gamma^{\Theta}(P) = 0. \tag{8}$$

Thus, the cost of the control design strategy Γ^{Θ} for each plant $P = (A, B, H)$ is upper-bounded as

$$\begin{aligned} J_P(\Gamma^{\Theta}(P))^2 &= \operatorname{tr}\left(H^T Z H\right) \\ &\leq \operatorname{tr}\left(H^T \left(A^T B^{-T} B^{-1} A + I\right) H\right). \end{aligned} \tag{9}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix. According to Theorem 3.2 in [10], it is evident that

$$A^T B^{-T} B^{-1} A \leq \left(1 + 1/\epsilon^2\right) (X - I),$$

and equivalently

$$\operatorname{tr}(H^T A^T B^{-T} B^{-1} A H) \leq \left(1 + 1/\epsilon^2\right) \operatorname{tr}(H^T (X - I) H), \tag{10}$$

where X is the unique positive definite solution of discrete algebraic Riccati equation

$$A^T XA - A^T XB(I + B^T XB)^{-1} B^T XA = X - I. \tag{11}$$

Putting (10) in (9), we get

$$\begin{aligned} J_P(\Gamma^\Theta(P))^2 &\leq \left(1 + 1/\epsilon^2\right) \operatorname{tr}(H^T XH) \\ &= \left(1 + 1/\epsilon^2\right) J_P(K_C^*(P))^2. \end{aligned}$$

Clearly, because $J_P(K_C^*(P)) \leq J_P(K^*(P))$, irrespective of the control graph $G_{\mathcal{K}}$, we have

$$J_P(\Gamma^\Theta(P))^2 \leq \left(1 + 1/\epsilon^2\right) J_P(K^*(P))^2,$$

and as a result

$$r_{\mathcal{P}}(\Gamma^\Theta) = \sup_{P=(A,B,H)\in\mathcal{P}} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*(P))} \leq \sqrt{1 + 1/\epsilon^2}.$$

To show that this upper-bound is tight, we should exhibit plants for which it is attained.

Part a: Condition (a) is satisfied. Since there is no isolated node in the plant graph, we can pick indices $1 \leq i \neq j \leq q$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. The rest of the proof is given in two different cases.

Case a.1: Node i is not a sink. Pick indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(s) = se_{i_1} e_{j_1}^T$, $B = \epsilon I$, and $H = I$. We get

$$r_{\mathcal{P}}(\Gamma^\Theta) \geq \lim_{s\to\infty} \sqrt{\frac{s^2/\epsilon^2 + n}{s^2/(1+\epsilon^2) + n}} = \sqrt{1 + 1/\epsilon^2},$$

since the unique positive definite solution of discrete algebraic Riccati equation in (11) is $X = I + [s^2/(1+\epsilon^2)] e_{j_1} e_{j_1}^T$, and as a result $J_P(K^*(P)) = \sqrt{s^2/(1+\epsilon^2) + n}$.

Case a.2: Node i is a sink. We know $(s_{\mathcal{P}})_{ii} \neq 0$ since all the self-loops are present. Pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r, s) = re_{i_1} e_{i_1}^T + se_{i_1} e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to Theorem 3.8 in [10], we get

$$J_P(\Gamma^\Theta(P)) = \sqrt{\beta_\Theta(s^2 + r^2) + n},$$

where

$$\beta_\Theta = \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2ar^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon^2 r^2}.$$

Again, using Theorem 3.8 in [10], the optimal closed-loop performance is

$$J_P(K^*(P)) = \sqrt{\beta_{K^*}(s^2 + r^2) + n},$$

where β_{K^*} is

$$\beta_{K^*} = \frac{\epsilon^2 s^2 + r^2(1+\epsilon^2) - (\epsilon^2+1)^2 + \sqrt{c_+ c_-}}{2\epsilon^2(\epsilon^2+1)(s^2+r^2)},$$

$$c_\pm = \epsilon^2 s^2 + (r^2 \pm 2r)(\epsilon^2+1) + (\epsilon^2+1)^2.$$

Then, we get

$$r_{\mathcal{P}}(\Gamma^\Theta) \geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*(P))} = \sqrt{1+1/\epsilon^2}.$$

Part b: Condition (b) is satisfied. Any acyclic directed graph has at least one sink. Let i denote a sink in plant graph $G_{\mathcal{P}}$. Since there is no isolated node in the plant graph, there exists an index $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r,s) = re_{i_1}e_{i_1}^T + se_{i_1}e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to Lemma 4.1 in [12], we get

$$J_P(K^*_{\mathcal{P}}(P)) = \sqrt{\beta_{K^*}s^2 + \beta_\Theta r^2 + n},$$

where $K^*_{\mathcal{P}}(P)$ is the optimal controller when $G_{\mathcal{K}}$ is equal to $G_{\mathcal{P}}$. This results in

$$\begin{aligned} r_{\mathcal{P}}(\Gamma^\Theta) &\geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*(P))} \\ &\geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*_{\mathcal{P}}(P))} = \sqrt{1+1/\epsilon^2} \end{aligned}$$

since clearly $J_P(K^*(P)) \leq J_P(K^*_{\mathcal{P}}(P))$. ∎

Lemma 2.1 shows that, if we apply the control design strategy Γ^Θ to a particular plant, the performance of the closed-loop system, at most, can be $\sqrt{1+1/\epsilon^2}$ times the cost of the optimal control design strategy K^*.

There is no loss of generality in assuming that the plant graph $G_{\mathcal{P}}$ contains no isolated node since it is always possible to design an optimal controller for an isolated subsystem without any model information about the other subsystems and without affecting them. In particular, this implies that there are $q \geq 2$ vertices in the plant graph.

4 Plant Graph Influence on Achievable Performance

In this section, we study the achievable closed-loop performance, in terms of the competitive ratio and the domination, for different plant interconnection pattern. The next theorem shows that the control design strategy Γ^Θ is an undominated minimizer of the competitive ratio for all given plant graphs $G_{\mathcal{P}}$ when the control graph $G_{\mathcal{K}}$ is a complete graph and the design graph $G_{\mathcal{C}}$ is fully disconnected.

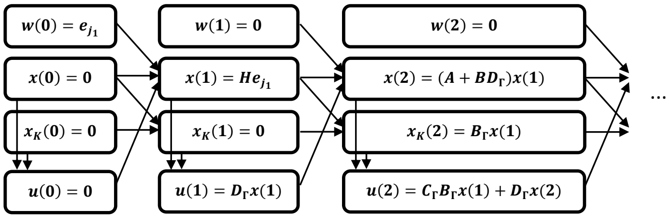

Figure 3: State transition of the closed-loop system and its controller as a function of time for the exogenous input $w(k) = \delta(k)e_{j_1}$.

Theorem 2.2 *Let the plant graph $G_\mathcal{P}$ contain no isolated node, the control graph $G_\mathcal{K}$ be a complete graph, and the design graph $G_\mathcal{C}$ be a totally disconnected graph. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_\mathcal{P}(\Gamma) \geq r_\mathcal{P}(\Gamma^\Theta)$. Furthermore, the control design strategy Γ^Θ is undominated by set of limited model information control design strategies with design graph $G_\mathcal{C}$.*

Proof: We use the following notation

$$\Gamma(P) = \left[\begin{array}{c|c} A_\Gamma(P) & B_\Gamma(P) \\ \hline C_\Gamma(P) & D_\Gamma(P) \end{array}\right],$$

to work with different parts of the state-space representation of a control design strategy Γ. The entries $A_\Gamma(P)$, $B_\Gamma(P)$, $C_\Gamma(P)$, and $D_\Gamma(P)$ are matrices with appropriate dimension for each plant $P = (A, B, H) \in \mathcal{P}$. The matrices $A_\Gamma(P)$ and $C_\Gamma(P)$ are block diagonal matrices since different subcontrollers should not share state variables (each controller should be implemented in a decentralized fashion). This realization is not necessarily a minimal realization.

Consider indices $1 \leq i \neq j \leq q$ such that $(s_\mathcal{P})_{ij} \neq 0$ (this is always possible since there is no isolated node in the plant graph). The rest of the proof is given in two different cases.

Case 1: Node i is not a sink. Therefore, there exists an index $\ell \neq i$ such that $(s_\mathcal{P})_{\ell i} \neq 0$. Pick indices $\ell_1 \in \mathcal{I}_\ell$, $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A(r,s) = se_{i_1}e_{j_1}^T + re_{\ell_1}e_{i_1}^T$ and $B = \epsilon I$. Let $H_{jj} = rI$ and $H_{tt} = I$ for all $t \neq j$. Using the exogenous impulse input $w(k) = \delta(k)e_{j_1}$ and the time-steps given in Figure 3, we get

$$\begin{aligned} J_P(\Gamma(P))^2 &\geq u_{\ell_1}(2)^2 + x_{\ell_1}(3)^2 \\ &= u_{\ell_1}(2)^2 + \left(r^2(s + \epsilon(d_\Gamma)_{i_1j_1}(s)) + \epsilon u_{\ell_1}(2)\right)^2 \\ &\geq r^4(s + \epsilon(d_\Gamma)_{i_1j_1}(s))^2/(\epsilon^2 + 1), \end{aligned}$$

because, irrespective of the choice of $u_{\ell_1}(2)$, the function $u_{\ell_1}(2)^2 + (r^2(s+\epsilon(d_\Gamma)_{i_1j_1}(s)) + \epsilon u_{\ell_1}(2))^2$ is lower-bounded by $r^4(s+\epsilon(d_\Gamma)_{i_1j_1}(s))^2/(\epsilon^2+1)$. It is worth mentioning that $(d_\Gamma)_{i_1j_1}(s)$ is only a function of the scalar s and it is independent of the scalar r, since r is in model parameters of subsystems $\ell, j \neq i$ and the design graph is fully disconnected. On the other hand

$$\begin{aligned} J_P(\Gamma^{\Delta}(P)) &= \sqrt{\operatorname{tr}\left(H^T\left((1/\epsilon^2)A^TA + I\right)H\right)} \\ &= \sqrt{(s^2r^2 + r^2)/\epsilon^2 + n - n_j + n_j r^2}, \end{aligned}$$

where Γ^{Δ} is the deadbeat control design strategy and it is defined as $\Gamma^{\Delta}(P) = -B^{-1}A$ [10]. Therefore

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &= \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(K^*(P))} \\ &= \sup_{P\in\mathcal{P}} \left[\frac{J_P(\Gamma(P))}{J_P(\Gamma^{\Delta}(P))}\frac{J_P(\Gamma^{\Delta}(P))}{J_P(K^*(P))}\right] \\ &\geq \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(\Gamma^{\Delta}(P))} \\ &\geq \lim_{r\to\infty} \sqrt{\frac{r^4(s+\epsilon(d_\Gamma)_{i_1j_1}(s))^2/(\epsilon^2+1)}{(s^2r^2+r^2)/\epsilon^2 + n - n_j + n_j r^2}}. \end{aligned} \tag{12}$$

since $J_P(\Gamma^{\Delta}(P)) \geq J_P(K^*(P))$ for all plants $P \in \mathcal{P}$. The competitive ratio $r_{\mathcal{P}}(\Gamma)$ is bounded only if $s + \epsilon(d_\Gamma)_{i_1j_1}(s) = 0$. Therefore, there is no loss of generality in assuming that $(d_\Gamma)_{i_1j_1}(s) = -s/\epsilon$ because otherwise the $r_{\mathcal{P}}(\Gamma)$ is infinity and the inequality $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$ is trivially satisfied. Now, let us redefine $A(s) = se_{i_1}e_{j_1}^T$, $H = I$ and $B = \epsilon I$. Since the parameters of the subsystem i is not changed, we have $(d_\Gamma)_{i_1j_1}(s) = -s/\epsilon$. Therefore, for the same impulse exogenous input $w(k) = \delta(k)e_{j_1}$, we have

$$J_P(\Gamma(P))^2 \geq u_{i_1}(1)^2 = (d_\Gamma)_{i_1j_1}(s)^2 = s^2/\epsilon^2,$$

and

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{s\to\infty} \sqrt{\frac{s^2/\epsilon^2}{s^2/(1+\epsilon^2)+n}} = \sqrt{1+1/\epsilon^2}, \tag{13}$$

since similar to Case a.1 in the proof of Lemma 2.1, we have $J_P(K^*(P)) = \sqrt{s^2/(1+\epsilon^2)+n}$.

Case 2: Node i is a sink. We have $(s_{\mathcal{P}})_{ii} \neq 0$ since all the self-loops are present. Let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r,s) = re_{i_1}e_{i_1}^T + se_{i_1}e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to the proof of the "only if" part of Theorem 3.6 in [10], for this particular family of plants, $\Gamma^{\Theta}(P)$ is the globally optimal H_2 state-feedback controller. Now using Case a.2 in the proof of Lemma 2.1, it is easy to see that $r_{\mathcal{P}}(\Gamma) \geq \sqrt{1+1/\epsilon^2}$.

To prove that the control design strategy Γ^{Θ} is undominated by set of limited model information control design strategies $\Gamma \in \mathcal{C}$, we construct plants $P =$

$(A, B, H) \in \mathcal{P}$ that satisfy $J_P(\Gamma(P)) > J_P(\Gamma^{\Theta}(P))$ for any control design method $\Gamma \in \mathcal{C} \setminus \{\Gamma^{\Theta}\}$. The detailed proof of this part is given in [12]. ■

As an example, consider the limited model information design problem given by the plant graph $G'_{\mathcal{P}}$ in Figure 1(a'), the control graph $G'_{\mathcal{K}}$ in Figure 1(b'), and the design graph $G'_{\mathcal{C}}$ in Figure 1(c'). Theorem 2.2 shows that the control design strategy Γ^{Θ} is the best control design strategy that one can propose based on the local model of subsystems since it is an undominated minimizer of the competitive ratio.

5 Control Graph Influence on Achievable Performance

In this section, we study the structured controllers and their influence on the achievable closed-loop performance of the limited model information control design strategies. Note that finding the optimal control design strategy $K^*(P)$ is numerically intractable for general plant and control graphs. We use the results in [6, 7] which give an explicit solution to the problem of designing optimal decentralized controller for some special classes of subsystems interconnection and controller structures. Therefore, we assume that the plant graph $G_{\mathcal{P}}$ is an acyclic directed graph and the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. Note that the control design strategy Γ^{Θ} is still applicable in this scenario.

Theorem 2.3 *Let the acyclic plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$. Furthermore, the control design strategy Γ^{Θ} is undominated by set of limited model information control design strategies with design graph $G_{\mathcal{C}}$.*

Proof: Any acyclic directed graph has at least one sink. Let i denote a sink in plant graph $G_{\mathcal{P}}$. Since there is no isolated node in the plant graph, there exists an index $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r, s) = r e_{i_1} e_{i_1}^T + s e_{i_1} e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to the proof of the "only if" part of Theorem 3.6 in [10], for this particular family of plants, $\Gamma^{\Theta}(P)$ is the globally optimal H_2 state-feedback controller. Now using Part b of the proof of Lemma 2.1, it is easy to see that $r_{\mathcal{P}}(\Gamma) \geq \sqrt{1 + 1/\epsilon^2}$.

The detailed proof of the part that control design strategy Γ^{Θ} is undominated is given in [12]. ■

For instance, consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 1(a), the control graph $G_{\mathcal{K}}$ in Figure 1(b), and the design graph $G'_{\mathcal{C}}$ in Figure 1(c'). Theorem 2.3 illustrates that the control design strategy Γ^{Θ} is again the best control design strategy that one can propose based on the local model of subsystems, because it is an undominated minimizer of the competitive ratio.

6 Design Graph Influence on Achievable Performance

In this section, we try to determine the amount of the model information that we need in each subsystem to be able to setup a control design strategy Γ with a smaller competitive ratio than the control design strategy Γ^{Θ}.

Theorem 2.4 *Let the plant graph $G_{\mathcal{P}}$ and the design graph $G_{\mathcal{C}}$ be given and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. If the plant graph $G_{\mathcal{P}}$ contains the path $j \to i \to \ell$ with distinct vertices i, j, and ℓ while $(\ell, i) \notin E_{\mathcal{C}}$, then $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$ for all $\Gamma \in \mathcal{C}$.*

Proof: Because of the path $j \to i \to \ell$ with distinct vertices i, j, and k, we have $(s_{\mathcal{P}})_{ij} \neq 0$ and $(s_{\mathcal{P}})_{\ell i} \neq 0$. Pick indices $\ell_1 \in \mathcal{I}_\ell$, $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A(r,s) = s e_{i_1} e_{j_1}^T + r e_{\ell_1} e_{i_1}^T$, $B = \epsilon I$, and $H = I$. Similar to the proof of Theorem 2.2, using the exogenous impulse input $w(k) = \delta(k) e_{j_1}$ and the time-steps given in Figure 3, we get

$$J_P(\Gamma(P))^2 \geq r^2 (s + \epsilon (d_\Gamma)_{i_1 j_1}(s))^2 / (\epsilon^2 + 1),$$

Again, it should be noted that $(d_\Gamma)_{i_1 j_1}(s)$ is only a function of the scalar s, and it is independent of the scalar r because r has appeared in model matrices of the subsystem $\ell \neq i$, and $(\ell, i) \notin E_{\mathcal{C}}$. We claim that for the competitive ratio to be bounded there should exist a positive constant $\theta \in \mathbb{R}$ independent of scalars s such that $|s + \epsilon (d_\Gamma)_{i_1 j_1}(s)| \leq \theta$. Assume this claim is not true, thus, there exist a sequence of scalars $\{s_z\}_{z=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{z \to \infty} |s_z + \epsilon (d_\Gamma)_{i_1 j_1}(s_z)| = +\infty.$$

Clearly, using (12) we get

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &\geq \lim_{z \to \infty, \frac{r}{s_z} \to \infty} \sqrt{\frac{r^2 |s_z + \epsilon (d_\Gamma)_{i_1 j_1}(s_z)|^2 / (\epsilon^2 + 1)}{(s_z^2 + r^2)/\epsilon^2 + n}} \\ &= +\infty. \end{aligned}$$

since $J_P(\Gamma^{\Delta}(P)) = \sqrt{(s_z^2 + r^2)/\epsilon^2 + n}$. Now, lets redefine $A(s) = s e_{i_1} e_{j_1}^T$. Since the model parameters of the subsystem i is not changed, and its controller is not a function of the model parameters of subsystem ℓ, the design entry $(d_\Gamma)_{i_1 j_1}(s)$ stays the same. Therefore, $|s + \epsilon (d_\Gamma)_{i_1 j_1}(s)| \leq \theta$ for all $s \in \mathbb{R}$, and as a result, for large enough $|s|$, we get $|(d_\Gamma)_{i_1 j_1}(s)| \geq (|s| - \theta)/\epsilon$. Therefore, using the exogenous impulse input $w(k) = \delta(k) e_{j_1}$, we get

$$J_P(\Gamma(P))^2 \geq u_{i_1}(1)^2 = (d_\Gamma)_{i_1 j_1}(s)^2 \geq (|s| - \theta)^2 / \epsilon^2,$$

and

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{s \to \infty} \sqrt{\frac{(|s| - \theta)^2 / \epsilon^2}{s^2/(1 + \epsilon^2) + n}} = \sqrt{1 + 1/\epsilon^2}.$$

For this special plant, we know $K_C^*(P) = -\epsilon/(1+\epsilon^2)A$ belongs to the set $\mathcal{K}(S_\mathcal{K})$ since the control graph $G_\mathcal{K} \supseteq G_\mathcal{P}$, and consequently $J_P(K^*(P)) \leq J_P(K_C^*(P))$ because $K^*(P)$ has a lower cost than any other controller is $\mathcal{K}(S_\mathcal{K})$. On the other hand, clearly, for any plant $J_P(K_C^*(P)) \leq J_P(K^*(P))$. Therefore, for this special plant

$$J_P(K^*(P)) = J_P(K_C^*(P)) = \sqrt{s^2/(1+\epsilon^2)+n}.$$

This concludes the proof. ■

Consider the limited model information design problem given by the plant graph $G'_\mathcal{P}$ in Figure 1(a'), the control graph $G'_\mathcal{K}$ in Figure 1(b'), and the design graph $G_\mathcal{C}$ in Figure 1(c). Note that there is a path $3 \to 2 \to 1$ in the plant graph $G_\mathcal{P}$ but the edge $1 \to 2$ is not present in the design graph $G_\mathcal{C}$. Therefore, using Theorem 2.4, it is easy see that $r_\mathcal{P}(\Gamma) \geq r_\mathcal{P}(\Gamma^\Theta)$ for any $\Gamma \in \mathcal{C}$.

7 Extensions

In this section, we relax the assumption that all the subsystems are required to be fully-actuated, that is, $B \in \mathcal{B}(\epsilon)$ is square invertible. To do so, we assume that plant graph $G_\mathcal{P}$ is an acyclic directed graph with $c \geq 1$ sinks since any acyclic graph has at least one sink. Accordingly, its adjacency matrix $S_\mathcal{P}$ is of the form

$$S_\mathcal{P} = \left[\begin{array}{c|c} (S_\mathcal{P})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_\mathcal{P})_{21} & (S_\mathcal{P})_{22} \end{array}\right], \tag{14}$$

where

$$(S_\mathcal{P})_{11} = \begin{bmatrix} (s_\mathcal{P})_{11} & \cdots & (s_\mathcal{P})_{1,q-c} \\ \vdots & \ddots & \vdots \\ (s_\mathcal{P})_{q-c,1} & \cdots & (s_\mathcal{P})_{q-c,q-c} \end{bmatrix},$$

$$(S_\mathcal{P})_{21} = \begin{bmatrix} (s_\mathcal{P})_{q-c+1,1} & \cdots & (s_\mathcal{P})_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ (s_\mathcal{P})_{q,1} & \cdots & (s_\mathcal{P})_{q,q-c} \end{bmatrix},$$

and $(S_\mathcal{P})_{22} = \operatorname{diag}((s_\mathcal{P})_{q-c+1,q-c+1}, \ldots, (s_\mathcal{P})_{qq})$, where we assume, without loss of generality, that the vertices are numbered such that the sinks are labeled $q-c+1, \ldots, q$. We define the set $\mathcal{P}'$ of plants of interest as the set of all triples $(A, B, H) \in \mathcal{A}(S_\mathcal{P}) \times \mathcal{B}'(\epsilon) \times \mathcal{H}$ where

$$\mathcal{B}'(\epsilon) = \{\bar{B} \in \mathbb{R}^{n\times m} \mid \underline{\sigma}(\bar{B}) \geq \epsilon,\, \bar{B}_{ij} = 0 \in \mathbb{R}^{n_i \times m_j} \text{ for all } 1 \leq i \neq j \leq q\}.$$

Each $m_i \in \mathbb{N}$ is the number of control inputs in subsystem i, and consequently $\sum_{i=1}^q m_i = m$. Let relax $m_i \leq n_i$ for all $q-c+1 \leq i \leq q$ but force $m_i = n_i$ otherwise. In addition, all matrices A and B must satisfy

(a) (A_{ii}, B_{ii}) is controllable, (*b*) $\text{span}(A_{ij}) \subseteq \text{span}(B_{ii})$ for all $j \neq i$ or equivalently there should exist a matrix $W_i \in \mathbb{R}^{m_i \times (n-n_i)}$ such that $[A_{i1} \ \cdots \ A_{i,i-1} \ A_{i,i+1} \ \cdots \ A_{iq}] = B_{ii}W_i$,

for all $q - c + 1 \leq i \leq q$. For this new set of plants, the control design strategy Γ^Θ is still applicable since it does not require B_{ii} to be invertible for $q - c + 1 \leq i \leq q$.

Now we are ready to solve the problem (5) for this set of underactuated plants $\mathcal{P}'$.

Theorem 2.5 *Let the acyclic plant graph $G_\mathcal{P}$ contain no isolated node, the control graph $G_\mathcal{K}$ be equal to the plant graph $G_\mathcal{P}$, and the design graph $G_\mathcal{C}$ be a totally disconnected graph. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_\mathcal{P}(\Gamma) \geq r_\mathcal{P}(\Gamma^\Theta) = \sqrt{1 + 1/\epsilon^2}$ if $(S_\mathcal{P})_{11}$ is not diagonal. Furthermore, the control design strategy Γ^Θ is undominated by set of limited model information control design strategies with design graph $G_\mathcal{C}$.*

Proof: Similar to (14), we can write any $A \in \mathcal{A}(S_\mathcal{P})$ as

$$A = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

where

$$\tilde{A}_{11} = \begin{bmatrix} A_{11} & \cdots & A_{1,q-c} \\ \vdots & \ddots & \vdots \\ A_{q-c,1} & \cdots & A_{q-c,q-c} \end{bmatrix},$$

$$\tilde{A}_{21} = \begin{bmatrix} A_{q-c+1,1} & \cdots & A_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ A_{q1} & \cdots & A_{q,q-c} \end{bmatrix},$$

and $\tilde{A}_{22} = \text{diag}(A_{q-c+1,q-c+1}, \ldots, A_{qq})$. Clearly, if we apply deadbeat to all subsystems that are not sinks, the other subsystems (i.e., sinks) become decoupled (see Theorem 3.6 in [10]), and as a result

$$J_P(\Gamma^\Theta(P))^2 = J^{(1)}(\tilde{A}_{11}, \tilde{B}_{11}, \tilde{H}_{11}) + J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22})$$

where $H = \text{diag}(\tilde{H}_{11}, \tilde{H}_{22})$, $B = \text{diag}(\tilde{B}_{11}, \tilde{B}_{22})$, $J^{(1)}(\tilde{A}_{11}, \tilde{B}_{11}, \tilde{H}_{11})$ is the cost of applying deadbeat control design to the nodes that are not sinks, and $J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22})$ is the cost of applying the same optimal control law as if the sinks were decoupled from the rest of the graph. Thus, we get

$$J^{(1)}(\tilde{A}_{11}, \tilde{B}_{11}, \tilde{H}_{11}) = \text{tr}(\tilde{H}_{11}^T \tilde{A}_{11}^T \tilde{B}_{11}^{-T} \tilde{B}_{11}^{-1} \tilde{A}_{11} \tilde{H}_{11})$$

and

$$J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22}) \leq \text{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}) + \text{tr}(\tilde{H}_{11}^T \tilde{A}_{21}^T \tilde{B}_{22}^{\dagger T} \tilde{B}_{22}^\dagger \tilde{A}_{21} \tilde{H}_{11}) \quad (15)$$

where $\tilde{B}_{22}^\dagger = (B_{22}^T B_{22})^{-1} B_{22}^T$. The inequality in (15) is true since $J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22})$ is the cost of the optimal control law as if the sinks were decoupled from the rest of the graph (see Theorem 3.6 in [10]), and it certainly has a lower cost than any other controller particularly

$$K_2 = -[\tilde{B}_{22}^{\dagger}\tilde{A}_{21} \;\; (I + \tilde{B}_{22}^T Y \tilde{B}_{22})^{-1}\tilde{B}_{22}^T Y \tilde{A}_{22}],$$

where Y is the unique positive definite solution of discrete algebraic Riccati equation

$$\tilde{A}_{22}^T Y \tilde{A}_{22} - \tilde{A}_{22}^T Y \tilde{B}_{22}(I + \tilde{B}_{22}^T Y \tilde{B}_{22})^{-1}\tilde{B}_{22}^T Y \tilde{A}_{22} - Y + I = 0.$$

Note that since $\tilde{A}_{22}$ is block diagonal, the positive definite matrix Y is also block diagonal, and each block is only a function the corresponding subsystem. Thus, we get

$$J_P(\Gamma^{\Theta}(P))^2 \leq \text{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}) + \text{tr}(\tilde{H}_{11}^T(\tilde{A}_{11}^T \tilde{B}_{11}^{-T}\tilde{B}_{11}^{-1}\tilde{A}_{11} + \tilde{A}_{21}^T \tilde{B}_{22}^{\dagger T}\tilde{B}_{22}^{\dagger}\tilde{A}_{21})\tilde{H}_{11}). \tag{16}$$

The optimal closed-loop performance is $J_P(K^*(P))^2 = \text{tr}(H^T U H)$ where $U = [I_{n\times n}\ 0]V[I_{n\times n}\ 0]^T$ and V is the unique positive definite solution of discrete algebraic Lyapunov equation

$$\begin{aligned} &\begin{bmatrix} A + BD^*(P) & BC^*(P) \\ B^*(P) & A^*(P) \end{bmatrix}^T V \begin{bmatrix} A + BD^*(P) & BC^*(P) \\ B^*(P) & A^*(P) \end{bmatrix} - V \\ &\qquad + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D^*(P)^T D^*(P) & D^*(P)^T C^*(P) \\ C^*(P)^T D^*(P) & C^*(P)^T C^*(P) \end{bmatrix} = 0 \end{aligned} \tag{17}$$

with $A^*(P)$, $B^*(P)$, $C^*(P)$, and $D^*(P)$ as the state-space realization matrices of the optimal control design strategy $K^*(P)$ for a given plant $P \in \mathcal{P}'$. Clearly, we have

$$J_P(K^*(P))^2 = \sum_{t=1}^{n} e_t^T H^T U H e_t = \sum_{t=1}^{n}\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k),$$

where for each t the vector $y^{(t)}(k)$ is the output of the system to the exogenous impulse input $w^{(t)}(k) = \delta(k)e_t$. This is true because for each t the summation $\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k)$ gives the diagonal element $e_t^T H^T U H e_t$. For any $P = (A, B, H) \in \mathcal{P}'$, we know that $H^T U H \geq H^T X H$ since centralized controller has the least performance cost over all other controllers either dynamic or static. Thus, for each $t \in \mathcal{N} = \bigcup_{z=1}^{q-c} \mathcal{I}_z$, we get $e_t^T H^T U H e_t \geq e_t^T H^T X H e_t$ which shows

$$\sum_{t\in\mathcal{N}}\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k) \geq \sum_{t\in\mathcal{N}} e_t^T(H^T X H)e_t.$$

According to [13], we have $X \geq A^T(I + BB^T)^{-1}A + I$ for any $P \in \mathcal{P}'$, and consequently

$$\begin{aligned}\sum_{t\in\mathcal{N}}\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k) \geq \text{tr}(\tilde{H}_{11}^T(&\tilde{A}_{11}^T(I + \tilde{B}_{11}\tilde{B}_{11}^T)^{-1}\tilde{A}_{11} \\ &+ \tilde{A}_{21}^T(I + \tilde{B}_{22}\tilde{B}_{22}^T)^{-1}\tilde{A}_{21})\tilde{H}_{11}).\end{aligned}$$

On the other hand, for each $t \in \mathcal{S} = \bigcup_{z=q-c+1}^{q} \mathcal{I}_z$, we know there exists a sink i such that $t \in \mathcal{I}_i$. For each $w^{(t)}(k)$, we get $\underline{x}_j = 0$ for any $j \neq i$ (since i is a sink in $G_{\mathcal{P}}$). The other subsystems cannot use state-measurements of subsystem i because $G_{\mathcal{K}}$ is equal to $G_{\mathcal{P}}$ (and consequently i is a sink in $G_{\mathcal{K}}$). Therefore, at best case scenario, the cost of controlling subsystem i is equal to the cost of optimal controller designed locally (independent of other subsystems). Thus, we get

$$\sum_{t\in\mathcal{S}}\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k) \geq \mathrm{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}).$$

Therefore, we get

$$J_P(K^*(P))^2 \geq \mathrm{tr}(\tilde{H}_{11}^T(\tilde{A}_{11}^T(I+\tilde{B}_{11}\tilde{B}_{11}^T)^{-1}\tilde{A}_{11} + \tilde{A}_{21}^T(I+\tilde{B}_{22}\tilde{B}_{22}^T)^{-1}\tilde{A}_{21})\tilde{H}_{11}) + \mathrm{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}). \quad (18)$$

Now, lets define the set

$$\mathcal{M} = \{\bar{\beta} \in \mathbb{R} \mid \bar{\beta} J_P(K^*(P)) - J_P(\Gamma^{\Theta}(P)) \geq 0 \;\forall P \in \mathcal{P}'\}.$$

Using the inequalities in (16) and in (18), it is evident if

$$\mathrm{tr}\left(\tilde{H}_{11}^T(\tilde{A}_{11}^T\left[\beta^2(I+\tilde{B}_{11}\tilde{B}_{11}^T)^{-1} - \tilde{B}_{11}^{-T}\tilde{B}_{11}^{-1}\right]\tilde{A}_{11} + \tilde{A}_{21}^T\left[\beta^2(I+\tilde{B}_{22}\tilde{B}_{22}^T)^{-1} - \tilde{B}_{22}^{\dagger T}\tilde{B}_{22}^{\dagger}\right]\tilde{A}_{21})\tilde{H}_{11}\right) \geq 0. \quad (19)$$

for some $\beta \in \mathbb{R}$, then β would belong to $\mathcal{M}$. Thus, $\{\bar{\beta} \in \mathbb{R} \mid \bar{\beta} \geq \sqrt{1+1/\epsilon^2}\} \subseteq \mathcal{M}$. This shows that $r_{\mathcal{P}}(\Gamma^{\Theta}) \leq \sqrt{1+1/\epsilon^2}$. Now if $(S_{\mathcal{P}})_{11}$ is not diagonal, with the same argument as in the proof of Case 1 in Theorem 2.3, we get $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta}) = \sqrt{1+1/\epsilon^2}$ for any $\Gamma \in \mathcal{C}$. This can be done because there are at least two fully-actuated subsystems and we can forget about the underactuated subsystems.

The proof of the part that the control design strategy Γ^{Θ} is undominated is similar to the one given in [12] for fully-actuated subsystems. ■

8 Conclusions

We considered optimal H_2 dynamic control design for interconnected linear systems under limited plant model information. We introduced control design strategies as functions from the set of plants to the set of structured dynamic controller and compared these control design strategies using the competitive ratio as a performance metric. For a large class of system interconnections, controller structure, and design information, we found an explicit undominated minimizer of the competitive ratio.

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PAPER 3

Decentralized Disturbance Accommodation with Limited Plant Model Information

Farhad Farokhi, Cédric Langbort, and Karl H. Johansson

Abstract–The design of optimal disturbance accommodation and servomechanism controllers with limited plant model information is studied in this paper. We consider discrete-time linear time-invariant systems that are fully actuated, and composed of scalar subsystems, each of which is controlled separately, and influenced by a scalar disturbance. Each disturbance is assumed to be generated by a system with known dynamics and unknown initial conditions. We restrict ourselves to control design methods that produce structured dynamic state feedback controllers where each subcontroller, at least, has access to the state-measurements of those subsystems that can affect its corresponding subsystem. The performance of such control design methods are compared using a metric called the competitive ratio which is the worst-case ratio of the cost of a given control design strategy to the cost of the optimal control design with full model information. We find an explicit minimizer of the competitive ratio and show that it is undominated, that is, there is no other control design strategy that performs better for all possible plants while having the same worst-case ratio. This optimal controller can be separated into a static feedback law and a dynamic disturbance observer. For step disturbances, it is shown that this structure corresponds to proportional-integral control.

1 Introduction

Advances in networked control systems have created new opportunities and challenges in controlling large-scale systems composed of several interacting subsystems. An example of a networked control system is shown in Figure 1. For such networked systems, many researchers have considered the problem of decentralized or distributed stabilization or optimal control as well as the effect of communication channel limitations on closed-loop performance [1–14]. However, at the heart of all these control methods lies the (sometimes implicit) assumption that the designer has access to the global plant model information when designing a local controller. In contrast, the broad goal of this paper, which continues our work started in [15–18], is to consider distributed control design problems where the full plant model is not globally available. In the next subsection, we discuss why such a situation might be at hand.

1.1 Motivation

There are several reasons why global plant model information may not be available in practice, and why a control designer may be constrained to compute local controllers for a large-scale systems in a distributed manner with access to only a limited or partial model of the plant. For example, (*i*) the designer wants the parameters of each local controller to only depend on local model information, so that the controllers do not need to be modified if the model parameters of a particular subsystem, which is not directly connected to them, change, (*ii*) the design of each local controller is done by a designer with no access to the global model of plant since at the time of design the complete plant model information is not available or might change later in the design process, or (*iii*) different subsystems belong to different individuals who refuse to share their model information since they consider it private. These situations are very common in practice. For instance, a chemical plant in process industry can have thousands of proportional-integral-derivative controllers. These processes well illustrate Case (*i*), as the tuning of each local controller does not typically involve model information from other control loops in order to simplify the maintenance and limit the controller complexity. Case (*ii*) is typical for cooperative driving such as vehicle platooning, where each vehicle has its own local (cruise) controller which cannot be designed based on model information of all possible vehicles that it may cooperate with in future traffic scenarios. Case (*iii*) can be illustrated by the control of the power grid, where economic incentives might limit the exchange of network model information across regional borders. Motivated by these important applications, we have started investigating the concept of limited model information control design for large-scale systems [15–18]. We briefly survey these studies in the next subsection.

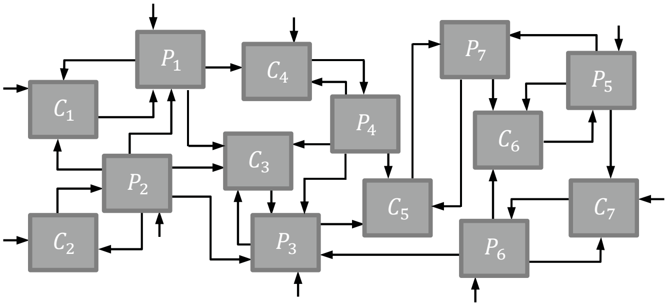

Figure 1: Illustrative example of a networked control system where P_i denotes the subsystems to be controlled and C_i denotes the controllers. The interactions between the subsystems and the controllers as well as the external disturbances and references are indicated by arrows.

1.2 Previous Studies

Recently, limitations of linear quadratic design under limited model information for a class of interconnected linear time-invariant dynamical system composed of scalar subsystems was studied in [15]. The authors introduced the competitive ratio as a metric for comparing control design strategies (i.e., mappings from the set of plants of interest to the set of applicable controllers) with various degrees of access to model information. The competitive ratio was simply defined as the worst-case ratio of the cost of a given control design strategy to the cost of the optimal control design with full model information. Showing that there is no control design strategy with a bounded competitive ratio when relying on local model information for continuous-time system, they justly concentrated on discrete-time systems. Then, they proved that the static deadbeat control design strategy attains the minimum competitive ratio among all strategies that only use local model information when designing a local controller. To distinguish between multiple possible minimizers of the competitive ratio, they introduced domination as a partial order on the set of limited model information control design strategies. They proved that the static deadbeat control design strategy is undominated, that is, there is no other control design strategy in the set of all limited model information design strategies with a better closed-loop performance for all possible plants while maintaining the same worst-case ratio.

This result was later extended to structured discrete-time fully-actuated linear time-invariant dynamical systems when the plant graph (i.e., directed graph that captures the interconnection pattern between different subsystems) contains no sink [16, 17]. In these studies, the set of applicable controllers was considered to be the set of structured static state feedback controllers. The structure of the controllers was captured using a control graph (i.e., directed graph that illustrates the state-measurement availability in subcontrollers) which was assumed to be a supergraph of the plant graph. In [16, 17], it was shown that the static deadbeat control design strategy is an undominated minimizer of the competitive ratio when the plant graph contains no sink. However, the design could be improved when the plant graph contains a sink. The choice of static controllers in these studies was justified, at first, by being the simplest case to explore [15–17], and then, maybe more surprisingly, by the recently proven fact that the best (in the sense of competitive ratio and domination) state-feedback structured H_2-controller for a plant with lower triangular information pattern that can be designed with limited model information is in fact static [18]. This is true even though the best such controller constructed with access to full model information is of course dynamic [9, 10]. In this paper, we study the problem of limited model information control design for optimal disturbance accommodation and servomechanism, and show that, contrary to the situations mentioned above, the best limited model information design method gives dynamic controllers. Optimal disturbance accommodation is a meaningful model for problems such as step disturbance rejection or step reference tracking, and has been well-studied in the literature [19–24], but with no attention being paid to the model information limitations in the design procedure.

1.3 Main Contributions

In this paper, specifically, we consider limited model information control design for interconnection of scalar discrete-time linear time-invariant subsystems being affected by scalar decoupled disturbances with a quadratic separable performance criterion. In each subsystem, the disturbance model is assumed to be known while its initial condition is unknown [24]. The motivation for such a cost function is given in the servomechanism and disturbance accommodation literature [19–24], and also stems from our interest in dynamically-coupled but cost-decoupled plants and their applications in supply chains and shared infrastructure [25, 26]. The assumptions on scalar subsystems and scalar disturbances are introduced to make the proofs shorter. Since we want each subsystem to be directly controllable (so that designing subcontrollers based on only local model information is possible), we assume that the overall system is fully-actuated (i.e., the same number of inputs as the state dimension). The results of this paper can be generalized to fully-actuated subsystems of arbitrary order. However, the generalization to under-actuated subsystems is non-trivial as explained in more detail in Remark 3.1. Note that we can also see this new model as a generalization of the problem formulation in [16–18] to under-actuated subsystems, since each subsystems can be considered as an aggregation of the original subsystem with its corresponding disturbance dynamics, however, only one of the states is in this case directly controlled and observed.

Our study in this paper starts with the case where each subcontroller is designed with the corresponding subsystem's information only. We prove that the so-called dynamic deadbeat control design strategy is an undominated minimizer of the competitive ratio when the plant graph contains no sink and the control graph is a supergraph of the plant graph. The fact that the dynamic deadbeat control design strategy is a minimizer of the competitive ratio is proved in Theorem 3.8 and the fact that it is undominated is proved in (the if part of) Theorem 3.9. For any fixed plant, the controller constructed by the dynamic deadbeat control design strategy can be separated into a static feedback law and a dynamic disturbance observer. For step disturbances, it is shown that this structure corresponds to a proportional-integral controller. However, the dynamic deadbeat control design strategy is dominated when the plant graph contains sinks. This is proved in (the only if part of) Theorem 3.9. We present an undominated limited model information control design method that takes advantage of the knowledge of the sinks' location to achieve a better closed-loop performance. We prove that this newly defined control design strategy is an undominated minimizer of the competitive ratio in Theorems 3.11 and 3.12. In Theorem 3.10, we further show that this control design strategy has the same competitive ratio as the dynamic deadbeat control design strategy. Later, in Theorem 3.13, we characterize the amount of model information needed to achieve a better competitive ratio than the dynamic deadbeat control design strategy. The amount of information is captured using the design graph (i.e., directed graph which indicates the dependency of each subcontroller on different parts of the global dynamical model). It turns out that, to achieve a better competitive ratio than the dynamic deadbeat control design strategy, each subsystem's controller should, at least, have access to the model of all those subsystems that can affect it.

1.4 Paper Outline

This paper is organized as follows. We formulate the problem and define the performance metric in Section 2. In Section 3, we introduce two specific control design strategies and study their properties. We characterize the best limited model information control design method as a function of the subsystems interconnection pattern in Section 4. In Section 5, we study the influence of the amount of the information available to each subsystem on the quality of the controllers that they can produce. We discuss special cases of step disturbance rejection, step reference tracking, and proportional-integral control in Section 6. Finally, we end with conclusions in Section 7.

1.5 Notation

The set of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. All other sets are denoted by calligraphic letters, such as $\mathcal{P}$ and $\mathcal{A}$. Particularly, the letter $\mathcal{R}$ denotes the set of proper real rational functions.

Matrices are denoted by capital roman letters such as A. A_j will denote the j^{th} row of A. A_{ij} denotes a submatrix of matrix A, the dimension and the position of which will be defined in the text. The entry in the i^{th} row and the j^{th} column of the matrix A is a_{ij}.

Let $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$. $A > (\geq) 0$ means that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq) B$ means that $A - B > (\geq) 0$.

$\underline{\sigma}(Y)$ and $\overline{\sigma}(Y)$ denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i denotes the column-vector with all entries zero except the i^{th} entry, which is equal to one.

All graphs considered in this paper are directed, possibly with self-loops, with vertex set $\{1, ..., q\}$ for some positive integer q. If $G = (\{1, ..., q\}, E)$ is a directed graph, we say that i is a sink if there does not exist $j \neq i$ such that $(i, j) \in E$. The adjacency matrix $S \in \{0, 1\}^{q\times q}$ of graph G is a matrix whose entries are defined as $s_{ij} = 1$ if $(j, i) \in E$ and $s_{ij} = 0$ otherwise. Since the set of vertices is fixed for all considered graphs, a subgraph of a graph G is a graph whose edge set is a subset of the edge set of G and a supergraph of a graph G is a graph of which G is a subgraph. We use the notation $G' \supseteq G$ to indicate that G' is a supergraph of G.

2 Mathematical Formulation

2.1 Plant Model

We are interested in discrete-time linear time-invariant dynamical systems described by

$$x(k+1) = Ax(k) + B(u(k) + w(k))\ ;\ x(0) = x_0, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^n$ is the control input, $w(k) \in \mathbb{R}^n$ is the disturbance vector and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are appropriate model matrices. Furthermore, we assume that the dynamic disturbance can be modeled as

$$w(k+1) = Dw(k)\ ;\ w(0) = w_0, \tag{2}$$

where $w_0 \in \mathbb{R}^n$ is unknown to the controller (and the control designer). Let a plant graph $G_{\mathcal{P}}$ with adjacency matrix $S_{\mathcal{P}}$ be given. We define the following set of matrices

$$\mathcal{A}(S_{\mathcal{P}}) = \{\bar{A} \in \mathbb{R}^{n\times n} \mid \bar{a}_{ij} = 0 \text{ for all } 1 \leq i, j \leq n \text{ such that } (s_{\mathcal{P}})_{ij} = 0\}.$$

Also, let us define

$$\mathcal{B}(\epsilon) = \{\bar{B} \in \mathbb{R}^{n\times n} \mid \underline{\sigma}(\bar{B}) \geq \epsilon, \bar{b}_{ij} = 0 \text{ for all } 1 \leq i \neq j \leq n\}, \tag{3}$$

for some given scalar $\epsilon > 0$ and

$$\mathcal{D} = \{\bar{D} \in \mathbb{R}^{n\times n} \mid \bar{d}_{ij} = 0 \text{ for all } 1 \leq i \neq j \leq n\}.$$

Now, we can introduce the set of plants of interest $\mathcal{P}$ as the set of all discrete-time linear time-invariant systems (1)–(2) with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $D \in \mathcal{D}$, $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathbb{R}^n$. With a slight abuse of notation, we will henceforth identify a plant $P \in \mathcal{P}$ with its corresponding tuple (A, B, D, x_0, w_0).

The variables $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$, and $w_i \in \mathbb{R}$ are the state, input, and disturbance of scalar subsystem i whose dynamics are given by

$$\begin{aligned} x_i(k+1) &= \sum_{j=1}^{n} a_{ij}x_j(k) + b_{ii}(u_i(k) + w_i(k)), \\ w_i(k+1) &= d_{ii}w_i(k). \end{aligned}$$

We call $G_{\mathcal{P}}$ the plant graph since it illustrates the interconnection structure between different subsystems, that is, subsystem j can affect subsystem i only if $(j,i) \in E_{\mathcal{P}}$. Note that we assume that the global system is fully-actuated; i.e., all the matrices $B \in \mathcal{B}(\epsilon)$ are square invertible matrices. This assumption is motivated by the fact that we need all subsystems to be directly controllable. Moreover, we make the standing assumption that the plant graph $G_{\mathcal{P}}$ contain no isolated node. There is no loss of generality in assuming that there is no isolated node in the plant graph $G_{\mathcal{P}}$, since it is always possible to design a controller for an isolated subsystem without any model information about the other subsystems and without influencing the overall system performance. Note that, in particular, this implies that there are $q \geq 2$ vertices in the graph because for $q = 1$ the only subsystem that exists is an isolated node in the plant graph.

Remark 3.1 *In this paper, we consider plants that are composed of scalar subsystems. Although this situation is admittedly restrictive, scalar subsystems can span a moderately rich family of physical or engineered systems (see [27–31] and references therein). In addition, the techniques and results presented here can be generalized to fully-actuated subsystems of arbitrary order. As will become clear later in Lemma 3.7, when dealing with fully-actuated subsystems, the designer should decouple the subsystems that are not a sink in the plant graph from the rest of the system. This can be intuitively justified since these subsystems should avoid affecting sensitive parts of the plant to achieve a bounded competitive ratio.*

Remark 3.2 *The special assumptions on the system and the disturbance in (1)–(2) enable us to estimate the initial condition of the disturbance using*

$$w_i(0) = \frac{1}{b_{ii}} \left[x_i(1) - b_{ii}u_i(0) - \sum_{j=1}^{n} a_{ij}x_j(0) \right], \tag{4}$$

which is called the deadbeat observer, since it recovers the initial-condition in just one time-step. Now that, for each $1 \leq i \leq n$, subsystem i has access to $w_i(0)$, it can easily cancel the effect of the disturbance by subtracting the terms $d_{ii}^k w(0)$ from its planned actuation input at each time step k. However, note that the problem of designing an optimal disturbance accommodation controller is a joint observer-controller design problem because one can always recover $w_i(0)$ using (4) irrespective of the value of $u_i(0)$, but by applying an erroneous $u_i(0)$, the competitive ratio would become infinite. In next section, through Lemma 3.7, we prove that given the control

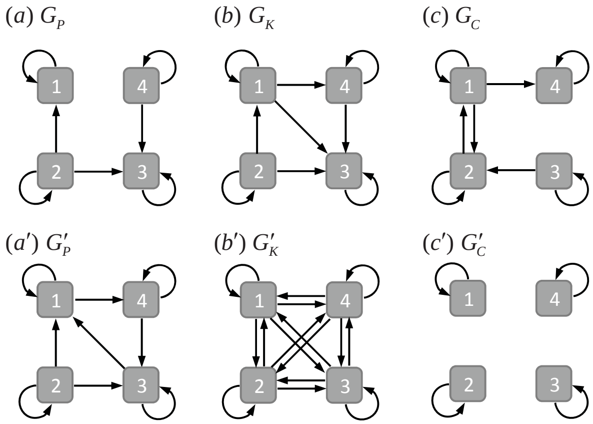

Figure 2: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs, $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs, and $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

action $u(0) = -B^{-1}(A+D)x(0)$, *the deadbeat observer introduced in (4) is the best control design strategy (in terms of competitive ratio and domination).*

Remark 3.3 *Because* $B \in \mathcal{B}(\epsilon)$ *is an invertible matrix, we can always rewrite the system dynamics so as* $u(k)$ *and* $w(k)$ *affect the system through the same matrix* B *as in (1). Note that even if* B *is not invertible, as long as the designer aims at stabilizing the origin (i.e.,* $\lim_{k\to\infty} x(k) = 0$*), it is no restriction to assume that* $u(k)$ *and* $w(k)$ *influence the system through the same* B*-matrix. It easily follows from considering the system*

$$x(k+1) = Ax(k) + Bu(k) + Ew(k)\ ;\ x(0) = x_0.$$

To achieve $\lim_{k\to\infty} x(k) = 0$*, it has to hold that* $\lim_{k\to\infty} Bu(k) + Ew(k) = 0$*. When* $\lim_{k\to\infty} w(k) \neq 0$*, this condition is satisfied if, and only if, there exists a matrix* M *such that* $E = BM$ *[23]. Defining* $\bar{w}(k) = Mw(k)$*, we get*

$$x(k+1) = Ax(k) + Bu(k) + B\bar{w}(k)\ ;\ x(0) = x_0.$$

Because we do not restrict D *in (2) to be stable, we thus have to make the assumption that* $E \in \text{image}(B)$ *as described above.*

Figure $2(a)$ shows an example of a plant graph $G_{\mathcal{P}}$. Each node represents a subsystem of the system. For instance, the second subsystem in this example affects the first subsystem and the third subsystem, that is, submatrices A_{12} and A_{32} can be nonzero. Note that the first subsystem in Figure $2(a)$ represents a sink of $G_{\mathcal{P}}$. The plant graph $G'_{\mathcal{P}}$ in Figure $2(a')$ has no sink.

2.2 Controller Model

The control laws of interest in this paper are discrete-time linear time-invariant dynamic state-feedback control laws of the form

$$x_K(k+1) = A_K x_K(k) + B_K x(k) \; ; \; x_K(0) = 0, \tag{5}$$

$$u(k) = C_K x_K(k) + D_K x(k). \tag{6}$$

Each controller can also be represented by a transfer function

$$K \triangleq \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = C_K(zI - A_K)^{-1}B_K + D_K,$$

where z is the symbol for the one time-step forward shift operator. Let a control graph $G_{\mathcal{K}}$ with adjacency matrix $S_{\mathcal{K}}$ be given. Each controller K belongs to

$$\mathcal{K}(S_{\mathcal{K}}) = \{K \in \mathcal{R}^{n\times n} \mid \quad k_{ij} = 0 \text{ for all } 1 \le i, j \le n \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}.$$

When the adjacency matrix $S_{\mathcal{K}}$ is not relevant or can be deduced from context, we refer to the set of controllers as $\mathcal{K}$. Since it makes sense for each subcontroller to use at least its corresponding subsystem state-measurements, we make the standing assumption that in each design graph $G_{\mathcal{K}}$, all the self-loops are present.

An example of a control graph $G_{\mathcal{K}}$ is given in Figure 2(b). Each node represents a subsystem–controller pair of the overall system. For instance, $G_{\mathcal{K}}$ shows that the first subcontroller can use state measurements of the second subsystem beside its corresponding subsystem state-measurements. Figure 2(b') shows a complete control graph $G'_{\mathcal{K}}$. This control graph indicates that each subcontroller has access to full state measurements of all subsystems, that is, $\mathcal{K}(S_{\mathcal{K}}) = \mathcal{R}^{n\times n}$.

2.3 Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}$. Any control design method Γ has the form

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix}, \tag{7}$$

where each entry γ_{ij} represents a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{D} \to \mathcal{R}$.

Let a design graph $G_{\mathcal{C}}$ with adjacency matrix $S_{\mathcal{C}}$ be given. We say that Γ has structure $G_{\mathcal{C}}$, if for all i, subcontroller i is computed with knowledge of the plant model of only those subsystems j such that $(j, i) \in E_{\mathcal{C}}$. Equivalently, Γ has structure $G_{\mathcal{C}}$, if for all i, the map $\Gamma_i = [\gamma_{i1} \;\cdots\; \gamma_{in}]$ is only a function of $\{[a_{j1} \;\cdots\; a_{jn}], b_{jj}, d_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}$. When $G_{\mathcal{C}}$ is not a complete graph, we refer to $\Gamma \in \mathcal{C}$ as being a "limited model information control design method". Since it makes sense for the designer of each subcontroller to have access to at least its corresponding subsystem model parameters, we make the standing assumption that in each design graph $G_{\mathcal{C}}$, all the self-loops are present.

The set of all control design strategies with structure $G_{\mathcal{C}}$ will be denoted by $\mathcal{C}$, which is considered as a subset of all maps from $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{D}$ to $\mathcal{K}(S_{\mathcal{K}})$ because a design method with structure $G_{\mathcal{C}}$ is not a function of the initial state x_0 or the initial disturbance w_0. We use the notation $\Gamma(A, B, D)$ instead of $\Gamma(P)$ for each plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ to emphasize this fact.

To simplify the notation, we assume that any control design strategy Γ has a state-space realization of the form

$$\Gamma(A, B, D) = \left[\begin{array}{c|c} A_\Gamma(A, B, D) & B_\Gamma(A, B, D) \\ \hline C_\Gamma(A, B, D) & D_\Gamma(A, B, D) \end{array}\right],$$

where $A_\Gamma(A, B, D)$, $B_\Gamma(A, B, D)$, $C_\Gamma(A, B, D)$, and $D_\Gamma(A, B, D)$ are matrices of appropriate dimension for each plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$. The matrices $A_\Gamma(A, B, D)$ and $C_\Gamma(A, B, D)$ are block diagonal matrices since subcontrollers do not share state variables. This realization is not necessarily minimal.

An example of a design graph $G_{\mathcal{C}}$ is given in Figure 2(c). Each node represents a subsystem–controller pair of the overall system. For instance, $G_{\mathcal{C}}$ shows that the second subsystem's model is available to the designer of the first subsystem's controller but not the third and the forth subsystems' model. Figure 2(c') shows a fully disconnected design graph $G'_{\mathcal{C}}$. A local designer in this case can only rely on the model of its corresponding subsystem.

2.4 Performance Metric

The goal of this paper is to investigate the influence of the plant graph on the properties of controllers derived from limited model information control design methods. We use two performance metrics to compare different control design methods, which are adapted from the notions of competitive ratio and domination recently introduced in [15–18]. Let us start with introducing the closed-loop performance criterion.

To each plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ and controller $K \in \mathcal{K}$, we associate the performance criterion

$$J_P(K) = \sum_{k=0}^{\infty} \left[x(k)^T Q x(k) + (u(k) + w(k))^T R (u(k) + w(k)) \right], \tag{8}$$

where $Q \in \mathcal{S}^n_{++}$ and $R \in \mathcal{S}^n_{++}$ are diagonal matrices. We make the following standing assumption:

Assumption 3.1 $Q = R = I$.

This is without loss of generality because the change of variables $(\bar{x}, \bar{u}, \bar{w}) = (Q^{1/2}x, R^{1/2}u, R^{1/2}w)$ transforms the closed-loop performance measure and state-space representation into

$$J_P(K) = \sum_{k=0}^{\infty} \left[\bar{x}(k)^T \bar{x}(k) + (\bar{u}(k) + \bar{w}(k))^T (\bar{u}(k) + \bar{w}(k))\right], \tag{9}$$

and

$$\begin{aligned} \bar{x}(k+1) &= Q^{1/2} A Q^{-1/2} \bar{x}(k) + Q^{1/2} B R^{-1/2} (\bar{u}(k) + \bar{w}(k)) \\ &= \bar{A}\bar{x}(k) + \bar{B}(\bar{u}(k) + \bar{w}(k)), \end{aligned}$$

without affecting the plant, control, or design graphs, due to Q and R being diagonal matrices.

Definition 3.1 (Competitive Ratio) *Let a plant graph $G_{\mathcal{P}}$, a control graph $G_{\mathcal{K}}$, and a constant $\epsilon > 0$ be given. Assume that, for every plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that*

$$J_P(K^*(P)) \leq J_P(K), \ \forall K \in \mathcal{K}.$$

The competitive ratio of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,D,x_0,w_0)\in\mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$" equals one.

Note that the optimal control design strategy (with full plant model information) K^* does not necessarily belong to the set $\mathcal{C}$.

Definition 3.2 (Domination) *A control design method Γ is said to dominate another control design method Γ' if*

$$J_P(\Gamma(A,B,D)) \leq J_P(\Gamma'(A,B,D)), \quad \forall\, P = (A,B,D,x_0,w_0) \in \mathcal{P}, \tag{10}$$

with strict inequality holding for at least one plant in $\mathcal{P}$. When $\Gamma' \in \mathcal{C}$ and no control design method $\Gamma \in \mathcal{C}$ exists that satisfies (10), we say that Γ' is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$.

In the remainder of this paper, we determine optimal control design strategies

$$\Gamma^* \in \arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma), \tag{11}$$

for a given plant, control, and design graph. Since several design methods may achieve this minimum, we are interested in determining which ones of these strategies are undominated.

3 Preliminary Results

Before stating the main results of the paper, we introduce two specific control design strategies and study their properties.

3.1 Optimal Centralized Control Design Strategy

The problem of designing optimal constant input-disturbance accommodation control for linear time-invariant continuous-time systems was solved earlier in [21, 23]. To the best of our knowledge, this was not the case for arbitrary dynamic disturbance accommodation when dealing with linear time-invariant discrete-time systems. As we need it later, we start by developing the optimal centralized (i.e, $G_{\mathcal{K}}$ is a complete graph) disturbance accommodation controller $K^*(P)$ for a given plant $P \in \mathcal{P}$. First, let us define the auxiliary variables $\xi(k) = u(k) + w(k)$ and $\bar{u}(k) = u(k+1) - Du(k)$. It then follows that

$$\begin{aligned} \xi(k+1) &= u(k+1) + w(k+1) \\ &= u(k+1) + Dw(k) \\ &= Du(k) + Dw(k) + \bar{u}(k) \\ &= D\xi(k) + \bar{u}(k). \end{aligned} \tag{12}$$

Augmenting the state-transition in (12) with the state-space representation of the system in (1) results in

$$\begin{bmatrix} x(k+1) \\ \xi(k+1) \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u}(k). \tag{13}$$

Besides, we can write the performance measure in (9) as

$$J_P(K) = \sum_{k=0}^{\infty} \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}. \tag{14}$$

To guarantee the existence and uniqueness of the optimal controller $K^*(P)$, we need the following lemma.

Lemma 3.1 *The pair $(\tilde{A}, \tilde{B})$, with*

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{15}$$

is controllable for any given $P = (A, B, D, x_0, w_0) \in \mathcal{P}$.

Proof: The pair $(\tilde{A}, \tilde{B})$ is controllable if and only if

$$\left[\begin{array}{c:c} \tilde{A} - \lambda I & \tilde{B} \end{array} \right] = \left[\begin{array}{cc:c} A - \lambda I & B & 0 \\ 0 & D - \lambda I & I \end{array} \right]$$

is full-rank for all $\lambda \in \mathbb{C}$. This condition is always satisfied since all matrices $B \in \mathcal{B}(\epsilon)$ are full-rank matrices. ■

Now the problem of minimizing the cost function in (14) subject to plant dynamics in (13) becomes a state-feedback linear quadratic optimal control with a unique solution of the form

$$\bar{u}(k) = G_1 x(k) + G_2 \xi(k),$$

where $G_1 \in \mathbb{R}^{n\times n}$ and $G_2 \in \mathbb{R}^{n\times n}$ satisfy

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} = -(\tilde{B}^T X \tilde{B})^{-1} \tilde{B}^T X \tilde{A} \tag{16}$$

and X is the unique positive-definite solution of the discrete algebraic Riccati equation

$$\tilde{A}^T X \tilde{B} (\tilde{B}^T X \tilde{B})^{-1} \tilde{B}^T X \tilde{A} - \tilde{A}^T X \tilde{A} + X - I = 0. \tag{17}$$

Therefore, we have

$$\begin{aligned} u(k+1) &= Du(k) + \bar{u}(k) \\ &= Du(k) + G_1 x(k) + G_2 \xi(k). \end{aligned} \tag{18}$$

Using the identity $\xi(k) = B^{-1}(x(k+1) - Ax(k))$ in (18), we get

$$\begin{aligned} u(k+1) &= Du(k) + G_1 x(k) + G_2 \xi(k) \\ &= Du(k) + G_1 x(k) + G_2 B^{-1}(x(k+1) - Ax(k)) \\ &= Du(k) + (G_1 - G_2 B^{-1} A) x(k) + G_2 B^{-1} x(k+1). \end{aligned} \tag{19}$$

Putting a control signal of the form $u(k) = x_K(k) + D_K x(k)$ in (19), we get

$$x_K(k+1) = Dx_K(k) + (DD_K + G_1 - G_2 B^{-1} A) x(k) + (G_2 B^{-1} - D_K) x(k+1).$$

Now, we enforce the condition $G_2 B^{-1} - D_K = 0$, as $x_K(k+1)$ can only be a function of $x(k)$ and $x_K(k)$, see (5). Therefore, the optimal controller $K^*(P)$ becomes

$$\begin{aligned} x_K(k+1) &= Dx_K(k) + [G_1 + DG_2 B^{-1} - G_2 B^{-1} A] x(k), \\ u(k) &= x_K(k) + G_2 B^{-1} x(k), \end{aligned}$$

with $x_K(0) = 0$.

Lemma 3.2 *Let the control graph $G_{\mathcal{K}}$ be a complete graph. Then, the cost of the optimal controller $K^*(P)$ for each plant $P \in \mathcal{P}$ is lower-bounded as*

$$J_P(K^*(P)) \geq \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2 B^{-2} & -D(W + B^{-2}) \\ -(W + B^{-2})D & W + B^{-2} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix},$$

where

$$W = A^T (I + B^2)^{-1} A + I.$$

Proof: Define

$$\bar{J}_P(K,\rho) = \sum_{k=0}^{\infty} \left(\begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} + \rho \bar{u}(k)^T \bar{u}(k) \right),$$

and

$$\bar{K}^*_\rho(P) = \arg\min_{K \in \mathcal{K}} \bar{J}_P(K,\rho).$$

Using Lemma 3.1, we know that $\bar{K}^*_\rho(P)$ exists and is unique. We can find $\bar{J}_P(\bar{K}^*_\rho(P),\rho)$ using $X(\rho)$ as the unique positive definite solution of the discrete algebraic Riccati equation

$$\tilde{A}^T X(\rho)\tilde{B}(\rho I + \tilde{B}^T X(\rho)\tilde{B})^{-1}\tilde{B}^T X(\rho)\tilde{A} - \tilde{A}^T X(\rho)\tilde{A} + X(\rho) - I = 0. \qquad (20)$$

According to [32], the positive-definite matrix $X(\rho)$ is lower-bounded by

$$\begin{aligned} X(\rho) - I &\geq \tilde{A}^T \left(\bar{X}(\rho)^{-1} + \rho^{-1}\tilde{B}\tilde{B}^T\right)^{-1} \tilde{A} \\ &= \tilde{A}^T \left(\bar{X}(\rho) - \bar{X}(\rho)\tilde{B}\left(\rho I + \tilde{B}^T \bar{X}(\rho)\tilde{B}\right)^{-1} \tilde{B}^T \bar{X}(\rho)\right) \tilde{A}, \end{aligned}$$

where

$$\bar{X}(\rho) = \tilde{A}^T \left(I + \rho^{-1}\tilde{B}\tilde{B}^T\right)^{-1} \tilde{A} = \begin{bmatrix} A^T A + I & A^T B \\ BA & B^2 + D^2 \frac{\rho}{\rho+1} + I \end{bmatrix}.$$

Basic algebraic calculations show that

$$\lim_{\rho \to 0} \left[\bar{X}(\rho) - \bar{X}(\rho)\tilde{B}\left(\rho I + \tilde{B}^T \bar{X}(\rho)\tilde{B}\right)^{-1} \tilde{B}^T \bar{X}(\rho)\right] = \begin{bmatrix} A^T(I+B^2)^{-1}A + I & 0 \\ 0 & 0 \end{bmatrix}.$$

According to [33], we know that

$$\lim_{\rho \to 0^+} \bar{J}_P(\bar{K}^*_\rho(P),\rho) = J_P(K^*(P)),$$

and as a result

$$X = \lim_{\rho \to 0} X(\rho) \geq \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^T \begin{bmatrix} A^T(I+B^2)^{-1}A + I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} + I. \qquad (21)$$

where X is the unique positive-definite solution of the discrete algebraic Riccati equation in (17) and consequently

$$J_P(K^*(P)) = \begin{bmatrix} x_0 \\ \xi(0) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ \xi(0) \end{bmatrix}$$

with X being partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}.$$

We know that

$$\xi(0) = u(0) + w_0 = G_2 B^{-1} x_0 + w_0 = -(X_{22}^{-1} X_{12}^T + DB^{-1}) x_0 + w_0.$$

Thus, the cost of the optimal control design $J_P(K^*(P))$ becomes

$$\begin{aligned}
&\begin{bmatrix} x_0 \\ -(X_{22}^{-1} X_{12}^T + DB^{-1}) x_0 + w_0 \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ -(X_{22}^{-1} X_{12}^T + DB^{-1}) x_0 + w_0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}^T \begin{bmatrix} X_{11} - X_{12} X_{22}^{-1} X_{12}^T + B^{-1} D X_{22} D B^{-1} & -B^{-1} D X_{22} \\ -X_{22} D B^{-1} & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}^T \begin{bmatrix} B^{-1}(X_{22} + D X_{22} D - I) B^{-1} & -B^{-1} D X_{22} \\ -X_{22} D B^{-1} & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}
\end{aligned} \tag{22}$$

The second equality is true because of the following equation extracted from the discrete algebraic Riccati equation in (17)

$$X_{22} = I + B X_{11} B - B X_{12} X_{22}^{-1} X_{12}^T B,$$

which is equivalent to

$$X_{11} - X_{12} X_{22}^{-1} X_{12}^T = B^{-1}(X_{22} - I) B^{-1}. \tag{23}$$

Using (21), it is evident that

$$X_{22} \geq B[A^T (I + B^2)^{-1} A + I] B + I = BWB + I,$$

and as a result, the inner-matrix in (22) is lower-bounded by

$$\begin{aligned}
&\begin{bmatrix} B^{-1}(X_{22} + D X_{22} D - I) B^{-1} & -B^{-1} D X_{22} \\ -X_{22} D B^{-1} & X_{22} \end{bmatrix} \\
&= \begin{bmatrix} B^{-1}(X_{22} - I) B^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B^{-1} D X_{22} D B^{-1} & -B^{-1} D X_{22} \\ -X_{22} D B^{-1} & X_{22} \end{bmatrix} \\
&= \begin{bmatrix} B^{-1}(X_{22} - I) B^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -B^{-1} D \\ I \end{bmatrix} X_{22} \begin{bmatrix} -B^{-1} D \\ I \end{bmatrix}^T \\
&\geq \begin{bmatrix} B^{-1}(BWB) B^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -B^{-1} D \\ I \end{bmatrix} (BWB + I) \begin{bmatrix} -B^{-1} D \\ I \end{bmatrix}^T \\
&= \begin{bmatrix} W + DWD + D^2 B^{-2} & -D(WB + B^{-1}) \\ -(BW + B^{-1}) D & BWB + I \end{bmatrix}
\end{aligned}$$

Finally, we get

$$\begin{aligned}
J_P(K^*(P)) &\geq \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2 B^{-2} & -D(WB + B^{-1}) \\ -(BW + B^{-1}) D & BWB + I \end{bmatrix} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2 B^{-2} & -D(W + B^{-2}) \\ -(W + B^{-2}) D & W + B^{-2} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}.
\end{aligned}$$

This statement concludes the proof. ∎

3.2 Deadbeat Control Design Strategy

In this subsection, we introduce the deadbeat control design strategy and calculate its competitive ratio.

Definition 3.3 *Let a plant graph $G_{\mathcal{P}}$ and a control graph $G_{\mathcal{K}}$ be given such that $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. The deadbeat control design strategy $\Gamma^{\Delta} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{D} \to \mathcal{K}$ is defined as*

$$\Gamma^{\Delta}(A, B, D) \triangleq \left[\begin{array}{c|c} D & -B^{-1}D^2 \\ \hline I & -B^{-1}(A+D) \end{array}\right].$$

Remark 3.4 *It should be noted that using the deadbeat control design strategy, the closed-loop system reaches the origin in just two time-steps irrespective of the value of the initial state x_0 and the initial disturbance w_0. Additionally, the deadbeat control design strategy is a limited model information control design method since*

$$\Gamma_i^{\Delta}(A, B, D) = -(z - d_{ii})^{-1} b_{ii}^{-1} d_{ii}^2 e_i^T - b_{ii}^{-1}(A_i + D_i),$$

for each $1 \leq i \leq n$, that is, subcontroller i uses only the plant model information of subsystem i, (A_i, B_i, D_i). Finally, when using the deadbeat control design strategy, $a_{ij} + b_{ii}(d_\Gamma)_{ij}(A, B, D) = 0$, $1 \leq i, j \leq n$, which, as shown later in Lemma 3.7, is a property that must necessarily be satisfied by nodes that are not a sink to obtain a finite competitive ratio.

The closed-loop system with deadbeat control design strategy is shown in Figure 3(a). This feedback loop can be rearranged as the one in Figure 3(b) which has two separate components. One component is a static deadbeat control design strategy for regulating the state of the plant and the other one is a deadbeat observer for canceling the disturbance. This structure is further discussed in Section 6, where it is shown that it corresponds to proportional-integral control in some cases. First, we need to calculate an expression for the cost of the deadbeat control design strategy.

Lemma 3.3 *The cost of the deadbeat control design strategy Γ^{Δ} for each plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ is*

$$J_P(\Gamma^{\Delta}(A, B, D)) = \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix},$$

where

$$Q_{11} = I + D^2(I + B^{-2}) + A^T B^{-2} A + D A^T B^{-2} A D + A^T B^{-2} D + D B^{-2} A, \tag{24}$$

$$Q_{12} = -D - A^T B^{-2} - D B^{-2} - D A^T B^{-2} A, \tag{25}$$

$$Q_{22} = A^T B^{-2} A + B^{-2} + I. \tag{26}$$

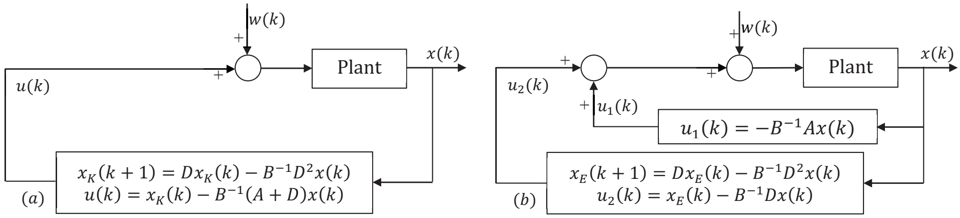

Figure 3: The closed-loop system with (a) the deadbeat controller corresponding to Γ^{Δ}, and (b) rearranging this controller as a static deadbeat controller and a deadbeat observer.

Proof: First, it should be noted that the state of the closed-loop system with $\Gamma^{\Delta}(A, B, D)$ in feedback reaches the origin in two time-steps. Now, using the system state transition, one can calculate the deadbeat control design strategy cost as

$$J_P(\Gamma^{\Delta}(A, B, D)) = x_0^T x_0 + (u(0) + w_0)^T (u(0) + w_0) \\ + x(1)^T x(1) + (u(1) + w(1))^T (u(1) + w(1)),$$

where $x(1) = -Dx_0 + Bw_0$, $u(0) = -B^{-1}(A + D)x_0$, and $u(1) = -B^{-1}(A + D)x(1) - B^{-1}D^2 x_0$. The rest of the proof is a trivial simplification. ■

We need the following lemma in order to calculate the competitive ratio of the deadbeat control design strategy Γ^{Δ} when the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. As the notation $K^*(P)$ is reserved for the optimal control design strategy for a given control graph $G_{\mathcal{K}}$, from now on, we will use K_C^* to denote the centralized optimal control design strategy (i.e., the optimal control design strategy with access to full-state measurement).

Lemma 3.4 *Let $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$, and $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ be a plant with A being a nilpotent matrix of degree two. Then, $J_P(K^*(P)) = J_P(K_C^*(P))$.*

Proof: When matrix A is nilpotent, the unique positive-definite solution of the discrete algebraic Riccati equation (17) is

$$X = \begin{bmatrix} A^T A + I & A^T B \\ BA & BA^T (I + B^2)^{-1} AB + I + B^2 \end{bmatrix}.$$

Consequently, the optimal centralized controller gains in (16) are

$$G_1 = 0, \qquad G_2 = -(I + B^2)^{-1} BAB - D,$$

and as a result, the optimal centralized controller $K_C^*(P)$ is

$$K_C^*(P) = \left[\begin{array}{c|c} D & D(I + B^2)^{-1} B^{-1} A - B^{-1} D^2 \\ \hline I & -(I + B^2)^{-1} BA - B^{-1} D \end{array}\right] \\ = (zI - D)^{-1} D(I + B^2)^{-1} B^{-1} A - B^{-1} D^2 - (I + B^2)^{-1} BA - B^{-1} D.$$

Thus, $K^*_C(P) \in \mathcal{K}(S_\mathcal{K})$ because the control graph $G_\mathcal{K}$ is a supergraph of the plant graph $G_\mathcal{P}$. Now, considering that $K^*(P)$ is the global optimal decentralized controller, it has a lower cost than any other decentralized controller $K \in \mathcal{K}(S_\mathcal{K})$, specially $K^*_C(P) \in \mathcal{K}(S_\mathcal{K})$ for this particular plant. Hence,

$$J_P(K^*(P)) \leq J_P(K^*_C(P)). \tag{27}$$

On the other hand, it is evident that

$$J_P(K^*_C(P)) \leq J_P(K^*(P)). \tag{28}$$

This concludes the proof. ■

Remark 3.5 *Finding the optimal structured controller is intractable in general, even when the global model is known. In this paper, we concentrate on the cases where the control graph $G_\mathcal{K}$ is a supergraph of the plant graph $G_\mathcal{P}$, because it is relatively easier to solve the optimal control design problem under limited model information in this case. In addition, although, in this paper, we may not be able to find the optimal structured controller $K^*(P)$ for a particular plant in some of the cases, we can still compute the competitive ratio $r_\mathcal{P}$. Thus, in a sense, this makes the competitive ratio a quite powerful tool.*

Next, we derive the competitive ratio of the deadbeat control design method.

Theorem 3.5 *Let $G_\mathcal{K} \supseteq G_\mathcal{P}$. Then, the competitive ratio of the deadbeat control design method Γ^Δ is equal to*

$$r_\mathcal{P}(\Gamma^\Delta) = \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2+1}}{2\epsilon^2}.$$

Proof: First, let us define the set of all real numbers that are greater than or equal to the competitive ratio of the deadbeat control design strategy

$$\mathcal{M} = \left\{ \beta \in \mathbb{R} \;\middle|\; \frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} \leq \beta \; \forall P \in \mathcal{P} \right\}.$$

It is evident that

$$J_P(K^*_C(P)) \leq J_P(K^*(P))$$

for each plant $P \in \mathcal{P}$ irrespective of the control graph $G_\mathcal{K}$, and as a result

$$\frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} \leq \frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*_C(P))}. \tag{29}$$

Using (29) and Lemmas 3.3 and 3.2, β belongs to the set $\mathcal{M}$ if

$$\frac{\begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}}{\begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2B^{-2} & -D(W+B^{-2}) \\ -(W+B^{-2})D & W+B^{-2} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}} \leq \beta, \tag{30}$$

for all $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $D \in \mathcal{D}$, $x_0 \in \mathbb{R}^n$, and $w_0 \in \mathbb{R}^n$ where Q_{11}, Q_{12}, and Q_{22} are matrices defined in (24)–(26). The condition (30) is satisfied, if and only if, for all $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$, we have

$$\begin{bmatrix} \beta(W + DWD + D^2B^{-2}) - Q_{11} & -\beta D(W + B^{-2}) - Q_{12} \\ -\beta(W + B^{-2})D - Q_{12}^T & \beta(W + B^{-2}) - Q_{22} \end{bmatrix} \geq 0.$$

Using Schur complement [34], β belongs to the set $\mathcal{M}$ if

$$\begin{aligned} Z =& \beta(W + B^{-2}) - Q_{22} \\ =& \beta(A^T(I + B^2)^{-1}A + I + B^{-2}) - A^TB^{-2}A - B^{-2} - I \\ =& A^T(\beta(I + B^2)^{-1} - B^{-2})A + (\beta - 1)(B^{-2} + I) \geq 0, \end{aligned} \tag{31}$$

and

$$\begin{aligned} - \left[-\beta D(W + B^{-2}) - Q_{12}\right] \left[\beta(W + B^{-2}) - Q_{22}\right]^{-1} \left[-\beta(W + B^{-2})D - Q_{12}^T\right] \\ + \beta(W + DWD + D^2B^{-2}) - Q_{11} \geq 0, \end{aligned} \tag{32}$$

for all $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$. We can do the simplification

$$\begin{aligned} -\beta D(W + B^{-2}) - Q_{12} &= -\beta D(A^T(I + B^2)^{-1}A + I + B^{-2}) \\ &\quad - (-D - A^TB^{-2} - DB^{-2} - DA^TB^{-2}A) \\ &= -(\beta - 1)D(I + B^{-2}) + A^TB^{-2} \\ &\quad - DA^T(\beta(I + B^2)^{-1} - B^{-2})A \\ &= -DZ + A^TB^{-2}, \end{aligned}$$

and as a result, the condition (32) is equivalent to

$$\beta(W+DWD+D^2B^{-2})-Q_{11}-[-DZ+A^TB^{-2}]Z^{-1}[-ZD+B^{-2}A] \geq 0, \tag{33}$$

where Z is defined in (31). Furthermore, we can simplify $\beta(W+DWD+D^2B^{-2})-Q_{11}$ as

$$\begin{aligned} A^T(\beta(I + B^2)^{-1} - B^{-2})A + (\beta - 1)[I + D^2B^{-2} + D^2] \\ +DA^T(\beta(I + B^2)^{-1} - B^{-2})AD - A^TB^{-2}D - DB^{-2}A, \end{aligned}$$

which helps us to expand condition (33) to

$$\begin{aligned} A^T \left(\beta(I + B^2)^{-1} - B^{-2}\right) A + (\beta - 1) \left(I + D^2B^{-2} + D^2\right) \\ + DA^T \left(\beta(I + B^2)^{-1} - B^{-2}\right) AD - A^TB^{-2}D - DB^{-2}A \\ - D \left(A^T \left(\beta(I + B^2)^{-1} - B^{-2}\right) A + (\beta - 1)(B^{-2} + I)\right) D \\ + A^TB^{-2}D + DB^{-2}A - A^TB^{-2}Z^{-1}B^{-2}A \geq 0. \end{aligned} \tag{34}$$

Hence, it follows from (34) that (33) can be simplified as

$$A^T\left(\beta(I+B^2)^{-1}-B^{-2}\right)A-A^TB^{-2}Z^{-1}B^{-2}A\geq 0. \tag{35}$$

The condition (31) is satisfied, for all plants $P\in\mathcal{P}$, if $\beta\geq 1+1/\epsilon^2$, since in this case $\beta(I+B^2)^{-1}-B^{-2}\geq 0$ (recall that any matrix B is diagonal and its diagonal elements are lower-bounded by ϵ). Furthermore, for all $\beta\geq 1+1/\epsilon^2$, it is easy to see that $Z\geq(\beta-1)(B^{-2}+I)$. As a result, it can be shown that the condition (35) is satisfied if

$$A^T\left(\beta(I+B^2)^{-1}-B^{-2}-(\beta-1)^{-1}B^{-2}(B^{-2}+I)^{-1}B^{-2}\right)A+(\beta-1)I\geq 0. \tag{36}$$

Now, the condition (36) is satisfied if

$$\beta(I+B^2)^{-1}-B^{-2}-(\beta-1)^{-1}B^{-2}(B^{-2}+I)^{-1}B^{-2}\geq 0. \tag{37}$$

Noting that the matrix $B=\text{diag}(b_{11},\dots,b_{nn})$, one can rewrite (37) as

$$\frac{\beta}{1+b_{ii}^2}-\frac{1}{b_{ii}^2}-\frac{1}{\beta-1}\frac{1}{b_{ii}^2(1+b_{ii}^2)}\geq 0. \tag{38}$$

for all $b_{ii}\geq\epsilon$. Retracing our steps backward, it easy to see that the set

$$\left\{\beta\mid\beta\geq 1+\frac{1}{\epsilon^2}\text{ and (38) satisfied}\right\}=\left\{\beta\geq\frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}\right\}\subseteq\mathcal{M}.$$

Therefore, we get

$$r_{\mathcal{P}}(\Gamma^\Delta)=\sup_{P\in\mathcal{P}}\frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))}\leq\frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}. \tag{39}$$

Now, we have to show that this upper bound can be achieved by a family of plants. Consider a one-parameter family of matrices $\{A(r)\}$ defined as $A(r)=re_je_i^T$ for each $r\in\mathbb{R}$. It is always possible to find indices i and j such that $i\neq j$ and $(s_{\mathcal{P}})_{ji}\neq 0$, because of the assumption that there be no isolated node in the plant graph. Let $B=\epsilon I$ and $D=I$. For each $r\in\mathbb{R}$, the matrix $A(r)$ is a nilpotent matrix of degree two, that is, $A(r)^2=0$. Thus, using Lemma 3.4, we get

$$J_P(K_C^*(P))=J_P(K^*(P))$$

for this special plant. The solution to the discrete algebraic Riccati equation in (17) is

$$X=\begin{bmatrix} A(r)^TA(r)+I & \epsilon A(r)^T \\ \epsilon A(r) & \epsilon^2/(1+\epsilon^2)A(r)^TA(r)+(\epsilon^2+1)I\end{bmatrix}.$$

Thus, if we assume that

$$x_0=\frac{(\epsilon^2+1)(\sqrt{4\epsilon^2+1}+1)}{2\epsilon r}e_i, \tag{40}$$

and

$$w_0 = \frac{(\epsilon^2+1)(\sqrt{4\epsilon^2+1}+1)}{2\epsilon^2 r} e_i - e_j, \tag{41}$$

the cost of the optimal control design strategy is

$$J_P(K^*(P)) = \frac{(\epsilon^2+1)\sqrt{4\epsilon^2+1}+5\epsilon^2+4\epsilon^4+1}{2\epsilon^2} + \frac{(2\epsilon^2+\sqrt{4\epsilon^2+1}+1)\sqrt{4\epsilon^2+1}}{2\epsilon^2 r^2}, \tag{42}$$

and the cost of the deadbeat control design strategy is

$$J_P(\Gamma^\Delta(A,B,D)) = \frac{(\epsilon^2+1)(3\epsilon^2\sqrt{4\epsilon^2+1}+5\epsilon^2+4\epsilon^4+\sqrt{4\epsilon^2+1}+1)}{2\epsilon^4} + \frac{(\epsilon^2+1)(\epsilon^2\sqrt{4\epsilon^2+1}+\epsilon^4\sqrt{4\epsilon^2+1}+\epsilon^2+3\epsilon^4+2\epsilon^6)}{2\epsilon^4 r^2}. \tag{43}$$

This results in

$$\lim_{r\to\infty} \frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} = \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}. \tag{44}$$

Equation (39) together with (44) conclude the proof. ■

Remark 3.6 *Consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 2(a) and the control graph $G_{\mathcal{K}}$ in Figure 2(b). Theorem 3.5 shows that, if we apply the deadbeat control design strategy to this particular problem, the performance of the deadbeat control design strategy, at most, can be $(2\epsilon^2+1+\sqrt{4\epsilon^2+1})/(2\epsilon^2)$ times the cost of the optimal control design strategy K^*. In fact, Theorem 3.5 states that this relationship between the performance of the deadbeat control design and the optimal control design with full model information holds for a rather general class of systems. For the case that $\mathcal{B}=\{I\}$, the relationship is given by $(3+\sqrt{5})/2 \approx 2.62$, so the deadbeat control design strategy is never worse than two or three times the optimal.*

Remark 3.7 *We only proved the results for the case where there is a uniform lower bound on the entries of matrices $B \in \mathcal{B}(\epsilon)$. A theorem similar to Theorem 3.5 can still be proved when we have a nonuniform lower bound on the diagonal entries of the matrices $B \in \mathcal{B}$ in (3). Let us in that case define the set of matrices*

$$\mathcal{B}'(\{\epsilon_i\}_{i=1}^n) = \{\bar{B} \in \mathbb{R}^{n\times n} \mid \bar{B} \geq \operatorname{diag}(\epsilon_1,\dots,\epsilon_n), \bar{b}_{ij}=0 \text{ for all } 1 \leq i \neq j \leq n\},$$

for given $\{\epsilon_i\}_{i=1}^n$, such that $\epsilon_i > 0$, for all $1 \leq i \leq n$. In the proof of Theorem 3.5, we can then use that

$$\left\{\beta \geq \frac{2\epsilon_*^2+1+\sqrt{4\epsilon_*^2+1}}{2\epsilon_*^2}\right\} \subseteq \mathcal{M},$$

where $\epsilon_* = \min_{1\le i\le n} \epsilon_i$*, based on the fact that inequality (37) should be satisfied for each diagonal entry* $b_{ii} \ge \epsilon_i$*. Using the definition of the set* $\mathcal{M}$ *in the proof of Theorem 3.5, we get*

$$r_{\mathcal{P}}(\Gamma^{\Delta}) \le \frac{2\epsilon_*^2 + 1 + \sqrt{4\epsilon_*^2 + 1}}{2\epsilon_*^2}.$$

The rest of the results in this paper can be similarly generalized to a non-uniform lower bound on the entries of the matrices $B \in \mathcal{B}'(\{\epsilon_i\}_{i=1}^n)$*.*

Remark 3.8 *In the proof of Theorem 3.5, we use a special family of plants to achieve the competitive ratio of the deadbeat control design strategy. In this family of plants, only a single entry of the A-matrix approaches infinity while the other entries are zero. As this specific entry goes to infinity, its corresponding subsystem becomes tightly coupled to another subsystem. Note that the overall system is controllable because* $\underline{\sigma}(B) \ge \epsilon$*. This special family of plants plays an important role in determining a lower bound for the competitive ratio of control design strategies for various design graphs also in later proofs of this paper.*

With this characterization of Γ^{Δ} in hand, we are now ready to tackle problem (11).

4 Plant Graph Influence on Achievable Performance

In this section, we study the relationship between the plant graph and the achievable closed-loop performance in terms of the competitive ratio as a performance metric and the domination as a partial order on the set of limited model information control design strategies. To this end, we first state and prove two lemmas which will simplify further developments.

Lemma 3.6 *Fix real numbers* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}$*. For any* $x \in \mathbb{R}$*, we have* $x^2 + (a + bx)^2 \ge a^2/(1 + b^2)$*.*

Proof: Consider the function $x \mapsto x^2 + (a + bx)^2$. Since this function is both continuously differentiable and strictly convex, we can find its unique minimizer as $\bar{x} = -ab/(1 + b^2)$ by setting its derivative to zero. As a result, we get

$$x^2 + (a + bx)^2 \ge \bar{x}^2 + (a + b\bar{x})^2 = a^2/(1 + b^2).$$

This concludes the proof. ■

Lemma 3.7 *Let the design graph* $G_{\mathcal{C}}$ *be a totally disconnected graph, and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$*. Furthermore, assume that node* i *is not a sink in the plant graph* $G_{\mathcal{P}}$*. Then, the competitive ratio of a control design strategy* $\Gamma \in \mathcal{C}$ *is bounded only if* $a_{ij} + b_{ii}(d_{\Gamma})_{ij}(A, B, D) = 0$ *for all* $j \ne i$ *and all matrices* $A \in \mathcal{A}(S_{\mathcal{P}})$*,* $B \in \mathcal{B}(\epsilon)$*, and* $D \in \mathcal{D}$*.*

Proof: The proof is by contrapositive. Let us assume that there exist matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $D \in \mathcal{D}$, and indices i and j such that $i \neq j$ and $\bar{a}_{ij} + b_{ii}(d_\Gamma)_{ij}(\bar{A}, B, D) \neq 0$. Let $1 \leq \ell \leq n$ be an index such that $\ell \neq i$ and $(s_{\mathcal{P}})_{\ell i} \neq 0$ (such an index always exists because node i is not a sink in the plant graph $G_{\mathcal{P}}$). Define matrix A such that $A_i = \bar{A}_i$, $A_\ell = re_i^T$, and $A_t = 0$ for all $t \neq i, \ell$. Because the design graph is a totally disconnected graph, we know that $\Gamma_i(\bar{A}, B, D) = \Gamma_i(A, B, D)$. Using the structure of the cost function in (9) and plant dynamics in (1), the cost of this control design strategy for $w_0 = e_j$ and $x_0 = 0$ is lower-bounded by

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A,B,D)) &\geq (u_\ell(2) + w_\ell(2))^2 + x_\ell(3)^2 \\ &= (u_\ell(2) + w_\ell(2))^2 + (rx_i(2) + b_{\ell\ell}[u_\ell(2) + w_\ell(2)])^2 . \end{aligned}$$

Based on Lemma 3.6 and the fact that $x_i(2) = (a_{ij} + b_{ii}(d_\Gamma)_{ij}(A, B, D))b_{jj}$ (see Figure 4), we get

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A,B,D)) &\geq r^2 x_i(2)^2/(1 + b_{\ell\ell}^2) \\ &= (a_{ij} + b_{ii}(d_\Gamma)_{ij}(A,B,D))^2 b_{jj}^2 r^2/(1 + b_{\ell\ell}^2). \end{aligned}$$

On the other hand, the cost of the deadbeat control design strategy is

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma^\Delta(A,B,D)) &= e_j^T B^T (A^T B^{-2} A + B^{-2} + I) B e_j \\ &= b_{jj}^2 + 1 + a_{ij}^2 b_{jj}^2 / b_{ii}^2. \end{aligned}$$

Note that the deadbeat control design strategy is applicable here since the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. This gives

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &= \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))} \\ &= \sup_{P \in \mathcal{P}} \left[\frac{J_P(\Gamma(A,B,D))}{J_P(\Gamma^\Delta(A,B,D))} \frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} \right] \\ &\geq \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(\Gamma^\Delta(A,B,D))} \\ &\geq \frac{(a_{ij} + b_{ii}(d_\Gamma)_{ij}(A,B,D))^2 b_{jj}^2/(1 + b_{\ell\ell}^2)}{b_{jj}^2 + 1 + a_{ij}^2 b_{jj}^2/b_{ii}^2} \lim_{r \to \infty} r^2 = \infty. \end{aligned} \tag{45}$$

This inequality proves the statement by contrapositive as the competitive ratio is not bounded in this case. ∎

4.1 Plant Graphs without Sinks

First, we assume that there is no sink in the plant graph and try to characterize the optimal control design strategy in terms of the competitive ratio and domination.

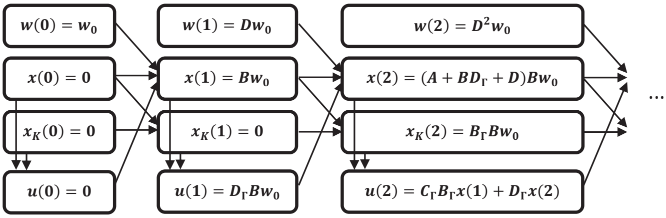

Figure 4: State evolution of the closed-loop system with any control design strategy Γ when $x_0 = 0$.

Theorem 3.8 *Let the plant graph $G_{\mathcal{P}}$ contain no sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2}.$$

Proof: Consider a one-parameter family of matrices $\{A(r)\}$ defined as $A(r) = re_j e_i^T$ for each $r \in \mathbb{R}$. It is always possible to find indices i and j such that $i \neq j$ and $(s_{\mathcal{P}})_{ji} \neq 0$, because of the assumption that there is no isolated node in the plant graph. Let $B = \epsilon I$ and $D = I$. Let $\Gamma \in \mathcal{C}$ be a control design strategy with design graph $G_{\mathcal{C}}$. Without loss of generality, we can assume that $(d_\Gamma)_{ji}(A, B, D) = -r/\epsilon$ since otherwise, using Lemma 3.7, we get that $r_{\mathcal{P}}(\Gamma)$ is infinity, and as a result the inequality in the theorem statement is trivially satisfied. Thus, for each $r \in \mathbb{R}$, the cost of the control design strategy Γ for x_0 in (40) and w_0 in (41) is lower-bounded by

$$\begin{aligned} J_P(\Gamma(A, B, D)) &\geq (u_j(0) + w_j(0))^2 + x_j(1)^2 \\ &= \left(\frac{(\epsilon^2 + 1)(\sqrt{4\epsilon^2 + 1} + 1)}{2\epsilon^2} + 1 \right)^2 + \epsilon^2 \\ &= \frac{(\epsilon^2 + 1)(3\epsilon^2\sqrt{4\epsilon^2 + 1} + 5\epsilon^2 + 4\epsilon^4 + \sqrt{4\epsilon^2 + 1} + 1)}{2\epsilon^4}. \end{aligned}$$

On the other hand, for each $r \in \mathbb{R}$, the matrix $A(r)$ is a nilpotent matrix of degree two, that is, $A(r)^2 = 0$. Consequently, using Lemma 3.4, the cost of the optimal control design strategy $K^*(P)$ for x_0 in (40) and w_0 in (41) is given by (42). This results in

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{r \to \infty} \frac{J_P(\Gamma(A, B, D))}{J_P(K^*(P))} = \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2}.$$

■

Theorem 3.8 shows that the deadbeat control design method Γ^Δ is a minimizer of the competitive ratio $r_{\mathcal{P}}$ as a function over the set of limited model information design methods $\mathcal{C}$. The following theorem shows that it is also undominated by methods of this type, if and only if, the plant graph $G_{\mathcal{P}}$ has no sink.

Theorem 3.9 *Let the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the control design strategy Γ^{Δ} is undominated if and only if there is no sink in the plant graph $G_{\mathcal{P}}$.*

Proof: First, we have to prove the sufficiency part of the theorem. Assume that there is no sink in the plant graph. For proving this claim, we are going to prove that for any control design method $\Gamma \in \mathcal{C} \setminus \{\Gamma^{\Delta}\}$, there exists a plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ such that $J_P(\Gamma(A, B, D)) > J_P(\Gamma^{\Delta}(A, B, D))$. First, assume that there exist matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$ and an index j such that $\bar{A}_j + b_{jj}(D_\Gamma)_j(\bar{A}, B, D) + d_{jj}e_j^T \neq 0$. Without loss of generality, we can assume that $\bar{a}_{jj} + b_{jj}(d_\Gamma)_{jj}(\bar{A}, B, D) + d_{jj} \neq 0$, because otherwise, using Equation (45) in the proof of Lemma 3.7, we know that, if there exists $\ell \neq j$ such that $\bar{a}_{j\ell} + b_{jj}(d_\Gamma)_{j\ell}(\bar{A}, B, D) \neq 0$, the ratio of the cost of the control design strategy Γ to the cost of the deadbeat design strategy Γ^{Δ} is unbounded. Therefore, the control design strategy Γ cannot dominate the deadbeat control design strategy Γ^{Δ}. Pick an index $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. It is always possible to pick such index i because there is no sink in the plant graph. Define matrix A such that $A_j = \bar{A}_j$, $A_i = re_j^T$, and $A_\ell = 0$ for all $\ell \neq i, j$. It should be noted that $\Gamma_j(A, B, D) = \Gamma_j(\bar{A}, B, D)$ because the design graph is a totally disconnected graph. We know that $r + b_{ii}(d_\Gamma)_{ij}(A, B, D) = 0$ because otherwise the control design strategy Γ cannot dominate the deadbeat control design strategy. The cost of this control design strategy for $w = e_j$ and $x_0 = 0$ satisfies

$$\begin{aligned} J_P(\Gamma(A, B, D)) &\geq (u_i(1) + w_i(1))^2 + (u_i(2) + w_i(2))^2 + x_i(3)^2 \\ &= r^2 b_{jj}^2 / b_{ii}^2 + (u_i(2) + w_i(2))^2 + (x_j(2) r + b_{ii}[u_i(2) + w_i(2)])^2, \end{aligned}$$

because of the structure of the cost function (9) and the plant dynamics (1). Now, using Lemma 3.6, we have

$$J_P(\Gamma(A, B, D)) \geq r^2 b_{jj}^2 / b_{ii}^2 + x_j(2)^2 r^2 / (1 + b_{ii}^2).$$

As a result

$$\begin{aligned} & J_P(\Gamma(A, B, D)) - J_P(\Gamma^{\Delta}(A, B, D)) \\ & \quad \geq (\bar{A}_{jj} + b_{jj}(d_\Gamma)_{jj}(\bar{A}, B, D) + d_{jj})^2 b_{jj}^2 r^2 / (1 + b_{ii}^2) - (b_{jj}^2 + 1 + a_{jj}^2), \end{aligned} \tag{46}$$

since $x_j(2) = (\bar{A}_{jj} + b_{jj}(d_\Gamma)_{jj}(\bar{A}, B, D) + d_{jj}) b_{jj}$ (see Figure 4) and

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma^{\Delta}(A, B, D)) &= e_j^T B^T (A^T B^{-2} A + B^{-2} + I) B e_j \\ &= b_{jj}^2 + 1 + r^2 b_{jj}^2 / b_{ii}^2 + a_{jj}^2. \end{aligned}$$

Thus, if we pick r large enough, the difference in (46) becomes positive, which shows that the control design strategy Γ cannot dominate the deadbeat control design strategy Γ^{Δ}. Now, assume that there exist matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $\bar{D} \in \mathcal{D}$ and an index j such that $\bar{A}_j + b_{jj}(D_\Gamma)_j(\bar{A}, B, \bar{D}) + \bar{d}_{jj} e_j^T = 0$ but $\Gamma_j(\bar{A}, B, \bar{D}) \neq \Gamma_j^{\Delta}(\bar{A}, B, \bar{D})$. Define matrix A such that $A_j = \bar{A}_j$ and $A_\ell = 0$ for all $\ell \neq j$ and matrix D as $d_{jj} = \bar{d}_{jj}$ and $d_{\ell\ell} = 0$ for all $\ell \neq j$. Let $x_0 = 0$. If there exists an index $i \neq j$ such that $\gamma_{ij}(\bar{A}, B, D) \neq \gamma_{ij}^{\Delta}(\bar{A}, B, D)$ pick $w_0 = e_i$, otherwise, pick $w_0 = e_j$. For this special case, the state of the closed-loop system with the controller $\Gamma(A, B, D)$ is equal to the state of the closed-loop system with the controller $\Gamma^{\Delta}(A, B, D)$ for the first and the second time-steps (see Figure 4 and Figure 5). As a result, the state of the subsystem j reaches zero in two time-steps. Now, since $\Gamma_j(\bar{A}, B, \bar{D}) \neq \Gamma_j^{\Delta}(\bar{A}, B, \bar{D})$, in the next time-step the state of the subsystem j becomes non-zero again. This results in a performance cost greater than the performance cost of the control design strategy Γ^{Δ}. Thus, the control design Γ^{Δ} is undominated by the control design method Γ.

Now, we have to prove the necessary part of the theorem. Proving this part is equivalent to proving that if there exists (a sink) j such that for every $i \neq j$, $(s_{\mathcal{P}})_{ij} = 0$, then there exists a control design strategy Γ which can dominate the deadbeat control design strategy. Without loss of generality, let $j = n$; i.e., assume that $(s_{\mathcal{P}})_{in} = 0$ for all $i \neq n$. In this situation, we can rewrite the matrix A as

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & 0 \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix},$$

Define $\bar{x}_0 = [x_1(0) \ \cdots \ x_{n-1}(0)]^T$ and $\bar{w}_0 = [w_1(0) \ \cdots \ w_{n-1}(0)]^T$. Let $\Gamma(A, B, D)$ be defined as $A_\Gamma(A, B, D) = D$, $C_\Gamma(A, B, D) = I$,

$$B_\Gamma(A, B, D) = \begin{bmatrix} -\frac{d_{11}^2}{b_{11}} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\frac{d_{n-1,n-1}^2}{b_{n-1,n-1}} & 0 \\ (b_\Gamma)_{n1} & \cdots & (b_\Gamma)_{n,n-1} & (b_\Gamma)_{nn} \end{bmatrix},$$

$$D_\Gamma(A, B, D) = \begin{bmatrix} -\frac{a_{11}+d_{11}}{b_{11}} & \cdots & -\frac{a_{1,n-1}}{b_{11}} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{a_{n-1,1}}{b_{n-1,n-1}} & \cdots & -\frac{a_{n-1,n-1}+d_{n-1,n-1}}{b_{n-1,n-1}} & 0 \\ (d_\Gamma)_{n1} & \cdots & (d_\Gamma)_{n,n-1} & (d_\Gamma)_{nn} \end{bmatrix},$$

where $\bar{B}_\Gamma = [(b_\Gamma)_{n1} \ \cdots \ (b_\Gamma)_{nn}]$ and $\bar{D}_\Gamma = [(d_\Gamma)_{n1} \ \cdots \ (d_\Gamma)_{nn}]$ are tunable gains for the last subsystem. We denote the cost of applying the deadbeat controller to subsystems $1, \ldots, n-1$ by $J^{(1)}_{(A,B,D,\bar{x}_0,\bar{w}_0)}$. This cost is independent of the control design parameters $\bar{B}_\Gamma$ and $\bar{D}_\Gamma$, because the last subsystem is a sink and it cannot affect the other subsystems. The overall cost of the controller is

$$J_{(A,B,x_0,w_0)}(\Gamma(A,B,D)) = J^{(1)}_{(A,B,D,\bar{x}_0,\bar{w}_0)} + J^{(2)}_{(A,B,D,x_0,w_0)}(\bar{B}_\Gamma, \bar{D}_\Gamma),$$

where $J^{(2)}_{(A,B,D,x_0,w_0)}(\bar{B}_\Gamma, \bar{D}_\Gamma)$ is the cost of the controller designed for the last subsystem. This cost $J^{(2)}_{(A,B,D,x_0,w_0)}(\bar{B}_\Gamma, \bar{D}_\Gamma)$ is independent of the rest of the system's model, because the deadbeat (for subsystems $1,\dots,n-1$) cancel out all dependencies in matrix A, thus, one can design the optimal controller for the lower part of the system without the model information of the upper part. Now, we can use the method mentioned in Subsection 3.1 to design the optimal controller for the lower part and find the optimal gains

$$\bar{B}_\Gamma = \frac{d_{nn}}{b_{nn}}\left((\alpha+1)A_n - D_n\right), \qquad \bar{D}_\Gamma = \frac{1}{b_{nn}}\left(\alpha A_n - D_n\right),$$

where

$$\alpha = \frac{2}{b_{nn}^2 + a_{nn}^2 + 1 + \sqrt{a_{nn}^4 + 2a_{nn}^2 b_{nn}^2 - 2a_{nn}^2 + b_{nn}^4 + 2b_{nn}^2 + 1}} - 1.$$

Note that this new control design strategy is always applicable since the control graph $G_\mathcal{K}$ is supergraph of the plant graph $G_\mathcal{P}$. Therefore, there exists a control design strategy which satisfies

$$J_{(A,B,D,x_0,w_0)}(\Gamma(A,B,D)) \le J_{(A,B,D,x_0,w_0)}(\Gamma^\Delta(A,B,D)),$$

for all matrices $A \in \mathcal{A}(S_\mathcal{P})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$ and all vectors $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathbb{R}^n$. Consider the matrix $A \in \mathcal{A}(S_\mathcal{P})$ such that $A_n = re_n^T$ and $A_\ell = 0$ for all $\ell \neq n$. Let $B = \epsilon I$ and $D = I$. For this special system, for all $r > 0$, we have

$$\begin{aligned} J_{(A,B,D,0,e_n)}(\Gamma(A,B,D)) &= \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 + \epsilon^2 + 1}{2} \\ &< r^2 + \epsilon^2 + 1 \\ &= J_{(A,B,D,0,e_n)}(\Gamma^\Delta(A,B,D)). \end{aligned}$$

Thus, the control design strategy Γ dominates the deadbeat control design strategy Γ^Δ. ■

Remark 3.9 *Consider the limited model information design problem given by the plant graph $G'_\mathcal{P}$ in Figure 2(a′), the control graph $G'_\mathcal{K}$ in Figure 2(b′), and the design graph $G'_\mathcal{C}$ in Figure 2(c′). Theorems 3.8 and 3.9 show that the deadbeat control design strategy Γ^Δ is the best control design strategy that one can propose based on local model of the subsystems and the plant graph, because the deadbeat control design strategy is the minimizer of the competitive ratio and it is undominated.*

We use the construction in proof of the "only if" part of Theorem 3.9 to build a control design strategy for the plant graphs with sinks in the next subsection.

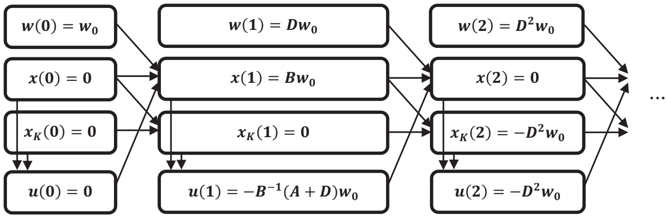

Figure 5: State evolution of the closed-loop system with deadbeat control design strategy Γ^{Δ} when $x_0 = 0$.

4.2 Plant Graphs with Sinks

In this section, we study the case where there are $c \geq 1$ sinks in the plant graph. By renumbering the sinks as subsystems number $n - c + 1, \cdots, n$ the matrix $S_{\mathcal{P}}$ can be written as

$$S_{\mathcal{P}} = \left[\begin{array}{c|c} (S_{\mathcal{P}})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_{\mathcal{P}})_{21} & (S_{\mathcal{P}})_{22} \end{array}\right], \tag{47}$$

where

$$(S_{\mathcal{P}})_{11} = \begin{bmatrix} (s_{\mathcal{P}})_{11} & \cdots & (s_{\mathcal{P}})_{1,n-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{n-c,1} & \cdots & (s_{\mathcal{P}})_{n-c,n-c} \end{bmatrix},$$

$$(S_{\mathcal{P}})_{21} = \begin{bmatrix} (s_{\mathcal{P}})_{n-c+1,1} & \cdots & (s_{\mathcal{P}})_{n-c+1,n-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{n,1} & \cdots & (s_{\mathcal{P}})_{n,n-c} \end{bmatrix},$$

and

$$(S_{\mathcal{P}})_{22} = \begin{bmatrix} (s_{\mathcal{P}})_{n-c+1,n-c+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (s_{\mathcal{P}})_{nn} \end{bmatrix}.$$

From now on, without loss of generality, we assume that the structure matrix is the one defined in (47). The control design method Γ^{Θ} for this type of systems is defined as

$$\Gamma^{\Theta}(A, B, D) = \left[\begin{array}{c|c} D & B^{-1}D(F(A,B) + I)A - B^{-1}D^2 \\ \hline I & B^{-1}(F(A,B)A - D) \end{array}\right], \; \forall P \in \mathcal{P}, \tag{48}$$

where

$$F(A, B) = \mathrm{diag}(0, \ldots, 0, f_{n-c+1}(A, B), \ldots, f_n(A, B))$$

and

$$f_i(A, B) = \frac{2}{b_{ii}^2 + a_{ii}^2 + 1 + \sqrt{a_{ii}^4 + 2a_{ii}^2 b_{ii}^2 - 2a_{ii}^2 + b_{ii}^4 + 2b_{ii}^2 + 1}} - 1 \tag{49}$$

for all $i = n - c + 1, \cdots, n$.

The control design strategy Γ^Θ applies the deadbeat to every subsystem that is not a sink and, for every sink, applies the same optimal control law as if the node was isolated. We will show that when the plant graph contains sinks, the control design method Γ^Θ has, in the worst case, the same competitive ratio as the deadbeat strategy. However, unlike the deadbeat strategy, it has the additional property of being undominated by limited model information methods for plants in $\mathcal{P}$ when the plant graph $G_\mathcal{P}$ has sinks.

Theorem 3.10 *Let the plant graph $G_\mathcal{P}$ contain at least one sink, and $G_\mathcal{K} \supseteq G_\mathcal{P}$. Then, the competitive ratio of the design method Γ^Θ introduced in (48) is*

$$r_\mathcal{P}(\Gamma^\Theta) = \begin{cases} \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}, & \text{if } (S_\mathcal{P})_{11} \neq 0 \text{ is not diagonal,} \\ 1, & \text{if both } (S_\mathcal{P})_{11} = 0 \text{ and } (S_\mathcal{P})_{22} = 0. \end{cases}$$

Proof: Based on Theorem 3.5, we know that

$$J_{(A,B,D,x_0,w_0)}(K^*(P)) \geq \frac{2\epsilon^2}{2\epsilon^2+1+\sqrt{4\epsilon^2+1}} J_{(A,B,D,x_0,w_0)}(\Gamma^\Delta(A,B,D)), \tag{50}$$

and by the proof of the "only if" part of Theorem 3.9, we know that

$$J_{(A,B,D,x_0,w_0)}(\Gamma^\Delta(A,B,D)) \geq J_{(A,B,D,x_0,w_0)}(\Gamma^\Theta(A,B,D)), \tag{51}$$

for all $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathbb{R}^n$. Putting (51) into (50) results in

$$J_{(A,B,D,x_0,w_0)}(K^*(P)) \geq \frac{2\epsilon^2}{2\epsilon^2+1+\sqrt{4\epsilon^2+1}} J_{(A,B,D,x_0,w_0)}(\Gamma^\Theta(A,B,D)),$$

and, therefore, in

$$\frac{J_{(A,B,D,x_0,w_0)}(\Gamma^\Theta(A,B,D))}{J_{(A,B,D,x_0,w_0)}(K^*(P))} \leq \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}, \ \forall P = (A,B,x_0,w) \in \mathcal{P}.$$

As a result

$$r_\mathcal{P}(\Gamma^\Theta) = \sup_{P\in\mathcal{P}} \frac{J_{(A,I,x_0,w)}(\Gamma^\Theta(A,B,D))}{J_{(A,I,x_0,w)}(K_*(P))} \leq \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}.$$

If $(S_\mathcal{P})_{11}$ has an off-diagonal entry, then there exist $1 \leq i,j \leq n-c$ and $i \neq j$ such that $(s_\mathcal{P})_{ij} \neq 0$. Define $A(r)$ such that $A(r) = re_je_i^T$. In this case, using the proof of Theorem 3.8, we know

$$r_\mathcal{P}(\Gamma^\Theta) = \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2},$$

because the control design Γ^Θ acts as the deadbeat controller on that part of the system. Using both these inequalities proves the statement.

If $(S_{\mathcal{P}})_{11} = 0$ and $(S_{\mathcal{P}})_{22} = 0$, every matrix A with structure matrix $(S_{\mathcal{P}})$ is a nilpotent matrix of degree two. Thus, using Lemma 3.4, we get

$$J_P(K^*(P)) = J_P(K_C^*(P)).$$

Now, based on the proof of Lemma 3.4, we also know that the optimal controller gain for this plant model is

$$K_C^*(P) = \left[\begin{array}{c|c} D & D(I+B^2)^{-1}B^{-1}A - B^{-1}D^2 \\ \hline I & -(I+B^2)^{-1}BA - B^{-1}D \end{array}\right].$$

For control design strategy Γ^{Θ}, we will have

$$\begin{aligned}\Gamma^{\Theta}(A,B,D) &= \left[\begin{array}{c|c} D & B^{-1}D(B(I+B^2)^{-1}B - I)A - B^{-1}D^2 \\ \hline I & B^{-1}(B(I+B^2)^{-1}BA - D) \end{array}\right] \\ &= \left[\begin{array}{c|c} D & D(I+B^2)^{-1}B^{-1}A - B^{-1}D^2 \\ \hline I & -(I+B^2)^{-1}BA - B^{-1}D \end{array}\right]\end{aligned}$$

based on (48). Thus, $r_{\mathcal{P}}(\Gamma^{\Theta}) = 1$. ■

Theorem 3.11 *Let the plant graph $G_{\mathcal{P}}$ contain at least one sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2+1}}{2\epsilon^2},$$

if $(S_{\mathcal{P}})_{11}$ is not diagonal.

Proof: First, suppose that $(S_{\mathcal{P}})_{11} \neq 0$ and $(S_{\mathcal{P}})_{11}$ is not a diagonal matrix, then there exist $1 \leq i, j \leq n-c$ and $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Consider the family of matrices $A(r)$ defined by $A(r) = re_ie_j^T$. Based on Lemma 3.7, if we want to have a bounded competitive ratio, the control design strategy should satisfy $r + b_{ii}(d_{\Gamma})_{ij}(A(r), B, D) = 0$ (because node $1 \leq i \leq n-c$ is not a sink). The rest of the proof is similar to the proof of Theorem 3.8. ■

Remark 3.10 *Combining Theorem 3.10 and Theorem 3.11 implies that if $(S_{\mathcal{P}})_{11} \neq 0$ is not diagonal (i.e., the nodes that are not sink can affect each other), control design method Γ^{Θ} is a minimizer of the competitive ratio over the set of limited model information control methods and consequently a solution to the problem (11). Furthermore, if $(S_{\mathcal{P}})_{11}$ and $(S_{\mathcal{P}})_{22}$ are both zero, then the Γ^{Θ} becomes equal to K^*, which shows that, Γ^{Θ} is a solution to the problem (11), in this case too. The rest of the cases are still open here.*

The next theorem shows that Γ^{Θ} is a more desirable control design method than the deadbeat when plant graph $G_{\mathcal{P}}$ has sinks, since it is then undominated by limited model information design methods for plants in $\mathcal{P}$.

Theorem 3.12 *Let the plant graph $G_{\mathcal{P}}$ contain at least one sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the control design method Γ^{Θ} is undominated by all limited model information control design methods.*

Proof: Assume that there are $c \geq 1$ sink in the plant graph. For proving this claim, we are going to prove that for any control design method $\Gamma \in \mathcal{C} \backslash \{\Gamma^{\Theta}\}$, there exits a plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ such that $J_P(\Gamma(A, B, D)) > J_P(\Gamma^{\Theta}(A, B, D))$. We will proceed in several steps, which require us to partition the set of limited model information control design strategies $\mathcal{C}$ as follows

$$\mathcal{C} = \mathcal{W}_2 \cup \mathcal{W}_1 \cup \mathcal{W}_0 \cup \{\Gamma^{\Delta}\},$$

where

$$\mathcal{W}_2 := \{\Gamma \in \mathcal{C} \mid \exists j, n-c+1 \leq j \leq n, \text{ such that } \Gamma_j(A, B, D) \neq \Gamma_j^{\Theta}(A, B, D)\},$$

$$\mathcal{W}_1 := \{\Gamma \in \mathcal{C} \setminus \mathcal{W}_2 \mid \exists j, 1 \leq j \leq n-c, \\ \text{and } \exists P \in \mathcal{P}, (D_{\Gamma})_j(A, B, D) \neq (D_{\Gamma}^{\Theta})_j(A, B, D)\},$$

and

$$\mathcal{W}_0 := \{\Gamma \in \mathcal{C} \setminus \mathcal{W}_2 \cup \mathcal{W}_1 \mid \exists j, 1 \leq j \leq n-c, \exists P \in \mathcal{P}, \\ \text{such that } \Gamma_j(A, B, D) \neq \Gamma_j^{\Theta}(A, B, D)\}.$$

First, we prove that the Γ^{Θ} is undominated by control design strategies in $\mathcal{W}_2$. We assume that there exist index $n-c+1 \leq j \leq n$ and matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $\bar{D} \in \mathcal{D}$ such that $\Gamma_j(\bar{A}, B, \bar{D}) \neq \Gamma_j^{\Theta}(\bar{A}, B, \bar{D})$. Consider matrices A and D defined as $A_j = \bar{A}_j$ and $A_i = 0$ for all $i \neq j$ and $d_{jj} = \bar{d}_{jj}$ and $d_{ii} = 0$. For this particular matrix A, any x_0, and any w_0, we know from the proof of the "only if" part of Theorem 3.9 that $\Gamma^{\Theta}(A, B, D, x_0, w_0)$ is the globally optimal controller with limited model information. Hence, every other control design method in $\mathcal{C}$ leads to a controller with greater performance cost than Γ^{Θ} for this particular type of plants. Therefore, the control design Γ^{Θ} is undominated by control design methods in $\mathcal{W}_2$.

Second, we prove that the control design strategy Γ^{Θ} is undominated by the control design strategies in $\mathcal{W}_1$. Let Γ be a control design strategy in $\mathcal{W}_1$ and let index $1 \leq j \leq n-c$ be such that $\bar{A}_j + b_{jj}(D_{\Gamma})_j(\bar{A}, B, \bar{D}) + \bar{d}_{jj} e_j^T \neq 0$ for some matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $\bar{D} \in \mathcal{D}$. It is always possible to pick an index $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$ because node j is not a sink in the plant graph. If $1 \leq i \leq n-c$, the proof is the same as the proof of the "if" part of Theorem 3.9, therefore, without any loss of generality, we assume that $n-c+1 \leq i \leq n$. Again, with the same argument as in the proof of the "if" part of Theorem 3.9, without loss of generality, we can assume that $a_{jj} + b_{jj}(d_{\Gamma})_{jj}(A, B, D) + d_{jj} \neq 0$ (because otherwise the ratio of the cost the control design strategy Γ to the cost of the control design strategy Γ^Θ becomes infinity). Define matrix A such that $A_j = \bar{A}_j$, $A_i = re_j^T$, and $A_\ell = 0$ for all $\ell \neq i, j$. Let $D \in \mathcal{D}$ be such that $d_{jj} = \bar{d}_{jj}$ and $d_\ell = 0$ for all $\ell \neq j$. It should be noted that $\Gamma_j(A, B, D) = \Gamma_j(\bar{A}, B, \bar{D})$ because the design graph is a totally disconnected graph. The cost of this control design strategy for $w_0 = e_j$ and $x_0 = 0$ would satisfy

$$\begin{aligned} J_P(\Gamma(A,B,D)) &\geq (u_i(1)+w_i(1))^2 + x_i(2)^2 + (u_i(2)+w_i(2))^2 + x_i(3)^2 \\ &= r^2 b_{jj}^2/(b_{ii}^2+1) + (u_i(2)+w_i(2))^2 + (x_j(2)r + b_{ii}[u_i(2)+w_i(2)])^2 \\ &\geq (r^2 b_{jj}^2 + x_j(2)^2 r^2)/(1+b_{ii}^2), \end{aligned}$$

This results in

$$\begin{aligned} &J_{(A,I,B,D,0,e_j)}(\Gamma(A,B,D)) - J_{(A,I,B,D,0,e_j)}(\Gamma^\Theta(A,B,D)) \\ &\qquad \geq (a_{jj} + b_{jj}(d_\Gamma)_{jj}(A,B,D) + d_{jj})^2 b_{jj}^2 r^2/(1+b_{ii}^2) - \kappa(A_j, b_{jj}). \end{aligned}$$

where $\kappa(A_j, b_{jj})$ is only a function A_j and b_{jj} and represents the part of the cost of the control design strategy Γ^Θ that is related to subsystem j only. If we pick r large enough, the difference would become positive, which shows that the control design strategy Γ cannot dominate the control design strategy Γ^Θ.

Finally, we prove that the control design strategy Γ^Θ is undominated by the control design strategies in $\mathcal{W}_0$. The same argument as in the proof of the "if" part of Theorem 3.9 holds here too. ■

Remark 3.11 *Consider the limited model information design problem given by the plant graph $G_\mathcal{P}$ in Figure 2(a), the control graph $G'_\mathcal{K}$ in Figure 2(b′), and the design graph $G'_\mathcal{C}$ in Figure 2(c′). Theorems 3.10, 3.11, and 3.12 together show that, the control design strategy Γ^Θ is the best control design strategy that one can propose based on local subsystems' model and the plant graph, because the control design strategy Γ^Θ is a minimizer of the competitive ratio and it is undominated.*

5 Design Graph Influence on Achievable Performance

In the previous section, we approached the optimal control design under limited model information when $G_\mathcal{C}$ is a totally disconnected graph. The next step is to determine the necessary amount of the model information needed in each subcontroller to be able to setup a control design strategy with a smaller competitive ratio than the deadbeat control design strategy. We tackle this question here.

Theorem 3.13 *Let the plant graph $G_\mathcal{P}$ and the design graph $G_\mathcal{C}$ be given, and $G_\mathcal{K} \supseteq G_\mathcal{P}$. Assume that the plant graph $G_\mathcal{P}$ contains the path $i \to j \to \ell$ with distinct nodes i, j, and ℓ while $(\ell, j) \notin E_\mathcal{C}$. Then, we have*

$$r_\mathcal{P}(\Gamma) \geq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2+1}}{2\epsilon^2}.$$

Proof: Let i, j, and k be three distinct nodes such that $(s_{\mathcal{P}})_{ji} \neq 0$ and $(s_{\mathcal{P}})_{\ell i} \neq 0$ (i.e., the path $i \to j \to \ell$ is contained in the plant graph $G_{\mathcal{P}}$). Define the 2-parameter family of matrices $A(r,s) = re_j e_i^T + se_\ell e_j^T$. Let $B = \epsilon I$, $D = I$, and $\Gamma \in \mathcal{C}$ be a limited model information with design graph $G_{\mathcal{C}}$. The cost of this control design strategy for $w_0 = e_i$ and $x_0 = 0$ satisfies

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A,B,D)) &\geq (u_\ell(2) + w_\ell(2))^2 + x_\ell(3)^2 \\ &= (u_\ell(2) + w_\ell(2))^2 + (sx_j(2) + \epsilon[u_\ell(2) + w_\ell(2)])^2, \end{aligned}$$

because of the structure of the cost function in (9) and the system dynamic in (1). Now, using Lemma 3.6 and the fact that $x_j(2) = (r + \epsilon(d_\Gamma)_{ji}(r))\epsilon$ (see Figure 4), we get

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A,B,D)) &\geq s^2 x_j(2)^2/(1+\epsilon^2) \\ &= (r + \epsilon(d_\Gamma)_{ji}(r))^2 \epsilon^2 s^2/(1+\epsilon^2). \end{aligned}$$

Note that $(d_\Gamma)_{ji}(r)$ is only a function of r and not s since $(\ell, j) \notin E_{\mathcal{C}}$. On the other hand, the cost of the deadbeat control design strategy is

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma^\Delta(A,B,D)) &= e_i^T B^T (A^T B^{-2} A + B^{-2} + I) B e_i \\ &= \epsilon^2 + 1 + r^2. \end{aligned}$$

Note that the deadbeat control design strategy is applicable here since the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. We have

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &= \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))} \\ &= \sup_{P \in \mathcal{P}} \left[\frac{J_P(\Gamma(A,B,D))}{J_P(\Gamma^\Delta(A,B,D))} \frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} \right] \\ &\geq \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(\Gamma^\Delta(A,B,D))} \\ &\geq \frac{(r + \epsilon(d_\Gamma)_{ji}(r))^2 \epsilon^2/(1+\epsilon^2)}{\epsilon^2 + 1 + r^2} \lim_{s \to \infty} s^2. \end{aligned} \tag{52}$$

Using (52) it is easy to see that the competitive ratio $r_{\mathcal{P}}(\Gamma)$ is bounded only if $r + \epsilon(d_\Gamma)_{ji}(r) = 0$, for all $r \in \mathbb{R}$. Therefore, there is no loss of generality in assuming that $(d_\Gamma)_{ji}(r) = -r/\epsilon$ because otherwise the $r_{\mathcal{P}}(\Gamma)$ is infinity and the inequality in the statement of the theorem is trivially satisfied. Now, let us fix $s = 0$ and use the notation $A(r) = re_j e_i^T$. Since the parameters of the subsystem j is not changed and $(\ell, j) \notin E_{\mathcal{C}}$, we have $(d_\Gamma)_{ji}(r) = -r/\epsilon$. Therefore, for each $r \in \mathbb{R}$, similar to the proof of Theorem 3.8, the cost of the control design strategy Γ for x_0 in (40) and w_0 in (41) is lower-bounded by

$$J_P(\Gamma(A,B,D)) \geq \frac{(\epsilon^2+1)(3\epsilon^2\sqrt{4\epsilon^2+1} + 5\epsilon^2 + 4\epsilon^4 + \sqrt{4\epsilon^2+1} + 1)}{2\epsilon^4},$$

On the other hand, for each $r \in \mathbb{R}$, the matrix $A(r)$ is a nilpotent matrix of degree two, that is, $A(r)^2 = 0$. Similar to the proof of Theorem 3.8, for x_0 in (40) and w_0 in (41), we get

$$J_P(K^*(P)) = \frac{(\epsilon^2+1)\sqrt{4\epsilon^2+1}+5\epsilon^2+4\epsilon^4+1}{2\epsilon^2} + \frac{(2\epsilon^2+\sqrt{4\epsilon^2+1}+1)\sqrt{4\epsilon^2+1}}{2\epsilon^2 r^2},$$

since $J_P(K^*(P)) = J_P(K_C^*(P))$ according to Lemma 3.4. This results in

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{r\to\infty} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))} = \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}.$$

This finishes the proof. ■

Remark 3.12 *Consider the limited model information design problem given by the plant graph $G'_{\mathcal{P}}$ in Figure 2(a′), the control graph $G_{\mathcal{K}}$ in Figure 2(b), and the design graph $G_{\mathcal{C}}$ in Figure 2(c). Theorem 3.13 shows that, because the plant graph $G_{\mathcal{P}}$ contains the path $2 \to 1 \to 4$ but the design graph $G_{\mathcal{C}}$ does not contain $4 \to 1$, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ would be greater than or equal to $r_{\mathcal{P}}(\Gamma^{\Delta})$.*

Remark 3.13 *Theorem 3.13 shows that, when $G_{\mathcal{P}}$ and $G_{\mathcal{K}}$ is a complete graph, achieving a better competitive ratio than the deadbeat design strategy requires each subsystem to have full knowledge of the plant model when constructing each subcontroller.*

6 Proportional-Integral Deadbeat Control Design Strategy

In this section, we use some of the results of the paper on familiar control design problems like step disturbance rejection and step reference tracking.

6.1 Step Disturbance Rejection

For the case of step disturbance rejection, we can model the disturbance as in (2) with matrix $D = I$. For each plant $P = (A, B, I, x_0, w_0) \in \mathcal{P}$, the deadbeat controller design strategy is

$$\Gamma^{\Delta}(A,B,I) \triangleq \left[\begin{array}{c|c} I & -B^{-1} \\ \hline I & -B^{-1}(A+I) \end{array}\right],$$

This controller can be realized as

$$u(k) = -B^{-1}Ax(k) - B^{-1}\sum_{i=0}^{k} x(i).$$

which is a proportional-integral controller. Thus, we call the restricted mapping $\Gamma^{\Delta}_{\text{step}} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathcal{K}(S_{\mathcal{K}})$, defined as $\Gamma^{\Delta}_{\text{step}}(A, B) = \Gamma^{\Delta}(A, B, I)$, the proportional-integral deadbeat control design strategy. The proportional term regulates the states of the system and the integral term compensates for the disturbance. For instance, in this case, Theorem 3.8 shows that when the plant graph $G_{\mathcal{P}}$ contains no sink and the design graph $G_{\mathcal{C}}$ is a totally disconnected graph, the deadbeat proportional-integral control design strategy is an undominated minimizer of the competitive ratio. Note that the integral part of this control design strategy is fully decentralized and the proportional part only needs the neighboring subsystems state-measurements.

6.2 Step Reference Tracking

Consider the case that we are interested in tracking a step reference signal $r \in \mathbb{R}^n$. We need to define the difference $\bar{x}(k) = x(k) - r$ which gives

$$\bar{x}(k+1) = x(k+1) - r = Ax(k) + Bu(k) - r = A\bar{x}(k) + Bu(k) + Ar - r.$$

Now if the subsystems do not want to share the reference points with each other, we can think of the additional term $Ar - r$ as the constant disturbance vector $w(k) = B^{-1}(Ar - r)$. Thus, we have

$$\bar{x}(k+1) = A\bar{x}(k) + B(u(k) + w(k)).$$

The subsystems only need to transmit the relative error between the state-measurements and reference points. In this case, we can use the cost function

$$J_P(K) = \sum_{k=0}^{\infty} [\bar{x}(k)^T \bar{x}(k) + (u(k) + w(k))^T (u(k) + w(k))], \tag{53}$$

to make sure that the error $\bar{x}(k)$ goes to zero as time tends to infinity. Note that if we want to have a complete state regulation $\lim_{k\to\infty} \bar{x}(k) = 0$, the control signal should have a limit as

$$\lim_{k\to\infty} u(k) = -B^{-1}(Ar - r).$$

Thus, the second term of the cost function (53) only penalizes the difference of the control signal and its steady-state value.

7 Conclusions

We considered the design of optimal disturbance rejection and servomechanism dynamic controllers under limited plant model information. We provided insight into the value of model information in control design and studied how local subsystem interaction, limited state measurements, and limited plant model information influenced the achievable closed-loop performance. To do so, we investigated the relationship between the closed-loop performance and the control design strategies with limited model information using the metric called competitive ratio. We found an explicit minimizer of the competitive ratio. The optimal controller is dynamic and composed of a static state feedback law and a dynamic disturbance observer. It was shown that this special structure corresponds to proportional-integral controllers when dealing with step disturbances. Possible future work will focus on extending the present framework to situations where the subsystems and disturbances are not scalar and extending the set of applicable controllers to include adaptive and nonlinear controllers to possibly achieve better closed-loop performance.

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8 Bibliography

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PAPER 4

Optimal Control Design under Structured Model Information Limitation Using Adaptive Algorithms

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Abstract–Networked control strategies based on limited information about the plant model usually results in worse closed-loop performance than optimal centralized control with full plant model information. Recently, this fact has been established by utilizing the concept of competitive ratio, which is defined as the worst case ratio of the cost of a control design with limited model information to the cost of the optimal control design with full model information. In this paper, we show that with an adaptive networked controller with limited plant model information, it is indeed possible to achieve a competitive ratio equal to one. We show that an adaptive controller introduced by Campi and Kumar asymptotically achieves closed-loop performance equal to the optimal centralized controller with full model information. The plant model considered in the paper belongs to a compact set of stochastic linear time-invariant systems and the closed loop performance measure is the ergodic mean of a quadratic function of the state and control input. We illustrate the applicability of the results numerically on a vehicle platooning problem.

1 Introduction

Networked control systems are often complex large-scale engineered systems, such as power grids [1], smart infrastructures [2], intelligent transportation systems [3–5], or future aerospace systems [6, 7]. These systems consists of several subsystems each one often having many unknown parameters. It is costly, or even unrealistic, to accurately identify all these plant model parameters offline. This fact motivates us to focus on optimal control design under structured parameter uncertainty and limited plant model information constraints.

There are some recent studies in optimal control design with limited plant model information [8–14]. The problem was initially addressed in [8] for designing static centralized controllers for a class of discrete-time linear time-invariant systems composed of scalar subsystem, where control strategies with various degrees of model information were compared using competitive ratio; i.e., the worst case ratio of the cost of a control design with limited model information scaled by the cost of the optimal control design with full model information. The result was generalized to static decentralized controller for a class of systems composed of fully-actuated subsystems of arbitrary order in [9, 10]. More recently, the problem of designing optimal H_2 dynamic controllers using limited plant model information was considered in [11]. It was shown that, when relying on local model information, the smallest competitive ratio achievable for any control design strategy for distributed linear time-invariant controllers is strictly greater than one; specifically, equal to square root of two when the B-matrix was assumed to be the identity matrix.

In this paper, we generalize the set of applicable controllers to include adaptive controllers. We use the ergodic mean of a quadratic function of the state and control as a performance measure of the closed-loop system. Choosing this closed-loop performance measure allows us to use certain adaptive algorithms available in the literature [15–18]. In particular, we consider an adaptive controller proposed by Campi and Kumar [15]. We prove that the smallest competitive ratio that a control design strategy using adaptive controllers can achieve is equal one. This shows that, although the design of each subcontroller is only relying on local model information, the closed-loop performance can still be as good as the optimal control design strategy with full model information.

The rest of the paper is organized as follows. In Section 2, we present the mathematical problem formulation. In Section 3, we introduce the Campi–Kumar adaptive controller using only local model information and we show that it achieves a competitive ration equal to one. We use this adaptive algorithm on a vehicle platooning problem to demonstrate its performance numerically in Section 4 and we conclude the paper in Section 5.

1.1 Notation

The sets of natural and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Additionally, all other sets are denoted by calligraphic letters such as $\mathcal{P}$ and $\mathcal{A}$. For any given sets $\mathcal{X}$ and $\mathcal{Y}$, the notation $\mathcal{M}(\mathcal{X}, \mathcal{Y})$ denotes the set of all mappings from the set $\mathcal{X}$ to the set $\mathcal{Y}$. In addition, for any $k \in \mathbb{N}_0$, we define $\mathcal{P}_k(\mathcal{X})$ as the set of all subsets of $\mathcal{X}$ containing exactly k elements.

Matrices are denoted by capital roman letters such as A. The entry in the i^{th} row and the j^{th} column of matrix A is a_{ij}. A_{ij} denotes a submatrix of matrix A, the dimension and the position of which will be defined in the text.

$A > (\geq) 0$ means symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite (positive semidefinite) and $A > (\geq) B$ means $A - B > (\geq) 0$. Let $\mathcal{S}_{++}^n$ ($\mathcal{S}_{+}^n$) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n \times n}$.

Let matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q \in \mathcal{S}_{+}^n$, and $R \in \mathcal{S}_{++}^m$ be given such that the pair (A, B) is controllable and the pair $(A, Q^{1/2})$ is observable. We define $\mathbf{P}(A, B, Q, R)$ as the unique positive definite solution of the discrete algebraic Riccati equation

$$P = A^\top P A - A^\top P B \left(B^\top P B + R\right)^{-1} B^\top P A + Q.$$

In addition, we define

$$\mathbf{L}(A, B, Q, R) = -\left(B^\top \mathbf{P}(A, B, Q, R) B + R\right)^{-1} B^\top \mathbf{P}(A, B, Q, R) A.$$

When matrices Q and R are not relevant or can be deduced from the text, we use $\mathbf{P}(A, B)$ and $\mathbf{L}(A, B)$ instead of $\mathbf{P}(A, B, Q, R)$ and $\mathbf{L}(A, B, Q, R)$, respectively.

A measurable function $f : \mathcal{Z} \to \mathbb{R}$ is said to be essentially bounded if there exists a constant $c \in \mathbb{R}$ such that $|f(z)| \leq c$ almost everywhere. The greatest lower bound of these constants is called the essential supremum of $f(z)$, which is denoted by $\operatorname{ess\,sup}_{z \in \mathcal{Z}} f(z)$.

All graphs G considered in this paper are directed with vertex set $\{1, ..., N\}$ for a given $N \in \mathbb{N}$. The adjacency matrix $S \in \{0, 1\}^{N \times N}$ of G is a matrix whose entry $s_{ij} = 1$ if $(j, i) \in E$ and $s_{ij} = 0$, otherwise, for all $1 \leq i, j \leq N$.

Let mappings $f, g : \mathbb{Z} \to \mathbb{R}$ be given. Denote $f(k) = O(g(k))$ if $\limsup_{k \to \infty} |f(k)/g(k)| < \infty$. Similarly, $f(k) = o(g(k))$ if $\limsup_{k \to \infty} |f(k)/g(k)| = 0$.

Finally, χ denotes the characteristic function, that is, it returns a value equal one if its statement is satisfied and a value equal zero otherwise.

2 Problem Formulation

2.1 Plant Model

Consider a discrete-time linear time-invariant dynamical system composed of N subsystems, such that the state-space representation of subsystems i, $1 \leq i \leq N$, is given by

$$x_i(k+1) = \sum_{j=1}^{N} A_{ij} x_j(k) + B_{ii} u_i(k) + w_i(k);\ x_i(0) = 0,$$

where $x_i(k) \in \mathbb{R}^{n_i}$, $u_i(k) \in \mathbb{R}^{m_i}$, and $w_i(k) \in \mathbb{R}^{n_i}$ are state, control input, and exogenous input vectors, respectively. We assume that $\{w_i(k)\}_{k=0}^{\infty}$ are independent and identically distributed Gaussian random variables with zero means $\mathbb{E}\{w_i(k)\} = 0$ and unit covariances $\mathbb{E}\{w_i(k)w_i(k)^\top\} = I$. In addition, let $w_i(k)$ and $w_j(k)$ be statistically independent for all $1 \leq i \neq j \leq N$. We introduce the augmented system as

$$x(k+1) = Ax(k) + Bu(k) + w(k);\; x(0) = 0,$$

where the augmented state, control input, and exogenous input vectors are

$$\begin{aligned} x(k)^\top &= [x_1(k)^\top \;\ldots\; x_N(k)^\top]^\top \in \mathbb{R}^n, \\ u(k)^\top &= [u_1(k)^\top \;\ldots\; u_N(k)^\top]^\top \in \mathbb{R}^m, \\ w(k)^\top &= [w_1(k)^\top \;\ldots\; w_N(k)^\top]^\top \in \mathbb{R}^n, \end{aligned}$$

with $n = \sum_{i=1}^{N} n_i$ and $m = \sum_{i=1}^{N} m_i$. In addition, the augmented model matrices are

$$B = \mathrm{diag}(B_{11}, \ldots, B_{NN}) \in \mathcal{B} \subset \mathbb{R}^{n\times m},$$

and

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix} \in \mathcal{A} \subset \mathbb{R}^{n\times n}.$$

Let a directed plant graph $G_{\mathcal{P}}$ with its associated adjacency matrix $S_{\mathcal{P}}$ be given.The plant graph $G_{\mathcal{P}}$ captures the interconnection structure of the plants, that is, $A_{ij} \neq 0$ only if $(s_{\mathcal{P}})_{ij} \neq 0$. Hence, the set $\mathcal{A}$ is structured by the plant graph:

$$\begin{aligned} \mathcal{A} \subseteq \{A \in \mathbb{R}^{n\times n} \mid (s_{\mathcal{P}})_{ij} = 0 \Rightarrow & A_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \\ & \text{for all } i, j \text{ such that } 1 \leq i, j \leq N\}. \end{aligned}$$

Note that the set $\mathcal{A}\times\mathcal{B}$ is isomorph to the set of all plants of interest $\mathcal{P}$, say. Hence, from now on, we present a plant with its pair of corresponding model matrices as $P = (A, B)$ and denote $\mathcal{P} = \mathcal{A} \times \mathcal{B}$. We make the following assumption on the set of all plants:

Assumption 4.1 *The set $\mathcal{A}\times\mathcal{B}$ is a compact set (with nonzero Lebesgue measure) and the pair (A, B) is controllable for all $(A, B) \in \mathcal{A} \times \mathcal{B}$ (except for possibly a set with zero Lebesgue measure).*

2.2 Adaptive Controller

We consider infinite-dimensional nonlinear time-invariant controllers $\mathbf{K}$ with control law

$$u(k) = \mathbf{K}(\mathcal{F}_k), \qquad \forall\, k \in \mathbb{N}_0,$$

where $\mathcal{F}_k = \sigma(\{x(t)\}_{t=0}^{k} \cup \{u(t)\}_{t=0}^{k-1})$ is the sigma algebra generated by the observation history. Hence, each controller is a mapping $\mathbf{K} : \mathcal{E}^{n,m} \to \mathbb{R}^m$, where

$$\mathcal{E}^{n,m} = \bigcup_{k=0}^{\infty} \bigcup_{\mathcal{X} \in \mathcal{P}_k(\mathbb{R}^n)} \bigcup_{\mathcal{U} \in \mathcal{P}_{k-1}(\mathbb{R}^m)} \sigma(\mathcal{X} \cup \mathcal{U}).$$

The set of all admissible controllers $\mathcal{K}$ can be captured as the set of mappings from $\mathcal{E}^{n,m}$ to $\mathbb{R}^m$, or, equivalently, $\mathcal{K} = \mathcal{M}(\mathcal{E}^{n,m}, \mathbb{R}^m)$.

2.3 Control Design Strategy

A control design strategy Γ is a mapping from the set of plants $\mathcal{P} = \mathcal{A} \times \mathcal{B}$ to the set of admissible controllers $\mathcal{K}$. We can partition Γ using the control input size as

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_N \end{bmatrix},$$

where, for each $1 \leq i \leq N$, we have $\Gamma_i : \mathcal{A} \times \mathcal{B} \to \mathcal{M}(\mathcal{E}^{n,m}, \mathbb{R}^{m_i})$. Let a directed design graph $G_\mathcal{C}$ with its associated adjacency matrix $S_\mathcal{C}$ be given. We say that the control design strategy Γ satisfies the limited model information constraint enforced by the design graph $G_\mathcal{C}$ if, for all $1 \leq i \leq N$, Γ_i is only a function of

$$\{[A_{j1} \ \ldots \ A_{jN}], B_{jj} \mid (s_\mathcal{C})_{ij} \neq 0\}.$$

The set of all control design strategies that obey the structure given by the design graph $G_\mathcal{C}$ is denoted by $\mathcal{C}$.

2.4 Performance Metric

In this paper, we are interested in minimizing the performance criterion

$$J_P(\mathbf{K}) = \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} x(k)^\top Q x(k) + u(k)^\top R u(k), \tag{1}$$

where $Q \in \mathcal{S}_{+}^{n}$ and $R \in \mathcal{S}_{++}^{n}$. We make the following assumption concerning the performance criterion:

Assumption 4.2 *For all $A \in \mathcal{A}$, the pair $(A, Q^{1/2})$ is observable (except for possibly a set with zero Lebesgue measure).*

Note that for linear controllers the performance measure (1) represents the H_2-norm of the closed-loop system from exogenous input $w(k)$ to output

$$y(k) = Q^{1/2} x(k) + R^{1/2} u(k).$$

Definition 4.1 *Let a plant graph $G_{\mathcal{P}}$ and a design graph $G_{\mathcal{C}}$ be given. Assume that, for every plant $P \in \mathcal{P}$, there exists an optimal controller $\mathbf{K}^*(P) \in \mathcal{K}$ such that*

$$J_P(\mathbf{K}^*(P)) \leq J_P(\mathbf{K}), \qquad \forall\, \mathbf{K} \in \mathcal{K}.$$

The average competitive ratio *of a control design method Γ is defined as*

$$r_{\mathcal{P}}^{\mathrm{ave}}(\Gamma) = \int_{\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(\mathbf{K}^*(P))} f(P)\, \mathrm{d}P, \tag{2}$$

where $f : \mathcal{P} \to \mathbb{R}$ is a positive-definite continuous function which shows the relative importance of plants in $\mathcal{P}$. Without loss of generality, we assume that $\int_{\mathcal{P}} f(P)\mathrm{d}P = 1$. The supremum competitive ratio *of a control design method Γ is defined as*

$$r_{\mathcal{P}}^{\mathrm{sup}}(\Gamma) = \operatorname*{ess\,sup}_{P \in \mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(\mathbf{K}^*(P))}. \tag{3}$$

The mapping $\mathbf{K}^*$ is not required to lie in the set $\mathcal{C}$, and is obtained by searching over the set of centralized controllers. Hence, $\mathbf{K}^*(P) = \mathbf{L}(A, B)$, for all plants $P = (A, B) \in \mathcal{P}$.

The supremum competitive ratio $r_{\mathcal{P}}^{\mathrm{sup}}$ is a modified version of the competitive ratio considered in [8–14]. Note that using essential supremum in (3), we are neglecting a subset of plants with zero Lebesgue measure. However, this is not crucial for practical purposes, because the probability of these plants appearing in a real situation is slim. As a starting point, let us prove a very interesting property relating the average and supremum competitive ratios.

Lemma 4.1 *For any control design strategy $\Gamma \in \mathcal{C}$, we have $1 \leq r_{\mathcal{P}}^{\mathrm{ave}}(\Gamma) \leq r_{\mathcal{P}}^{\mathrm{sup}}(\Gamma)$.*

Proof: See Appendix A. ■

In this paper, we are interested in solving the optimization problem

$$\arg\min_{\Gamma \in \mathcal{C}} r_{\mathcal{P}}(\Gamma), \tag{4}$$

where $r_{\mathcal{P}}$ is either $r_{\mathcal{P}}^{\mathrm{ave}}$ or $r_{\mathcal{P}}^{\mathrm{sup}}$. This problem was studied in [11] when the set of plants is fully-actuated discrete-time linear time-invariant systems and the set of admissible controllers is finite-dimensional discrete-time linear dynamic time-invariant systems. It was shown that a modified deadbeat control strategy (which constructs static controllers) is a minimizer of the competitive ratio. Specifically, it was proved that the smallest competitive ratio that a control design strategy which gives decentralized linear time-invariant controllers can achieve is strictly greater than one when relying on local model information. Note that since the optimal control design with full model information is unique (due to Assumption 4.2), even when considering a compact set of plants, the competitive ratio is strictly larger than one for limited model information control design strategies. In this paper, we generalize the formulation of [11] to include adaptive controllers. We prove in next section that we can achieve a competitive ratio equal to one for adaptive controllers. Therefore, we can achieve the optimal performance asymptotically, even if the complete model of the system is not known in advance when designing the subcontrollers.

3 Main Results

We introduce a specific control design strategy Γ^*, and subsequently, prove that Γ^* is a minimizer of both the average and supremum competitive ratios $r_{\mathcal{P}}^{\text{ave}}$ and $r_{\mathcal{P}}^{\text{sup}}$. For each plant $P \in \mathcal{P}$, this control design strategy constructs an adaptive controller $\Gamma^*(P)$ using a modified version of the Campi–Kumar adaptive algorithm [15], see Algorithm 1. Note that in the Campi–Kumar adaptive algorithm, a central controller estimates the model of the system and control the system. However, in our modified Campi–Kumar adaptive algorithm in Algorithm 1, each subcontroller estimates the model of the system independently and control its corresponding subsystem separately. Hence, each adaptive subcontroller arrives at different model estimates.

In Algorithm 1, we use the notation $(A^{(i)}(k), B^{(i)}(k))$, at each time step $k \in \mathbb{N}_0$, to denote subsystem i's estimate of the global system model $P = (A, B)$. Furthermore, for each $1 \leq i \leq N$, we use the mapping $\mathbf{T}_i : \mathbb{R}^{m \times n} \to \mathbb{R}^{m_i \times n}$ defined as

$$\mathbf{T}_i \begin{bmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & \ddots & \vdots \\ X_{N1} & \cdots & X_{NN} \end{bmatrix} = \begin{bmatrix} X_{i1} & \cdots & X_{iN} \end{bmatrix},$$

where $X_{\ell j} \in \mathbb{R}^{m_\ell \times n_j}$ for each $1 \leq \ell, j \leq N$. Let us also, for all $k \in \mathbb{N}_0$, introduce the notation

$$K(k) = \begin{bmatrix} \mathbf{T}_1 K^{(1)}(k) \\ \vdots \\ \mathbf{T}_N K^{(N)}(k) \end{bmatrix} \in \mathbb{R}^{m \times n},$$

where matrices $K^{(i)}(k)$ are defined in Algorithm 1. For each $\delta > 0$, we introduce

$$\mathcal{W}_\delta(A, B) := \{(\bar{A}, \bar{B}) \in \mathcal{A} \times \mathcal{B} \mid \|[A + B\mathbf{L}(\bar{A}, \bar{B})] - [\bar{A} + \bar{B}\mathbf{L}(\bar{A}, \bar{B})]\| \geq \delta\}.$$

Let us start by presenting a result on the convergence of the global plant model estimates to the correct value.

Lemma 4.2 *Let $\mathbf{K} = \Gamma^*(P)$ be defined in Algorithm 1 for a given plant $P = (A, B) \in \mathcal{P}$. There exists a set $\mathcal{N} \subset \mathcal{P}$ with zero Lebesgue measure such that, if $P \notin \mathcal{N}$, then*

$$\lim_{k \to \infty} \mathbf{P}(A^{(i)}(k)\ B^{(i)}(k)) \overset{as}{\leq} \mathbf{P}(A, B), \tag{5}$$

Algorithm 1 Control design strategy $\Gamma^*(P)$.

Parameter: $\{\mu(k)\}_{k=0}^{\infty}$ such that $\lim_{k\to\infty} \mu(k) = \infty$ but $\mu(k) = o(\log(k))$.
Initialize $(A^{(i)}(0), B^{(i)}(0))$ for all $i \in \{1, \ldots, N\}$.
for $k = 1, 2, \ldots$ **do**
  for $i = 1, 2, \ldots, N$ **do**
    if $k = 2, 4, \ldots$ **then**
      Update subsystem i estimate as

$$(A^{(i)}(k), B^{(i)}(k)) = \underset{(\hat{A},\hat{B})\in\mathcal{A}\times\mathcal{B}}{\arg\min} \ \mathbf{W}(\hat{A}, \hat{B}, \mathcal{F}_k),$$
$$\text{subject to } \hat{A}_{\ell j} = A_{\ell j}, \hat{B}_{\ell\ell} = B_{\ell\ell},$$
$$\forall j, \ell \in \{1, \ldots, N\}, (s_{\mathcal{C}})_{\ell i} \neq 0,$$
$$\hat{A}_{zq} = 0, \forall z, q \in \{1, \ldots, N\}, (s_{\mathcal{P}})_{zq} = 0,$$

      where

$$\mathbf{W}(\hat{A},\hat{B}, \mathcal{F}_k) = \mu(k) \operatorname{tr}(\mathbf{P}(\hat{A}, \hat{B})) + \sum_{t=1}^{k} \|x(t) - \hat{A}x(t-1) - \hat{B}u(t-1)\|_2^2.$$

    else
      $(A^{(i)}(k), B^{(i)}(k)) \leftarrow (A^{(i)}(k-1), B^{(i)}(k-1))$.
    end if
    $K^{(i)}(k) \leftarrow \mathbf{L}(A^{(i)}(k)\ B^{(i)}(k))$.
    $u_i(k) \leftarrow \mathbf{T}_i K^{(i)}(k)x(k)$.
  end for
end for

$$\sum_{t=0}^{k} \chi((A^{(i)}(k), B^{(i)}(k)) \in \mathcal{W}_\delta(A, B)) \overset{as}{=} O(\mu(k)), \tag{6}$$
$$\sum_{t=0}^{k} \chi(\|K^{(i)}(k) - \mathbf{L}(A, B)\| > \rho) \overset{as}{=} O(\mu(k)), \tag{7}$$
$$\sum_{t=0}^{k} \chi(\|K(k) - \mathbf{L}(A, B)\| > \rho) \overset{as}{=} O(\mu(k)), \tag{8}$$

for all $\delta > 0$, where $x \overset{as}{=} y$ and $x \overset{as}{\leq} y$ mean $\mathbb{P}\{x = y\} = 1$ and $\mathbb{P}\{x \leq y\} = 1$, respectively. In addition, we get

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \|x(k)\|^p + \|u(k)\|^p \overset{as}{<} \infty, \ \forall\, p \geq 1. \tag{9}$$

Proof: See Appendix B. ∎

Note that, according to Lemma 4.2, we know that there exists a set $\mathcal{N} \subset \mathcal{P}$ with zero Lebesgue measure such that, if $P \notin \mathcal{N}$, the estimates in the modified Campi–Kumar adaptive algorithm (Algorithm 1) converge to the correct global plant model. This fact is a direct consequence of the use of regularized maximum likelihood estimators in the Campi–Kumar algorithm [19]. We need the following lemma.

Lemma 4.3 *For any matrices $X, P, Y \in \mathbb{R}^{n\times n}$, we have*

$$\|X^\top PX - Y^\top PY\| \leq \|P\|\|X-Y\|(\|X\| + \|Y\|).$$

Proof: See Appendix C. ∎

Now, we are ready to present the main result of this section.

Theorem 4.4 *Let $\mathbf{K} = \Gamma^*(P)$ be defined in Algorithm 1 for a given plant $P = (A, B) \in \mathcal{P}$. There exists a set $\mathcal{N} \subset \mathcal{P}$ with zero Lebesgue measure such that, if $P \notin \mathcal{N}$, then*

$$J_P(\Gamma^*(P)) \overset{as}{=} J_P(\mathbf{K}^*(P)).$$

Proof: The proof follows the same reasoning as in [15]. According to [20, p.158], for all $1 \leq i \leq N$, we get the set of equations in

$$\begin{aligned}
&\operatorname{tr}\{\mathbf{P}(A^{(i)}(k), B^{(i)}(k))\} + x(k)^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))x(k)\\
&\quad = x(k)^\top Qx(k) + u^{(i)}(k)^\top Ru^{(i)}(k)\\
&\qquad + \mathbb{E}\{(A^{(i)}(k)x(k) + B^{(i)}(k)u^{(i)}(k) + w(k))^\top\\
&\qquad\quad \times \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A^{(i)}(k)x(k) + B^{(i)}(k)u^{(i)}(k) + w(k)) \mid \mathcal{F}_{k-1}\}\\
&\quad = x(k)^\top Qx(k) + u^{(i)}(k)^\top Ru^{(i)}(k)\\
&\qquad + \mathbb{E}\{x(k+1)^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))x(k+1) \mid \mathcal{F}_{k-1}\}\\
&\qquad + (A^{(i)}(k)x(k) + B^{(i)}(k)u^{(i)}(k))^\top\\
&\qquad\quad \times \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A^{(i)}(k)x(k) + B^{(i)}(k)u^{(i)}(k))\\
&\qquad - (Ax(k) + Bu(k))^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(Ax(k) + Bu(k)),
\end{aligned} \tag{10}$$

with $u^{(i)}(k) = K^{(i)}(k)x(k)$ and $u(k) = K(k)x(k)$. Averaging both sides of (10) over time and all subsystems, we get

$$\begin{aligned}
&\zeta_1(T) + \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} x(k)^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))x(k)\\
&= \frac{1}{T}\sum_{k=0}^{T-1} x(k)^\top Qx(k) + \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} u^{(i)}(k)^\top Ru^{(i)}(k)\\
&\quad + \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} \mathbb{E}\{x(k+1)^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))x(k+1) \mid \mathcal{F}_{k-1}\} + \zeta_2(T),
\end{aligned} \tag{11}$$

where

$$\zeta_1(T) = \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} \mathrm{tr}\{\mathbf{P}(A^{(i)}(k), B^{(i)}(k))\},$$

and $\zeta_2(T)$ is given in

$$\begin{aligned}\zeta_2(T) = \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} \Big[&(A^{(i)}(k)x(k) + B^{(i)}(k)u^{(i)}(k))^\top \\ &\times \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A^{(i)}(k)x(k) + B^{(i)}(k)u^{(i)}(k)) \\ &- (Ax(k) + Bu(k))^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(Ax(k) + Bu(k))\Big].\end{aligned} \tag{12}$$

Subtracting $\frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N}\mathbb{E}\{x(k+1)^\top \mathbf{P}(A^{(i)}(k+1), B^{(i)}(k+1))x(k+1) \mid \mathcal{F}_{k-1}\}$, from both sides of (11) while adding and subtracting $\frac{1}{T}\sum_{k=0}^{T-1} u(k)^\top Ru(k)$ from right-hand side of (11), we get

$$\frac{1}{T}\sum_{k=0}^{T-1}[x(k)^\top Qx(k) + u(k)^\top Ru(k)] + \zeta_4(T) + \zeta_5(T) + \zeta_2(T) = \zeta_1(T) + \zeta_3(T), \tag{13}$$

where

$$\begin{aligned}\zeta_3(T) =& \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} x(k)^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))x(k) \\ &- \mathbb{E}\{x(k+1)^\top \mathbf{P}(A^{(i)}(k+1), B^{(i)}(k+1))x(k+1) \mid \mathcal{F}_{k-1}\},\end{aligned}$$

$$\zeta_4(T) = \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} u^{(i)}(k)^\top Ru^{(i)}(k) - u(k)^\top Ru(k),$$

and

$$\begin{aligned}\zeta_5(T) =& \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} \mathbb{E}\{x(k+1)^\top [\mathbf{P}(A^{(i)}(k), B^{(i)}(k)) \\ &- \mathbf{P}(A^{(i)}(k+1), B^{(i)}(k+1))]x(k+1) \mid \mathcal{F}_{k-1}\}.\end{aligned}$$

In the rest of the proof, we study the asymptotic behavior of the sequences $\{\zeta_\ell(k)\}_{k=0}^{\infty}$ for all $1 \le \ell \le 5$.

• Asymptotic behavior of $\zeta_1(T)$: First, note that

$$\begin{aligned}\limsup_{T\to\infty} \zeta_1(T) &= \limsup_{T\to\infty} \frac{1}{NT}\sum_{k=0}^{T-1}\sum_{i=1}^{N} \mathrm{tr}\{\mathbf{P}(A^{(i)}(k), B^{(i)}(k))\} \\ &= \frac{1}{N}\sum_{i=1}^{N} \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \mathrm{tr}\{\mathbf{P}(A^{(i)}(k), B^{(i)}(k))\}.\end{aligned}$$

Using (5) inside the above identity, we get

$$\limsup_{T\to\infty} \zeta_1(T) \overset{as}{\leq} \operatorname{tr}\{\mathbf{P}(A,B)\}.$$

• Asymptotic behavior of $\zeta_3(T)$: With a similar strategy as in case (B) in the proof of Theorem 6 in [15], we can prove that

$$0 \overset{as}{=} \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1}\Big[x(k)^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))x(k) \\ - \mathbb{E}\{x(k+1)^\top \mathbf{P}(A^{(i)}(k+1), B^{(i)}(k+1))x(k+1) \mid \mathcal{F}_{k-1}\}\Big].$$

Hence, $\limsup_{T\to\infty} \zeta_3(T) \overset{as}{=} 0$.
• Asymptotic behavior of $\zeta_4(T)$: In this case, we have

$$\left|\frac{1}{T}\sum_{k=0}^{T-1} u^{(i)}(k)^\top R u^{(i)}(k) - u(k)^\top R u(k)\right| \\ \leq \frac{1}{T}\sum_{k=0}^{T-1}\left|u^{(i)}(k)^\top R u^{(i)}(k) - u(k)^\top R u(k)\right| \\ \leq \frac{1}{T}\sum_{k=0}^{T-1}\|K^{(i)}(k)^\top R K^{(i)}(k) - K(k)^\top R K(k)\|\|x(k)\|^2.$$

According to Lemma 4.3, we have $\|K^{(i)}(k)^\top RK^{(i)}(k) - K(k)^\top RK(k)\| \leq \|R\| \times \|K^{(i)}(k) - K(k)\|(\|K^{(i)}(k)\| + \|K(k)\|)$. Considering that $\mathbf{L}(\cdot,\cdot)$ is a continuous function of its arguments (see [21]) and $\mathcal{P}$ is a compact set, we know that $\|K^{(i)}(k)\|$ and $\|K(k)\|$ are uniformly bounded. Hence, $\|K^{(i)}(k)\| + \|K(k)\| \leq M$. Now, using Cauchy–Schwartz inequality [22], we get

$$\left|\frac{1}{T}\sum_{k=0}^{T-1} u^{(i)}(k)^\top R u^{(i)}(k) - u(k)^\top R u(k)\right|^2 \\ \leq \|R\| M\left(\frac{1}{T}\sum_{k=0}^{T-1}\|K^{(i)}(k) - K(k)\|^2\right)\left(\frac{1}{T}\sum_{k=0}^{T-1}\|x(k)\|^4\right).$$

Let us introduce the notation $K^o = \mathbf{L}(A,B)$. Note that, for all $\rho > 0$, we have

$$\frac{1}{T}\sum_{k=0}^{T-1}\|K^{(i)}(k) - K^o\|^2 \leq \rho^2 + \frac{1}{T}\sum_{k=0}^{T-1}\|K^{(i)}(k) - K^o\|^2\chi(\|K^{(i)}(k) - K^o\| > \rho).$$

Again, considering the facts that $\mathbf{L}(\cdot,\cdot)$ is a continuous function of its arguments and $\mathcal{P}$ is a compact set, we know that $\|K^{(i)}(k) - K^o\|$ is uniformly bounded. Thus, using (7) from Theorem 4.2, we can show that $\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1}\|K^{(i)}(k) - K^o\| \overset{as}{\leq}$

ρ^2, for all $\rho > 0$. Since the choice of ρ was arbitrary, we get $\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \|K^{(i)}(k) - K^o\|^2 \overset{as}{=} 0$. With a similar reasoning, we can also prove that $\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \|K(k) - K^o\|^2 \overset{as}{=} 0$. Therefore, considering that $\|K^{(i)}(k) - K(k)\|^2 \leq \|K^{(i)}(k)-K^o\|^2 + \|K(k)-K^o\|^2$, we have $\limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T-1} \|K^{(i)}(k)-K(k)\|^2 \overset{as}{=} 0$. Hence, $\limsup_{T\to\infty} \zeta_4(T) \overset{as}{=} 0$ due to the fact that $\limsup_{T\to\infty} \|x(k)\|^4 \overset{as}{<} \infty$ according to (9).

• Asymptotic behavior of $\zeta_5(T)$: With the same approach as in case (C) in the proof of Theorem 6 in [15], we can prove $\limsup_{T\to\infty} \zeta_5(T) \overset{as}{=} 0$.

• Asymptotic behavior of $\zeta_2(T)$: Let us start with studying the asymptotic behavior of the sequence $\{\hat{\zeta}_2^{(i)}(T)\}_{T=0}^{\infty}$ in

$$\begin{aligned}\hat{\zeta}_2^{(i)}(T) = \frac{1}{T}\sum_{k=0}^{T-1} \Big[& x(k)^\top (A^{(i)}(k) + B^{(i)}(k)K^{(i)}(k))^\top \\ & \times \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A^{(i)}(k) + B^{(i)}(k)K^{(i)}(k))x(k) \\ & - x(k)^\top (A + BK(k))^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A + BK(k))x(k)\Big].\end{aligned} \tag{14}$$

Using Lemma 4.3, we can upper bound each term as in

$$\begin{aligned}& x(k)^\top (A^{(i)}(k)+B^{(i)}(k)K^{(i)}(k))^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A^{(i)}(k)+B^{(i)}(k)K^{(i)}(k))x(k) \\ & \quad - x(k)^\top (A + BK(k))^\top \mathbf{P}(A^{(i)}(k), B^{(i)}(k))(A + BK(k))x(k) \\ & \leq \|x(k)\| \left\|\mathbf{P}(A^{(i)}(k), B^{(i)}(k))\right\| \left\|[A^{(i)}(k) + B^{(i)}(k)K^{(i)}(k)] - [A + BK(k)]\right\| \\ & \quad \times \left\|[A^{(i)}(k) + B^{(i)}(k)K^{(i)}(k)] + [A + BK(k)]\right\|.\end{aligned} \tag{15}$$

Considering again that $\mathbf{L}(\cdot,\cdot)$ and $\mathbf{P}(\cdot,\cdot)$ are continuous functions of their arguments (see [21]) and $\mathcal{P}$ is a compact set, we know that

$$\begin{aligned}\left\|\mathbf{P}(A^{(i)}(k), B^{(i)}(k))\right\| &\leq M_1, \\ \left\|[A^{(i)}(k) + B^{(i)}(k)K^{(i)}(k)] + [A + BK(k)]\right\| &\leq M_2.\end{aligned}$$

Using Cauchy–Schwartz inequality, we get the inequality in

$$\begin{aligned}\hat{\zeta}_2^{(i)}(T) &\leq M_1 M_2 \frac{1}{T}\sum_{k=0}^{T-1} \|x(k)\|^2 \left\|[A^{(i)}(k) + B^{(i)}(k)K^{(i)}(k)] - [A + BK(k)]\right\| \\ &\leq M_1 M_2 \left(\frac{1}{T}\sum_{k=0}^{T-1}\|x(k)\|^4\right)^{\frac{1}{2}} \left(\frac{1}{T}\sum_{k=0}^{T-1}\left\|[A^{(i)}(k)+B^{(i)}(k)K^{(i)}(k)] - [A+BK(k)]\right\|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{16}$$

Now, note that

$$\begin{aligned}&\left\|[A^{(i)}(k)+B^{(i)}(k)K^{(i)}(k)]-[A+BK(k)]\right\|^2\\&\quad\le\left\|[A^{(i)}(k)+B^{(i)}(k)K^{(i)}(k)]-[A+BK^{(i)}(k)]\right\|^2\\&\qquad+\|[A+BK^{(i)}(k)]-[A+BK^o]\|^2+\|[A+BK^o]-[A+BK(k)]\|^2\\&\quad\le\left\|[A^{(i)}(k)+B^{(i)}(k)K^{(i)}(k)]-[A+BK^{(i)}(k)]\right\|^2\\&\qquad+\|B\|^2\left(\|K^{(i)}(k)-K^o\|^2+\|K(k)-K^o\|^2\right).\end{aligned}$$

Hence, with similar argument as above, we can prove that $\limsup_{T\to\infty}\hat{\zeta}_2^{(i)}(T)\overset{as}{=}0$, and as a result $\limsup_{T\to\infty}\zeta_2(T)=\limsup_{T\to\infty}\frac{1}{N}\sum_{i=1}^N\hat{\zeta}_2^{(i)}(T)\overset{as}{=}0$. Now, we are ready to prove the statement of this theorem. From the asymptotic behavior of sequences $\zeta_1(T)$ and $\zeta_3(T)$, we know that

$$\mathrm{tr}\{\mathbf{P}(A,B)\}\overset{as}{\ge}\limsup_{T\to\infty}\zeta_1(T)+\zeta_3(T). \tag{17}$$

Using identity (13) inside inequality (17) shows that

$$\begin{aligned}\mathrm{tr}\{\mathbf{P}(A,B)\}&\overset{as}{\ge}\limsup_{T\to\infty}\zeta_4(T)+\zeta_5(T)+\zeta_2(T)\\&\qquad+\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}[x(k)^\top Qx(k)+u(k)^\top Ru(k)],\end{aligned}$$

which result in

$$\mathrm{tr}\{\mathbf{P}(A,B)\}\overset{as}{\ge}\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}[x(k)^\top Qx(k)+u(k)^\top Ru(k)].$$

This inequality finishes the proof. ■

Now, we are ready to present the solution of problem (4).

Corollary 4.5 *For any plant graph $G_{\mathcal{P}}$ and design graph $G_{\mathcal{C}}$, we get $r_{\mathcal{P}}^{\mathrm{ave}}(\Gamma^*)\overset{as}{=}1$, and $r_{\mathcal{P}}^{\sup}(\Gamma^*)\overset{as}{=}1$.*

Proof: The proof is a direct consequence of Theorem 4.4 and Lemma 4.1. ■

Corollary 4.5 shows that, irrespective of the plant graph $G_{\mathcal{P}}$ and design graph $G_{\mathcal{C}}$, there exists a limited model information control design strategy that can achieve a competitive ratio equal one. This control design strategy gives adaptive controllers achieving asymptotically the closed-loop performance of optimal control design strategy with full model information. Note that earlier results stated that such competitive ratio cannot be achieved by static or linear time-invariant dynamic controllers [8–14].

4 Example

As a simple numerical example, let us consider the problem of regulating the distance between N vehicles in a platoon. We model vehicle i, $1 \leq i \leq N$, as

$$\begin{bmatrix} x_i(k+1) \\ v_i(k+1) \end{bmatrix} = \left(I + \Delta T \begin{bmatrix} 0 & 1 \\ 0 & -\alpha_i/m_i \end{bmatrix} \right) \begin{bmatrix} x_i(k) \\ v_i(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta T \beta_i / m \end{bmatrix} \bar{u}_i(k) + \begin{bmatrix} \bar{w}_1^i(k) \\ \bar{w}_2^i(k) \end{bmatrix},$$

where $x_i(k)$ is the vehicle position, $v_i(k)$ its velocity, m_i the mass, α_i the viscous drag coefficient, β_i the power conversion quality coefficient, and ΔT the sampling time. For each vehicle, stochastic exogenous inputs $\bar{w}_j^i(k) \in \mathbb{R}^n$, $j = 1, 2$, capture the effect of wind, road quality, friction, etc. For simplicity of presentation, let us consider the case of $N = 2$ vehicles. In addition, assume that $\Delta T = 1$. As performance objective, the designer wants to minimize the cost function

$$J = \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \left[q_d (x_1(k) - x_2(k) - d^*)^2 + \sum_{i=1,2} q_v (v_i(k) - v^*)^2 + r(\bar{u}_i(k) - \bar{u}_i^*)^2 \right],$$

to regulate the distance between the trucks with minimum control effort. Note that $\bar{u}_i^* = \alpha_i v^* / \beta_i$ is the average control signal. We can write the reduced-order system using the distance between vehicles and their velocities as state variables in the form

$$z(k+1) = Az(k) + Bu(k) + w(k),\ z(0) = 0, \tag{18}$$

where

$$\begin{aligned} z(k) &= [v_1(k) - v^*\ ,\ x_1(k) - x_2(k) - d^*\ ,\ v_2(k) - v^*]^\top, \\ u(k) &= [\bar{u}_1(k) - \bar{u}_1^*\ ,\ \bar{u}_2(k) - \bar{u}_2^*]^\top, \\ w(k) &= [\bar{w}_2^1(k)\ ,\ \bar{w}_1^1(k) + \bar{w}_1^2(k)\ ,\ \bar{w}_2^2(k)]^\top, \end{aligned}$$

and

$$A = \begin{bmatrix} 1 - \alpha_1/m_1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 - \alpha_2/m_2 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1/m_1 & 0 \\ 0 & 0 \\ 0 & \beta_2/m_2 \end{bmatrix}.$$

This model leads to

$$J = \limsup_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} z(k)^T Q z(k) + u(k)^T R u(k), \tag{19}$$

where $Q = \text{diag}(q_v, q_d, q_v)$ and $R = \text{diag}(r, r)$. To simplify the presentation, let $Q = I$ and $R = I$.

Note that $z(0) = 0$ in (18) indicates that the vehicles start from desired distance d^* of each other and with velocity v^*. However, due to the exogenous inputs $w(k)$, the vehicles will drift away from this ideal situation. By minimizing the closed-loop performance criterion in (19), the designer minimizes this drift using the least control effort possible.

We define the first subsystem as $\underline{z}_1(k) = z_1(k)$ and the second subsystem as $\underline{z}_2(k) = [z_2(k)\ z_3(k)]^T$. Therefore, we get

$$\underline{z}_1(k+1) = a_{11}\underline{z}_1(k) + b_{11}u_1(k) + w_1(k),$$

and

$$\underline{z}_2(k+1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \underline{z}_1(k) + \begin{bmatrix} 1 & -1 \\ 0 & a_{22} \end{bmatrix} \underline{z}_2(k) + b_{22}u_2(k) + \begin{bmatrix} w_2(k) \\ w_3(k) \end{bmatrix},$$

where (a_{ii}, b_{ii}) are local parameters of subsystem i. Assume that

$$\mathcal{A} = \left\{ A \in \mathbb{R}^{3\times 3} \,\middle|\, A = \begin{bmatrix} a_{11} & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & a_{22} \end{bmatrix}, a_{11}, a_{22} \in [0,1] \right\},$$

$$\mathcal{B} = \left\{ B \in \mathbb{R}^{3\times 2} \,\middle|\, B = \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ 0 & b_{22} \end{bmatrix}, b_{11}, b_{22} \in [0.5, 1.5] \right\}.$$

Note that we can verify Assumptions 4.1 and 4.2 using the observability and controllability of structured system, see [23, 24].

We compare the performance of the introduced adaptive controller with a deadbeat control design strategy $\Gamma^{\Delta} : \mathcal{P} \to \mathbb{R}^{2\times 3}$ for this special family of systems as

$$\Gamma^{\Delta}(P) = \begin{bmatrix} -a_{11}/b_{11} & 0 & 0 \\ 1/b_{22} & 1/b_{22} & -(1+a_{22})/b_{22} \end{bmatrix},$$

for all $P = (A, B) \in \mathcal{P}$. Note that Γ^{Δ} is a limited model information control design strategy, because each local controller i is based on only parameters of subsystem i, $i = 1, 2$.

Figure 1 illustrates the running cost of the closed-system with the optimal control design with full model information $\mathbf{K}^*(P)$ (solid curve), the modified Campi-Kumar adaptive controller $\Gamma^*(P)$ (dashed curve), and the deadbeat control design strategy $\Gamma^{\Delta}(P)$ (dotted curve). The running costs of the closed-system with $\Gamma^*(P)$ and the optimal control design with full model information $\mathbf{K}^*(P)$ both converge to $\text{tr}\{\mathbf{P}(A, B)\}$ (the horizontal line) as time goes to infinity. The cost of the optimal control design strategy with global model knowledge is always lower than the cost of the adaptive controllers. The simulation is done for parameters $(a_{11}, b_{11}) = (0.4360, 1.0497)$ and $(a_{22}, b_{22}) = (0.0259, 0.9353)$. Figure 2 illustrates the convergence of the individual model parameters (a_{ii}, b_{ii}), $i = 1, 2$, for the adaptive subcontrollers. Note that only one of the subsystems need to estimate each parameter (as each one has access to its own model parameters).

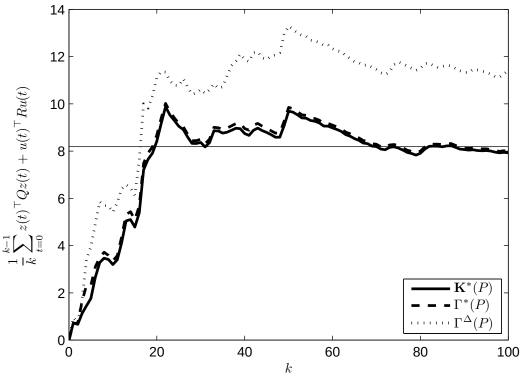

Figure 1: The running cost of the closed-system with different controllers.

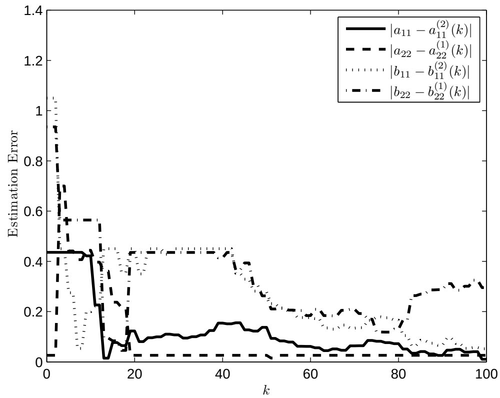

Figure 2: Estimation error of model parameters.

5 Conclusion

In this paper, as a generalization of earlier results in optimal control design with limited model information, we searched over the set of control design strategies that construct adaptive controllers. We found a minimizer of the competitive ratio both in average and supremum senses. We used the Campi–Kumar adaptive algorithm to setup an adaptive control design strategy that achieves a competitive ratio equal to one, that is, this adaptive controller asymptotically achieves closed-loop performance equal to the optimal centralized controller with full model information. We illustrated the applicability of this adaptive controller on a vehicle platooning problem. As a future work, we suggest studying decentralized adaptive controllers.

6 Bibliography

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A Proof of Lemma 4.1

Let us assume, without loss of generality, that $r_{\mathcal{P}}^{\sup}(\Gamma) < \infty$ since otherwise, the desired inequality is trivially satisfied. First, note that using Theorem 2.10.1 in [22], function $J_P(\Gamma(P))/J_P(\mathbf{K}^*(P))$ is integrable on $\mathcal{P}$ since we assumed $r_{\mathcal{P}}^{\sup}(\Gamma) = \operatorname{ess\,sup} J_P(\Gamma(P))/J_P(\mathbf{K}^*(P)) < \infty$ (and $\mathcal{P}$ is a compact set thanks to Assumption 4.1). Then, using Theorem 2.7.1 in [22], we get

$$r_{\mathcal{P}}^{\mathrm{ave}}(\Gamma) = \int_{\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(\mathbf{K}^*(P))} f(P)\,\mathrm{d}P \leq \int_{\mathcal{P}} r_{\mathcal{P}}^{\sup}(\Gamma) f(P)\,\mathrm{d}P = r_{\mathcal{P}}^{\sup}(\Gamma).$$

This completes the proof.

B Proof of Lemma 4.2

Equations (5)–(7) are direct consequences of Theore-ms 2 and 3 in [15]. We start with proving (8). To do so, let us prove $\|K(k) - \mathbf{L}(A,B)\| > \rho$ implies that there exists at least an index i such that $\|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\| > \rho/\sqrt{N}$. We can prove this fact by contradiction. Assume that there does not exists any index i such that $\|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\| > \rho/\sqrt{N}$. Therefore, for all $1 \leq i \leq N$, we have $\|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\| \leq \rho/\sqrt{N}$, and as a result, according to Theorem 1 in [25], we get

$$\|K(k) - \mathbf{L}(A,B)\|^2 \leq \sum_{i=1}^{N} \|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\|^2 \leq \rho^2.$$

This is contradictory to the assumption that $\|K(k) - \mathbf{L}(A,B)\| > \rho$. Hence, we proved the implication. Based on this property, it is easy to see that

$$\sum_{t=0}^{k} \chi(\|K(k) - \mathbf{L}(A,B)\| > \rho) \sum_{t=0}^{k} \sum_{i=1}^{N} \chi(\|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\| > \rho/\sqrt{N}). \quad (20)$$

Now, note that $\|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\| > \rho/\sqrt{N}$ implies that $\|K^{(i)}(k) - \mathbf{L}(A,B)\| > \rho/\sqrt{N}$. Thus, we get

$$\sum_{t=0}^{k} \chi(\|\mathbf{T}_i K^{(i)}(k) - \mathbf{T}_i \mathbf{L}(A,B)\| > \rho/\sqrt{N}) \leq \sum_{t=0}^{k} \chi(\|K^{(i)}(k) - \mathbf{L}(A,B)\| > \rho/\sqrt{N}). \quad (21)$$

Substituting (21) inside (20), we get

$$\sum_{t=0}^{k} \chi(\|K(k) - \mathbf{L}(A,B)\| > \rho) \leq \sum_{t=0}^{k} \sum_{i=1}^{N} \chi(\|K^{(i)}(k) - \mathbf{L}(A,B)\| > \rho/\sqrt{N}).$$

Now, using (7), we can show that

$$\sum_{t=0}^{k} \chi(\|K^{(i)}(k) - \mathbf{L}(A,B)\| > \rho/\sqrt{N}) \overset{as}{=} O(\mu(k)),$$

for all $1 \leq i \leq N$. Therefore, we have

$$\sum_{t=0}^{k} \chi(\|K(k) - \mathbf{L}(A,B)\| > \rho) \overset{as}{=} O(\mu(k)).$$

Finally, note that the proof of (9) is a direct result of applying (8) to the proof of Theorem 5 in [15]. This concludes the proof.

C Proof of Lemma 4.3

First, note that

$$(X-Y)^\top P(X+Y) + (X+Y)^\top P(X-Y) = 2(X^\top PX - Y^\top PY).$$

Hence, we get

$$\begin{aligned} 2\|(X^\top PX - Y^\top PY)\| &= \|(X-Y)^\top P(X+Y) + (X+Y)^\top P(X-Y)\| \\ &\leq \|(X-Y)^\top P(X+Y)\| + \|(X+Y)^\top P(X-Y)\| \\ &\leq 2\|P\|\|X-Y\|\|X+Y\| \\ &\leq 2\|P\|\|X-Y\|(\|X\| + \|Y\|). \end{aligned}$$

This concludes the proof.

PAPER 5

Optimal H_∞ Control Design under Model Information Limitations and State Measurement Constraints

Farhad Farokhi, Henrik Sandberg, and Karl H. Johansson

Abstract–We present a suboptimal control design algorithm for a family of continuous-time parameter-dependent linear systems that are composed of interconnected subsystems. We are interested in designing the controller for each subsystem such that it only utilizes partial state measurements (characterized by a directed graph called the control graph) and limited model parameter information (characterized by the design graph). The algorithm is based on successive local minimizations and maximizations (using the subgradients) of the H_∞–norm of the closed-loop transfer function with respect to the controller gains and the system parameters. We use a vehicle platooning example to illustrate the applicability of the results.

1 Introduction

Distributed and decentralized control design problem is a classical topic in the control literature (e.g., see [1–3]). Most of the available approaches in this field implicitly assume that the design procedure is done in a centralized fashion using the complete knowledge of the model parameters. However, this assumption is not realistic when dealing with large-scale systems due to several reasons. For instance, the overall system might be assembled from modules that are designed by separate entities without access to the entire set of model parameters because at the time of design this information was unavailable. Another reason could be that we want to keep the system maintenance simple by making it robust to nonlocal parameter changes; i.e., if a controller is designed knowing only local parameters, we do not need to redesign it whenever the parameters of a subsystem not in its immediate neighborhood change. Financial gains, for instance, in the case of power network control, could also be a motivation for limited access to model knowledge since competing companies are typically reluctant to share information on their production with each other. For a more detailed survey of the motivations behind control design using local model parameter information, see [4, Ch. 1].

Recently, there have been some studies on control design with limited model information [4–7]. For instance, the authors in [6] introduce control design strategies as mappings from the set of plants to the set of structured static state-feedback controllers. They compare the control design strategies using a measure called the competitive ratio, which is defined to be the worst case ratio (over the set of all possible plants) of the closed-loop performance of the control design strategy in hand scaled by the best performance achievable having access to global model parameter information. Then, they seek a minimizer of the competitive ratio over a family of control design strategies that use only the parameters of their corresponding subsystems when designing controllers. Noting that, in those studies, the plants can vary over an unbounded set, the results are somewhat conservative. Additionally, all the aforementioned studies deal with discrete-time system as it was proved that the competitive ratio is unbounded when working with continuous-time systems [5]. Not much have been done in optimal control design under limited model information for continuous-time systems.

In this paper, contrary to previous studies, we investigate continuous-time systems with parameters in a compact set. Specifically, we propose a numerical algorithm for calculating suboptimal H_∞ control design strategies (i.e., mappings from the set of parameters to the set of structured static state-feedback controllers) for a set of parameter-dependent linear continuous-time systems composed of interconnected subsystems. We consider the case where each subsystem has access to a (possibly strict) subset of the system parameters when designing and implementing its local controller. Additionally, we assume that each local controller uses partial state measurements to close the feedback loop. To solve the problem, we first expand the control design strategies in terms of the system parameters (using a fixed set of basis functions) in such a way that each controller only uses its available parameters. Following the approach in [8], we calculate the subgradient of the H_∞–norm of the closed-loop transfer function with respect to the controller gains and the system parameters. Then, we propose a numerical optimization algorithm based on successive local minimizations and maximizations of this performance measure with respect to the controller gains and the system parameters. Designing parameter-dependent controllers has a very rich history in the control literature, specially in gain scheduling and supervisory control; e.g., see [9–15]. However, most of these studies implicitly assume that the overall controller has access to all the parameters. Contrary to these studies, we assume that local controllers have access to only subsets of the system parameters.

The rest of the paper is organized as follows. In Section 2, we introduce the problem formulation. We propose a numerical algorithm for calculating a suboptimal H_∞ control design strategy in Section 3. We illustrate the approach on a vehicle platooning example in Section 4. Finally, we present the conclusions in Section 5.

1.1 Notation

Let the sets of integer and real numbers be denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. Let $\mathbb{Z}_{>(\geq)n} = \{m \in \mathbb{Z} \mid m > (\geq)n\}$ and $\mathbb{R}_{>(\geq)x} = \{y \in \mathbb{R} \mid y > (\geq)x\}$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

We use capital roman letters to denote matrices. The notation $A > (\geq)0$ shows that the symmetric matrix A is positive (semi-)definite. For any $q, m \in \mathbb{Z}_{\geq 1}$, we define the notation $\mathbb{B}^q_m = \{(Y_1, \dots, Y_q) \mid Y_i \in \mathbb{R}^{m\times m}, Y_i \geq 0, \sum_{i=1}^q \mathrm{tr}(Y_i) = 1\}$. We use $\mathbb{B}^q$ whenever the dimension m is irrelevant (or can be deduced from the text). For any $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$, we use $A \otimes B \in \mathbb{R}^{np\times mq}$ to denote the Kronecker product of these matrices.

Let an ordered set of real functions $(\xi_\ell)_{\ell=1}^L$ be given such that $\xi_\ell : \mathbb{R}^p \to \mathbb{R}$, $1 \leq \ell \leq L$, are continuous functions with continuous first derivatives. We define $\mathrm{span}((\xi_\ell)_{\ell=1}^L)$ as the set composed of all linear combinations of the functions $(\xi_\ell)_{\ell=1}^L$; i.e., for any $f \in \mathrm{span}((\xi_\ell)_{\ell=1}^L)$, there exists at least one ordered set of real numbers $(x_\ell)_{\ell=1}^L$ such that $f(\alpha) = \sum_{\ell=1}^L x_\ell \xi_\ell(\alpha)$ for all $\alpha \in \mathbb{R}^p$. For any $n, m \in \mathbb{Z}_{\geq 1}$, $\mathrm{span}((\xi_\ell)_{\ell=1}^L)^{n\times m}$ denotes the set of all functions $A : \mathbb{R}^p \to \mathbb{R}^{n\times m}$ such that $A(\alpha) = \sum_{\ell=1}^L \xi_\ell(\alpha) A^{(\ell)}$ with $A^{(\ell)} \in \mathbb{R}^{n\times m}$ for all $1 \leq \ell \leq L$.

We consider directed graphs with vertex set $\mathcal{V} = \{1, \dots, N\}$ for a fixed $N \in \mathbb{Z}_{\geq 1}$. For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ denotes its edge set, we define the adjacency matrix $S \in \{0, 1\}^{N\times N}$ such that $s_{ij} = 1$ if $(j, i) \in \mathcal{E}$, and $s_{ij} = 0$ otherwise. We define the set of structured matrices $\mathcal{X}(S, (n_i)_{i=1}^N, (m_i)_{i=1}^N)$ as the set of all matrices $X \in \mathbb{R}^{n\times m}$ with $n = \sum_{i=1}^N n_i$ and $m = \sum_{i=1}^N m_i$ such that $X_{ij} = 0 \in \mathbb{R}^{n_i \times n_j}$ whenever $s_{ij} = 0$ for $1 \leq i, j \leq N$.

For any function $f : \mathcal{U} \to \mathcal{Y}$, we call $\mathcal{U}$ the domain of f and $\mathcal{Y}$ the codomain of f. Additionally, we define its image $f(\mathcal{U})$ as the set of all $y \in \mathcal{Y}$ such that $y = f(x)$ for a $x \in \mathcal{U}$.

For any $n \in \mathbb{Z}_{\geq 1}$, I_n denotes the $n \times n$ identity matrix. To simplify the presentation, we use I whenever the dimension can be inferred from the text. For any $n, m \in \mathbb{Z}_{\geq 1}$, we define $0_{n\times m}$ as the $n \times m$ zero matrix. Finally, let $\mathbf{1}_n \in \mathbb{R}^n$ be a vector of ones.

2 Mathematical Problem Formulation

In this section, we introduce the underlying system model, the controller structure, and the closed-loop performance criterion.

2.1 System Model

Consider a continuous-time linear parameter-dependent system composed of $N \in \mathbb{Z}_{\geq 1}$ subsystems. Let subsystem i, $1 \leq i \leq N$, be described as

$$\dot{x}_i(t) = \sum_{j=1}^{N} \big[A_{ij}(\alpha_i)x_j(t) + (B_w)_{ij}(\alpha_i)w_i(t) + (B_u)_{ij}(\alpha_i)u_i(t)\big], \tag{1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state vector, $w_i(t) \in \mathbb{R}^{m_{w,i}}$ is the exogenous input, $u_i(t) \in \mathbb{R}^{m_{u,i}}$ is the control input, and lastly, $\alpha_i \in \mathbb{R}^{p_i}$ is the parameter vector. Let us introduce the augmented state, control input, exogenous input, and parameter vector as

$$\begin{aligned}
x(t) &= \begin{bmatrix} x_1(t)^\top & \cdots & x_N(t)^\top \end{bmatrix}^\top \in \mathbb{R}^n,\\
w(t) &= \begin{bmatrix} w_1(t)^\top & \cdots & w_N(t)^\top \end{bmatrix}^\top \in \mathbb{R}^{m_w},\\
u(t) &= \begin{bmatrix} u_1(t)^\top & \cdots & u_N(t)^\top \end{bmatrix}^\top \in \mathbb{R}^{m_u},\\
\alpha(t) &= \begin{bmatrix} \alpha_1(t)^\top & \cdots & \alpha_N(t)^\top \end{bmatrix}^\top \in \mathbb{R}^p,
\end{aligned}$$

where $n = \sum_{i=1}^N n_i$, $m_w = \sum_{i=1}^N m_{w,i}$, $m_u = \sum_{i=1}^N m_{u,i}$, and $p = \sum_{i=1}^N p_i$. This results in

$$\dot{x}(t) = A(\alpha)x(t) + B_w(\alpha)w(t) + B_u(\alpha)u(t).$$

We use the notation $\mathcal{A}$ to denote the set of all eligible parameter vectors α. We make the following standing assumption concerning the model matrices:

Assumption 5.1 *There exists a basis set $(\xi_\ell)_{\ell=1}^L$ such that $A(\alpha) \in \operatorname{span}((\xi_\ell)_{\ell=1}^L)^{n\times n}$, $B_w(\alpha) \in \operatorname{span}((\xi_\ell)_{\ell=1}^L)^{n\times m_w}$, and $B_u(\alpha) \in \operatorname{span}((\xi_\ell)_{\ell=1}^L)^{n\times m_u}$.*

Example 5.1 *Consider a parameter-dependent system described by*

$$\begin{aligned}
\dot{x}_1(t) &= (-2.0+\alpha_1)x_1(t)+(0.1+0.4\sin(\alpha_1))x_2(t)+(0.6-0.3\sin(\alpha_1))u_1(t) + w_1(t),\\
\dot{x}_2(t) &= +0.3x_1(t)+(-1.0-\alpha_2)x_2(t)+(1.0+0.1\cos(\alpha_2))u_2(t) + w_2(t),
\end{aligned}$$

where $x_i(t) \in \mathbb{R}$, $u_i(t) \in \mathbb{R}$, $w_i(t) \in \mathbb{R}$, and $\alpha_i \in \mathbb{R}$ are respectively the state, the control input, the exogenous input, and the parameter of subsystem $i = 1, 2$. We define the set of eligible parameters as

$$\mathcal{A} = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in \mathbb{R}^2 \mid \alpha_i \in [-1, +1] \text{ for } i = 1, 2 \right\}.$$

Clearly, this system satisfies Assumption 5.1 with basis functions $\xi_1(\alpha) = 1$, $\xi_2(\alpha) = \alpha_1$, $\xi_3(\alpha) = \sin(\alpha_1)$, $\xi_4(\alpha) = \cos(\alpha_2)$, and $\xi_5(\alpha) = \alpha_2$. ◀

2.2 Measurement Model and Controller

Let a control graph $\mathcal{G}_\mathcal{K}$ with adjacency matrix $S_\mathcal{K}$ be given. We consider the case where each subsystem has access to a (potentially parameter-dependent) observation vector $y_i(t) \in \mathbb{R}^{o_{y,i}}$ that can be described by

$$y_i(t) = \sum_{j=1}^{N} \left[(C_y)_{ij}(\alpha_i) x_j(t) + (D_{yw})_{ij}(\alpha_i) w_j(t) \right].$$

Now, we can define the augmented observation vector as

$$y(t) = \begin{bmatrix} y_1(t)^\top & \cdots & y_N(t)^\top \end{bmatrix}^\top \in \mathbb{R}^{o_y},$$

where $o_y = \sum_{i=1}^{N} o_{y,i}$. Thus,

$$y(t) = C_y(\alpha) x(t) + D_{yw}(\alpha) w(t).$$

We say that the measurement vector $y(t)$ obeys the structure given by the control graph $\mathcal{G}_\mathcal{K}$ if $C_y(\mathcal{A}) \in \mathcal{X}(S_\mathcal{K}, (o_{y,i})_{i=1}^N, (n_i)_{i=1}^N)$ and $D_{yw}(\mathcal{A}) \in \mathcal{X}(S_\mathcal{K}, (o_{y,i})_{i=1}^N, (m_{w,i})_{i=1}^N)$, where the definition of the structured set $\mathcal{X}$ can be found in the notation subsection. We make the following standing assumption concerning the observation matrices:

Assumption 5.2 *For the same basis set $(\xi_\ell)_{\ell=1}^L$ as in Assumption 5.1, $C_y(\alpha) \in \operatorname{span}((\xi_\ell)_{\ell=1}^L)^{o_y \times n}$ and $D_{yw}(\alpha) \in \operatorname{span}((\xi_\ell)_{\ell=1}^L)^{o_y \times m_w}$.*

In this paper, we are interested in linear static state-feedback controllers of the form

$$u(k) = K y(k), \tag{2}$$

where $K \in \mathcal{K} = \mathcal{X}(I, (m_{u,i})_{i=1}^N, (o_{y,i})_{i=1}^N)$. Note that fol-lowing the same reasoning as in [8, 16], the extension to fixed-order dynamic controllers is trivial (using just a change of variable).

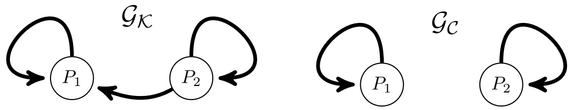

Figure 1: The control graph $\mathcal{G}_\mathcal{K}$ and the design graph $\mathcal{G}_\mathcal{C}$ utilized in the recurring numerical example.

Example 5.1 (Cont'd) *Let the control graph $\mathcal{G}_\mathcal{K}$ in Figure 1 represent the state-measurement availability. Consider the observation vectors*

$$y_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2, \qquad y_2(t) = x_2(t) \in \mathbb{R}.$$

Clearly, the augmented observation vector obeys the structure dictated by $\mathcal{G}_\mathcal{K}$. Furthermore, since the measurement matrices are constant, they obviously satisfy Assumption 5.2. Finally, the controller (2) *is given by*

$$\begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix},$$

where $K_{11} \in \mathbb{R}^{1\times 2}$ and $K_{22} \in \mathbb{R}$. ◀

2.3 Control Design Strategy

Following [6], we define a control design strategy Γ as a mapping from $\mathcal{A}$ to $\mathcal{K}$. Let a control design strategy $\Gamma : \mathcal{A} \to \mathcal{K}$ be partitioned following the measurement vector and the control input dimensions as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Gamma_{NN} \end{bmatrix},$$

where each block Γ_{ii} represents a map $\mathcal{A} \to \mathbb{R}^{m_{u,i}\times o_{y,i}}$. Let a directed graph $\mathcal{G}_\mathcal{C}$ with adjacency matrix $S_\mathcal{C}$ be given. We say that the control design strategy Γ has structure $\mathcal{G}_\mathcal{C}$ if Γ_{ii}, $1 \le i \le N$, is only a function of $\{\alpha_j \mid (s_\mathcal{C})_{ij} \neq 0\}$. Let $\mathcal{C}$ denote the set of all control design strategies Γ with structure $\mathcal{G}_\mathcal{C}$. We make the following standing assumption:

Assumption 5.3 *There exists a basis set $(\eta_{\ell'})_{\ell'=1}^{L'}$ such that $\Gamma \in \text{span}((\eta_{\ell'})_{\ell'=1}^{L'})^{m_u\times o_y}$.*

Now, we define $\mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'}) = \mathcal{C} \cap \text{span}((\eta_{\ell'})_{\ell'=1}^{L'})^{m_u\times o_y}$ as the set of all control design strategies over which we optimize the closed-loop performance.

Example 5.1 (Cont'd) *The design graph $\mathcal{G}_\mathcal{C}$ in Figure 1 illustrates the available plant model information. We use the basis functions $\eta_1(\alpha) = 1$, $\eta_2(\alpha) = \alpha_1$, $\eta_3(\alpha) = \alpha_1^2$, and $\eta_4(\alpha) = \alpha_2$ for parameterizing the control design strategies. Clearly, any $\Gamma \in \mathcal{C}(\{\eta_{\ell'}\}_{\ell'=1}^4)$ can be expressed in the form*

$$\Gamma(\alpha) = \sum_{\ell'=1}^{4} G^{(\ell')}\eta_{\ell'}(\alpha),$$

with

$$G^{(1)} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad G^{(2)} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$G^{(3)} = \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G^{(4)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix},$$

*where * denotes the nonzero entries of these matrices. Note that the functions $\{\eta_{\ell'}\}_{\ell'=1}^4$ are indeed design choices and we can improve the closed-loop performance by increasing the number of the basis functions. However, this can only be achieved at the price of a higher computational time.* ◀

2.4 Performance Metric

Let us introduce the performance measure output vector

$$z(t) = C_z x(t) + D_{zw}w(t) + D_{zu}u(t) \in \mathbb{R}^{o_z}. \tag{3}$$

We are interested in finding a control design method Γ that solves the optimization problem

$$\min_{\Gamma \in \mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'})} \max_{\alpha \in \mathcal{A}} \|T_{zw}(s;\Gamma,\alpha)\|_\infty, \tag{4}$$

where $T_{zw}(s;\Gamma,\alpha)$ denotes the closed-loop transfer function from the exogenous input $w(t)$ to the performance measurement vector $z(t)$ for $\alpha \in \mathcal{A}$. We make the following assumptions to avoid singularities in the optimal control problem:

Assumption 5.4 *$D_{zu}^\top D_{zu} = I$ and $D_{yw}D_{yw}^\top = I$.*

These assumptions are common in the H_∞-control design literature [17, p. 288]. However, notice that the conditions in Assumption 5.4 are only sufficient (and not necessary). For instance, although $D_{yw} = 0$ in Example 5.1, as we will see later, a nontrivial solution indeed exists and the optimal control problem is in fact well-posed.

To simplify the presentation in what follows, we define the notation

$$J(\Gamma,\alpha) = \|T_{zw}(s;\Gamma,\alpha)\|_\infty.$$

Now, noting that there may exist many local solutions to the optimization problem (4), it is difficult to find the global solution of this problem. Hence, we define:

Definition 5.1 *A pair* $(\Gamma^*, \alpha^*) \in \mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'}) \times \mathcal{A}$ *is a saddle point of* $J:\mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'}) \times \alpha \rightarrow \mathbb{R}_{\geq 0}$ *if there exists a constant* $\epsilon \in \mathbb{R}_{>0}$ *such that*

$$J(\Gamma^*, \alpha) \leq J(\Gamma^*, \alpha^*) \leq J(\Gamma, \alpha^*),$$

for any $(\Gamma, \alpha) \in \mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'}) \times \mathcal{A}$ *where* $\|\Gamma - \Gamma^*\| \leq \epsilon$ *and* $\|\alpha - \alpha^*\| \leq \epsilon$*.*

Evidently, the global solution of the minimax optimization problem (4) is also a saddle point of J. However, there might be many more saddle points. In the rest of this paper, we focus on finding a saddle point (Γ^*, α^*) of J. To make sure that the set of saddle points is nonempty, we make the following standing assumption:

Assumption 5.5 *The set of all eligible parameters* $\mathcal{A}$ *is a compact subset of* $\mathbb{R}^p$*. In addition, for any* $\alpha \in \mathcal{A}$*, the pair* $(A(\alpha), B_u(\alpha))$ *is stabilizable and the pair* $(A(\alpha), C_y(\alpha))$ *is detectable.*

Notice that Assumption 5.5 is only a necessary condition for the existence of a saddle point solution since we are solving a decentralized control design problem rather than a centralized one. Therefore, throughout the paper, we are going to assume that, at least, one such saddle point exists. If we switch the stabilizability and the detectability conditions with the absence of unstable fixed modes, this assumption becomes more realistic (but still not sufficient because of the asymmetric parameter dependencies).

Example 5.1 (Cont'd) *In this example, we are interested in minimizing the closed-loop transfer function from the exogenous inputs to the performance measurement vector with* $C_z = [I_2 \;\; 0_{2\times2}]^\top$*,* $D_{zu} = [0_{2\times2} \;\; I_2]^\top$*, and* $D_{zw} = 0$*. Clearly, the choice of* D_{zu} *satisfies Assumption 5.4. It is easy to check that the system satisfies Assumption 5.5 as well.* ◀

3 Optimization Algorithm

In this section, we develop a numerical algorithm for finding a saddle point (Γ^*, α^*) of J. We start by calculating subgradients[1] $\Delta\Gamma \in \partial_\Gamma J(\Gamma, \alpha)$ and $\Delta\alpha \in \partial_\alpha J(\Gamma, \alpha)$ for any $(\Gamma, \alpha) \in \mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'}) \times \mathcal{A}$.

[1] We say that a vector $g \in \mathcal{X}$ is a subgradient of $f : \mathcal{X} \rightarrow \mathbb{R}$ at $x \in \mathcal{X}$ if for all $x' \in \mathcal{X}$, $f(x') \geq f(x) + g^\top(x' - x)$. Let $\partial f(x)$ denote the set of subgradients of f at the point $x \in \mathcal{X}$. If f is convex, then $\partial f(x)$ is nonempty and bounded. We would like refer interested readers to [18, 19] (and the references therein) for a detailed review of the subgradients and numerical optimization algorithm using them.

Lemma 5.1 *Let us define the transfer functions in*

$$\begin{bmatrix} T_{zw}(s;\Gamma,\alpha) & G_{12}(s;\Gamma,\alpha) \\ G_{21}(s;\Gamma,\alpha) & \bullet \end{bmatrix} = \begin{bmatrix} C'_{\rm cl}(\Gamma,\alpha) \\ C_{y'}(\alpha) \end{bmatrix} (sI - A'_{\rm cl}(\Gamma,\alpha))^{-1} \begin{bmatrix} B'_{\rm cl}(\Gamma,\alpha) & B_u(\alpha) \end{bmatrix} + \begin{bmatrix} D'_{\rm cl}(\Gamma,\alpha) & D_{zu} \\ D_{y'w}(\alpha) & \bullet \end{bmatrix}, \tag{5}$$

with

$$\begin{aligned} A'_{\rm cl}(\Gamma,\alpha) &= A(\alpha) + B_u(\alpha)K'C_{y'}(\alpha), \\ B'_{\rm cl}(\Gamma,\alpha) &= B_w(\alpha) + B_u(\alpha)K'D_{y'w}(\alpha) \\ C'_{\rm cl}(\Gamma,\alpha) &= C_z(\alpha) + D_{zu}(\alpha)K'C_{y'}(\alpha), \\ D'_{\rm cl}(\Gamma,\alpha) &= D_{zw}(\alpha) + D_{zu}(\alpha)K'D_{y'w}(\alpha), \end{aligned}$$

where $K' = [G^{(1)} \ \cdots \ G^{(L')}]$ *and*

$$C_{y'}(\alpha) = \begin{bmatrix} \eta_1(\alpha)C_y(\alpha) \\ \vdots \\ \eta_{L'}(\alpha)C_y(\alpha) \end{bmatrix}, \quad D_{y'w}(\alpha) = \begin{bmatrix} \eta_1(\alpha)D_{yw}(\alpha) \\ \vdots \\ \eta_{L'}(\alpha)D_{yw}(\alpha) \end{bmatrix}.$$

Furthermore, let $\Delta\Gamma = \sum_{\ell'=1}^{L'} \Delta G^{(\ell')}$ *be such that* $\Delta G^{(\ell')} \in \mathbb{R}^{m\times o_y}$ *are defined in*

$$\begin{bmatrix} \Delta G^{(1)} & \cdots & \Delta G^{(L')} \end{bmatrix} = \|T_{zw}(s;\Gamma,\alpha)\|_\infty^{-1} \sum_{\nu=1}^{q} \mathrm{Re}\big\{ G_{21}(j\omega_\nu;\Gamma,\alpha) T_{zw}(j\omega_\nu;\Gamma,\alpha)^* \times Q_\nu Y_\nu Q_\nu^* G_{12}(j\omega_\nu;\Gamma,\alpha) \big\}^\top, \tag{6}$$

where $\|T_{zw}(s;\Gamma,\alpha)\|_\infty$ *is attained at a finite number of frequencies* $(\omega_1,\ldots,\omega_q)$ *and* $(Y_1,\ldots,Y_q) \in \mathbb{B}^q$. *In addition, the columns of* Q_ν, $1 \le \nu \le q$, *are chosen so as to form an orthonormal basis for the eigenspace of* $T_{zw}(j\omega_\nu;\Gamma,\alpha)T_{zw}(j\omega_\nu;\Gamma,\alpha)^*$ *associated with the leading eigenvalue* $\|T_{zw}(s;\Gamma,\alpha)\|_\infty$. *Then,* $\Delta\Gamma \in \partial_\Gamma J(\Gamma,\alpha)$.

Proof: Due to space constraints, we only present a sketch of the proof here. First, we prove that the closed-loop system

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B_w(\alpha)w(t) + B_u(\alpha)u(t), \\ z(t) = C_z x(t) + D_{zw}w(t) + D_{zu}u(t), \\ y'(t) = C_{y'}(\alpha)x(t) + D_{y'w}(\alpha)w(t), \\ u(t) = K'y'(t), \end{cases}$$

is equivalent to the closed-loop system that we introduced in the previous section. Then, we can use the method presented in [8] for calculating the subgradients of the closed-loop performance with respect to the controller gain. Doing so, we find $\Delta G^{(\ell')} \in \partial_{G^{\ell'}} J(\Gamma,\alpha)$ for $1 \le \ell' \le L'$. Finally, we get $\sum_{\ell'=1}^{L'} \Delta G^{(\ell')}\eta_{\ell'} \in \partial_\Gamma J(\Gamma,\alpha)$. ■

Lemma 5.2 *Let us define the transfer functions in*

$$\begin{bmatrix} T_{zw}(s;\Gamma,\alpha) & H_{12}(s;\Gamma,\alpha) \\ H_{21}(s;\Gamma,\alpha) & \bullet \end{bmatrix} = \begin{bmatrix} C''_{\mathrm{cl}}(\Gamma,\alpha) \\ C_{y''} \end{bmatrix} (sI - A''_{\mathrm{cl}}(\Gamma,\alpha))^{-1} \begin{bmatrix} B''_{\mathrm{cl}}(\Gamma,\alpha) & B_{u''} \end{bmatrix} + \begin{bmatrix} D''_{\mathrm{cl}}(\Gamma,\alpha) & D_{zu''} \\ D_{y''w} & \bullet \end{bmatrix}, \tag{7}$$

with

$$\begin{aligned} A''_{\mathrm{cl}}(\Gamma,\alpha) &= B_{u''}K''(\alpha)C_{y''}, \\ B''_{\mathrm{cl}}(\Gamma,\alpha) &= B_{u''}K''(\alpha)D_{y''w}, \\ C''_{\mathrm{cl}}(\Gamma,\alpha) &= C_z + D_{zu''}K''(\alpha)C_{y''}, \\ D''_{\mathrm{cl}}(\Gamma,\alpha) &= D_{zw} + D_{zu''}K''(\alpha)D_{y''w}, \end{aligned}$$

where

$$C_{y''} = \begin{bmatrix} A^{(1)} \\ \vdots \\ A^{(L)} \\ \mathbf{1}_{L+1} \otimes \begin{bmatrix} G^{(1)}C_y^{(1)} \\ G^{(1)}C_y^{(2)} \\ \vdots \\ G^{(1)}C_y^{(L)} \\ \vdots \\ G^{(L')}C_y^{(L)} \end{bmatrix} \\ 0_{(nL+m_uL(L+1)L')\times n} \end{bmatrix},$$

$$D_{y''w} = \begin{bmatrix} 0_{(nL+m_uL(L+1)L')\times m_w} \\ B_w^{(1)} \\ \vdots \\ B_w^{(L)} \\ \mathbf{1}_{L+1} \otimes \begin{bmatrix} G^{(1)}D_{yw}^{(1)} \\ G^{(1)}D_{yw}^{(2)} \\ \vdots \\ G^{(1)}D_{yw}^{(L)} \\ \vdots \\ G^{(L')}D_{yw}^{(L)} \end{bmatrix} \end{bmatrix},$$

$$D_{zu''} = \begin{bmatrix} 0_{(nL+m_uL^2L')\times o_z} \\ \mathbf{1}_{LL'} \otimes D_{zu}^\top \\ 0_{(nL+m_uL^2L')\times o_z} \\ \mathbf{1}_{LL'} \otimes D_{zu}^\top \end{bmatrix}^\top,$$

and

$$B_{u''} = \left[\mathbf{1}_L^\top \otimes I_{n\times n}\ \Upsilon\ \ 0_{n\times nLL'}\ \mathbf{1}_L^\top \otimes I_{n\times n}\ \Upsilon\ \ 0_{n\times nLL'}\right]^\top,$$

with

$$\Upsilon = \left[\mathbf{1}_{LL'}^\top \otimes B_u^{(1)\top}\ \ \cdots\ \ \mathbf{1}_{LL'}^\top \otimes B_u^{(L)\top}\right].$$

Additionally, we have

$$\begin{aligned}K''(\alpha) = \mathrm{diag}(&\Xi(\alpha)\otimes I_n,\ \Xi(\alpha)\otimes\Psi(\alpha)\otimes\Xi(\alpha)\otimes I_{m_u}, \Psi(\alpha)\otimes\Xi(\alpha)\otimes I_{m_u},\\ &\Xi(\alpha)\otimes I_n, \Xi(\alpha)\otimes\Psi(\alpha)\otimes\Xi(\alpha)\otimes I_{m_u}, \Psi(\alpha)\otimes\Xi(\alpha)\otimes I_{m_u}).\end{aligned}$$

where, for all $\alpha\in\mathbb{R}^p$, $\Xi(\alpha) = \mathrm{diag}(\xi_1(\alpha),\dots,\xi_L(\alpha))$ and $\Psi(\alpha) = \mathrm{diag}(\eta_1(\alpha),\dots,\eta_{L'}(\alpha))$. Furthermore, let $\Delta\alpha = [\Delta\alpha_1\ \cdots\ \Delta\alpha_p]^\top$ be such that the scalars $\Delta\alpha_i \in \mathbb{R}$, $1\le i\le p$, are calculated using

$$\begin{aligned}\Delta\alpha_i = \|T_{zw}(s;\Gamma,\alpha)\|_\infty^{-1}\sum_{\nu=1}^{q}\mathrm{Re}\Big\{\ \mathrm{tr}\Big[&H_{21}(j\omega_\nu;\Gamma,\alpha)T_{zw}(j\omega_\nu;\Gamma,\alpha)^*\\ &\times Q_\nu Y_\nu Q_\nu^* H_{12}(j\omega_\nu;\Gamma,\alpha)\frac{\partial}{\partial\alpha_i}K''(\alpha)\Big]\Big\},\end{aligned}\tag{8}$$

where $\|T_{zw}(s;\Gamma,\alpha)\|_\infty$ is attained at a finite number of frequencies $(\omega_1,\dots,\omega_q)$ and $(Y_1,\dots,Y_q)\in\mathbb{B}^q$. In addition, the columns of Q_ν, $1\le\nu\le q$, form an orthonormal basis of the eigenspace of $T_{zw}(j\omega_\nu;\Gamma,\alpha)T_{zw}(j\omega_\nu;\Gamma,\alpha)^*$ associated with the leading eigenvalue $\|T_{zw}(s;\Gamma,\alpha)\|_\infty$. Then, $\Delta\alpha\in\partial_\alpha J(\Gamma,\alpha)$.

Proof: The proof follows the same line of reasoning as in the proof of Lemma 5.1. ■

Algorithm 2 introduces a numerical algorithm for finding a saddle point of J, or equivalently, a local solution of the optimization problem in (4).

Theorem 5.3 *In Algorithm 2, let $\{\mu_k\}_{k=0}^\infty$ be chosen such that $\lim_{k\to\infty}\sum_{z=1}^k \mu_z = \infty$ and $\lim_{k\to\infty}\sum_{z=1}^k\mu_z^2 < \infty$. Assume that there exists $C\in\mathbb{R}$ such that $\|g_{k,\tau}\|_2\le C$ and $\|\Delta G^{(\ell')}(k)\|_2 \le C$ for all $k,\tau\ge 0$ and $1\le\ell'\le L'$. Then, if $\lim_{k\to\infty}(\Gamma^{(k)}, \alpha(k))$ exists, it is a saddle point of J.*

Proof: The proof follows from the convergence properties of subgradient optimization algorithms. ■

Example 5.1 (Cont'd) *Let us initialize Algorithm 2 at $\alpha(0) = [0.0\ -0.0]^\top$ and*

$$\Gamma^0(\alpha) = \begin{bmatrix} +0.0 & +0.0 & +0.0\\ +0.0 & +0.0 & -0.5\end{bmatrix}.$$

Algorithm 2 A numerical algorithm for calculating a saddle point (Γ^*, α^*) of J.

Input: $\{G^{(\ell')}(0)\}_{\ell'=0}^{L'}$, $\alpha(0)$, $\epsilon, \varepsilon \in \mathbb{R}_{>0}$, $\{\mu_k\}_{k=1}^{\infty}$
Output: Γ^*, α^*

```
1:  k ← 0
2:  repeat
3:      Γ^(k) ← Σ_{ℓ'=1}^{L'} G^(ℓ')(k) η_ℓ'
4:      ᾱ(0) ← α(k)
5:      τ ← 0
6:      repeat
7:          ᾱ(τ+1) ← P_A(ᾱ(τ) + μ_τ g_{k,τ}) where g_{k,τ} ∈ ∂_α J(Γ^(k), α) calculated at ᾱ(τ)
            and P_A(·) is the projection to A
8:          τ ← τ + 1
9:      until |J(Γ^(k), ᾱ(τ)) − J(Γ^(k), ᾱ(τ − 1))| ≤ ε
10:     α(k + 1) ← ᾱ(τ)
11:     for ℓ' = 1, . . . , L' do
12:         G^(ℓ')(k + 1) ← P_C(G^(ℓ')(k) − μ_k ΔG^(ℓ')(k)) where ΔG^(ℓ')(k) ∈
            ∂_{G^ℓ'} J(Γ, α(k + 1)) calculated at Γ^(k) and P_C(·) is the projection to
            C((η_ℓ')_{ℓ'=1}^{L'})
13:     end for
14:     k ← k + 1
15: until |J(Γ^(k−1), α(k − 1)) − J(Γ^(k), α(k))| ≤ ϵ
16: Γ* ← Σ_{ℓ'=1}^{L'} G^(ℓ')(k) η_ℓ'
17: α* ← α(k)
```

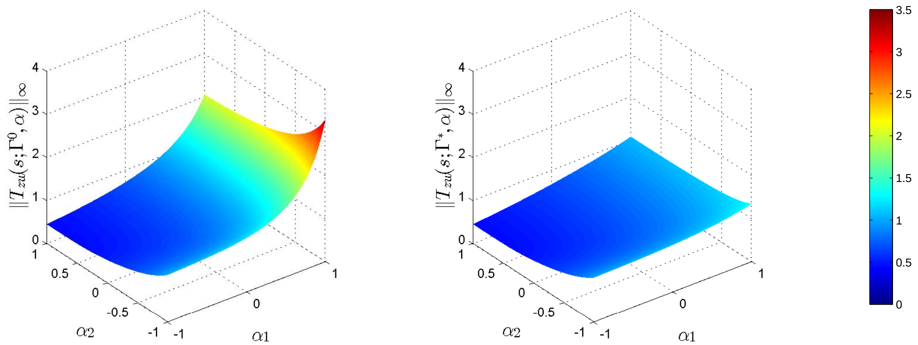

Figure 2: The initial closed-loop performance $\|T_{zw}(s;\Gamma^0,\alpha)\|_\infty$ (left) and the optimal closed-loop performance $\|T_{zw}(s;\Gamma^*,\alpha)\|_\infty$ (right) as function of the parameters α_i, $i = 1, 2$.

Furthermore, we pick $\epsilon = \varepsilon = 10^{-3}$ *and* $\mu_k = 0.1/k$ *for all* $k \in \mathbb{Z}_{\geq 1}$. *This results in* $\Gamma^*(\alpha) = G^{(1)} + G^{(2)}\alpha_1 + G^{(3)}\alpha_1^2 + G^{(4)}\alpha_2$, *where*

$$G^{(1)} = \begin{bmatrix} -0.1892 & -1.008 & 0.0 \\ 0.0 & 0.0 & -7.1070 \end{bmatrix},$$

$$G^{(2)} = \begin{bmatrix} -0.1892 & -1.008 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix},$$

$$G^{(3)} = \begin{bmatrix} -0.1892 & -1.008 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix},$$

$$G^{(4)} = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 6.6070 \end{bmatrix}.$$

Figure 2 illustrates the closed-loop performance measure $\|T_{zw}(s;\Gamma^0,\alpha)\|_\infty$ *for the initial control design strategy* Γ^0 *(left) and the suboptimal closed-loop performance measure* $\|T_{zw}(s;\Gamma^*,\alpha)\|_\infty$ *(right) as a function of the system parameters* α_i, $i = 1, 2$. ◀

Now, we adapt the definition of the competitive ratio (see [5, 6]) to our problem formulation. Using this measure, we can characterize the value of the model parameter information in the control design. Assume that for every $\alpha \in \mathcal{A}$, there exists an optimal controller $K^*(\alpha) \in \mathcal{K}$ such that

$$J(K^*(\alpha),\alpha) \leq J(K,\alpha),\ \forall K \in \mathcal{K}.$$

Notice that $K^* : \mathcal{A} \to \mathcal{K}$ is not necessarily in $\mathcal{C}$ or $\mathcal{C}((\eta_{\ell'})_{\ell'=1}^{L'})$ since its entries might depend on all the parameters in the vector α (and not just some specific subset of them). Now, we define the competitive ratio of a control design method Γ as

$$r(\Gamma) = \sup_{\alpha \in \mathcal{A}} \frac{J(\Gamma(\alpha),\alpha)}{J(K^*(\alpha),\alpha)},$$

with the convention that "$\frac{0}{0}$" equals one. Let us calculate this ratio for our numerical example.

Example 5.1 (Cont'd) *For the definition of the competitive ratio, we need to calculate* $K^*(\alpha)$. *To do so, we assume that the control graph* $\mathcal{G}_\mathcal{K}$ *is a complete graph. Consider the output vectors*

$$y_1(t) = y_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2.$$

Hence, we are dealing with full state feedback, but it is still a parameter-dependent control design problem. For any $\alpha \in \mathcal{A}$, $K^*(\alpha)$ *is a static controller, which can* *be derived from a convex optimization problem [17]. For this setup, let us run Algorithm 2 with* $\alpha(0) = [0 \quad 0]^\top$ *and*

$$\Gamma^0(\alpha) = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.5 \end{bmatrix}.$$

Then, we get $\Gamma^*(\alpha) = G^{(1)} + G^{(2)}\alpha_1 + G^{(3)}\alpha_1^2 + G^{(4)}\alpha_2$*, where*

$$G^{(1)} = \begin{bmatrix} -0.0624 & -0.1023 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.3992 & -1.1650 \end{bmatrix},$$
$$G^{(2)} = \begin{bmatrix} -0.0624 & -0.1023 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$
$$G^{(3)} = \begin{bmatrix} -0.0624 & -0.1023 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$
$$G^{(4)} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3992 & 0.6650 \end{bmatrix}.$$

To calculate the competitive ratio, we grid the set of all eligible parameters $\mathcal{A}$ *and calculate* K^* *(and its closed-loop performance) for each grid point. This results in*

$$r(\Gamma^*) = \sup_{\alpha \in \mathcal{A}} \frac{J(\Gamma(\alpha), \alpha)}{J(K^*(\alpha), \alpha)} = 1.1475.$$

Hence, the closed-loop performance of Γ^* *can be at most 15% worse than the performance of the control design strategy with access to the full parameter vector. We can also infer that, although using gradient descent optimization,* Γ^* *is close to the global solution of the optimization problem* (4) *since the performance cost of the global solution must lay somewhere between the performances of* Γ^* *and* K^**, which are very close to each other thanks to the relatively small* $r(\Gamma^*)$*. The 15% performance degradation is partly due to using local model information, but it is also due to the use of the basis functions* $\{\eta_{\ell'}\}_{\ell'=1}^4$ *to expand the control design strategies (since* $\mathrm{span}(\eta_{\ell'})_{\ell'=1}^4$ *is not dense in* $\mathcal{C}$*). To portray this fact quantitatively, let us assume that the design graph* $\mathcal{G}_\mathcal{C}$ *is a complete graph and use Algorithm 2 to calculate a saddle point* $(\Gamma^\bullet, \alpha^\bullet)$ *of* J*. Doing so, we get*

$$r(\Gamma^\bullet) = 1.1344,$$

so about 13% of the performance degradation is caused by the choice of the basis functions $\{\eta_{\ell'}\}_{\ell'=1}^4$*. This amount can be certainly reduced by increasing* L' *(i.e., adding to the number of basis functions employed to describe the control design strategies).* ◀

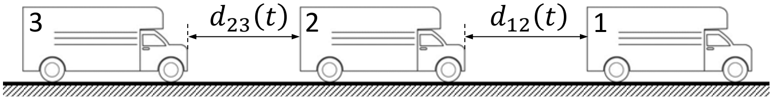

Figure 3: Regulating the distance between three vehicles in a platoon.

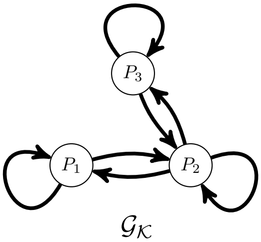

Figure 4: The control graph in the vehicle platooning.

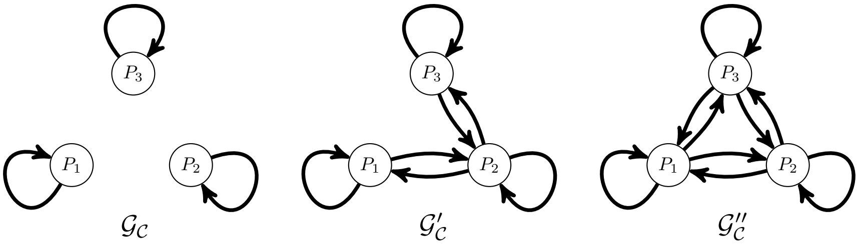

Figure 5: The design graphs utilized in the vehicle platooning.

4 Application to Vehicle Platooning

Consider a physical example where three heavy-duty vehicles are following each other closely in a platoon (see Figure 3). We can model this system as

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t) + w(t),$$

where

$$x(t) = \begin{bmatrix} v_1(t) & d_{12}(t) & v_2(t) & d_{23}(t) & v_3(t) \end{bmatrix}^\top \in \mathbb{R}^5,$$

is the state vector with $v_i(t)$ denoting the velocity of vehicle i and $d_{ij}(t)$ denoting the distance between vehicles i and j (see Figure 3). Additionally, $u(t) \in \mathbb{R}^3$ is the control input (i.e., the acceleration of the vehicles), $w(t) \in \mathbb{R}^5$ is the exogenous input (i.e., the effect of wind, road quality, friction, etc), and $\alpha = [m_1 \; m_2 \; m_3]^\top \in \mathbb{R}^3$ is the vector of parameters with m_i denoting the mass of vehicle i (scaled by its maximum allowable mass). We define the state of each subsystem as

$$x_1(t) = \begin{bmatrix} v_1(t) \\ d_{12}(t) \end{bmatrix}, \quad x_2(t) = v_2(t), \quad x_3(t) = \begin{bmatrix} d_{23}(t) \\ v_3(t) \end{bmatrix}.$$

Furthermore, we have

$$A(\alpha) = \begin{bmatrix} -\varrho_1/m_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\varrho_2/m_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\varrho_3/m_3 \end{bmatrix},$$

and

$$B(\alpha) = \begin{bmatrix} b_1/m_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2/m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_3/m_3 \end{bmatrix},$$

where ϱ_i is the viscous drag coefficient of vehicle i and b_i is the power conversion quality coefficient. These parameters are all scaled by the maximum allowable mass of each vehicle. Let us fix $\varrho_i = 0.1$ and $b_i = 1$ for all $i = 1, 2, 3$. We assume that

$$\mathcal{A} = \{\alpha \in \mathbb{R}^3 \,|\, \alpha_i \in [0.5, 1.0] \text{ for all } i = 1, 2, 3\}.$$

Clearly, we can satisfy Assumption 5.1 with the choice of basis functions $\xi_1(\alpha) = 1$, $\xi_2(\alpha) = 1/m_1$, $\xi_3(\alpha) = 1/m_2$, and $\xi_4(\alpha) = 1/m_3$. Now, we assume that each vehicle only has access to the state measurements of its neighbors. This pattern is captured by the control graph $\mathcal{G}_\mathcal{K}$ in Figure 4. Hence, we get

$$y_1(t)=\begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \end{bmatrix},\ y_2(t)=\begin{bmatrix} v_1(t) \\ d_{12}(t) \\ v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix},\ y_3(t)=\begin{bmatrix} v_2(t) \\ d_{23}(t) \\ v_3(t) \end{bmatrix},$$

Notice that the choice of these particular observation vectors is convenient as the vehicles can measure them directly (using velocity and distance sensors mounted on the front and the back of the vehicles) and they do not need to relay these measurements to each other through a communication medium. For safety reasons, we would like to ensure that the exogenous inputs do not significantly influence the distances between the vehicles. However, we would like to guarantee this fact using as little control action as possible. We capture this goal by minimizing the H_∞-norm of the closed-loop transfer function from the exogenous inputs $w(t)$ to

$$z(t) = \begin{bmatrix} d_{12}(t) & d_{23}(t) & u_1(t) & u_2(t) & u_3(t) \end{bmatrix}^\top.$$

Let us use the basis functions $\eta_1(\alpha) = 1$, $\eta_2(\alpha) = m_1$, $\eta_3(\alpha) = m_1^2$, $\eta_4(\alpha) = m_2$, $\eta_5(\alpha) = m_2^2$, $\eta_6(\alpha) = m_3$, and $\eta_7(\alpha) = m_3^2$ to expand the control design strategies. We use Algorithm 2 to compute the optimal control design strategy. Notice that the open-loop system has two poles on the imaginary axis for all $\alpha \in \mathcal{A}$. To eliminate this problem, we initialize the algorithm with an stabilizing control design strategy

$$\Gamma^0(\alpha)=\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & -5 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -5 \end{bmatrix}.$$

We pick $\alpha(0) = [0.5\ 0.5\ 0.5]^\top$, $\varepsilon = 10^{-2}$, $\epsilon = 10^{-3}$, and $\mu_k = 1/k$ for all $k \in \mathbb{Z}_{\geq 1}$. For comparisons, note that

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^0,\alpha\right)\right\|_\infty = 11.9626.$$

In the following subsections, we calculate optimal control design strategy under three different information regimes. Note that the importance of communicating parameter information for vehicle platooning was also considered in [20], where the authors designed decentralized linear quadratic controllers.

4.1 Local Model Information Availability

We start with the case where each local controller only relies on the mass of its own vehicle. This model information availability corresponds to the design graph $\mathcal{G}_\mathcal{C}$ in Figure 5. For this case, we get the performance

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^{\text{local}},\alpha\right)\right\|_\infty = 4.7905,$$

where Γ^{local} is the outcome of Algorithm 2 with the described initialization.

4.2 Limited Model Information Availability

Here, we let the neighboring vehicles communicate their mass to each other. This model information availability corresponds to the design graph $\mathcal{G}'_\mathcal{C}$ in Figure 5. For this information regime, we get

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^{\text{limited}},\alpha\right)\right\|_\infty = 3.5533,$$

where Γ^{limited} is the outcome of Algorithm 2. Clearly, we get a 25% improvement in comparison to Γ^{local}.

4.3 Full Model Information Availability

Finally, we consider the case where each local controller has access to all the model parameters (i.e., the mass of all other vehicles). This model information availability corresponds to the design graph $\mathcal{G}''_\mathcal{C}$ in Figure 5. We get

$$\max_{\alpha\in\mathcal{A}} \left\|T_{zw}\left(s;\Gamma^{\text{full}},\alpha\right)\right\|_\infty = 3.3596,$$

where Γ^{full} is the outcome of Algorithm 2. It is interesting to note that with access to full model information, we only improve the closed-loop performance by another 5% in comparison to Γ^{limited}. This might be caused by the fact that the first and the third vehicles are not directly interacting.

5 Conclusions

In this paper, we studied optimal static control design under limited model information and partial state measurements for continuous-time linear parameter-dependent systems. We defined the control design strategies as mappings from the set of parameters to the set of controllers. Then, we expanded these mappings using basis functions. We proposed a numerical optimization method based on consecutive local minimizations and maximizations of the H_∞–norm of the closed-loop transfer function with respect to the control design strategy gains and the system parameters. The optimization algorithm relied on using the subgradients of this closed-loop performance measure. As future work, we will focus on finding the best basis functions for expanding the control design strategies. We will also study the rate at which the closed-loop performance improves when increasing the number of the basis functions.

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PAPER 6

Optimal Control Design under Limited Model Information for Discrete-Time Linear Systems with Stochastically-Varying Parameters

Farhad Farokhi and Karl H. Johansson

Abstract–The value of plant model information available in the control design process is discussed. We design optimal state-feedback controllers for interconnected discrete-time linear systems with stochastically-varying parameters. The parameters are assumed to be independently and identically distributed random variables in time. The design of each controller relies only on (*i*) exact local plant model information and (*ii*) statistical beliefs about the model of the rest of the system. We consider both finite-horizon and infinite-horizon quadratic cost functions. The optimal state-feedback controller is derived in both cases. The optimal controller is shown to be linear in the state and to depend on the model parameters and their statistics in a particular way. Furthermore, we study the value of model information in optimal control design using the performance degradation ratio which is defined as the supremum (over all possible initial conditions) of the ratio of the cost of the optimal controller with limited model information scaled by the cost of the optimal controller with full model information. An upper bound for the performance degradation ratio is presented for the case of fully-actuated subsystems. Comparisons are made between designs based on limited, statistical, and full model information. Throughout the paper, we use a power network example to illustrate concepts and results.

1 Introduction

1.1 Motivation

Large-scale systems such as automated highways [1, 2], aircraft and satellite formations [3, 4], supply chains [5, 6], power grids and other shared infrastructures [7, 8] are typically composed of several locally controlled subsystems that are connected to each other either through the physical dynamics, the communication infrastructure, or the closed-loop performance criterion. The problem of designing these local controllers, widely known as distributed or decentralized control design, is an old and well-studied problem in the literature [9–15]. Although the controller itself is highly structured for these large-scale systems, it is commonly assumed that the complete model of the system is available and the design is done in a centralized fashion using the global plant model information. However, this assumption is usually not easily satisfied in practice. For instance, this might be because the design of each local controller is done by a separate designer with no access to the global plant model because the full plant model information is not available at the time of design or it might change later. Recently, this concern has become more important as engineers implement large-scale systems using off-the-shelf components which are designed in advance with limited prior knowledge of their future operating condition. Another reason to consider control design based on only local information is to simplify the tuning and the maintenance of the system. For instance, dependencies between cyber components in a large system can cause complex interactions influencing the physical plant, not present without the controller. Privacy concerns could also be a motivation for designing control actions using only local information. For further motivations behind optimal control design using local model information, see [16].

As an illustrative physical example, let us consider a power network control problem with power being generated in generators and distributed throughout the network via transmission lines (e.g., [17, 18]). It is fairly common to assume that the power consumption of the loads in such a network can be modeled stochastically with a priori known statistics, such as, mean and variance extracted from long term observations [19–21]. When the load variations are "small enough", local generators meet these demand variations. These variations shift the generators operating points, and consequently, change their model parameters. If the loads are modeled as impedances, they change the system model by changing the transmission line impedances. As power networks are typically implemented over a vast geographical area, it is inefficient or even impossible to gather all these model information variations or to identify all the parameters globally. Even if we could gather all the information and identify the whole system based on them, it might take very long and by then the information might be outdated (noting that the model parameters vary stochastically over time). This motivates the interest in designing local controllers for these systems based on only local model information and statistical model information of the rest of the system. We revisit this power network problem in detail for a small example in the paper. A recurring example is used to explain the underlying definitions as well as the mathematical results. It is not difficult to see that similar examples can also be derived for process control, intelligent transportation, irrigation systems, and other shared infrastructures.

1.2 Related Studies

Optimal control design under limited model information has recently attracted attention. The authors in [22] introduced control design strategies as mappings from the set of plants of interest to the set of eligible controllers. They studied the quality of these control design strategies using a performance metric called the competitive ratio; i.e., the worst case ratio of the closed-loop performance of a given control design strategy to the closed-loop performance of the optimal control design with full model information. Clearly, the smaller the competitive ratio is, the more desirable the control design strategy becomes since it can closely replicate the performance of the optimal control design strategy with full model information while only relying on local plant model information. They showed that for discrete-time systems composed of scalar subsystems, the deadbeat control design strategy is a minimizer of the competitive ratio. Additionally, the deadbeat control design strategy is undominated; i.e., there is no other control design strategy that performs always better while having the same competitive ratio. This work was later generalized to limited model information control design methods for inter-connected linear time-invariant systems of arbitrary order in [23]. In that study, the authors investigated the best closed-loop performance that is achievable by structured static state-feedback controllers based on limited model information. It was shown that the result depends on the subsystems interconnection pattern and availability of state measurements. Whenever there is no subsystem that cannot affect any other subsystem and each controller has access to at least the state measurements of its neighbors, the deadbeat strategy is the best limited model information control design method. However, the deadbeat control design strategy is dominated (i.e., there exists another control design strategy that outperforms it while having the same competitive ratio) when there is a subsystem that cannot affect any other subsystem. These results were generalized to structured dynamic controllers when the closed-loop performance criterion is set to be the H_2-norm of the closed-loop transfer function [24]. In this case, the optimal control design strategy with limited model information is static even though the optimal structured state-feedback controller with full model information is dynamic [25, 26]. Later in [27], the design of dynamic controllers for optimal disturbance accommodation was discussed. It was shown that in some cases an observer-based-controller is the optimal architecture also under limited model information. Finally, in [28], it was shown that using an adaptive control design strategy, the designer can achieve a competitive ratio equal to one when the considered plant model belongs to a compact set of linear time-invariant systems and the closed-loop performance measure is the ergodic mean of a quadratic function of the state and control input (which is a natural extension of the H_2-norm of the closed-loop system considering that the closed-loop system in this case is nonlinear due to the adaptive controller).

In all these studies, the model information of other subsystems are assumed to be completely unknown which typically results in conservative controllers because it forces the designer to study the worst-case behavior of the control design methods. In this paper, we take a new approach by assuming that a statistical model is available for the parameters of the other subsystems. There have been many studies of optimal control design for linear discrete-time systems with stochastically-varying parameters [29–33]. In these papers, the optimal controller is typically calculated as a function of model parameter statistics. Considering a different problem formulation, in this paper, we assume each controller design is done using the exact model information of its corresponding subsystem and the other subsystems' model statistics.

Note that studying the worst-case behavior of the system using the competitive ratio is not the only approach for optimal control design under limited model information. For instance, the authors in [34–36] developed methods for designing near-optimal controllers using only local model information whenever the coupling between the subsystems is negligible. However, not even the closed-loop stability can be guaranteed when the coupling grows. As a different approach, in a recent study [37], the authors used an iterative numerical optimization algorithm to solve a finite-horizon linear quadratic problem in a distributed way using only local model information and communication with neighbors. However, this approach (and similarly [38, 39]) require many rounds of communication between the subsystems to converge to a reasonable neighborhood of the optimal controller. To the best of our knowledge, there is also no stopping criteria (for terminating the numerical optimization algorithm) that uses only local information. There have been some studies in developing stopping criteria but these studies require global knowledge of the system [40, 41]. Recently, there has been an attempt for designing optimal controllers using only local model information for linear systems with stochastically-varying parameters [42]. However, that setup is completely different from the problem that is considered in this paper. First, the authors of [42] considered the case where the B-matrix was parameterized with stochastic variables but in our setup the A-matrix is assumed to be stochastic. Additionally, in [42], the infinite-horizon problem was only considered for the case of two subsystems, while here we present all the results for arbitrary number of subsystems. In this paper, we introduce the concept of performance degradation ratio as a measure to study the value information in optimal control design. Furthermore, the proof techniques are different since the authors of [42] use a team-theoretic approach to solve the problem opposed to the approach presented in this paper.

1.3 Main Contribution

The main contribution of this paper is to study the value of plant model information available in the control design process. To do so, we consider limited model information control design for discrete-time linear systems with stochastically-varying parameters. First, in Theorem 6.1, we design the optimal finite-horizon controller based on exact local model information and global model parameter statistics. We generalize these results to infinite-horizon cost functions in Theorem 6.2 assuming that the underlying system is mean square stabilizable; i.e., there exists a constant matrix that can mean square stabilizes the system [29]. However, in Corollary 6.3, we partially relax the assumptions of Theorem 6.2 to calculate the infinite-horizon optimal controller whenever the underlying system is mean square stabilizable under limited model information. This new concept is defined through borrowing the idea of control design strategies from [22, 23]. We define a special class of control design strategies to construct time-varying control gains for each subsystem. We say that a system is mean square stabilizable under limited model information if the intersection of this special class of control design strategies (that use only local model information) and the set of mean square stabilizing control design strategies is nonempty; i.e., there exists a control design strategy that uses only local model information and it can mean square stabilizes the system (see Definition 6.3 for more details).

Using the closed-loop performance of the optimal controller with limited model information, we study the effect of lack of full model information on the closed-loop performance. Specifically, we study the ratio of the cost of the optimal control design strategy with limited model information scaled by the cost of the optimal control design strategy with full model information (which is introduced in Theorems 6.4 and 6.5 for finite-horizon and infinite-horizon cost functions, respectively). We call the supremum of this ratio over the set of all initial conditions, the performance degradation ratio. In Theorem 6.6, we find an upper bound for the performance degradation ratio assuming the underlying systems are fully-actuated (i.e., they have the same number of inputs as the state dimension). As a future direction for research, one might be able to generalize these results to designing structured state-feedback controllers following the same line of reasoning as in [43].

An early and brief version of the paper was presented as [44]. The current paper is a considerable extension of [44] as the results have been generalized, a new literature survey has been included, and a power network example has been introduced to illustrate concepts and results throughout the paper.

1.4 Paper Outline

The rest of the paper is organized as follows. We start with introducing the system model in Section 2. In Section 3, we design optimal controller for each subsystem based on limited model information (i.e., using its own model information and the statistical belief about the other subsystems). We start by the finite-horizon optimal control problem and then generalize the results to infinite-horizon cost functions. In Section 4, we introduce the optimal controller for both finite-horizon and infinite-horizon cost functions when using the full model information. In Section 5, we study the value of plant model information in optimal control design using the performance degradation ratio. Finally, the conclusions and directions for future research are presented in Section 6.

1.5 Notation

The sets of integers and reals are denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. We denote all other sets with calligraphic letters such as $\mathcal{A}$ and $\mathcal{X}$. Specifically, we define $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$) as the set of all symmetric matrices in $\mathbb{R}^{n\times n}$ that are positive definite (positive semidefinite). Matrices are denoted by capital roman letters such as A. We use the notation A_{ij} to denote a submatrix of matrix A (its dimension and position will be defined in the text). The entry in the i^{th} row and the j^{th} column of the matrix A is denoted a_{ij}. We define $A > (\geq) 0$ as $A \in \mathcal{S}^n_{++}(\mathcal{S}^n_{+})$ and $A > (\geq) B$ as $A - B > (\geq) 0$. Let $A \otimes B \in \mathbb{R}^{np\times qm}$ denote the Kronecker product between matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$; i.e.,

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}.$$

For any positive integers n and m, we define the mapping vec : $\mathbb{R}^{n\times m} \to \mathbb{R}^{nm}$ as $\text{vec}(A) = [A_1^\top \; A_2^\top \; \cdots A_m^\top]^\top$ where A_i, $1 \leq i \leq m$, denotes the i^{th} column of A. The mapping $\text{vec}^{-1} : \mathbb{R}^{nm} \to \mathbb{R}^{n\times m}$ is the inverse of $\text{vec}(\cdot)$, where the dimension of the matrix will be clear from the context. It is useful to note that both vec and vec^{-1} are linear operators. Finally, for any given positive integers n and m, we define the discrete Riccati operator $\mathbf{R} : \mathbb{R}^{n\times n} \times \mathcal{S}^n_{+} \times \mathbb{R}^{n\times m} \times \mathcal{S}^m_{++} \to \mathcal{S}^n_{+}$ as $\mathbf{R}(A, P, B, R) = A^\top(P - PB(R + B^\top PB)^{-1}B^\top P)A$ for any $A \in \mathbb{R}^{n\times n}$, $P \in \mathcal{S}^n_{+}$, $B \in \mathbb{R}^{n\times m}$, and $R \in \mathcal{S}^m_{++}$.

2 Control Systems with Stochastically-Varying Parameters

Consider a discrete-time linear system with stochastically-varying parameters composed of N subsystems with each subsystem represented in state-space form as

$$x_i(k+1) = \sum_{j=1}^{N} A_{ij}(k)x_j(k) + B_{ii}(k)u_i(k), \tag{1}$$

where $x_i(k) \in \mathbb{R}^{n_i}$ and $u_i(k) \in \mathbb{R}^{m_i}$ denote subsystem i, $1 \leq i \leq N$, state vector and control input, respectively.

Remark 6.1 *Linear systems with stochastically-varying parameters have been studied in many applications including power networks [19, 20], process control [45], finance [46], and networked control [33, 47]. Various system theoretic properties and control design methods have been developed for these systems [29–32].*

We make the following two standing assumptions:

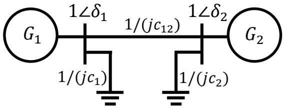

Figure 1: Schematic diagram of the power network in Example 6.1.

Assumption 6.1 *The submatrices $A_{ij}(k)$, $1 \leq i, j \leq N$, are independently distributed random variables in time; i.e., $\mathbb{P}\{A_{ij}(k_1) \in \mathcal{X} \mid A_{ij}(k_2)\} = \mathbb{P}\{A_{ij}(k_1) \in \mathcal{X}\}$ for any $\mathcal{X} \subseteq \mathbb{R}^{n_i \times n_j}$ whenever $k_1 \neq k_2$.*

Assumption 6.2 *The subsystems are statistically independent of each other; i.e., $\mathbb{P}\{A_{ij}(k) \in \mathcal{X} \mid A_{i'j'}(k)\} = \mathbb{P}\{A_{ij}(k) \in \mathcal{X}\}$ for any $\mathcal{X} \subseteq \mathbb{R}^{n_i \times n_j}$ and $1 \leq j, j' \leq N$ whenever $i \neq i'$.*

We illustrate these properties on a small power network example. We will frequently revisit this example to demonstrate the developed results as well as their implications.

Example 6.1 *Let us consider the power network composed of two generators shown in Figure 1 from [48, pp. 64–65], see also [17]. We can model this power network as*

$$\begin{aligned}
\dot{\delta}_1(t) &= \omega_1(t),\\
\dot{\omega}_1(t) &= \frac{1}{M_1}\big[P_1(t) - c_{12}^{-1}\sin(\delta_1(t) - \delta_2(t)) - c_1^{-1}\sin(\delta_1(t)) - D_1\omega_1(t)\big],\\
\dot{\delta}_2(t) &= \omega_2(t),\\
\dot{\omega}_2(t) &= \frac{1}{M_2}\big[P_2(t) - c_{12}^{-1}\sin(\delta_2(t) - \delta_1(t)) - c_2^{-1}\sin(\delta_2(t)) - D_2\omega_2(t)\big],
\end{aligned}$$

where $\delta_i(t)$, $\omega_i(t)$, and $P_i(t)$ are, respectively, the phase angle of the terminal voltage of generator i, its rotation frequency, and its input mechanical power. We assume that $P_1(t) = 1.6 + v_1(t)$ and $P_2(t) = 1.2 + v_2(t)$, where $v_1(t)$ and $v_2(t)$ are the continuous-time control inputs of this system. The power network parameters can be found in Table 6.1 (see [17, 48] and references therein for a discussion on these parameters). Now, we can find the equilibrium point (δ_1^*, δ_2^*) of this system and linearize it around this equilibrium. Let us discretize the linearized system by applying Euler's constant step scheme with sampling time $\Delta T = 300\,\text{ms}$, which results in

$$\begin{bmatrix}\Delta\delta_1(k+1)\\ \Delta\omega_1(k+1)\\ \Delta\delta_2(k+1)\\ \Delta\omega_2(k+1)\end{bmatrix} = \begin{bmatrix} 1 & \Delta T & 0 & 0\\ \xi_1 & 1-\frac{\Delta T D_1}{M_1} & \frac{\Delta T\cos(\delta_1^*-\delta_2^*)}{c_{12}M_1} & 0\\ 0 & 0 & 1 & \Delta T\\ \frac{\Delta T\cos(\delta_2^*-\delta_1^*)}{c_{12}M_2} & 0 & \xi_2 & 1-\frac{\Delta T D_2}{M_2}\end{bmatrix}\begin{bmatrix}\Delta\delta_1(k)\\ \Delta\omega_1(k)\\ \Delta\delta_2(k)\\ \Delta\omega_2(k)\end{bmatrix} + \begin{bmatrix}0\\ u_1(k)\\ 0\\ u_2(k)\end{bmatrix},$$

with $\xi_1 = -\Delta T(c_{12}^{-1}\cos(\delta_1^* - \delta_2^*) + c_1^{-1}\cos(\delta_1^*))/M_1$ and $\xi_2 = -\Delta T(c_{12}^{-1}\cos(\delta_2^* - \delta_1^*) + c_2^{-1}\cos(\delta_2^*))/M_2$, where $\Delta\delta_1(k)$, $\Delta\delta_2(k)$, $\Delta\omega_1(k)$, and $\Delta\omega_2(k)$ denote the deviation *of $\delta_1(t)$, $\delta_2(t)$, $\omega_1(t)$, and $\omega_2(t)$ from their equilibrium points at time instances $t = k\Delta T$. Additionally, let the actuators be equipped with a zero order hold unit which corresponds to $v_i(t) = u_i(k)$ for all $k\Delta T \leq t < (k+1)\Delta T$. Let us assume that we have connected impedance loads to each generator locally, such that the parameters c_1 and c_2 vary stochastically over time according to the load profiles. Furthermore, assume that each generator changes its input mechanical power according to these local load variations (to meet their demand and avoid power shortage). Doing so, we would not change the equilibrium point (δ_1^*, δ_2^*). For this setup, we can model the system as a discrete-time linear system with stochastically-varying parameters*

$$x(k+1) = A(k)x(k) + Bu(k),$$

where

$$x(k) = \begin{bmatrix} \Delta\delta_1(k) \\ \Delta\omega_1(k) \\ \Delta\delta_2(k) \\ \Delta\omega_2(k) \end{bmatrix}, \qquad u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$A(k) = \begin{bmatrix} 1.0000 & 0.3000 & 0 & 0 \\ -45.6923 - 6.9297\alpha_1(k) & 0.9250 & 29.3953 & 0 \\ 0 & 0 & 1.0000 & 0.3000 \\ 23.5163 & 0 & -37.3757 - 8.1485\alpha_2(k) & 0.9400 \end{bmatrix},$$

where $\alpha_i(k)$, $i = 1, 2$, denotes the deviation of the admittance c_i^{-1} from its nominal value in Table 6.1. Let us assume that $\alpha_1(k)$ and $\alpha_2(k)$ are independently and identically distributed random variables in time with $\alpha_1(k) \sim \mathcal{N}(0, 0.1)$ and $\alpha_2(k) \sim \mathcal{N}(0, 0.3)$. Note that in this example, $\alpha_i(k)$ is a stochastically-varying parameter of subsystem i describing the dynamics of the local power consumption. It only appears in the model of subsystem i; i.e., in $\{A_{ij}(k) | 1 \leq j \leq N\}$. In the rest of the paper when discussing this example and for designing controller i, we assume that we only have access to the exact realization of $\alpha_i(k)$ in addition to the statistics of the other subsystem. This is motivated by the fact that the controller of the other generator might not have access to this model information. $\triangleright$

We define the concatenated system from (1) as

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{2}$$

where $x(k) = [x_1(k)^\top \ \cdots \ x_N(k)^\top]^\top \in \mathbb{R}^n$ and $u(k) = [u_1(k)^\top \ \cdots \ u_N(k)^\top]^\top \in \mathbb{R}^m$, with $n = \sum_{i=1}^N n_i$ and $m = \sum_{i=1}^N m_i$. Let $x_0 = x(0)$. We also use the notations $\bar{A}_{ij}(k) = \mathbb{E}\{A_{ij}(k)\}$, $\tilde{A}_{ij}(k) = A_{ij}(k) - \bar{A}_{ij}(k)$, $\bar{A}(k) = \mathbb{E}\{A(k)\}$, and

Table 6.1: Nominal values of power system parameters in Example 6.1.

Parameters	Nominal Value (p.u.)
M_1	2.6×10^{-2}
M_2	3.2×10^{-2}
c_{12}	0.40
c_1	0.50
c_2	0.50
D_1	6.4×10^{-3}
D_2	6.4×10^{-3}

$\tilde{A}(k) = A(k) - \bar{A}(k)$. Furthermore, for all $1 \leq i \leq N$, we introduce the notations

$$B_i(k) = \begin{bmatrix} 0_{(\sum_{j=1}^{i-1} n_j)\times m_i} \\ B_{ii}(k) \\ 0_{(\sum_{j=i+1}^{N} n_j)\times m_i} \end{bmatrix}, \quad \tilde{A}_i(k) = \begin{bmatrix} 0_{(\sum_{j=1}^{i-1} n_j)\times n_1} & \cdots & 0_{(\sum_{j=1}^{i-1} n_j)\times n_N} \\ \tilde{A}_{i1}(k) & \cdots & \tilde{A}_{iN}(k) \\ 0_{(\sum_{j=i+1}^{N} n_j)\times n_1} & \cdots & 0_{(\sum_{j=i+1}^{N} n_j)\times n_N} \end{bmatrix}.$$

Now, we are ready to calculate the optimal controller under model information constraints.

3 Optimal Control Design with Limited Model Information

In this section, we study the finite-horizon and infinite-horizon optimal control design using exact local model information and statistical beliefs about other subsystems. We consider state-feedback control laws $u_i(k) = F_i(x(0), \ldots, x(k))$ where in the design of F_i only limited model information is available about the overall system (2). We formalize the notion of what model information is available in the design of controller i, $1 \leq i \leq N$, through the following definition.

Definition 6.1 *The design of controller i, $1 \leq i \leq N$, has limited model information if (*a*) the exact local realizations $\{A_{ij}(k) \mid 1 \leq j \leq N, \forall k\}$ are available together with (*b*) the first- and the second-order moments of the system parameters (i.e., $\mathbb{E}\{A(k)\}$ and $\mathbb{E}\{\tilde{A}(k) \otimes \tilde{A}(k)\}$ for all k).*

Remark 6.2 *Note that the assumption that the exact realizations $\{A_{ij}(k) \mid 1 \leq j \leq N\}$ are available to designer of controller i (and not the rest of the submatrices) is reasonable in the context of interconnected systems where the coupling strengths are known (stochastically-varying or not) and the uncertainties are arising in each subsystem independently. For instance, such systems occur naturally when studying power network control since the power grid, which determines the coupling strengths between the generators and the consumers, is typically accurately modeled, however, the loads and the generators are stochastically varying and uncertain. A direction for future research could be to consider the case where also the coupling strengths are uncertain.*

3.1 Finite-Horizon Cost Function

In the finite-horizon optimal control design problem, for a fixed $T > 0$, we minimize the cost function

$$J_T(x_0, \{u(k)\}_{k=0}^{T-1}) = \mathbb{E}\Bigg\{ x(T)^\top Q(T) x(T) + \sum_{k=0}^{T-1} \left(x(k)^\top Q(k) x(k) + \sum_{j=1}^{N} u_j(k)^\top R_{jj}(k) u_j(k) \right) \Bigg\}, \quad (3)$$

subject to the system dynamics in (2) and the model information constraints in Definition 6.1. In (3), we assume that $Q(k) \in \mathcal{S}_+^n$ for all $0 \leq k \leq T$ and $R(k) = \mathrm{diag}(R_{11}(k), \dots, R_{NN}(k)) \in \mathcal{S}_{++}^m$ for all $0 \leq k \leq T-1$. The following theorem presents the solution of the finite-horizon optimal control problem.

Theorem 6.1 *The solution of the finite-horizon optimal control design problem with limited model information is given by*

$$u(k) = -(R(k) + B(k)^\top P(k+1) B(k))^{-1} B(k)^\top P(k+1) \bar{A}(k) x(k) - \begin{bmatrix} (R_{11}(k) + B_1(k)^\top P(k+1) B_1(k))^{-1} B_1(k)^\top P(k+1) \tilde{A}_1(k) \\ \vdots \\ (R_{NN}(k) + B_N(k)^\top P(k+1) B_N(k))^{-1} B_N(k)^\top P(k+1) \tilde{A}_N(k) \end{bmatrix} x(k), \quad (4)$$

where the sequence of matrices $\{P(k)\}_{k=0}^{T}$ can be calculated using the backward difference equation

$$P(k) = Q(k) + \mathbf{R}(\bar{A}(k), P(k+1), B(k), R) + \sum_{i=1}^{N} \mathbb{E}\left\{ \mathbf{R}(\tilde{A}_i(k), P(k+1), B_i(k), R_{ii}) \right\}, \quad (5)$$

with the boundary condition $P(T) = Q(T)$. Furthermore, $\inf_{\{u(k)\}_{k=0}^{T-1}} J_T(x_0, \{u(k)\}_{k=0}^{T-1}) = x_0^\top P(0) x_0$.

Proof: We solve the finite-horizon optimal control problem using dynamic programming

$$V_k(x(k)) = \inf_{u(k)} \mathbb{E}\Bigg\{ x(k)^\top Q(k) x(k) + u(k)^\top R(k) u(k) + V_{k+1}(A(k)x(k) + B(k)u(k)) \big| x(k) \Bigg\}, \quad (6)$$

where $V_T(x(T)) = x(T)^\top Q(T) x(T)$. The proof strategy is to (a) show $V_k(x(k)) = x(k)^\top P(k) x(k)$ for all k using backward induction, (b) find a lower bound for

$\mathbb{E}\{x(k)^\top Q(k)x(k) + u(k)^\top R(k)u(k) + V_{k+1}(A(k)x(k) + B(k)u(k))\big|x(k)\}$ which is attained by $u(k)$ in (4), and (c) using optimal controller calculate a recursive equation for $P(k)$, $0 \leq k \leq T$, starting from $P(T) = Q(T)$. Note that because of Definition 6.1, in each step of the dynamic programming, the infimum is taken over the set of all control signals $u(k)$ of the form

$$\begin{bmatrix} u_1(k) \\ \vdots \\ u_N(k) \end{bmatrix} = \begin{bmatrix} \psi_1(A_{11}(k),\dots,A_{1N}(k);x(0),\cdots,x(k)) \\ \vdots \\ \psi_N(A_{N1}(k),\dots,A_{NN}(k);x(0),\cdots,x(k)) \end{bmatrix}, \tag{7}$$

where $\psi_i : \mathbb{R}^{n_i\times n_1}\times\cdots\times\mathbb{R}^{n_i\times n_N}\times\mathbb{R}^n \to \mathbb{R}^{m_i}$, $1 \leq i \leq N$, can be any mapping (i.e., it is not necessarily a linear mapping, a smooth one, etc). Let us assume, for all k, that $V_k(x(k)) = x(k)^\top P(k)x(k)$ where $P(k) \in \mathcal{S}_+^n$. This is without loss of generality since $V_T(x(T)) = x(T)^\top Q(T)x(T)$ is a quadratic function of the state vector $x(T)$ and using dynamic programming, $V_k(x(k))$ remains a quadratic function of $x(k)$ if $V_{k+1}(x(k+1))$ is a quadratic function of $x(k+1)$ and $u(k)$ is a linear function of $x(k)$. This can be easily proved using mathematical induction. For the control input of the form in (7), we define

$$\bar{G}(k) = \begin{bmatrix} \mathbb{E}\left\{\psi_1(A_{11}(k),\dots,A_{1N}(k);x(0),\cdots,x(k))|x(k)\right\} \\ \vdots \\ \mathbb{E}\left\{\psi_N(A_{N1}(k),\dots,A_{NN}(k);x(0),\cdots,x(k))|x(k)\right\} \end{bmatrix} - \bar{K}(k)x(k),$$

and

$$\begin{aligned}\tilde{g}_i(k) =& \psi_i(A_{i1}(k),\dots,A_{iN}(k);x(0),\cdots,x(k)) \\ & - \mathbb{E}\{\psi_i(A_{i1}(k),\dots,A_{iN}(k);x(0),\cdots,x(k))|x(k)\} - \tilde{K}_i(k)x(k),\end{aligned}$$

where $\bar{K}(k) = -(R(k) + B(k)^\top P(k+1)B(k))^{-1}B(k)^\top P(k+1)\bar{A}(k)$ and $\tilde{K}_i(k) = -(R_{ii}(k) + B_i(k)^\top P(k+1)B_i(k))^{-1}B_i(k)^\top P(k+1)\tilde{A}_i(k)$ are the gains in (4). By definition, we have $\mathbb{E}\{\tilde{g}_i(k)|x(k)\} = 0$. Furthermore, let us define the notation

$$C_i = \begin{bmatrix} 0_{(\sum_{j=1}^{i-1} m_j)\times m_i} \\ I \\ 0_{(\sum_{j=i+1}^{N} m_j)\times m_i} \end{bmatrix},$$

for all $1 \leq i \leq N$. Evidently, we have

$$\begin{aligned}
&\begin{bmatrix} \psi_1(A_{11}(k), \ldots, A_{1N}(k); x(0), \cdots, x(k)) \\ \vdots \\ \psi_N(A_{N1}(k), \ldots, A_{NN}(k); x(0), \cdots, x(k)) \end{bmatrix} \\
&\qquad = \bar{G}(k) + \begin{bmatrix} \tilde{g}_1(k) \\ \vdots \\ \tilde{g}_N(k) \end{bmatrix} + \bar{K}(k)x(k) + \begin{bmatrix} \tilde{K}_1(k)x(k) \\ \vdots \\ \tilde{K}_N(k)x(k) \end{bmatrix} \\
&\qquad = \bar{G}(k) + \bar{K}(k)x(k) + \sum_{i=1}^{N} C_i \tilde{g}_i(k) + \sum_{i=1}^{N} C_i \tilde{K}_i(k)x(k).
\end{aligned}$$

By rearranging the terms, we can easily show that

$$\begin{aligned}
&\mathbb{E}\{u(k)^\top R(k)u(k)|x(k)\} \\
&=\mathbb{E}\Bigg\{(\bar{K}(k)x(k) + \bar{G}(k))^\top R(k)(\bar{K}(k)x(k) + \bar{G}(k)) \\
&\quad + (\bar{K}(k)x(k) + \bar{G}(k))^\top R(k) \left(\sum_{i=1}^{N} C_i(\tilde{g}_i(k) + \tilde{K}_i(k)x(k))\right) \\
&\quad + \left(\sum_{i=1}^{N} C_i(\tilde{g}_i(k) + \tilde{K}_i(k)x(k))\right)^\top R(k)(\bar{K}(k)x(k) + \bar{G}(k)) \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{g}_i(k) + \tilde{K}_i(k)x(k))^\top C_i^\top R(k)C_j(\tilde{g}_j(k) + \tilde{K}_j(k)x(k))\Big|x(k)\Bigg\} \\
&=(\bar{K}(k)x(k) + \bar{G}(k))^\top R(k)(\bar{K}(k)x(k) + \bar{G}(k)) \\
&\quad + \sum_{i=1}^{N} \mathbb{E}\Big\{(\tilde{g}_i(k) + \tilde{K}_i(k)x(k))^\top R_{ii}(k)(\tilde{g}_i(k) + \tilde{K}_i(k)x(k))\Big|x(k)\Big\},
\end{aligned} \tag{8}$$

where the second equality holds due to that $\mathbb{E}\{\tilde{g}_i(k) + \tilde{K}_i(k)x(k)|x(k)\} = 0$ and $C_i^\top RC_j = R_{ij}$ (while recalling that $R_{ij} = 0$ if $i \neq j$). Following the same line of reasoning, we show that

$$
\begin{aligned}
\mathbb{E}\big\{(A(k)x(k)+&B(k)u(k))^\top P(k+1)(A(k)x(k)+B(k)u(k))\big|x(k)\big\} \\
=\mathbb{E}\Bigg\{ &\left(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\right)^\top P(k+1) \\
&\times \left(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\right) \\
&+\left(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\right)^\top P(k+1) \\
&\times \left(\sum_{i=1}^N \tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\right) \\
&+\left(\sum_{i=1}^N \tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\right)^\top P(k+1) \\
&\times \left(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\right) \\
&+\sum_{i=1}^N\sum_{j=1}^N\left(\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\right)^\top P(k+1) \\
&\times \left(\tilde{A}_j(k)x(k)+B_j(k)(\tilde{g}_j(k)+\tilde{K}_j(k)x(k))\right)\big|x(k)\Bigg\},
\end{aligned}
$$

where the equality follows from

$$
\begin{aligned}
A(k)x(k)+B(k)u(k)=&\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k)) \\
&+\sum_{i=1}^N \tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k)).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\mathbb{E}\big\{(A(k)x(k)&+B(k)u(k))^\top P(k+1)(A(k)x(k)+B(k)u(k))\big|x(k)\big\} \\
&=\left(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\right)^\top P(k+1) \\
&\quad\times \left(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\right) \\
&\quad+\sum_{i=1}^N \mathbb{E}\Bigg\{ \left(\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\right)^\top P(k+1) \\
&\quad\times \left(\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\right)\big|x(k)\Bigg\},
\end{aligned}
\tag{9}
$$

because random variables $\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))$ and $\tilde{A}_j(k)x(k)+B_j(k)(\tilde{g}_j(k)+\tilde{K}_j(k)x(k))$ are independent for $i\neq j$ (see Assumption 6.1 and Definition 6.1) and $\mathbb{E}\{\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))|x(k)\}=0$ for all $1\leq i\leq N$.

Now, note that

$$
\begin{aligned}
&\big(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\big)^\top P(k+1)\big(\bar{A}(k)x(k)+B(k)(\bar{K}(k)x(k)+\bar{G}(k))\big)\\
&\quad + (\bar{K}(k)x(k)+\bar{G}(k))^\top R(k)(\bar{K}(k)x(k)+\bar{G}(k))\\
&= x(k)^\top \bar{K}(k)^\top R(k)\bar{K}(k)x(k)\\
&\quad + x(k)^\top(\bar{A}(k)+B(k)\bar{K}(k))^\top P(k+1)(\bar{A}(k)+B(k)\bar{K}(k))x(k)\\
&\quad + \bar{G}(k)^\top \left(B(k)^\top P(k+1)(\bar{A}(k)+B(k)\bar{K}(k))+R(k)\bar{K}(k)\right)x(k)\\
&\quad + x(k)^\top \left(B(k)^\top P(k+1)(\bar{A}(k)+B(k)\bar{K}(k))+R(k)\bar{K}(k)\right)^\top \bar{G}(k)\\
&\quad + \bar{G}(k)^\top R(k)\bar{G}(k)+\bar{G}(k)^\top B(k)^\top P(k+1)B(k)\bar{G}(k)\\
&= x(k)^\top \bar{K}(k)^\top R(k)\bar{K}(k)x(k)\\
&\quad + x(k)^\top(\bar{A}(k)+B(k)\bar{K}(k))^\top P(k+1)(\bar{A}(k)+B(k)\bar{K}(k))x(k)\\
&\quad + \bar{G}(k)^\top R(k)\bar{G}(k)+\bar{G}(k)^\top B(k)^\top P(k+1)B(k)\bar{G}(k)\\
&\geq x(k)^\top \bar{K}(k)^\top R(k)\bar{K}(k)x(k)\\
&\quad + x(k)^\top(\bar{A}(k)+B(k)\bar{K}(k))^\top P(k+1)(\bar{A}(k)+B(k)\bar{K}(k))x(k),
\end{aligned}
\tag{10}
$$

where the second equality follows from that $B(k)^\top P(k+1)(\bar{A}(k)+B(k)\bar{K}(k))+R(k)\bar{K}(k)=0$ using the definition of $\bar{K}(k)$ and the inequality holds due to that $\bar{G}(k)^\top(R(k)+B(k)^\top P(k+1)B(k))\bar{G}(k)\geq 0$ for any $\bar{G}(k)\in\mathbb{R}^m$ since $R(k)+B(k)^\top P(k+1)B(k)$ is a positive-definite matrix. Similarly, for each $1\leq i\leq N$, we conclude that

$$
\begin{aligned}
&\mathbb{E}\big\{(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))^\top R_{ii}(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\\
&\qquad + \big(\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\big)^\top P(k+1)\\
&\qquad\quad \times \big(\tilde{A}_i(k)x(k)+B_i(k)(\tilde{g}_i(k)+\tilde{K}_i(k)x(k))\big)\,\big|x(k)\big\}\\
&\quad = \mathbb{E}\big\{\tilde{g}_i(k)^\top R_{ii}(k)\tilde{g}_i(k)+x(k)^\top \tilde{K}_i(k)^\top R_{ii}(k)\tilde{K}_i(k)x(k)\\
&\qquad + \tilde{g}_i(k)^\top B_i(k)^\top P(k+1)B_i(k)\tilde{g}_i(k)\\
&\qquad + \tilde{g}_i(k)^\top\big(B_i(k)^\top P(k+1)(\tilde{A}_i(k)+B_i(k)\tilde{K}_i(k))+R_{ii}(k)\tilde{K}_i(k)\big)x(k)\\
&\qquad + x(k)^\top\big(B_i(k)^\top P(k+1)(\tilde{A}_i(k)+B_i(k)\tilde{K}_i(k))+R_{ii}(k)\tilde{K}_i(k)\big)^\top \tilde{g}_i(k)\\
&\qquad + x(k)^\top(\tilde{A}_i(k)+B_i(k)\tilde{K}_i(k))^\top P(k+1)(\tilde{A}_i(k)+B_i(k)\tilde{K}_i(k))x(k)\big|x(k)\big\}\\
&\quad \geq \mathbb{E}\big\{x(k)^\top \tilde{K}_i(k)^\top R_{ii}(k)\tilde{K}_i(k)x(k)\\
&\qquad + x(k)^\top(\tilde{A}_i(k)+B_i(k)\tilde{K}_i(k))^\top P(k+1)(\tilde{A}_i(k)+B_i(k)\tilde{K}_i(k))x(k)\big|x(k)\big\}.
\end{aligned}
\tag{11}
$$

Combining identities (8)–(9) with inequalities (10)–(11) results in

$$\begin{aligned}
&\mathbb{E}\big\{x(k)^\top Q(k)x(k) + u(k)^\top R(k)u(k) \\
&\qquad + (A(k)x(k) + B(k)u(k))^\top P(k+1)(A(k)x(k) + B(k)u(k))\big|x(k)\big\} \\
&\quad \geq x(k)^\top Q(k)x(k) + x(k)^\top \bar{K}(k)^\top R(k)\bar{K}(k)x(k) \\
&\qquad + x(k)^\top(\bar{A}(k) + B(k)\bar{K}(k))^\top P(k+1)(\bar{A}(k) + B(k)\bar{K}(k))x(k) \\
&\qquad + \sum_{i=1}^{N} \mathbb{E}\big\{x(k)^\top \tilde{K}_i(k)^\top R_{ii}(k)\tilde{K}_i(k)x(k)\big|x(k)\big\} \\
&\qquad + \sum_{i=1}^{N} \mathbb{E}\big\{x(k)^\top(\tilde{A}_i(k) + B_i(k)\tilde{K}_i(k))^\top B_i(k)^\top P(k+1) \\
&\qquad\qquad \times (\tilde{A}_i(k) + B_i(k)\tilde{K}_i(k))x(k)\big|x(k)\big\} \\
&\quad = \mathbb{E}\big\{x(k)^\top Q(k)x(k) + u^*(k)^\top R(k)u^*(k)\big|x(k)\big\} \\
&\qquad + \mathbb{E}\big\{(A(k)x(k) + B(k)u^*(k))^\top P(k+1)(A(k)x(k) + B(k)u^*(k))\big|x(k)\big\},
\end{aligned}$$

where

$$u^*(k) = \bar{K}(k)x(k) + \sum_{i=1}^{N} C_i\tilde{K}_i(k)x(k).$$

This inequality proves that $u^*(k)$ is the solution of (6) since any other controller results in a larger or equal cost. By substituting this optimal controller inside the recursion (6), we get the cost function update equation

$$\begin{aligned}
&x(k)^\top P(k)x(k) \\
&\quad = x(k)^\top Q(k)x(k) + x(k)^\top \bar{K}(k)^\top R(k)\bar{K}(k)x(k) \\
&\qquad + x(k)^\top(\bar{A}(k) + B(k)\bar{K}(k))^\top P(k+1)(\bar{A}(k) + B(k)\bar{K}(k))x(k) \\
&\qquad + \sum_{i=1}^{N} x(k)^\top \mathbb{E}\left\{\tilde{K}_i(k)^\top R_{ii}(k)\tilde{K}_i(k)\right\} x(k) \\
&\qquad + \sum_{i=1}^{N} x(k)^\top \mathbb{E}\left\{(\tilde{A}_i(k) + B_i(k)\tilde{K}_i(k))^\top P(k+1)(\tilde{A}_i(k) + B_i(k)\tilde{K}_i(k))\right\} x(k),
\end{aligned} \tag{12}$$

By expanding and reordering the terms, we can simplify this equation as

$$\begin{aligned}
x(k)^\top P(k)x(k) = {} & x(k)^\top Q(k)x(k) + x(k)^\top \mathbf{R}(\bar{A}(k), P(k+1), B(k), R)x(k) \\
& + \sum_{i=1}^{N} x(k)^\top \mathbb{E}\left\{\mathbf{R}(\tilde{A}_i(k), P(k+1), B_i(k), R_{ii})\right\} x(k).
\end{aligned} \tag{13}$$

Now, since the equality in (13) is true irrespective of the value of the state vector $x(k)$, we get the recurrence relation in (5). This concludes the proof. ∎

Remark 6.3 *Theorem 6.1 shows that the optimal controller (4) is a linear state-feedback controller and that it is composed of two parts. The first part is a function of only the parameter statistics (i.e.,* $\mathbb{E}\{A(k)\}$ *and* $\mathbb{E}\{\tilde{A}(k)\otimes\tilde{A}(k)\}$*) while the second part is a function of exact local model parameters (i.e.,* $\{A_{ij}(k) \mid 1 \leq j \leq N\}$ *for controller* i*). Note that the optimal controller does not assume any specific probability distribution for the model parameters. It is worth mentioning whenever* $n \gg 1$*, for computing the optimal controller, we need to perform arithmetic operations on very large matrices (since* $\mathbb{E}\{A(k)\} \in \mathbb{R}^{n\times n}$ *and* $\mathbb{E}\{\tilde{A}(k) \otimes \tilde{A}(k)\} \in \mathbb{R}^{n^2\times n^2}$*) which might be numerically difficult (except for special cases where the statistics of the underlying system follows a specific structure or sparsity pattern).*

Remark 6.4 *Note that the optimal controller in Theorem 6.1 is not structured in terms of the state measurement availability, i.e., controller* i *accesses the full state measurement* $x(k)$*. This situation can be motivated for many applications by the rise of fast communication networks that can guarantee the availability of full state measurements in moderately large systems. However, in many scenarios, the model information is simply not available due the fact that each module is being designed separately for commercial purposes without any specific information about its future setup (except the average behavior of other components). A viable direction for future research is to optimize the cost function over the set of structured control laws.*

Remark 6.5 *It might seem computationally difficult to calculate* $\mathbb{E}\{\tilde{A}_i(k)^\top Z\tilde{A}_i(k)\}$ *for each time-step* k *and any given matrix* Z*. However, as pointed out in [29], it suffices to calculate* $\mathbb{E}\{\tilde{A}_i(k) \otimes \tilde{A}_i(k)\}$ *once, and then use the identity*

$$\begin{aligned}\mathbb{E}\{\tilde{A}_i(k)^\top Z(k)\tilde{A}_i(k)\} &= vec^{-1}\left(\mathbb{E}\left\{\left(\tilde{A}_i(k)\otimes\tilde{A}_i(k)\right)^\top vec\left(Z(k)\right)\right\}\right)\\ &= vec^{-1}\left(\mathbb{E}\left\{\tilde{A}_i(k)\otimes\tilde{A}_i(k)\right\}^\top vec\left(Z(k)\right)\right).\end{aligned}$$

3.2 Infinite-Horizon Cost Function

In this subsection, we use Theorem 6.1 to minimize the infinite-horizon performance criterion

$$J_\infty(x_0, \{u(k)\}_{k=0}^{\infty}) = \lim_{T\to\infty} J_T(x_0, \{u(k)\}_{k=0}^{T-1}),$$

where $Q(k) = Q \in \mathcal{S}_{++}^n$ and $R(k) = R \in \mathcal{S}_{++}^m$ for all $0 \leq k \leq T-1$ and $Q(T) = 0$. For this case, we make the following standing assumption concerning the system parameters statistics:

Assumption 6.3 *For all time steps* k*, the stochastic processes generating the model parameters of the system in (2) satisfy*

- $\bar{A}(k) = \bar{A} \in \mathbb{R}^{n\times n}$ *and* $\mathbb{E}\{A(k) \otimes A(k)\} = \Sigma \in \mathbb{R}^{n^2\times n^2}$*;*
- $B(k) = B \in \mathbb{R}^{n\times m}$*.*

These assumptions are in place to make sure that we are dealing with stationary parameter processes, as otherwise the infinite-horizon optimal control problem could lack physical meaning. We borrow the following technical definition and assumption from [29]. We refer interested readers to [29] for numerical methods for checking this condition.

Definition 6.2 *System (2) is called mean square stabilizable if there exists a matrix $L \in \mathbb{R}^{m\times n}$ such that the closed-loop system with controller $u(k) = Lx(k)$ is mean square stable; i.e., $\lim_{k\to+\infty} \mathbb{E}\{x(k)^\top x(k)\} = 0$.*

With this definition in hand, we are ready to present the solution of the infinite-horizon optimal control design problem with limited model information.

Theorem 6.2 *Suppose (2) satisfies Assumption 6.3 and is mean square stabilizable. The solution of the infinite-horizon optimal control design problem with limited model information is then given by*

$$\begin{aligned} u(k) = &-(R + B^\top PB)^{-1}B^\top P\bar{A}x(k) \\ &- \begin{bmatrix} (R_{11} + B_1^\top PB_1)^{-1}B_1^\top P\tilde{A}_1(k) \\ \vdots \\ (R_{NN} + B_N^\top PB_N)^{-1}B_N^\top P\tilde{A}_N(k) \end{bmatrix} x(k), \end{aligned} \tag{14}$$

where P is the unique positive-definite solution of the modified discrete algebraic Riccati equation

$$P = Q + \mathbf{R}(\bar{A}, P, B, R) + \sum_{i=1}^{N} \mathbb{E}\left\{\mathbf{R}(\tilde{A}_i(k), P, B_i, R_{ii})\right\}. \tag{15}$$

Furthermore, the closed-loop system (2) and (14) is mean square stable and

$$\inf_{\{u(k)\}_{k=0}^\infty} J_\infty(x_0, \{u(k)\}_{k=0}^\infty) = x_0^\top Px_0.$$

Proof: Note that the proof of this theorem follows the same line of reasoning as in [29]. We extend the result of [29] to hold for the Riccati-like backward difference equation presented in (5). First, let us define the mapping $f : \mathcal{S}_+^n \to \mathcal{S}_+^n$ such that, for any $X \in \mathcal{S}_+^n$,

$$\begin{aligned} f(X) = Q + \bar{A}^\top &\left(X - XB(R + B^\top XB)^{-1}B^\top X\right)\bar{A} \\ &+ \sum_{i=1}^{N} \mathbb{E}\left\{\tilde{A}_i^\top \left(X - XB_i(R_{ii} + B_i^\top XB_i)^{-1}B_i^\top X\right)\tilde{A}_i\right\}. \end{aligned}$$

Using part 2 of Subsection 3.5.2 in [49], we have the matrix inversion identity

$$X - XW(Z + W^\top XW)^{-1}W^\top X = (X^{-1} + WZ^{-1}W^\top)^{-1},$$

for any matrix W and positive-definite matrices X and Z. Therefore, for any $X \in \mathcal{S}_{++}^n$, we have

$$f(X) = Q + \bar{A}^\top (X^{-1} + BR^{-1}B^\top)^{-1}\bar{A} + \sum_{i=1}^{N} \mathbb{E}\left\{\tilde{A}_i^\top (X^{-1} + B_i R_{ii}^{-1} B_i^\top)^{-1}\tilde{A}_i\right\}. \tag{16}$$

Note that, if $X \geq Y \geq 0$, then

$$(X^{-1} + WZ^{-1}W^\top)^{-1} \geq (Y^{-1} + WZ^{-1}W^\top)^{-1},$$

for any matrix W and positive-definite matrix Z. Therefore, if $X \geq Y \geq 0$, we get

$$f(X) \geq f(Y) > 0.$$

For any given $T \geq 0$, we define the sequence of matrices $\{X_i\}_{i=0}^T$ such that $X_0 = 0$ and $X_{i+1} = f(X_i)$. We have

$$X_1 = f(X_0) = f(0) = Q > 0 = X_0.$$

Similarly,

$$X_2 = f(X_1) \geq f(X_0) = X_1 > 0. \tag{17}$$

The left-most inequality in (17) is true because $X_1 \geq X_0$. We can repeat the same argument, and show that for all $1 \leq i \leq T-1$, $X_{i+1} \geq X_i > 0$. Using Theorem 6.1, we know that

$$x_0^\top X_T x_0 = \inf_{\{u(k)\}_{k=0}^{T-1}} J_T(x_0, \{u(k)\}_{k=0}^{T-1}).$$

According to Theorem 5.1 in [29] (using the assumption that the underlying system is mean square stabilizable), the sequence $\{X_i\}_{i=0}^\infty$ is uniformly upper-bounded; i.e., there exists $W \in \mathbb{R}^{n\times n}$ such that $X_i \leq W$ for all $i \geq 0$. Therefore, we get

$$\lim_{T\to+\infty} X_T = X \in \mathbb{R}^{n\times n} \tag{18}$$

since $\{X_i\}_{i=0}^\infty$ is an increasing and bounded sequence. In addition, we have $X \in \mathcal{S}_{++}^n$ since $X_i \in \mathcal{S}_{++}^n$ for all $i \geq 2$ and $\{X_i\}_{i=0}^\infty$ is an increasing sequence. Now, we need to prove that the limit X in (18) is the unique positive definite solution of the modified discrete algebraic Riccati equation (15). This is done by a contrapositive argument. Assume that there exists $Z \in \mathcal{S}_+^n$ such that $f(Z) = Z$. For this matrix Z, we have

$$Z = f(Z) \geq f(0) = X_1$$

since $Z \geq 0$. Similarly, noting that $Z \geq X_1$, we get

$$Z = f(Z) \geq f(X_1) = X_2.$$

Repeating the same argument, we get $Z \geq X_i$ for all $i \geq 0$. Therefore, for each $T > 0$, we have the inequality

$$\begin{aligned}
&\inf_{\{u(k)\}_{k=0}^{T-1}} J_T(x_0, \{u(k)\}_{k=0}^{T-1}) \\
&\qquad = x_0^\top X_T x_0 \\
&\qquad \leq x_0^\top Z x_0 \\
&\qquad = \inf_{\{u(k)\}_{k=0}^{T-1}} \mathbb{E}\left\{ x(T)^\top Z x(T) + \sum_{k=0}^{T-1} x(k)^\top Q x(k) + u(k)^\top R u(k) \right\}.
\end{aligned} \tag{19}$$

Note that the last equality in (19) is a direct consequence of Theorem 6.1 and the fact that $Z = f^q(Z)$ for any positive $q \in \mathbb{Z}$. Let us define $\{u^*(k)\}_{k=0}^{T-1} = \arg\inf_{\{u(k)\}_{k=0}^{T-1}} J_T(x_0, \{u(k)\}_{k=0}^{T-1})$, and $x^*(k)$ as the state of the system when the control sequence $u^*(k)$ is applied. Now, we get the inequality

$$\begin{aligned}
&\inf_{\{u(k)\}_{k=0}^{T-1}} \mathbb{E}\left\{ x(T)^\top Z x(T) + \sum_{k=0}^{T-1} x(k)^\top Q x(k) + u(k)^\top R u(k) \right\} \\
&\qquad \leq \mathbb{E}\left\{ x^*(T)^\top Z x^*(T) + \sum_{k=0}^{T-1} x^*(k)^\top Q x^*(k) + u^*(k)^\top R u^*(k) \right\},
\end{aligned} \tag{20}$$

since, by definition, $\{u^*(k)\}_{k=0}^{T-1}$ is not the minimizer of this cost function. It is easy to see that the right-hand side of (20) is equal to $J_T(x_0, \{u^*(k)\}_{k=0}^{T-1}) + \mathbb{E}\left\{x^*(T)^\top Z x^*(T)\right\}$. Thus, using (19) and (20), we get

$$\begin{aligned}
x_0^\top X_T x_0 &\leq x_0^\top Z x_0 \\
&\leq J_T(x_0, \{u^*(k)\}_{k=0}^{T-1}) + \mathbb{E}\left\{x^*(T)^\top Z x^*(T)\right\} \\
&= x_0^\top X_T x_0 + \mathbb{E}\left\{x^*(T)^\top Z x^*(T)\right\}.
\end{aligned} \tag{21}$$

Finally, thanks to the facts that $Q > 0$ and

$$\lim_{T \to +\infty} \mathbb{E}\left\{ \sum_{k=0}^{T-1} x^*(k)^\top Q x^*(k) + u^*(k)^\top R u^*(k) \right\} = \lim_{T \to +\infty} x_0^\top X_T x_0 = x_0^\top X x_0 < \infty,$$

we get that $\lim_{T\to\infty} \mathbb{E}\left\{x^*(T)^\top x^*(T)\right\} = 0$. Therefore, we have $\lim_{T\to\infty} \mathbb{E}\left\{x^*(T)^\top Z x^*(T)\right\} = 0$. Letting T go to infinity in (21), results in $x_0^\top X x_0 = x_0^\top Z x_0$ for all $x_0 \in \mathbb{R}^n$. Thus, $X = Z$. This concludes the proof. ∎

Remark 6.6 *Note that we can use the procedure introduced in the proof of Theorem 6.2 to numerically compute the unique positive-definite solution of the modified discrete algebraic Riccati equation in (15); i.e., we can construct a sequence of matrices $\{X_i\}_{i=0}^{\infty}$, such that $X_{i+1} = f(X_i)$ with $X_0 = 0$ where $f(\cdot)$ is defined as in (16).*

Because of (18), it is evident that, for each $\delta > 0$, there exists a positive integer $q(\delta)$ such that $X_{q(\delta)}$ is in the δ-neighborhood of the unique positive-definite solution of the modified discrete algebraic Riccati equation (15). Hence, the procedure generates a solution with any desired precision.

Note that Definition 6.2 requires the existence of a fixed feedback gain L that ensures the closed-loop mean square stability. This might result in conservative results. In what follows, we relax this assumption to time-varying matrices.

Definition 6.3 *System (2) is called mean square stabilizable under limited model information if there exist mappings $\Gamma_i : \mathbb{R}^{n_i \times n_1} \times \cdots \times \mathbb{R}^{n_i \times n_N} \to \mathbb{R}^{m_i \times n}$, $1 \le i \le N$, such that the closed-loop system with controller*

$$u(k) = \begin{bmatrix} \Gamma_1(A_{11}(k), \dots, A_{1N}(k)) \\ \vdots \\ \Gamma_N(A_{N1}(k), \dots, A_{NN}(k)) \end{bmatrix} x(k),$$

is mean square stable.

Clearly, if a discrete-time linear system with stochastically-varying parameters is mean square stabilizable, it is also mean square stabilizable under limited model information.

Remark 6.7 *All fully-actuated systems (i.e., systems where $m_i = n_i$ for all $1 \le i \le N$) are mean square stabilizable under limited model information because, for each $1 \le i \le N$, the deadbeat controller*

$$\Gamma_i(A_{i1}(k), \dots, A_{iN}(k)) = -B_{ii}^{-1}[A_{i1}(k) \ \cdots \ A_{iN}(k)],$$

is based on limited model information and mean square stabilizes the system.

As a price of relaxing this assumption to time-varying matrices, we need to strengthen Assumption 6.3.

Assumption 6.4 *The stochastic processes generating the model parameters of system (2) satisfy that*

- *The probability distribution of the matrices $\{A(k)\}_{k=0}^{\infty}$ is constant in time;*
- *$B(k) = B \in \mathbb{R}^{n \times m}$ for all $k \ge 0$.*

Note that in Assumption 6.3 we only needed the first and the second moments of the system parameters to be constant. However, in Assumption 6.4 all the moments are constant.

Corollary 6.3 *Suppose (2) satisfies Assumption 6.4 and is mean square stabilizable under limited model information. The solution of the infinite-horizon optimal control design problem with limited model information is then given by (14) where P is the unique finite positive-definite solution of the modified discrete algebraic Riccati equation in (15). Furthermore, the closed-loop system (2) and (14) is mean square stable and* $\inf_{\{u(k)\}_{k=0}^{\infty}} J_\infty(x_0, \{u(k)\}_{k=0}^{\infty}) = x_0^\top P x_0$.

Proof: The only place in the proof of Theorem 6.2 where we used the assumption that the underlying system is mean square stabilizable, was to show that the sequence $\{X_i\}_{i=0}^{\infty}$ is upper bounded; i.e., there exists $W \in \mathcal{S}_+^n$ such that $X_i \leq W$ for all $i \geq 0$. We just need to prove this fact considering the assumption that the system is mean square stabilizable under limited model information. Note that for any $T > 0$, we have

$$\begin{aligned}\inf_{\{u(k)\}_{k=0}^{T-1}} J_T(x_0, \{u(k)\}_{k=0}^{T-1}) &= x_0^\top X_T x_0 \\ &\leq \mathbb{E}\left\{\sum_{k=0}^{T-1} x(k)^\top Q x(k) + \bar{u}(k)^\top R \bar{u}(k)\right\},\end{aligned} \tag{22}$$

where $x(k)$ is the system state when it is initialized at $x(0) = x_0$ and the control law $\bar{u}(k) = \Gamma(k)x(k)$ is in effect with

$$\Gamma(k) = \begin{bmatrix} \Gamma_1(A_{11}(k), \dots, A_{1N}(k)) \\ \vdots \\ \Gamma_N(A_{N1}(k), \dots, A_{NN}(k)) \end{bmatrix}$$

that satisfies the condition of Definition 6.3. Note that, at each time step k, $\Gamma(k)$ is independent of $x(k)$ because of Assumption 6.1. Therefore, we have

$$\begin{aligned}\mathbb{E}\left\{\sum_{k=0}^{T-1} x(k)^\top Q x(k) + \bar{u}(k)^\top R \bar{u}(k)\right\} &= \mathbb{E}\left\{\sum_{k=0}^{T-1} x(k)^\top (Q + \Gamma(k)^\top R \Gamma(k)) x(k)\right\} \\ &= \mathbb{E}\left\{\sum_{k=0}^{T-1} x(k)^\top (Q + \mathbb{E}\{\Gamma(k)^\top R \Gamma(k)\}) x(k)\right\}.\end{aligned}$$

Furthermore, we can see that $\mathbb{E}\{\Gamma(k)^\top R \Gamma(k)\} = \bar{R} \in \mathcal{S}_+^n$ due to Assumption 6.4. Now, let us define the sequence $\{W_i\}_{i=0}^{\infty}$ such that $W_0 = Q + \bar{R}$ and $W_{i+1} = \mathbb{E}\{(A(i) + B\Gamma(i))^\top W_i (A(i) + B\Gamma(i))\}$ which results in

$$\mathbb{E}\left\{\sum_{k=0}^{T-1} x(k)^\top Q x(k) + \bar{u}(k)^\top R \bar{u}(k)\right\} = \mathbb{E}\left\{\sum_{k=0}^{T-1} x_0^\top W_k x_0\right\} = x_0^\top \mathbb{E}\left\{\sum_{k=0}^{T-1} W_k\right\} x_0.$$

Notice that by construction, $W_i \geq 0$ for all i. In what follows, we prove that $\lim_{T\to\infty} \sum_{k=0}^{T-1} W_k = W < \infty$. Notice that using Assumption 6.4, we have $\mathbb{E}\{(A(i) + B\Gamma(i))^\top \otimes (A(i) + B\Gamma(i))^\top\} = \bar{U}$ for a fixed matrix $\bar{U} \in \mathbb{R}^{n^2 \times n^2}$.

Claim 1: $\max_j |\lambda_j(\bar{U})| < 1$ where $\lambda_j(\cdot)$ denotes the eigenvalues of a matrix.

To prove this claim, construct a sequence $\{\bar{W}_i\}_{i=0}^{\infty}$ such that $\bar{W}_{i+1} = \mathbb{E}\{(A(i) + B\Gamma(i))^\top \bar{W}_i (A(i) + B\Gamma(i))\}$ and $\bar{W}_0$ can be an arbitrary matrix (note that the difference between $\{W_i\}_{i=0}^{\infty}$ and $\{\bar{W}_i\}_{i=0}^{\infty}$ is the initial condition). Now, using an inductive argument, we prove that $\bar{W}_k = \mathrm{vec}^{-1}(\bar{U}^k \mathrm{vec}(\bar{W}_0))$. Firstly,

$$\begin{aligned}
\bar{W}_1 &= \mathbb{E}\left\{(A(1) + B\Gamma(1))^\top \bar{W}_0 (A(1) + B\Gamma(1))\right\} \\
&= \mathbb{E}\left\{\mathrm{vec}^{-1}\left((A(1) + B\Gamma(1))^\top \otimes (A(1) + B\Gamma(1))^\top \mathrm{vec}(\bar{W}_0)\right)\right\} \\
&= \mathrm{vec}^{-1}\left(\mathbb{E}\left\{(A(1) + B\Gamma(1))^\top \otimes (A(1) + B\Gamma(1))^\top\right\}\mathrm{vec}(\bar{W}_0)\right)\Bigg\} \\
&= \mathrm{vec}^{-1}(\bar{U}\mathrm{vec}(\bar{W}_0)).
\end{aligned}$$

where the second equality follows from the fact that for any three compatible matrices A, B, C, we have $ABC = \mathrm{vec}^{-1}((C^\top \otimes A)\mathrm{vec}(B))$ and the third equality holds because vec^{-1} is a linear operator. Now, let us show that $\bar{W}_{k+1} = \mathrm{vec}^{-1}(\bar{U}^{k+1}\mathrm{vec}(\bar{W}_0))$ if $\bar{W}_k = \mathrm{vec}^{-1}(\bar{U}^k \mathrm{vec}(\bar{W}_0))$. To do so, notice that

$$\begin{aligned}
\bar{W}_{k+1} &= \mathbb{E}\left\{(A(k+1) + B\Gamma(k+1))^\top \bar{W}_k (A(k+1) + B\Gamma(k+1))\right\} \\
&= \mathbb{E}\left\{\mathrm{vec}^{-1}\left((A(k+1) + B\Gamma(k+1))^\top \otimes (A(k+1) + B\Gamma(k+1))^\top \mathrm{vec}(\bar{W}_k)\right)\right\} \\
&= \mathrm{vec}^{-1}\left(\mathbb{E}\left\{(A(k+1) + B\Gamma(k+1))^\top \otimes (A(k+1) + B\Gamma(k+1))^\top\right\}\bar{U}^k\mathrm{vec}(\bar{W}_0)\right)\Bigg\} \\
&= \mathrm{vec}^{-1}(\bar{U}^{k+1}\mathrm{vec}(\bar{W}_0)).
\end{aligned}$$

This conclude the induction. Now, notice that $\lim_{k\to\infty} x_0^\top \bar{W}_k x_0 = \lim_{k\to\infty} \mathbb{E}\{x(k)^\top \bar{W}_0 x(k)\} = 0$ for any $x_0 \in \mathbb{R}^n$ because $\Gamma(k)$ satisfies the condition of Definition 6.3. As a result, $\lim_{k\to\infty} \bar{W}_k = 0$. Therefore, we get $\lim_{k\to\infty} \bar{U}^k \mathrm{vec}(\bar{W}_0) = 0$ irrespective of the choice of $\bar{W}_0$ which, in turn, implies that $\lim_{k\to\infty} \bar{U}^k = 0$. Using Theorem 4 [50, p. 14], we get $\max_j |\lambda_j(\bar{U})| < 1$.

Now that we have proved Claim 1, we are ready to show that $\lim_{T\to\infty}\sum_{k=0}^{T-1} W_k = W < \infty$. Recalling the proof of Claim 1 while setting $\bar{W}_0 = W_0 = Q + \bar{R}$, we get that $W_k = \mathrm{vec}^{-1}(\bar{U}^k \mathrm{vec}(Q + \bar{R}))$ and as a result

$$\lim_{T\to\infty}\sum_{k=0}^{T-1} W_k = \lim_{T\to\infty}\sum_{k=0}^{T-1} \mathrm{vec}^{-1}(\bar{U}^k \mathrm{vec}(Q + \bar{R})) = \mathrm{vec}^{-1}\left(\left[\lim_{T\to\infty}\sum_{k=0}^{T-1} \bar{U}^k\right]\mathrm{vec}(Q + \bar{R})\right).$$

Now, notice that using Claim 1, $\lim_{T\to\infty}\sum_{k=0}^{T-1} \bar{U}^k = (I - \bar{U})^{-1}$. Let $\bar{U}^\infty = (I - \bar{U})^{-1} \in \mathbb{R}^{n^2 \times n^2}$. Hence, we get $\lim_{T\to\infty}\sum_{k=0}^{T-1} W_k = \mathrm{vec}^{-1}(\bar{U}^\infty \mathrm{vec}(Q + \bar{R})) < \infty$. Let us define $W = \mathrm{vec}^{-1}(\bar{U}^\infty \mathrm{vec}(Q + \bar{R}))$. Using (22), we get

$$\begin{aligned}
x_0^\top X_T x_0 \le \mathbb{E}\left\{\sum_{k=0}^{T-1} x(k)^\top Q x(k) + \bar{u}(k)^\top R \bar{u}(k)\right\} = \sum_{k=0}^{T-1} x_0^\top W_k x_0 &\le \sum_{k=0}^{\infty} x_0^\top W_k x_0 \\
&\le x_0^\top W x_0.
\end{aligned}$$

This inequality is indeed true irrespective of the initial condition x_0 and the time horizon T. Therefore, $X_i \leq W$ for all $i \geq 0$. The rest of the proof is similar to that of Theorem 6.2. ∎

Example 6.1 (Cont'd) *Let us introduce the quadratic cost function*

$$J_\infty(x_0, \{u(k)\}_{k=0}^{\infty}) = \mathbb{E}\left\{\sum_{k=0}^{\infty} x(k)^\top x(k) + u(k)^\top u(k)\right\}.$$

Following Theorem 6.2, we can easily calculate the optimal controller with limited model information as

$$u^{\mathrm{LMI}}(k) = \begin{bmatrix} 42.7701 + 8.0694\alpha_1(k) & -1.6741 & -29.1868 & 0.1041 \\ -23.2274 & 0.1757 & 34.4246 + 6.8698\alpha_2(k) & -1.7331 \end{bmatrix} x(k).$$

Clearly, the control gain $L_i \in \mathbb{R}^{1\times 4}$ of controller $u_i(k) = L_i(k)x(k)$, $i = 1, 2$, is a function of only its corresponding subsystem's model parameter $\alpha_i(k)$. ▷

An interesting question is what is the value of model information when designing an optimal controller; i.e., having only access to local model information how much does the closed-loop performance degrade in comparison to having access to global model information. To answer this question for the setting considered in this paper, we need to introduce the optimal control design with full model information.

4 Control Design with Full Model Information

In this section, we consider the case where we have access to the full model information when designing each subcontroller. Hence, we make the following definition:

Definition 6.4 *The design of controller i, $1 \leq i \leq N$, has full model information if (a) the entire model parameters $\{A_{ij}(k) \mid 1 \leq i, j \leq N, \forall k\}$ are available together with (b) the first- and the second-order moments of the system parameters (i.e., $\mathbb{E}\{A(k)\}$ and $\mathbb{E}\{\tilde{A}(k) \otimes \tilde{A}(k)\}$ for all k).*

We have the following result for the finite-horizon case.

Theorem 6.4 *The solution of the finite-horizon optimal control design problem with full model information is given by*

$$u(k) = -(R + B(k)^\top P(k+1)B(k))^{-1}B(k)^\top P(k+1)A(k)x(k), \tag{23}$$

where $\{P(k)\}_{k=0}^{T}$ can be found using the backward difference equation

$$P(k) = Q(k) + \mathbf{R}(\bar{A}(k), P(k+1), B(k), R) + \sum_{i=1}^{N} \mathbb{E}\left\{\mathbf{R}(\tilde{A}_i(k), P(k+1), B(k), R)\right\}, \tag{24}$$

with the boundary condition $P(T) = Q(T)$. Furthermore, $\inf_{\{u(k)\}_{k=0}^{T-1}} J_T(x_0, \{u(k)\}_{k=0}^{T-1}) = x_0^\top P(0)x_0$.

Proof: The proof is similar to the proof of Theorem 6.1 and is therefore omitted. See [51] for the detailed proof. ∎

This result can be extended to the infinite-horizon cost function. However, we first need to present the following definition.

Definition 6.5 *System (2) is called mean square stabilizable under full model information if there exists a mapping* $\Gamma : \mathbb{R}^{n\times n} \to \mathbb{R}^{m\times n}$ *such that the closed-loop system with controller* $u(k) = \Gamma(A(k))x(k)$ *is mean square stable.*

Theorem 6.5 *Suppose (2) satisfies Assumption 6.3 and is mean square stabilizable under full model information. The solution of the infinite-horizon optimal control design problem with full model information is then given by*

$$u(k) = -(R + B^\top PB)^{-1}B^\top PA(k)x(k), \tag{25}$$

where P *is the unique finite positive-definite solution of the modified discrete algebraic Riccati equation*

$$P = Q + \mathbf{R}(\bar{A}, P, B, R) + \sum_{i=1}^{N} \mathbb{E}\left\{\mathbf{R}(\tilde{A}_i(k), P, B, R)\right\}. \tag{26}$$

Furthermore, this controller mean square stabilizes the system and $\inf_{\{u(k)\}_{k=0}^{\infty}} J_\infty(x_0, \{u(k)\}_{k=0}^{\infty}) = x_0^\top P x_0$.

Proof: The proof is similar to the proofs of Theorem 6.2 and Corollary 6.3. ∎

Example 6.1 (Cont'd) *Following Theorem 6.5, the optimal control design with full model information is*

$$u^{\mathrm{FMI}}(k) = \begin{bmatrix} 42.7701+7.9708\alpha_1(k) & -1.6741 & -29.1868-0.1035\alpha_2(k) & 0.1041 \\ -23.2274-0.1215\alpha_1(k) & 0.1757 & 34.4246+6.7725\alpha_2(k) & -1.7330 \end{bmatrix} x(k).$$

Note that the gain of controller i *depends on the global model parameters.* ▷

5 Performance Degradation under Model Information Limitation

In this section, we study the value of the plant model information using the closed-loop performance degradation caused by lack of full model information in the control design procedure. The performance degradation is captured using the ratio of the closed-loop performance of the optimal controller with limited model information to the closed-loop performance of the optimal controller with global plant model information. Let $\{u^{\mathrm{LMI}}(k)\}_{k=0}^{\infty}$ and $\{u^{\mathrm{FMI}}(k)\}_{k=0}^{\infty}$ denote the optimal controller with limited model information (Theorem 6.2) and the optimal controller with full model information (Theorem 6.5), respectively. We define the performance degradation ratio as

$$r = \sup_{x_0 \in \mathbb{R}^n} \frac{J_\infty(x_0, \{u^{\text{LMI}}(k)\}_{k=0}^{\infty})}{J_\infty(x_0, \{u^{\text{FMI}}(k)\}_{k=0}^{\infty})}.$$

Note that $r \geq 1$ since the optimal controller with full model information always outperforms the optimal controller with limited model information.

Example 6.1 (Cont'd) *In this example, we compare the closed-loop performance of the optimal controllers under different information regimes. We have already calculated the optimal controller with limited model information as well as the optimal controller with full model information for this numerical example. Now, let us find the optimal controller using statistical model information based on [29]. Using Theorem 5.2 from [29], we get*

$$u^{\text{SMI}}(k) = \begin{bmatrix} 41.9043 & -1.7873 & -29.3969 & -0.0121 \\ -23.3180 & 0.0435 & 32.7901 & -1.8779 \end{bmatrix} x(k).$$

Note how these three control laws depend on the plant model parameters. The control $u^{\text{SMI}}(k)$ *has a static gain depending on the statistical information of the A-matrix, while* $u^{\text{FMI}}(k)$ *and* $u^{\text{LMI}}(k)$ *depend on the actual realizations of the stochastic parameters. Now, we can explicitly compute the performance degradation ratio*

$$r = \sup_{x_0 \in \mathbb{R}^n} \frac{x_0^\top P^{\text{LMI}} x_0}{x_0^\top P^{\text{FMI}} x_0} = 1 + 2.266 \times 10^{-4}.$$

This shows that the performance of the optimal controller with limited model information is practically the same as the performance of the optimal controller with full model information. It is interesting to note that with access to (precise) local model information, one can expect a huge improvement in the closed-loop performance in comparison to the optimal controller with only statistical model information because

$$\sup_{x_0 \in \mathbb{R}^n} \frac{x_0^\top P^{\text{SMI}} x_0}{x_0^\top P^{\text{LMI}} x_0} = 5.8790.$$

$\triangleright$

Next we derive an upper bound for the performance degradation ratio r. We do that for fully-actuated systems.

Assumption 6.5 *All subsystems (1) are fully-actuated; i.e.,* $B_{ii} \in \mathbb{R}^{n_i \times n_i}$ *and* $\underline{\sigma}(B_{ii}) \geq \epsilon > 0$ *for all* $1 \leq i \leq N$*, where* $\underline{\sigma}(\cdot)$ *denotes the smallest singular value of a matrix.*

To simplify the presentation, we also assume that $Q = R = I$. This is without loss of generality since the change of variables $(x', u') = (Q^{1/2}x, R^{1/2}u)$ transforms the cost function and state space representation into

$$J_\infty(x_0, \{u'(k)\}_{k=0}^\infty) = \lim_{T\to\infty} \mathbb{E}\left\{\sum_{k=0}^{T-1} x'(k)^\top x'(k) + u'(k)^\top u'(k)\right\},$$

and

$$x'(k+1) = Q^{1/2}A(k)Q^{-1/2}x'(k) + Q^{1/2}BR^{-1/2}u'(k) = A'(k)x'(k) + B'u'(k).$$

The next theorem presents an upper bound for the performance degradation.

Theorem 6.6 *Suppose (2) satisfies Assumptions 6.3 and 6.5 and is mean square stabilizable under limited model information. The performance degradation ratio is then upper bounded as $r \le 1 + 1/\epsilon^2$ where $\epsilon > 0$ is defined in Assumption 6.5.*

Proof: Using the modified discrete algebraic Riccati equation (26) in Theorem 6.5, the cost of the optimal control design with full model information $J_\infty(x_0, \{u^{\mathrm{FMI}}(k)\}_{k=0}^\infty) = x_0^\top P^{\mathrm{FMI}} x_0$ is equal to

$$\begin{aligned} x_0^\top P^{\mathrm{FMI}} x_0 &= x_0^\top Q x_0 + x_0^\top \mathbf{R}(\bar{A}, P^{\mathrm{FMI}}, B, I)x_0 \\ &\quad + \sum_{i=1}^N x_0^\top \mathbb{E}\left\{\mathbf{R}(\tilde{A}_i(k), P^{\mathrm{FMI}}, B, I)\right\} x_0. \end{aligned} \tag{27}$$

In addition, we know that $P^{\mathrm{FMI}} \ge Q = I$, which (using the proof of Theorem 6.2) results in

$$\mathbf{R}(\bar{A}, P^{\mathrm{FMI}}, B, I) \ge \mathbf{R}(\bar{A}, I, B, I), \tag{28}$$

$$\mathbf{R}(\tilde{A}_i(k), P^{\mathrm{FMI}}, B, I) \ge \mathbf{R}(\tilde{A}_i(k), I, B, I). \tag{29}$$

Substituting (28)–(29) inside (27) gives

$$\begin{aligned} x_0^\top P^{\mathrm{FMI}} x_0 \ge & x_0^\top (I + \bar{A}^\top (I + BB^\top)^{-1}\bar{A})x_0 \\ & + \sum_{i=1}^N x_0^\top \mathbb{E}\left\{\tilde{A}_i(k)^\top (I + BB^\top)^{-1}\tilde{A}_i(k)\right\} x_0 \\ = & x_0^\top x_0 + x_0^\top \mathbb{E}\{A(k)^\top (I + BB^\top)^{-1} A(k)\} x_0, \end{aligned}$$

where the equality follows from the fact that $\tilde{A}_i(k)$ and $\tilde{A}_j(k)$ for $i \ne j$ are independent random variables with zero mean. On the other hand, for a given $x_0 \in \mathbb{R}^n$, the cost of the optimal control design with limited model information $J_\infty(x_0, \{u^{\mathrm{LMI}}(k)\}_{k=0}^\infty) = x_0^\top P^{\mathrm{LMI}} x_0$ is upper-bounded by

$$x_0^\top P^{\mathrm{LMI}} x_0 \le \mathbb{E}\left\{\sum_{k=0}^{+\infty} x(k)^\top x(k) + u(k)^\top u(k)\right\},$$

where $u(k) = -B^{-1}A(k)x(k)$ and $x(k)$ is the state vector of the system when this control sequence is applied to the system. This is true since the deadbeat control design strategy $u(k) = -B^{-1}A(k)x(k)$ uses only local model information for designing each controller [16]. Therefore,

$$x_0^\top P^{\mathrm{LMI}} x_0 \leq \mathbb{E}\left\{x_0^\top (I + A(k)^\top B^{-\top} B^{-1} A(k)) x_0\right\}.$$

Let us define the set $\mathcal{M}_r = \left\{\bar{\beta} \in \mathbb{R} \mid r \leq \bar{\beta}\right\}$ where r is the performance degradation ratio. If $\beta \in \mathbb{R}$ satisfy $\beta P^{\mathrm{FMI}} - P^{\mathrm{LMI}} \geq 0$, then $\beta \in \mathcal{M}_r$. We have

$$\beta P^{\mathrm{FMI}} - P^{\mathrm{LMI}} \geq (\beta - 1)I + \mathbb{E}\{A(k)^\top \left[\beta(I + BB^\top)^{-1} - B^{-\top}B^{-1}\right] A(k)\}. \quad (30)$$

Note that if $\beta \geq 1 + 1/\epsilon^2$, we get $\beta(I + B_{ii}B_{ii}^\top)^{-1} - B_{ii}^{-\top}B_{ii}^{-1} \geq 0$ and therefore, $\beta(I + BB^\top)^{-1} - B^{-\top}B^{-1} \geq 0$. As a result, if $\beta \geq 1 + 1/\epsilon^2$, the right hand side of (30) is a positive-semidefinite matrix and, subsequently, $\beta P^{\mathrm{FMI}} - P^{\mathrm{LMI}} \geq 0$. Hence, $[1 + 1/\epsilon^2, +\infty) \subseteq \mathcal{M}_r$. This shows that

$$r = \sup_{x_0 \in \mathbb{R}^n} \frac{x_0^\top P^{\mathrm{LMI}} x_0}{x_0^\top P^{\mathrm{FMI}} x_0} \leq 1 + \frac{1}{\epsilon^2}.$$

■

As the power network in Example 6.1 is not fully-actuated, we consider another power network example to the illustrate the previous result.

Example 6.2 *Consider DC power generators, such as solar farms and batteries. Suppose these sources are connected to AC transmission lines through DC/AC converters that are equipped with a droop-controller [52, 53]. Let us assume that both power generators in Figure 1 are such DC power generators equipped with droop-controlled converters. We can then model this power network as*

$$\dot{\delta}_1(t) = \frac{1}{D_1}\left[P_1(t) - c_{12}^{-1}\sin(\delta_1(t) - \delta_2(t)) - c_1^{-1}\sin(\delta_1(t)) - D_1\omega_1(t)\right],$$
$$\dot{\delta}_2(t) = \frac{1}{D_2}\left[P_2(t) - c_{12}^{-1}\sin(\delta_2(t) - \delta_1(t)) - c_2^{-1}\sin(\delta_2(t)) - D_2\omega_2(t)\right],$$

where $\delta_i(t)$, $1/D_i > 0$, and $P_i(t)$ are respectively the phase angle of the terminal voltage of converter i, its converter droop-slope, and its input power. The power network parameters in this example are the same as the ones in Example 6.1, except $D_1 = D_2 = 1.0$. Now, similarly to Example 6.1, we find the equilibrium point of this nonlinear system, linearize it around this equilibrium, and then, discretize the system with sampling time $\Delta T = 300\,\mathrm{ms}$ to get

$$\begin{bmatrix} \Delta\delta_1(k+1) \\ \Delta\delta_2(k+1) \end{bmatrix} = \begin{bmatrix} \zeta_1 & \frac{\Delta T \cos(\delta_1^* - \delta_2^*)}{c_{12}D_1} \\ \frac{\Delta T \cos(\delta_2^* - \delta_1^*)}{c_{12}D_2} & \zeta_2 \end{bmatrix} \begin{bmatrix} \Delta\delta_1(k) \\ \Delta\delta_2(k) \end{bmatrix} + \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix},$$

where $\zeta_1 = 1-\Delta T(c_{12}^{-1}\cos(\delta_1^*-\delta_2^*)+c_1^{-1}\cos(\delta_1^*))/D_1$ *and* $\zeta_2 = 1-\Delta T(c_{12}^{-1}\cos(\delta_2^*+\delta_1^*)-c_2^{-1}\cos(\delta_2^*))/D_2$*. Consider the same variation of the local loads as in Example 6.1. We get the discrete-time linear with stochastically-varying parameters*

$$x(k+1) = Ax(k) + Bu(k)$$

where $x(k) = [\Delta\delta_1(k)\ \Delta\delta_2(k)]^\top$, $u(k) = [u_1(k)\ u_2(k)]^\top$, *and*

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(k) = \begin{bmatrix} -0.1635 - 0.2075\alpha_1(k) & 0.7486 \\ 0.7486 & -0.1897 - 0.0877\alpha_2(k) \end{bmatrix}.$$

where $\alpha_1(k) \sim \mathcal{N}(0, 0.1)$ *and* $\alpha_2(k) \sim \mathcal{N}(0, 0.3)$*. The goal is to optimize the performance criterion*

$$J = \mathbb{E}\left\{\sum_{k=0}^{\infty} x(k)^\top x(k) + u(k)^\top u(k)\right\}.$$

Following Theorem 6.5, we can calculate the optimal controller with full model information as

$$u^{\mathrm{FMI}}(k) = \begin{bmatrix} 0.1166 + 0.1185\alpha_1(k) & -0.4334 - 0.0027\alpha_2(k) \\ -0.4334 - 0.0064\alpha_1(k) & 0.1317 + 0.0502\alpha_2(k) \end{bmatrix} x(k).$$

Furthermore, using Theorem 6.2, we can calculate the optimal controller with limited model information as

$$u^{\mathrm{LMI}}(k) = \begin{bmatrix} 0.1166 + 0.1190\alpha_1(k) & -0.4334 \\ -0.4334 & 0.1317 + 0.0504\alpha_2(k) \end{bmatrix} x(k).$$

It is easy to see that

$$r = \sup_{x_0 \in \mathbb{R}^n} \frac{x_0^\top P^{\mathrm{LMI}} x_0}{x_0^\top P^{\mathrm{FMI}} x_0} = 1 + 1.2660 \times 10^{-6} \le 1 + 1/\epsilon^2 = 2,$$

since $\epsilon = 1$*. In this example, the upper bound computed in Theorem 6.6 is not tight.* $\triangleright$

Remark 6.8 *Under Assumption 6.5, when the variances of the plant model parameters tend to infinity, the optimal controller with limited model information (introduced in Theorem 6.2) approaches the deadbeat control law. The intuition behind this result is that when the model information of the other subsystems is inaccurate, the deadbeat control law (which decouples our subsystem from the rest of the plant) is the best controller to use. The presented approach balances in a natural way the use of statistical information about the plant parameters with precise knowledge of their realizations.*

Example 6.2 (Cont'd) *Let us consider the case where variances of the plant model parameters are very large. Hence, we assume* $\alpha_1(k) \sim \mathcal{N}(0, 1000)$ *and* $\alpha_2(k) \sim \mathcal{N}(0, 3000)$*. Now, the optimal controller with limited model information is given by*

$$u^{\mathrm{LMI}}(k) = \begin{bmatrix} 0.1635 + 0.2075\alpha_1(k) & -0.7485 \\ -0.7485 & 0.1897 + 0.0877\alpha_2(k) \end{bmatrix} x(k),$$

which is practically equal to the deadbeat control law in Remark 6.7. ▷

6 Conclusion

We presented a statistical framework for the study of control design under limited model information. We found the best performance achievable by a limited model information control design method. We also studied the value of information in control design using the performance degradation ratio. Possible future work will focus on generalizing the results to discrete-time Markovian jump linear systems and to decentralized controllers.

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Part 2:

Strategic Decision Making in Transportation Systems

PAPER 7

When Do Potential Functions Exist in Heterogeneous Routing Games?

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Abstract–We study a heterogeneous routing game in which vehicles might belong to more than one type. The type determines the cost of traveling along an edge as a function of the flow of various types of vehicles over that edge. We relax the assumptions needed for the existence of a Nash equilibrium in this heterogeneous routing game. We extend the available results to present necessary and sufficient conditions for the existence of a potential function. We characterize a set of tolls that guarantee the existence of a potential function when only two types of users are participating in the game. We present an upper bound for the price of anarchy (i.e., the worst-case ratio of the social cost calculated for a Nash equilibrium over the social cost for a socially optimal flow) for the case in which only two types of players are participating in a game with affine edge cost functions. A heterogeneous routing game with vehicle platooning incentives is used as an example throughout the article to clarify the concepts and to validate the results.

1 Introduction

1.1 Motivation

Routing games are of special interest in transportation networks [1–3] and communication networks [4–6] because they allow us to study cases in which a desirable global behavior (e.g., achieving a socially optimal solution) can emerge from simple local strategies (e.g., imposing tolls on each road based on only local information). For the purpose of this article, routing games can be decomposed into two categories. In the first category, namely, homogeneous routing games, drivers or vehicles are of the same type and, therefore, experience the same cost when using an edge in the network. Such an assumption is primarily motivated by transportation networks for which the drivers only worry about the travel time (and indeed under the assumption that all the drivers are equally sensitive to latency through considering their average behavior) or packet routing in communication networks where all the packets that are using a particular link experience the same delay or reliability. In the second category, namely, heterogeneous routing games (a.k.a., multi-class routing games [7, 8]), drivers or vehicles belong to more than one type due to the following reasons:

- *Fuel Consumption*: In a transportation network, if we include the fuel consumption of the vehicles in the cost functions, two vehicles (of different types) may experience different costs for using a road even if their travel times are equal. For instance, [9] studied this phenomenon in atomic congestion games in which heavy-duty vehicles experience an increased efficiency when a higher number of heavy-duty vehicles are present on the same road, because of a higher possibility of platooning and, therefore, a higher fuel efficiency, while such an increased efficiency may not be true for cars. For an experimental study of improvements in the fuel efficiency caused by platooning in heavy-duty vehicles, see [10].

- *Sensitivity to Latency*: It is known that drivers on a road have different sensitivities to the latency under different circumstances as well as depending on their personality and background [11, 12]. In addition, due to economic advantages, heavy-duty vehicles might be more sensitive to latency in comparison to cars (because they need to deliver their goods at specific times).

- *Sensitivity to Tolls*: Drivers generally react differently to road tolls, e.g., based on the reason of the trip or their socioeconomic background. For instance, in 2001, by the request of the Swedish Institute for Transport and Communications Analysis, the consulting firm Inregia compiled a survey to estimate the value of time for the drivers in Stockholm [13]. According to that study, drivers valued their time as 0.98, 3.30, and 0.19 SEK/min for work and school commuting trips, business trips, and other trips, respectively.

These examples motivate our interest for studying heterogeneous routing games in which the drivers or the vehicles might belong to more than one type and their type determines the cost of traveling along an edge as a function of the flow of all types of vehicles over that edge.

1.2 Related Work

In the context of transportation networks, routing games were originally studied in [2]. This study also formulated the definition of Nash equilibrium in routing games[1]. Later, [16] showed that under some mild conditions, the routing game admits a potential function and the minimizers of this potential function are Nash equilibria of the routing game. This result guarantees the existence of a Nash equilibrium for the routing game. The problem of bounding the inefficiency of Nash equilibria has been extensively studied; see [14, 17–21] for a survey of these results.

Heterogeneous routing games have been studied extensively over the past starting with pioneering works in [7, 8]. In these studies, a routing game with multi-class users were introduced and the definition of equilibrium was given. Furthermore, in [8], the author introduced a *sufficient* condition for transforming the problem of finding an equilibrium to that of an optimization (i.e., equivalent to the existence of a potential function [22, 23]). The sufficient condition is that over each edge, the increased cost of a user of the first type due to addition of one more user of the second type is equal to the increased cost of a user of the second type due to addition of one more user of the first type, i.e., the users of the first and the second type influence each other equally [8]. This condition was considered later in [24] in which it was also noted that satisfaction of this symmetry condition may depend on the units (e.g., time or money) adopted for representing the cost functions for the case in which the users' types are determined by their value of time (i.e., a scalar factor that balances the relationship between the latency and the imposed tolls). This results is of special interest since the equilibrium does not change by using different units for the cost functions (if the latency only depends on the sum of the flows of various types over the edge, not the individual flows, and the value of time appears linearly in the cost functions) [25]. Necessary and sufficient conditions for the existence of potential functions in games with finite number of players was recently investigated in [26]; however, these results were not generalized to games with a continuum of players as in heterogeneous routing games. The authors in [27] studied the existence of an equilibrium in heterogeneous routing games even if such a symmetry condition does not hold. In contrast to these articles that assumed a finite set of types to which the users may belong, a wealth of studies also considered the case in which the users may belong to a continuum of types [28, 29]. The problem of finding tolls for general heterogeneous routing games as well as the case in which types of the users is determined by their value of time have been considered

[1] Throughout this article, following the convention of [14, 15], we use the term Nash equilibrium to refer to this equilibrium. See Remark 7.1 for more information regarding this matter.

extensively [30–35]. For instance, in [30], the problem of determining tolls on each edge or path for heterogeneous routing games was studied. Guarantees were provided for that the socially optimal solution (also referred to as system-optimizing flow [8]) is indeed an equilibrium of the game. However, in that article, the users were assumed to be equally sensitive to the imposed tolls. The problem of finding optimal tolls for routing game in which the users' value of time belongs to a continuum was studied in [31].

1.3 Contributions of the Article

In this article, we formulate a general heterogeneous routing game in which the vehicles[2] might belong to more than one type. The vehicle's type determines the mapping that specifies the cost for using an edge based on the flow of all types of vehicles over that edge.

We prove that the problem of characterizing the set of Nash equilibria for a heterogeneous routing game is equivalent to the problem of determining the set of pure strategy Nash equilibria in a game with finitely many players (in which each player represents one of the types in the original heterogeneous routing game). Doing so, we can employ classic results in game theory and economics literature, specially regarding the existence of an equilibrium in games and abstract economies [36, 37] (which is an extension of games to a situation where the actions of other players can modify the set of feasible actions for a player), to show that under mild conditions, a heterogeneous routing game admits at least one Nash equilibrium.

Then, we present a necessary condition for the existence of a potential function for the heterogeneous routing game. We show that this condition is also sufficient for the case in which only two types of players are participating in the routing game. In this case, we show, following the potential game literature [22], that the problem of finding a Nash equilibrium in the heterogeneous routing game can be posed as an optimization problem (which is numerically tractable if the potential function is convex). Motivated by the sufficient condition, in the rest of this article, we focus on heterogeneous routing games in which only two types of users are participating. Note that in contrast to the results of [8, 24], here, we present a *necessary and sufficient* condition for the existence of potential function through which minimization we can recover an equilibrium. However, the price of providing this tighter condition is that we can only treat routing games with two distinct types contrary to the sufficient condition in [8, 24].

If the problem of finding a Nash equilibrium in the heterogeneous routing game is numerically intractable[3], it might be unlikely for the drivers to figure out a Nash equilibrium in finite time (let alone an efficient one) and utilize it. This might result in inefficient utilization of the transportation network resources. Therefore,

[2] We use the terms players, drivers, users, and vehicles interchangeably to denote an infinitesimal part of the flow that strategically tries to minimize its own cost for using the road.

[3] In general, the problem of finding a pure strategy Nash equilibrium is not numerically tractable; e.g., [38–40].

we present a set of tolling policies for distinguishable types (i.e., a routing game in which we may impose different tolls for different user types) and indistinguishable types (i.e., when we cannot impose type-dependent tolls) to guarantee the existence of a potential function for heterogeneous routing games. The idea of proposing tolls for indistinguishable types has been previously studied in [41]. However, in that study, the tolls were introduced to minimize the total travel time and the total travel cost (as a bi-objective optimization problem). In addition, in [41], the users' types corresponded to socio-economic characteristics and, therefore, the cost functions of various types of users were the latency (which the function of the total flow and not individual flows of each type) plus the tolls multiplied by the value of time.

Finally, because a Nash equilibrium is typically inefficient (i.e., it does not minimize the social cost function[4]), we study the price of anarchy[5] (a measure of the inefficiency of a Nash equilibrium which can be defined as the worst-case ratio of the social cost at a Nash equilibrium over the social cost at a socially optimal flow). We prove that for the case in which a convex potential function exists, the price of anarchy is bounded from above by two for affine edge cost functions, that is, the social cost of a Nash equilibrium can be at most twice as much as the cost of a socially optimal solution.

1.4 Article Organization

The rest of the article is organized as follows. We formulate the heterogeneous routing game in Section 2. In Section 3, we prove that a Nash equilibrium may indeed exist in this routing game. We present a set of necessary and sufficient conditions to guarantee the existence of a potential function in Section 4. In Section 5, a set of tolling policies is presented to satisfy the aforementioned conditions. We bound the price of anarchy for affine cost functions in Section 6. A numerical example motivated by a heterogeneous routing game with platooning incentives is studied in Section 7. Finally, we conclude the article and present directions for future research in Section 8.

2 A Heterogeneous Routing Game

2.1 Notation

Let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real and integer numbers, respectively. Furthermore, define $\mathbb{Z}_{\geq(\leq)a} = \{n \in \mathbb{Z} \,|\, n \geq (\leq) a\}$ and $\mathbb{R}_{\geq(\leq)a} = \{x \in \mathbb{R} \,|\, x \geq (\leq) a\}$. For simplicity of presentation, let $\mathbb{N} = \mathbb{Z}_{\geq 1}$. We use the notation $[\![N]\!]$ to denote $\{1, \ldots, N\}$.

[4] We use a utilitarian social cost function (i.e., summation of the individual cost functions of all the players) as opposed to a Rawlsian social cost function (i.e., the worst-case cost function of the players); see [42, p. 413] for more information regarding the difference between these two categories of social cost functions. We present the definition of the social cost function in Section 6.

[5] The notion of price of anarchy was first introduced in [43, 44]. Later, it was utilized in various games including routing games [14, 17, 45–47].

All other sets are denoted by calligraphic letters. Specifically, $\mathcal{C}^k$ consists of all k-times continuously differentiable functions.

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a set such that $0 \in \mathcal{X}$. A mapping $f : \mathcal{X} \to \mathbb{R}$ is called positive definite if $f(x) \geq 0$ for all $x \in \mathcal{X}$.

A set-valued mapping $f : \mathcal{X} \rightrightarrows \mathcal{Y}$ is said to be continuous at $x^0 \in \mathcal{X}$ if for every $y^0 \in f(x^0)$ and every sequence $\{x^k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x^k = x^0$, there exists a sequence $\{y^k\}_{k\in\mathbb{N}}$ such that $y^k \in f(x^k)$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} y^k = y^0$.

We use the notation $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to denote a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Each entry $(i, j) \in \mathcal{E}$ denotes an edge from vertex $i \in \mathcal{V}$ to vertex $j \in \mathcal{V}$. A directed path of length z from vertex i to vertex j is a set of edges $\{(i_0, i_1), (i_1, i_2), \ldots, (i_{z-1}, i_z)\} \subseteq \mathcal{E}$ such that $i_0 = i$ and $i_z = j$.

2.2 Problem Formulation

We propose an extension of the routing game introduced in [2] to admit more than one type of players. To be specific, we assume that the type of a player θ belongs to a finite set Θ.

Let us assume that a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ models the transportation network and that a set of source–destination pairs $\{(s_k, t_k)\}_{k=0}^K$ for some constant $K \in \mathbb{N}$ are given. Each pair (s_k, t_k) is called a commodity. We use the notation $\mathcal{P}_k$ to denote the set of all admissible paths over the graph $\mathcal{G}$ that connect vertex $s_k \in \mathcal{V}$ (i.e., the source of this commodity) to vertex $t_k \in \mathcal{V}$ (i.e., the destination of this commodity). Let $\mathcal{P} = \cup_{k=1}^K \mathcal{P}_k$. We assume that each commodity $k \in [\![K]\!]$ needs to transfer a flow equal to $(F_k^\theta)_{\theta\in\Theta} \in \mathbb{R}_{\geq 0}^{|\Theta|}$.

We use the notation $f_p^\theta \in \mathbb{R}_{\geq 0}$ to denote the flow of players of type $\theta \in \Theta$ that use a given path $p \in \mathcal{P}$. We use the notation $f = (f_p^\theta)_{p\in\mathcal{P},\theta\in\Theta} \in \mathbb{R}^{|\mathcal{P}|\cdot|\Theta|}$ to denote the aggregate vector of flows[6]. A flow vector $f \in \mathbb{R}^{|\mathcal{P}|\cdot|\Theta|}$ is feasible if $\sum_{p\in\mathcal{P}_k} f_p^\theta = F_k^\theta$ for all $k \in [\![K]\!]$ and $\theta \in \Theta$. We use the notation $\mathcal{F}$ to denote the set of all feasible flows. To ensure that the set of feasible flows is not an empty set, we assume that $\mathcal{P}_k \neq \emptyset$ if $F_k^\theta \neq 0$ for any $\theta \in \Theta$. Notice that the constraints associated with each type are independent of the rest. Therefore, the flows of a specific type can be changed without breaking the feasibility of the flows associated with the rest of the types.

A vehicle of type $\theta \in \Theta$ that travels along an edge $e \in \mathcal{E}$ experiences a cost equal to $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta})$, where for any $\theta \in \Theta$, ϕ_e^θ denotes the total flow of drivers of type θ that are using this specific edge, i.e., $\phi_e^\theta = \sum_{p\in\mathcal{P}:e\in p} f_p^\theta$. This cost can encompass aggregates of the latency, fuel consumption, etc. For notational convenience, we assume that we can change the order with which the edge flows $\phi_e^{\theta'}$ appear as arguments of the cost function $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta})$. A driver of type $\theta \in \Theta$ from commodity $k \in [\![K]\!]$ that uses path $p \in \mathcal{P}_k$ (connecting s_k to t_k) experiences a total cost of $\ell_p^\theta(f) = \sum_{e\in p} \tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta})$.

[6] Note that there is a one-to-one correspondence between the elements of $\mathcal{P} \times \Theta$ and the set of integers $\{1, \ldots, |\mathcal{P}| \cdot |\Theta|\}$.

Each player is an infinitesimal part of the flow that tries to minimize its own cost (i.e., each player is inclined to choose the path that has the least cost). Now, based on this model, we can define the Nash equilibrium.

Definition 7.1 (Nash Equilibrium in Heterogeneous Routing Games) *A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium if for all $k \in [\![K]\!]$ and $\theta \in \Theta$, $f_p^\theta > 0$ for a path $p \in \mathcal{P}_k$ implies that $\ell_p^\theta(f) \leq \ell_{p'}^\theta(f)$ for all $p' \in \mathcal{P}_k$.*

This definition implies that for a commodity $k \in [\![K]\!]$ and type $\theta \in \Theta$, all paths with a nonzero flow for vehicles of type θ have equal costs and the rest (i.e., paths with a zero flow for vehicles of type θ) have larger or equal costs.

Remark 7.1 *Note that various articles use different names for the equilibrium such as, user-optimizing flow [8, 27], Wardrop equilibrium [3, 27, 48], Wardrop first principle [3], and Nash equilibrium [14, 15]. The term Wardrop equilibrium is common, specially in transportation literature, due to the pioneering work of [2] as well as the fact that the term pure strategy Nash equilibrium is primarily utilized in the context of games with finitely many players [48]. It is vital to note that the definition of Nash equilibrium in this paper is indeed different from that of [48], which shows that by increasing the number of users (in a game with finitely many players), the Nash equilibrium converges to the Wardrop equilibrium under appropriate assumptions. Throughout this article, following the convention of [14, 15], we use the term Nash equilibrium to refer to this equilibrium.*

We make the following standing assumption regarding the edge latency functions for all the types.

Assumption 7.1 *For all $\theta \in \Theta$ and $e \in \mathcal{E}$, the edge cost function $\tilde{\ell}_e^\theta$ satisfies the following properties:*

(i) $\tilde{\ell}_e^\theta \in \mathcal{C}^1$;

(ii) $\tilde{\ell}_e^\theta$ *is positive definite;*

(iii) $\int_0^{\phi_e^\theta} \tilde{\ell}_e^\theta(u, (\phi_e^{\theta'})_{\theta' \in \Theta \setminus \{\theta\}}) \mathrm{d}u$ *is a convex function in* ϕ_e^θ *for any given* $(\phi_e^{\theta'})_{\theta' \in \Theta \setminus \{\theta\}}$.

Assumption 7.1 (*iii*) is equivalent to[7]:

(*iii*)' $\tilde{\ell}_e^\theta(\phi_e^\theta, (\phi_e^{\theta'})_{\theta' \in \Theta \setminus \{\theta\}})$ *is an increasing function of* ϕ_e^θ *for any given* $(\phi_e^{\theta'})_{\theta' \in \Theta \setminus \{\theta\}}$.

We start by proving the existence of a Nash equilibrium and, then, study the computational complexity of finding such an equilibrium. However, before that, we present an example of a heterogeneous routing game in the next subsection.

[7] Consult [17] for the proof of the equivalence when $|\Theta| = 1$. The proof in the heterogeneous case follows the same line of reasoning.

2.3 Example: Routing Game with Platooning Incentives

Let $\Theta = \{\mathrm{c}, \mathrm{t}\}$, where t denotes trucks (or, equivalently, heavy-duty vehicles) and c denotes cars (or, equivalently, light vehicles). Let the edge cost functions be characterized as

$$\tilde{\ell}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = \xi_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}),$$
$$\tilde{\ell}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = \xi_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}) + \zeta_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}),$$

where mappings $\xi_e : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\zeta_e : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote respectively the latency for using edge e as a function of the total flow of vehicles over that edge and the fuel consumption of trucks as a function of the flow of each type. These costs imply that cars only observe the latency $\xi_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}})$ when using the roads (which is only a function of the total flow over that edge and not the individual flows of each type). However, the cost associated with trucks encompasses an additional term which models their fuel consumption. Following this interpretation, $\zeta_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})$ is a decreasing function in ϕ_e^{t} since by having a higher flow of trucks over a given road (i.e., larger ϕ_e^{t}) each truck gets a higher probability for collaboration such as platooning (and as a result, a higher chance of decreasing its fuel consumption).

Let us give examples of these functions. Based on the traffic data measurements available from [2, p. 366] (see [9] for a case study on the relationship between the average velocity and the number of the vehicles on the road in Stockholm), we know that whenever the traffic on a road is in free-flow mode, we can model the average velocity of traveling along that road as an affine function of the flow of vehicles over that edge according to

$$\bar{v}_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = a_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}) + b_e.$$

In this model, $b_e \in \mathbb{R}_{\geq 0}$ and $a_e \in \mathbb{R}_{\leq 0}$ for $e \in \mathcal{E}$. Therefore, if the length of edge $e \in \mathcal{E}$ is equal to $L_e \in \mathbb{R}_{\geq 0}$, we can calculate the latency of using that edge as

$$\xi_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}) = \frac{L_e}{\bar{v}_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})} = \frac{L_e}{a_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}) + b_e}.$$

Now, in cases where $a_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}) \ll b_e$, we can use a linearized[8] model for the latency

$$\xi_e(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}) = \frac{L_e}{b_e} - \frac{L_e a_e}{b_e^2}(\phi_e^{\mathrm{c}} + \phi_e^{\mathrm{t}}).$$

[8] Notice that such a linearization is certainly not valid for a wide range of traffic flows, however, it models the latency functions well-enough for small flows. The authors in [49, 50] proposed a piecewise linear mapping (based on numerical data from the Toronto metropolitan area) for modeling the latency as a function of the flow of vehicles. This model justifies using a linear model for small flows (i.e., at the beginning what they call the feasible region), however, it also points out that a linear approximation is not valid for large flows. For a comprehensive comparison of different latency mappings (linear as well as nonlinear ones), see [51].

In addition, using [52], we know that the total fuel consumption of a truck which is traveling with velocity $\bar{v}_e$ for distance L_e over a flat road can be modeled by

$$\zeta_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = \frac{c_0 L_e}{\bar{\eta}_{\mathrm{eng}} \rho_d} \left(\frac{1}{2} \rho_a A_a c_D \bar{v}_e^2(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) + mgc_r \right), \tag{1}$$

where $\bar{\eta}_{\mathrm{eng}}$ is the engine efficiency, ρ_d is the energy density of diesel fuel, c_D is the air drag coefficient, A_a is the frontal area of the truck, ρ_a is the air density, m is the mass of the truck, g is the gravitational acceleration, and c_r is the the roll resistance coefficient. In addition, we have multiplied the fuel consumption by c_0 to balance the trade-off between the latency and fuel consumption in the aggregate cost function of the trucks. Following [10], we know that the air drag coefficient c_D decreases if the trucks are platooning (e.g., two identical trucks can achieve 4.7%–7.7% reduction in the fuel consumption caused by the air drag reduction when platooning at 70 km/h depending on the distance between them). Let us model these changes as $c_D = c_D' \gamma(\phi_e^{\mathrm{t}})$ where $\gamma : \mathbb{R}_{\geq 0} \to [0, 1]$ is the probability of forming platoons (which is a function of the flow of trucks ϕ_e^{t}) multiplied by the improvements in the air drag coefficient upon platooning. Let us define parameters

$$\alpha = \frac{L_e \rho_a A_a c_D'}{2 \bar{\eta}_{\mathrm{eng}} \rho_d}, \qquad \beta = \frac{L_e mgc_r}{\bar{\eta}_{\mathrm{eng}} \rho_d}.$$

Now, again if we linearize (1) around $\phi_e^{\mathrm{t}} = 0$, we get

$$\begin{aligned} \zeta_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = {} & \left(c_0 \alpha \frac{\mathrm{d}}{\mathrm{d}u} \gamma(u)|_{u=0} b_e^2 + 2 c_0 \alpha \gamma(0) b_e a_e \right) \phi_e^{\mathrm{t}} \\ & + (2 c_0 \alpha \gamma(0) b_e a_e) \, \phi_e^{\mathrm{c}} + \left(c_0 \beta + c_0 \alpha \gamma(0) b_e^2 \right). \end{aligned}$$

Combing all these terms results in

$$\begin{aligned} \tilde{\ell}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) &= \frac{L_e}{b_e} + \left(-\frac{L_e a_e}{b_e^2} \right) \phi_e^{\mathrm{c}} + \left(-\frac{L_e a_e}{b_e^2} \right) \phi_e^{\mathrm{t}}, \\ \tilde{\ell}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) &= \frac{L_e}{b_e} + c_0 \beta + c_0 \alpha \gamma(0) b_e^2 + \left(-\frac{L_e a_e}{b_e^2} + 2 c_0 \alpha \gamma(0) b_e a_e \right) \phi_e^{\mathrm{c}} \\ & \quad + \left(-\frac{L_e a_e}{b_e^2} + c_0 \alpha \frac{\mathrm{d}\gamma(0)}{\mathrm{d}u} b_e^2 + 2 c_0 \alpha \gamma(0) b_e a_e \right) \phi_e^{\mathrm{t}}. \end{aligned}$$

Notice that Assumption 7.1 (*i*) and (*ii*) are easily satisfied. However, Assumption 7.1 (*iii*) is only satisfied if

$$-\frac{L_e a_e}{b_e^2} + c_0 \alpha \frac{\mathrm{d}\gamma(0)}{\mathrm{d}u} b_e^2 + 2 c_0 \alpha \gamma(0) b_e a_e \geq 0.$$

This is indeed true because of the observation that Assumption 7.1 (*iii*) and (*iii*)' are equivalent.

3 Existence of Nash Equilibrium

In this section, we show that the heterogeneous routing game admits a Nash equilibrium. Before stating the result, we need to introduce some concepts from [36] which uses results of [37] to prove that an abstract economy (an extension of a game) admits an equilibrium under appropriate conditions[9].

3.1 Existence of Nash Equilibrium in Games

Let us define an abstract economy[10] as follows. Let $\mathcal{X}_i \subseteq \mathbb{R}^n$ (for some $n \in \mathbb{N}$) denote the action set of player $i \in [\![N]\!]$ in an abstract economy with N players. We use the notation $x_i \in \mathcal{X}_i$ to denote the action of player i. In contrast to a game, the feasible set of actions that player i can choose from is a function of actions of other players $x_{-i} = (x_j)_{j \neq i}$. Let $\mathcal{Z}_i : \times_{j \neq i} \mathcal{X}_j \rightrightarrows \mathcal{X}_i$ be a set-valued mapping that determines the set of feasible actions for player i. The utility of player i is governed by a real-valued function $U_i : \times_{j=1}^N \mathcal{X}_j \to \mathbb{R}$. In this setup (opposed to the one presented in [36]), we assume the players are seeking to minimize their utility.

Definition 7.2 (Equilibrium of an Abstract Economy [36]) *x^* is an equilibrium point of an abstract economy if, for all $i \in [\![N]\!]$, $x_i^* \in \arg\min_{x_i \in \mathcal{Z}_i(x^*_{-i})} U_i(x_i, x^*_{-i})$.*

For any $i \in [\![N]\!]$, we say that $\mathcal{Z}_i$ has a closed graph at $x_{-i} \in \times_{j \neq i} \mathcal{X}_j$ if the set $\{(x_j)_{j \in [\![N]\!]} | x_i \in \mathcal{Z}_i(x_{-i})\}$ is a closed set. Now, we can state the result of [36] regarding the existence of such an equilibrium.

Theorem 7.1 ([36]) *If, for each $i \in [\![N]\!]$, $\mathcal{X}_i$ is a compact convex set, $U_i(x_i, x_{-i})$ is continuous on $\times_{j=1}^N \mathcal{X}_j$ and quasi-convex in x_i for each $x_{-i} \in \times_{j \neq i} \mathcal{X}_j$, $\mathcal{Z}_i$ is a continuous set-valued mapping that has a closed graph, and $\mathcal{Z}_i(x_{-i})$ is a nonempty convex set for each $x_{-i} \in \times_{j \neq i} \mathcal{X}_j$, then the abstract economy admits an equilibrium.*

Note that when $\mathcal{Z}_i(x_{-i}) = \mathcal{X}_i$ for all $x_{-i} \in \times_{j \neq i} \mathcal{X}_j$, and all i, we have a game with finitely many players. Therefore, an abstract economy can be considered as a generalization of a game.

Definition 7.3 (Pure Strategy Nash Equilibrium in Games with Finitely Many Players [53]) *x^* is a pure strategy Nash equilibrium if, for all $i \in [\![N]\!]$, $x_i^* \in \arg\min_{x_i \in \mathcal{X}_i} U_i(x_i, x^*_{-i})$.*

Theorem 7.1 results now in the following useful corollary.

[9] Note that we could alternatively follow the definition and results of [37] in a direct manner, however, in that case, we need more background material presented which might be distracting to the audience.

[10] An abstract economy was originally defined in [36]. It is an extension of a game.

Corollary 7.2 *If, for each $i \in [\![N]\!]$, $\mathcal{X}_i$ is a compact convex set and $U_i(x_i, x_{-i})$ is continuous on $\times_{j=1}^{N} \mathcal{X}_j$ and quasi-convex in x_i for each $x_{-i} \in \times_{j \neq i} \mathcal{X}_j$, then the game admits a pure strategy Nash equilibrium.*

Proof: The proof follows from utilizing Theorem 7.1 when, for all $i \in [\![N]\!]$, $\mathcal{Z}_i(x_{-i}) = \mathcal{X}_i$ for all $x_{-i} \in \times_{j \neq i} \mathcal{X}_j$.

■

With this result in hand, we can go ahead and prove the existence of a Nash equilibrium in the heterogeneous routing game. In the next subsection, we first prove that the problem of finding a Nash equilibrium for the heterogeneous routing game is equivalent to the problem of finding a pure strategy Nash equilibrium in an abstract game[11] with finitely many players. Then, we use Corollary 7.2 to show that this game admits a Nash equilibrium under Assumption 7.1.

3.2 Existence of Nash Equilibrium in Heterogeneous Routing Games

For the sake of simplicity of presentation and without loss of generality (since Θ is finite), we can assume that $\Theta = \{\theta_1, \ldots, \theta_N\}$ where $N = |\Theta|$. Now, let us define the abstract game.

Definition 7.4 *An abstract game is a game with N players in which player $i \in [\![N]\!]$ corresponds to type $\theta_i \in \Theta$ in the heterogeneous routing game. The action of player i is $a_i = (f_{p'}^{\theta_i})_{p' \in \mathcal{P}}$ which belongs to the action set*

$$\mathcal{A}_i = \left\{ (f_{p'}^{\theta_i})_{p' \in \mathcal{P}} \in \mathbb{R}^{|\mathcal{P}|} \;\middle|\; \sum_{p' \in \mathcal{P}_k} f_{p'}^{\theta_i} = F_k^{\theta_i} \right\}.$$

Additionally, the utility of player i is defined as

$$U_i(a_i, a_{-i}) = \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \mathrm{d}u, \tag{2}$$

where a_{-i} represents the actions of the rest of the players $(a_j)_{j \in [\![N]\!] \setminus \{i\}}$ and $\phi_e^{\theta_i} = \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_i}$ is the edge flow of type θ_i for each $i \in [\![N]\!]$.

The following result establishes an interesting relationship between the introduced abstract game and the underlying heterogeneous routing game.

Lemma 7.3 *A flow vector $(f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium of the heterogeneous routing game if and only if $((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, \ldots, (f_{p'}^{\theta_N})_{p' \in \mathcal{P}})$ is a pure strategy Nash equilibrium of the abstract game.*

[11] We use the term "abstract" to emphasize the fact that the introduced game does not have any physical intuition and it is simply a mathematical concept defined for proving the results of this paper. This expression should not be confused with that of an "abstract economy".

Proof: Notice that $((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},\dots,(f_{p'}^{\theta_N})_{p'\in\mathcal{P}})$ being a pure strategy Nash equilibrium (see Definition 7.3) of the abstract game is equivalent to that for all $i\in[\![N]\!]$, $a_i=(f_{p'}^{\theta_i})_{p'\in\mathcal{P}}$ is the best response of player i to the tuple of actions $a_{-i}=((f_{p'}^{\theta_j})_{p'\in\mathcal{P}})_{\theta_j\in\Theta\setminus\{\theta_i\}}$ or, equivalently,

$$\begin{aligned} a_i\in\operatorname*{arg\,min}_{(f_{p'}^{\theta_i})_{p'\in\mathcal{P}}} & \sum_{e\in\mathcal{E}}\int_0^{\phi_e^{\theta_i}}\tilde{\ell}_e^{\theta_i}(u,(\phi_e^{\theta_j})_{\theta_j\in\Theta\setminus\{\theta_i\}})\mathrm{d}u,\\ \text{s.t.} & \sum_{p\in\mathcal{P}:e\in p} f_p^{\theta_i}=\phi_e^{\theta_i},\ \forall e\in\mathcal{E},\\ & \sum_{p\in\mathcal{P}_k} f_p^{\theta_i}=F_k^{\theta_i},\ \forall k\in[\![K]\!],\\ & f_p^{\theta_i}\geq 0,\ \forall p\in\mathcal{P}. \end{aligned}$$

where $\phi_e^{\theta_j}=\sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_j}$ for all $j\in[\![N]\!]\setminus\{i\}$. Notice that due to Assumption 7.1 (*iii*), this problem is indeed a convex optimization problem. Let us define the Lagrangian as

$$\begin{aligned} L_i((\phi_{e'}^{\theta_i})_{e'\in\mathcal{E}},(f_{p'}^{\theta_i})_{p'\in\mathcal{P}}) = & \sum_{e\in\mathcal{E}}\int_0^{\phi_e^{\theta_i}}\tilde{\ell}_e^{\theta_i}(u,(\phi_e^{\theta_j})_{\theta_j\in\Theta\setminus\{\theta_i\}})\mathrm{d}u+\sum_{e\in E}v_e^i\left(\sum_{p\in\mathcal{P}:e\in p}f_p^{\theta_i}-\phi_e^{\theta_i}\right)\\ & -\sum_{k=1}^{K}w_k^i\left(\sum_{p\in\mathcal{P}_k}f_p^{\theta_i}-F_k^{\theta_i}\right)-\sum_{p\in\mathcal{P}}\lambda_p^i f_p^{\theta_i}, \end{aligned}$$

where $(v_e^i)_{e\in\mathcal{E}}\in\mathbb{R}^{|\mathcal{E}|}$, $(w_k^i)_{k\in[\![K]\!]}\in\mathbb{R}^K$, and $(\lambda_p^i)_{p\in\mathcal{P}}\in\mathbb{R}_{\geq 0}^{|\mathcal{P}|}$ are Lagrange multipliers. Now, using Karush–Kuhn–Tucker theorem [54, p. 244], optimality conditions are

$$\frac{\partial}{\partial\phi_e^{\theta_i}}L_i((\phi_{e'}^{\theta_i})_{e'\in\mathcal{E}},(f_{p'}^{\theta_i})_{p'\in\mathcal{P}})=\tilde{\ell}_e^{\theta_i}(\phi_e^{\theta_i},(\phi_e^{\theta_j})_{\theta_j\in\Theta\setminus\{\theta_i\}})-v_e^i=0,\ \ \forall e\in\mathcal{E}, \tag{3}$$

and

$$\frac{\partial}{\partial f_p^{\theta_i}}L_i((\phi_{e'}^{\theta_i})_{e'\in\mathcal{E}},(f_{p'}^{\theta_i})_{p'\in\mathcal{P}})=\left(\sum_{e\in p}v_e^i\right)-w_k^i-\lambda_p^i=0,\ \ \forall p\in\mathcal{P}_k,\forall k\in[\![K]\!]. \tag{4}$$

Additionally, the complimentary slackness conditions (for inequality constraints) result in $\lambda_p^i f_p^i=0$ for all $p\in\mathcal{P}$. Hence, for all k and $p\in\mathcal{P}_k$, we have

$$\begin{aligned} \ell_p^{\theta_i}((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}) &= \sum_{e\in p}\tilde{\ell}_e^{\theta_i}(\phi_e^{\theta_i},(\phi_e^{\theta_j})_{\theta_j\in\Theta\setminus\{\theta_i\}})\\ &=\sum_{e\in p}v_e^i && \text{by (3)}\\ &=w_k^i+\lambda_p^i. && \text{by (4)} \end{aligned}$$

Therefore, for any $p_1, p_2 \in \mathcal{P}_k$, if $f_{p_1}^{\theta_i}, f_{p_2}^{\theta_i} > 0$, we have $\lambda_{p_1}^{\theta_i} = \lambda_{p_2}^{\theta_i} = 0$ (because of the complimentary slackness conditions), which results in

$$\begin{aligned}\ell_{p_1}^{\theta_i}((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}) &= w_k^i \\ &= \ell_{p_2}^{\theta_i}((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}).\end{aligned}$$

Furthermore, for any $p_3 \in \mathcal{P}_k$ such that $f_{p_3}^{\theta_i} = 0$, we get $\lambda_{p_3}^{\theta_i} \geq 0$ (because of dual feasibility, i.e., the Lagrange multipliers associated with inequality constraints must be non-negative), which results in

$$\begin{aligned}\ell_{p_3}^{\theta_i}((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}) &= w_k^i + \lambda_{p_3}^{\theta_i} \\ &\geq w_k^i \\ &= \ell_{p_1}^{\theta_i}((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}).\end{aligned}$$

This completes the proof. ■

Theorem 7.4 *Under Assumption 7.1, the heterogeneous routing game admits at least one Nash equilibrium.*

Proof: Following the result of Lemma 7.3, proving the statement of this theorem is equivalent to showing the fact that the abstract game introduced in Definition 7.4 admits at least one pure strategy Nash equilibrium. First, notice that for all $i \in [\![N]\!]$, $\mathcal{A}_i$ is a non-empty, convex, and compact subset of the Euclidean space $\mathbb{R}^{|\mathcal{P}|}$. Second, $U_i(a_i, a_{-i})$ is continuous in all its arguments (because it is defined as an integral of a real-valued measurable function). Finally, because of Assumption 7.1 (*iii*), $U_i(a_i, a_{-i})$ is a convex function in a_i. Now, it follows from Corollary 7.2 that the abstract game admits at least one pure strategy Nash equilibrium. ■

Remark 7.2 *Theorem 7.4 can be seen as an extension of [27]. In that study, the authors assume that the cost functions are monotone, that is, $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta}) \leq \tilde{\ell}_e^\theta((\bar{\phi}_e^{\theta'})_{\theta'\in\Theta})$ for all $\theta \in \Theta$ if $\phi_e^{\theta'} \leq \bar{\phi}_e^{\theta'}$ if $\theta' \in \Theta$; see [27, p. 58]. This condition, in turn, implies that $\tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta'\in\Theta})$ is a non-decreasing function of all its arguments which is stronger than Assumption 7.1 (iii)'.*

4 Finding a Nash Equilibrium

A family of games that are relatively easy to analyze are potential games. In this section, we give conditions for when the introduced abstract game is a potential game.

Definition 7.5 (Potential Game [22]) *The abstract game is a potential game with potential function* $V : \times_{i=1}^{N} \mathcal{A}_i \to \mathbb{R}$ *if for all* $i \in [\![N]\!]$,

$$V(a_i, a_{-i}) - V(\bar{a}_i, a_{-i}) = U_i(a_i, a_{-i}) - U_i(\bar{a}_i, a_{-i}), \qquad \forall a_i, \bar{a}_i \in \mathcal{A}_i \text{ and } a_{-i} \in \times_{j \in [\![N]\!] \setminus \{i\}} \mathcal{A}_j.$$

The next lemma provides a necessary condition for the existence of a potential function in $\mathcal{C}^2$.

Lemma 7.5 *If the abstract game admits a potential function* $V \in \mathcal{C}^2$*, then*

$$\sum_{e \in p_1 \cap p_2} \left[\frac{\partial}{\partial \phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta' \in \Theta}) - \frac{\partial}{\partial \phi_e^{\theta_j}} \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta}) \right] = 0,$$

for all $i, j \in [\![N]\!]$ *and* $p_1, p_2 \in \mathcal{P}$.

Proof: Since $V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, \ldots, (f_{p'}^{\theta_N})_{p' \in \mathcal{P}})$ is a potential function for the abstract game, it satisfies, for all $i \in [\![N]\!]$,

$$\begin{aligned} &V((f_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}) - V((\bar{f}_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \\ &\quad = U_i((f_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}) - U_i((\bar{f}_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}), \end{aligned}$$

which results in the identity

$$\begin{aligned} &\frac{\partial V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i}} \\ &\quad = \lim_{\epsilon \to 0} \frac{V((f_{p'}^{\theta_i} + \epsilon \delta_{p_1 p'})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}) - V((f_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}})}{\epsilon} \\ &\quad = \lim_{\epsilon \to 0} \frac{U_i((f_{p'}^{\theta_i} + \epsilon \delta_{p_1 p'})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}}) - U_i((f_{p'}^{\theta_i})_{p' \in \mathcal{P}}, ((f_{p'}^{\theta_j})_{p' \in \mathcal{P}})_{\theta_j \in \Theta \setminus \{\theta_i\}})}{\epsilon} \\ &\quad = \frac{\partial U_i((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i}}. \end{aligned} \tag{5}$$

in which δ_{ij} denotes the Kronecker index (or delta) defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Hence, we get

$$\begin{aligned} \frac{\partial V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i}} &= \frac{\partial}{\partial f_{p_1}^{\theta_i}} \sum_{e \in \mathcal{E}} \int_0^{\phi_e^{\theta_i}} \tilde{\ell}_e^{\theta_i}(u, (\phi_e^{\theta_j})_{\theta_j \in \Theta \setminus \{\theta_i\}}) \mathrm{d}u \\ &= \sum_{e \in p_1} \tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta' \in \Theta}). \end{aligned}$$

Now, because of Clairaut-Schwarz theorem [55, p. 1067], we know that the following equality must hold since $V \in \mathcal{C}^2$,

$$\frac{\partial^2 V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_1}^{\theta_i} \partial f_{p_2}^{\theta_j}} = \frac{\partial^2 V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_{p_2}^{\theta_j} \partial f_{p_1}^{\theta_i}}. \tag{6}$$

Let us calculate

$$\begin{aligned}\frac{\partial^2 V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_{p_1}^{\theta_i}\partial f_{p_2}^{\theta_j}} &= \frac{\partial}{\partial f_{p_1}^{\theta_i}}\left[\frac{\partial V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_{p_2}^{\theta_j}}\right]\\ &= \frac{\partial}{\partial f_{p_1}^{\theta_i}}\left[\sum_{e\in p_2}\tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta'\in\Theta})\right]\\ &= \sum_{e\in p_1\cap p_2}\frac{\partial\tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta'\in\Theta})}{\partial\phi_e^{\theta_i}},\end{aligned} \tag{7}$$

and, similarly,

$$\frac{\partial^2 V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_{p_2}^{\theta_j}\partial f_{p_1}^{\theta_i}} = \sum_{e\in p_1\cap p_2}\frac{\partial\tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta'\in\Theta})}{\partial\phi_e^{\theta_j}}. \tag{8}$$

Substituting (6) and (7) into (8) results in

$$\sum_{e\in p_1\cap p_2}\left[\frac{\partial}{\partial\phi_e^{\theta_j}}\tilde{\ell}_e^{\theta_i}((\phi_e^{\theta'})_{\theta'\in\Theta}) - \frac{\partial}{\partial\phi_e^{\theta_i}}\tilde{\ell}_e^{\theta_j}((\phi_e^{\theta'})_{\theta'\in\Theta})\right] = 0,$$

for all $p_1, p_2 \in \mathcal{P}$ and $\theta_i, \theta_j \in \Theta$. ■

Interestingly, we can prove that this condition is also a sufficient condition for the existence of a potential function (that belongs to $\mathcal{C}^2$) whenever two types of players are participating in the heterogeneous routing game.

Lemma 7.6 *Assume that* $|\Theta| = 2$. *If*

$$\sum_{e\in p_1\cap p_2}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right] = 0,$$

for all $p_1, p_2 \in \mathcal{P}$, *then*

$$\begin{aligned}V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) = \sum_{e\in\mathcal{E}}\Bigg[&\int_0^{\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2\\ &-\int_0^{\phi_e^{\theta_2}}\int_0^{\phi_e^{\theta_1}}\frac{\partial}{\partial u_2}\tilde{\ell}_e^{\theta_1}(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2\Bigg]\end{aligned}$$

is a potential function for the abstract game.

Proof: Notice that for all $p \in \mathcal{P}$, we get

$$\begin{aligned}\frac{\partial V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_p^{\theta_1}} &= \frac{\partial}{\partial f_p^{\theta_1}}\Bigg(\sum_{e\in\mathcal{E}}\Bigg[\int_0^{\phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1+\int_0^{\phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2 \\ &\quad -\int_0^{\phi_e^{\theta_2}}\int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u_2}\tilde{\ell}_e^{\theta_1}(u_1,u_2)\mathrm{d}u_1\mathrm{d}u_2\Bigg]\Bigg) \\ &= \sum_{e\in p}\Bigg[\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}} \frac{\partial}{\partial \phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2 \\ &\quad -\int_0^{\phi_e^{\theta_2}} \frac{\partial}{\partial u_2}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2\Bigg] \\ &= \sum_{e\in p}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) \\ &\quad +\sum_{e\in p}\int_0^{\phi_e^{\theta_2}}\left[\frac{\partial}{\partial \phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2) - \frac{\partial}{\partial u_2}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u_2)\right]\mathrm{d}u_2. \qquad (9)\end{aligned}$$

Now, let us define

$$\Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},(\phi_e^{\theta_2})_{e\in\mathcal{E}}) = \sum_{e\in p}\int_0^{\phi_e^{\theta_2}}\left[\frac{\partial}{\partial \phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u) - \frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u)\right]du.$$

We have

$$\frac{\partial \Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},(\phi_e^{\theta_2})_{e\in\mathcal{E}})}{\partial f_{\hat{p}}^{\theta_2}} = \sum_{e\in p\cap\hat{p}}\left[\frac{\partial}{\partial \phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right]=0,$$

for all $\hat{p} \in \mathcal{P}$. Noticing that $\phi_e^{\theta_2} = \sum_{\hat{p}\in\mathcal{P}:e\in\hat{p}} f_{\hat{p}}^{\theta_2}$ for all $e \in \mathcal{E}$, we get

$$\frac{\partial \Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},(\phi_e^{\theta_2})_{e\in\mathcal{E}})}{\partial \phi_e^{\theta_2}} = \sum_{\hat{p}\in\mathcal{P}:e\in\hat{p}}\frac{\partial \Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},(\phi_e^{\theta_2})_{e\in\mathcal{E}})}{\partial f_{\hat{p}}^{\theta_2}} = 0, \quad \forall e \in \mathcal{E}.$$

Thus, $\Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},(\phi_e^{\theta_2})_{e\in\mathcal{E}}) = \Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},0) = 0$. Setting $\Psi((\phi_e^{\theta_1})_{e\in\mathcal{E}},(\phi_e^{\theta_2})_{e\in\mathcal{E}}) = 0$ (see definition above) inside (9) results in

$$\frac{\partial V((f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta})}{\partial f_p^{\theta_1}} = \sum_{e\in p}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) = \frac{\partial U_1((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})}{\partial f_p^{\theta_1}}, \qquad (10)$$

where the partial derivatives of U_1 can be computed from its definition in (2). Let $(f_{p'}^{\theta_1})_{p'\in\mathcal{P}}$ and $(\bar{f}_{p'}^{\theta_1})_{p'\in\mathcal{P}}$ be arbitrary points in set of actions $\mathcal{A}_1$. Furthermore, let $r : [0,1] \to \mathcal{A}_1$ be a continuously differentiable mapping (i.e., $r \in \mathcal{C}^1$) such that $r(0) = (\bar{f}_{p'}^{\theta_1})_{p'\in\mathcal{P}}$ and $r(1) = (f_{p'}^{\theta_1})_{p'\in\mathcal{P}}$ which remains inside $\mathcal{A}_1 \subseteq \mathbb{R}^{|\mathcal{P}|}$ for all

$t \in (0,1)$. We define graph(r) as the collection of all ordered pairs $(t, r(t))$ for all $t \in [0,1]$, which denotes a continuous path that connects $(f_{p'}^{\theta_1})_{p' \in \mathcal{P}}$ and $(\bar{f}_{p'}^{\theta_1})_{p' \in \mathcal{P}}$. We know that at least one such mapping exists because $\mathcal{A}_1$ is a simply connected set for all $i \in [\![N]\!]$. Hence, we have

$$\begin{aligned}
\int_{\mathrm{graph}(r)} \left[\frac{\partial V(a_1,a_2)}{\partial a_1}\bigg|_{a_1=r} \right]^\top \mathrm{d}r &= \int_0^1 \left[\frac{\partial V(a_1,a_2)}{\partial a_1}\bigg|_{a_1=r(t)} \right]^\top \frac{\partial r(t)}{\partial t} \mathrm{d}t \\
&= \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}t} V(r(t), a_2) \right] \mathrm{d}t \\
&= V(r(1), a_2) - V(r(0), a_2) \\
&= V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}) - V((\bar{f}_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}),
\end{aligned}$$

where the second to last equality is a direct consequence of the fundamental theorem of calculus [55, p. 1257]. Note that this equality holds irrespective of the selected path. Therefore,

$$\begin{aligned}
V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, &(f_{p'}^{\theta_2})_{p' \in \mathcal{P}}) - V((\bar{f}_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}) \\
&= \int_{\mathrm{graph}(r)} \left[\frac{\partial V(a_1,a_2)}{\partial a_1}\bigg|_{a_1=r} \right]^\top \mathrm{d}r \\
&= \int_{\mathrm{graph}(r)} \left[\frac{\partial U_1(a_1,a_2)}{\partial a_1}\bigg|_{a_1=r} \right]^\top \mathrm{d}r \qquad \text{by (10)} \\
&= U_1((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}) - U_1((\bar{f}_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}),
\end{aligned}$$

Similarly, we can also prove

$$\begin{aligned}
\frac{\partial V((f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta})}{\partial f_p^{\theta_2}} &= \frac{\partial}{\partial f_p^{\theta_2}} \Bigg(\sum_{e \in \mathcal{E}} \Bigg[\int_0^{\phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) \mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u_2) \mathrm{d}u_2 \\
&\qquad - \int_0^{\phi_e^{\theta_2}} \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u_2} \tilde{\ell}_e^{\theta_1}(u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 \Bigg] \Bigg) \\
&= \sum_{e \in p} \Bigg[\int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial \phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) \mathrm{d}u_1 + \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) \\
&\qquad - \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial \phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) \mathrm{d}u_1 \Bigg] \\
&= \sum_{e \in p} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}), \qquad (11)
\end{aligned}$$

which results in

$$\frac{\partial V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}})}{\partial f_p^{\theta_2}} = \sum_{e \in p} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \frac{\partial U_2((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}})}{\partial f_p^{\theta_2}},$$

and, consequently,

$$V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})-V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(\bar{f}_{p'}^{\theta_2})_{p'\in\mathcal{P}})$$
$$=U_2((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})-U_2((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(\bar{f}_{p'}^{\theta_2})_{p'\in\mathcal{P}}).$$

This concludes the proof. ■

Now, combing the previous two lemmas results in the main result of this section.

Theorem 7.7 *Assume that* $|\Theta| = 2$. *The abstract game admits a potential function* $V \in \mathcal{C}^2$ *if and only if*

$$\sum_{e\in p_1\cap p_2}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2})-\frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right]=0,$$

for all $p_1, p_2 \in \mathcal{P}$.

Proof: The proof easily follows from Lemmas 7.5 and 7.6. Note that the potential function presented in Lemma 7.6 belongs to $\mathcal{C}^2$ due to Assumption 7.1 (i). ■

Following a basic property of potential games, it is easy to prove the following corollary which shows that the process of finding a Nash equilibrium of the heterogeneous routing game is equivalent to solving an optimization problem.

Corollary 7.8 *Assume that* $|\Theta| = 2$. *Furthermore, let*

$$\sum_{e\in p}\left[\frac{\partial}{\partial\phi_e^{\theta_1}}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2})-\frac{\partial}{\partial\phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2})\right]=0,$$

for all $p \in \mathcal{P}$. *If* $f = (f_{p'}^{\theta'})_{p'\in\mathcal{P},\theta'\in\Theta}$ *is a solution of the optimization problem*

$$\begin{aligned}
\min\ & V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}),\\
\text{s.t.}\ & \sum_{p\in\mathcal{P}:e\in p} f_p^{\theta_1}=\phi_e^{\theta_1}\ \textit{and}\sum_{p\in\mathcal{P}:e\in p} f_p^{\theta_2}=\phi_e^{\theta_2},\ \forall e\in\mathcal{E},\\
& \sum_{p\in\mathcal{P}_k} f_p^{\theta_1}=F_k^{\theta_1}\ \textit{and}\sum_{p\in\mathcal{P}_k} f_p^{\theta_2}=F_k^{\theta_2},\ \forall k\in[\![K]\!],\\
& f_p^{\theta_1},f_p^{\theta_2}\geq 0,\ \forall p\in\mathcal{P},
\end{aligned}$$

where $V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}})$ *is defined in Lemma 7.6, then* $f = (f_p^{\theta})_{p\in\mathcal{P},\theta\in\Theta}$ *is a Nash equilibrium of the heterogeneous routing game.*

Proof: The proof is consequence of the fact that a minimizer of the potential function is a pure strategy Nash equilibrium of a potential game; see [22]. ■

Notice that so far we have proved that a minimizer of the potential function is a Nash equilibrium but not the other way round. Now, we are ready to prove this whenever the potential function is convex.

Corollary 7.9 *Let* $|\Theta| = 2$ *and*

$$\frac{\partial}{\partial \phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \frac{\partial}{\partial \phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}),$$

for all $e \in \mathcal{E}$. *Furthermore, assume that the potential function* $V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}})$, *defined in Lemma 7.6, is a convex function. Then* $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ *is a Nash equilibrium of the heterogeneous routing game if and only if it is a solution of the convex optimization problem*

$$\begin{aligned} \min \quad & V((f_{p'}^{\theta_1})_{p' \in \mathcal{P}}, (f_{p'}^{\theta_2})_{p' \in \mathcal{P}}), \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_1} = \phi_e^{\theta_1} \text{ and } \sum_{p \in \mathcal{P}: e \in p} f_p^{\theta_2} = \phi_e^{\theta_2}, \ \forall e \in \mathcal{E}, \\ & \sum_{p \in \mathcal{P}_k} f_p^{\theta_1} = F_k^{\theta_1} \text{ and } \sum_{p \in \mathcal{P}_k} f_p^{\theta_2} = F_k^{\theta_2}, \ \forall k \in [\![K]\!], \\ & f_p^{\theta_1}, f_p^{\theta_2} \geq 0, \ \forall p \in \mathcal{P}. \end{aligned}$$

Proof: See Appendix A. ■

Remark 7.3 *Note that Corollary 7.9 is proved at the price of a more conservative condition because the conditions in Corollary 7.8 requires the summation of the differences between the derivatives of the cost functions to be equal to zero while Corollary 7.9 needs the individual differences to be equal to zero. Notice that Corollary 7.9 provides the same sufficient condition for characterizing the set of all equilibria as in [8, 24], but these references handle the general case of* $|\Theta| \geq 2$ *(specifically, see Proposition 1 and Theorem 1 in [24]). Therefore, we can see that the presented condition in Corollary 7.8 is tighter than the results of [8, 24] (since it is also a necessary condition); however, it is only valid for* $|\Theta| = 2$ *in contrast.*

4.1 Example: Routing Game with Platooning Incentives

Let us examine the implications of Corollary 7.9 in the routing game with platooning incentives in Subsection 2.3. For the linearized model, we can easily calculate that

$$\frac{\partial \tilde{\ell}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})}{\partial \phi_e^{\mathrm{t}}} = -L_e a_e / b_e^2, \tag{12}$$

$$\begin{aligned} \frac{\partial \tilde{\ell}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})}{\partial \phi_e^{\mathrm{c}}} &= -L_e a_e / b_e^2 + 2 c_0 \alpha \gamma(0) b_e a_e \\ &= -L_e a_e / b_e^2 + 2 c_0 \alpha b_e a_e. \end{aligned} \tag{13}$$

where the second equality follows from $\gamma_e(0) = 1$, which holds because from the definition of the mapping $\gamma : \mathbb{R}_{\geq 0} \to [0, 1]$, we know that in this case (i.e., when no trucks are using that edge) the air drag coefficient is equal to its nominal value. Therefore, the condition of Corollary 7.9 does not hold (unless $c_0 = 0$). Noting that if the problem of finding a Nash equilibrium in the heterogeneous routing game is numerically intractable, it might be highly unlikely for the drivers to figure out a Nash equilibrium in reasonable time (let alone an efficient one) and utilize it, which might result in wasting parts of the transportation network resources. Therefore, a natural question that comes to mind is whether it is possible to guarantee the existence of a potential function for a heterogeneous routing game by imposing appropriate tolls.

5 Imposing Tolls to Guarantee the Existence of a Potential Function

5.1 Definition and Results

Let us assume that a vehicle of type $\theta \in \Theta$ must pay a toll $\tilde{\tau}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$ for using an edge $e \in \mathcal{E}$, where $\phi_e^\theta = \sum_{p \in \mathcal{P}: e \in p} f_p^\theta$. Therefore, a vehicle using path $p \in \mathcal{P}_k$ endures a total cost of $\ell_p^\theta(f) + \tau_p^\theta(f)$, where $\tau_p(f)$ is the total amount of money that the vehicle must pay for using path p and can be calculated as $\tau_p^\theta(f) = \sum_{e \in p} \tilde{\tau}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta})$. The definition of a Nash equilibrium is slightly modified to account for the tolls.

Definition 7.6 (Nash Equilibrium in Heterogeneous Routing Game with Tolls) *A flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ is a Nash equilibrium for the routing game with tolls if, for all $k \in [\![K]\!]$ and $\theta \in \Theta$, whenever $f_p^\theta > 0$ for some path $p \in \mathcal{P}_k$, then $\ell_p^\theta(f) + \tau_p^\theta(f) \leq \ell_{p'}^\theta(f) + \tau_{p'}^\theta(f)$ for all $p' \in \mathcal{P}_k$.*

Before stating the main result of this section, note that we can have both distinguishable and indistinguishable types. This characterization is of special interest when considering the implementation of tolls. For distinguishable types, we can impose individual tolls for each type. However, for indistinguishable types, the tolls are independent of the type. To give an example, if $\Theta = \{\text{cars}, \text{trucks}\}$, we can impose different tolls for each group of vehicles while if $\Theta = \{\text{patient drivers}, \text{impatient drivers}\}$, we cannot. Notice that in the case of indistinguishable types, one might argue that we cannot measure $\phi_e^{\theta_i}$ for each $\theta_i \in \Theta$ individually (because as we motivated the type of user may not be identified from physical traits). However, we can use surveys and historical data to extract the statistics of each type (e.g, to realize what ratio of the actual flow belongs to each type) but when calculating the tolls for each user we cannot force that user to participate in a survey. We treat these two cases separately.

Proposition 7.10 (DISTINGUISHABLE TYPES) *Assume that* $|\Theta| = 2$. *The abstract game admits the potential function*

$$V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}}, (f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) = \sum_{e\in\mathcal{E}} \Bigg[\int_0^{\phi_e^{\theta_1}} (\tilde{\ell}_e^{\theta_1}(u_1, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta_1}(u_1, \phi_e^{\theta_2}))\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}} (\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, u_2) + \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, u_2))\mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}} \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u_2} (\tilde{\ell}_e^{\theta_1}(u_1, u_2) + \tilde{\tau}_e^{\theta_1}(u_1, u_2))\mathrm{d}u_1\mathrm{d}u_2 \Bigg]$$

if

$$\frac{\partial \tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} - \frac{\partial \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} = \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}},$$

for all $e \in \mathcal{E}$.

Proof: See Appendix B. ∎

Proposition 7.11 (INDISTINGUISHABLE TYPES) *Assume that* $|\Theta| = 2$. *The abstract game admits the potential function* $V \in \mathcal{C}^2$ *in Proposition 7.10 with*

$$\tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})$$

if

$$\frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} - \frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} = \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}},$$

for all $e \in \mathcal{E}$.

Proof: The proof immediately follows from using Proposition 7.10 with the constraint that the tolls may not depend on the type, i.e., $\tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})$. ∎

In general, we can prove the following corollary concerning the type-independent tolls.

Corollary 7.12 (INDISTINGUSHABLE TYPES) *Assume that* $|\Theta| = 2$. *The abstract game admits a potential function* $V \in \mathcal{C}^2$ *if the imposed tolls are of the following form*

$$\tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2}) = c_e + \int_0^{\phi_e^{\theta_2}} f_e(q, \phi_e^{\theta_1} + \phi_e^{\theta_2} - q)\mathrm{d}q + \psi_e(\phi_e^{\theta_1} + \phi_e^{\theta_2}),$$

where $c_e \in \mathbb{R}_{\geq 0}$, $\psi_e \in \mathcal{C}^1$, *and* $f_e(x, y) = \partial \tilde{\ell}_e^{\theta_2}(y, x)/\partial y - \partial \tilde{\ell}_e^{\theta_1}(y, x)/\partial x$ *for all* $e \in \mathcal{E}$.

Proof: See Appendix C. ∎

Throughout this subsection, we assumed that all the drivers portray similar sensitivity to the imposed tolls. This is indeed a source of conservatism, specially when dealing with routing games in which the heterogeneity is caused by the fact that the drivers react differently to the imposed tolls. Certainly, an avenue for future research is to develop tolls for a more general setup.

5.2 Example: Routing Game with Platooning Incentives

Let us examine the possibility of finding a set of tolls that satisfies the conditions of Propositions 7.10 and 7.11 for the heterogeneous routing game introduced in Subsection 2.3.

- **Distinguishable Types-Case 1:** Substituting (12) and (13) into the condition of Proposition 7.10 results in

$$\frac{\partial \tilde{\tau}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})}{\partial \phi_e^{\mathrm{t}}} - \frac{\partial \tilde{\tau}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}})}{\partial \phi_e^{\mathrm{c}}} = 2c_0\alpha b_e a_e. \tag{14}$$

  Following simple algebraic calculations, we can check that the tolls $\tilde{\tau}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = 0$ and $\tilde{\tau}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = (2c_0\alpha b_e a_e)\phi_e^{\mathrm{c}}$ satisfy (14). Noticing that $\tilde{\tau}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) \leq 0$ because by definition $a_e \in \mathbb{R}_{\leq 0}$, these terms can be interpreted as subsidies paid to the trucks to compensate for the fuel that is wasted due to presence of the cars on that specific edge.

- **Distinguishable Types-Case 2:** Another example of appropriate tolls is $\tilde{\tau}_e^{\mathrm{t}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = 0$ and $\tilde{\tau}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = (-2c_0\alpha b_e a_e)\phi_e^{\mathrm{t}}$. Now, we have $\tilde{\tau}_e^{\mathrm{c}}(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) \geq 0$. In this case, the cars pay directly for the increased fuel consumption of the trucks and, therefore, they are inclined to travel along the edges that trucks do not use.

- **Indistinguishable Types:** For this case, using Corollary 7.12, it is easy to see that tolls $\tilde{\tau}_e(\phi_e^{\mathrm{c}}, \phi_e^{\mathrm{t}}) = (2c_0\alpha b_e a_e)\phi_e^{\mathrm{t}}$ work fine. We use these tolls in the numerical example developed in Section 7.

6 Price of Anarchy for Affine Cost Functions

In the routing game literature, it is a widely known fact that generally, a Nash equilibrium is inefficient even when dealing with homogeneous routing games; see [14, 17, 21]. To quantify this inefficiency, many studies have used *Price of Anarchy* (PoA) as a metric.

6.1 Social Cost Function

First, let us define the social cost of a flow vector $f = (f_{p'}^{\theta'})_{p' \in \mathcal{P}, \theta' \in \Theta}$ as

$$\begin{aligned} C(f) &\triangleq \sum_{p \in \mathcal{P}} \sum_{\theta \in \Theta} f_p^\theta \ell_p^\theta(f) \\ &= \sum_{e \in \mathcal{E}} \sum_{\theta \in \Theta} \phi_e^\theta \tilde{\ell}_e^\theta((\phi_e^{\theta'})_{\theta' \in \Theta}), \end{aligned}$$

where the second equality can be easily obtained by rearranging the terms. Using this social cost, we can define the optimal flow that we will use later for comparison with the Nash equilibrium.

Definition 7.7 (Socially Optimal Flow) *$f \in \mathcal{F}$ is a socially optimal flow if $C(f) \leq C(\bar{f})$ for all $\bar{f} \in \mathcal{F}$.*

Definition 7.8 (PoA) *The price of anarchy is defined as*

$$\text{PoA} = \sup_{f^{\text{Nash}} \in \mathcal{N}} \frac{C(f^{\text{Nash}})}{\min_{f \in \mathcal{F}} C(f)},$$

where $\mathcal{N}$ denotes the set of Nash equilibria of the heterogeneous routing game. In this definition, we follow the convention that "$\frac{0}{0} = 1$".

6.2 Bounding the Price of Anarchy for Two Types with Affine Cost Functions

Here, we present an upper bound for the inefficiency of the Nash equilibrium in heterogeneous routing games when $|\Theta| = 2$. The edge cost functions are taken to be affine functions of the form

$$\begin{aligned} \ell_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) &= \alpha_{\theta_1\theta_1}^e \phi_e^{\theta_1} + \alpha_{\theta_1\theta_2}^e \phi_e^{\theta_2} + \beta_{\theta_1}^e, \\ \ell_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) &= \alpha_{\theta_2\theta_1}^e \phi_e^{\theta_1} + \alpha_{\theta_2\theta_2}^e \phi_e^{\theta_2} + \beta_{\theta_2}^e, \end{aligned}$$

where $\alpha_{\theta_1\theta_1}^e, \alpha_{\theta_1\theta_2}^e, \alpha_{\theta_2\theta_1}^e, \alpha_{\theta_2\theta_2}^e, \beta_{\theta_1}^e, \beta_{\theta_2}^e \in \mathbb{R}_{\geq 0}$ are parameters of the routing game for each edge $e \in \mathcal{E}$. Notice that the condition $\alpha_{\theta_1\theta_1}^e, \alpha_{\theta_1\theta_2}^e, \alpha_{\theta_2\theta_1}^e, \alpha_{\theta_2\theta_2}^e \in \mathbb{R}_{\geq 0}$ implies that the cost of using an edge is increasing in each flow separately (i.e., when a driver of any type switches to an edge, she cannot decrease the cost of the users on this new edge) while $\beta_{\theta_1}^e, \beta_{\theta_2}^e \in \mathbb{R}_{\geq 0}$ implies that the starting cost of using a road is non-negative. This assumption is certainly stronger than Assumption 7.1. Subsection 2.3 presents a motivating example for affine cost functions.

Theorem 7.13 *Let*

$$\alpha_{\theta_2\theta_1}^e = \alpha_{\theta_1\theta_2}^e \tag{15a}$$

$$\begin{bmatrix} \alpha_{\theta_1\theta_1}^e & \alpha_{\theta_1\theta_2}^e \\ \alpha_{\theta_2\theta_1}^e & \alpha_{\theta_2\theta_2}^e \end{bmatrix} \geq 0, \tag{15b}$$

for all $e \in \mathcal{E}$. Then, PoA ≤ 2.

Proof: First, note that if $\alpha^e_{\theta_2\theta_1} = \alpha^e_{\theta_1\theta_2}$ for all $e \in \mathcal{E}$, the condition of Corollary 7.8 is satisfied. Therefore, we can easily calculate the potential function as

$$\begin{aligned} V(f) =& \sum_{e\in\mathcal{E}} \Big[\frac{1}{2}\alpha^e_{\theta_1\theta_1}(\phi_e^{\theta_1})^2 + (\alpha^e_{\theta_1\theta_2}\phi_e^{\theta_2} + \beta^e_{\theta_1})\phi_e^{\theta_1} + \frac{1}{2}\alpha^e_{\theta_2\theta_2}(\phi_e^{\theta_2})^2 \\ &+ (\alpha^e_{\theta_2\theta_1}\phi_e^{\theta_1} + \beta^e_{\theta_2})\phi_e^{\theta_2} - \alpha^e_{\theta_1\theta_2}\phi_e^{\theta_1}\phi_e^{\theta_2}\Big] \\ =& \sum_{e\in\mathcal{E}} \Big[\frac{1}{2}\phi_e^{\theta_1}\ell_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \frac{1}{2}\beta^e_{\theta_1}\phi_e^{\theta_1} + \frac{1}{2}\phi_e^{\theta_2}\ell_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \frac{1}{2}\beta^e_{\theta_2}\phi_e^{\theta_2}\Big] \\ =& \frac{1}{2}C(f) + \sum_{e\in\mathcal{E}}\frac{1}{2}\Big[\beta^e_{\theta_1}\phi_e^{\theta_1} + \beta^e_{\theta_2}\phi_e^{\theta_2}\Big], \end{aligned} \tag{16}$$

Furthermore, following the argument of [54, p. 71], we know that the social cost function is a convex function if and only if (15b) is satisfied. Notice that (16) shows that the potential function V is a convex function if the social cost function C is a convex function (because the summation of a convex function and a linear function is a convex function). Let us use $\bar{f}$ and f to denote the Nash equilibrium and the socially optimal flow, respectively. Now, we can prove inequality

$$\begin{aligned} C(\bar{f}) &\leq 2V(\bar{f}) && \text{by (16)}, \beta^e_{\theta_1}, \beta^e_{\theta_2} \geq 0 \\ &\leq 2V(f) && \text{by Corollary 7.9} \\ &\leq 2\sum_{e\in\mathcal{E}}\Big(\int_0^{\phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2 \\ &\quad - \int_0^{\phi_e^{\theta_2}}\int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(t,u)\mathrm{d}t\mathrm{d}u\Big) && \text{by Definition of } V \\ &\leq 2\sum_{e\in\mathcal{E}}\Big(\int_0^{\phi_e^{\theta_1}} \tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\mathrm{d}u_1 + \int_0^{\phi_e^{\theta_2}} \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\mathrm{d}u_2\Big) && \text{by } \alpha^e_{\theta_1\theta_2}, \alpha^e_{\theta_2\theta_1} \geq 0 \\ &\leq 2\Big(\sum_{e\in\mathcal{E}}\int_0^{\phi_e^{\theta_1}} \Big[\tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2}) + u_1\frac{\partial}{\partial u_1}\tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})\Big]\mathrm{d}u_1 \\ &\quad + \sum_{e\in\mathcal{E}}\int_0^{\phi_e^{\theta_2}} \Big[\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2) + u_2\frac{\partial}{\partial u_2}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)\Big]\mathrm{d}u_2\Big) && \text{by } \alpha^e_{\theta_1\theta_1}, \alpha^e_{\theta_2\theta_2} \geq 0 \\ &\leq 2\Big(\sum_{e\in\mathcal{E}}\phi_e^{\theta_1}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \sum_{e\in\mathcal{E}}\phi_e^{\theta_2}\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2})\Big) \\ &= 2C(f). \end{aligned} \tag{17}$$

This completes the proof. ■

Notice that in many practical situations (such as the one presented in Subsection 4.1 for routing games with platooning incentives), $\alpha^e_{\theta_2\theta_1} \neq \alpha^e_{\theta_1\theta_2}$. Therefore, we may not be able to use Theorem 7.13 to find an upper bound for the PoA.

Table 7.1: Parameters of the heterogeneous routing game in the numerical example.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
α_{aa}	1.0	2.0	3.0	1.0	4.0	0.5	1.0	1.0	2.0	1.0	4.0	1.0
α_{at}	0.6	0.4	0.1	0.1	0.5	0.1	0.7	0.1	0.1	0.2	0.1	0.3
α_{tt}	2.0	3.0	1.0	0.8	1.0	1.0	1.5	3.0	1.7	3.0	1.0	1.3
β_{a}	2.0	2.0	4.5	2.0	2.0	4.5	2.0	2.0	4.5	2.0	2.0	4.5
β_{t}	4.0	4.0	1.5	4.0	4.0	1.5	4.0	4.0	1.5	4.0	4.0	1.5

However, as also discussed in Section 5, in some cases, we might be able to manipulate these gains through appropriate tolls to make sure (15a) holds. In addition, condition (15b) is equivalent to the condition that $\alpha^e_{\theta_1\theta_1}\alpha^e_{\theta_2\theta_2} \geq \alpha^e_{\theta_2\theta_1}\alpha^e_{\theta_1\theta_2}$ for all $e \in \mathcal{E}$. This condition intuitively means that cost function of each type of vehicles is more influenced by the flow of its own type than the flow of the other type. This condition may not hold in general in transportation networks. In such case, instead of using Corollary 7.9, we may use Corollary 7.8 in the proof of Theorem 7.13 (that is the only place that we use the convexity of the potential function which we proved using the convexity of the social decision function). However, doing so, we cannot bound the ratio $C(f^{\mathrm{Nash}})/\min_f C(f)$ for all $f^{\mathrm{Nash}} \in \mathcal{N}$. Therefore, instead of showing that PoA is bounded from above by two, we can then only show that the Price of Stability[12] is upper bounded by two (because we can show that the ratio is bounded by two for only one Nash equilibrium and not for all Nash equilibria).

7 Numerical Example

In this section, we present a numerical example motivated by the routing game with platooning incentives in Subsection 2.3. We use the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 1. We have three commodities $(s_1, t_1) = (0, 1)$, $(s_2, t_2) = (2, 3)$, and $(s_3, t_3) = (7, 8)$. The corresponding paths for the commodities are

$$\begin{aligned}
\mathcal{P}_1 &= \{\{e_1\},\{e_2, e_4, e_3\},\{e_2, e_7, e_5\}\},\\
\mathcal{P}_2 &= \{\{e_{10}\},\{e_9, e_7, e_8\},\{e_9, e_4, e_6\}\},\\
\mathcal{P}_3 &= \{\{e_{11}, e_{10}, e_0\},\{e_{11}, e_9, e_7, e_8, e_0\},\{e_{11}, e_9, e_4, e_6, e_0\}\}.
\end{aligned}$$

The edge cost functions are taken to be affine functions of the form

$$\begin{aligned}
\tilde{\ell}^{\mathrm{c}}_{e_i}(\phi^{\mathrm{c}}_{e_i}, \phi^{\mathrm{t}}_{e_i}) &= \alpha^{(i)}_{\mathrm{cc}}\phi^{\mathrm{c}}_{e_i} + \bar{\alpha}^{(i)}_{\mathrm{ct}}\phi^{\mathrm{t}}_{e_i} + \beta^{(i)}_{\mathrm{c}},\\
\tilde{\ell}^{\mathrm{t}}_{e_i}(\phi^{\mathrm{c}}_{e_i}, \phi^{\mathrm{t}}_{e_i}) &= \alpha^{(i)}_{\mathrm{tc}}\phi^{\mathrm{c}}_{e_i} + \bar{\alpha}^{(i)}_{\mathrm{tt}}\phi^{\mathrm{t}}_{e_i} + \beta^{(i)}_{\mathrm{t}},
\end{aligned}$$

where the definitions and the physical intuition of the parameters $\alpha^{(i)}_{\mathrm{cc}}, \alpha^{(i)}_{\mathrm{tc}}, \bar{\alpha}^{(i)}_{\mathrm{tt}}, \bar{\alpha}^{(i)}_{\mathrm{ct}}$, $\beta^{(i)}_{\mathrm{c}}, \beta^{(i)}_{\mathrm{t}}$ can be found in Subsection 2.3. Recalling that $\bar{\alpha}^{(i)}_{\mathrm{tc}} \neq \bar{\alpha}^{(i)}_{\mathrm{ct}}$ (see Subsec-

[12] Price of Stability (PoS), or commonly known as the optimistic Price of Anarchy, is defined as $\inf_{f^{\mathrm{Nash}} \in \mathcal{N}} C(f^{\mathrm{Nash}})/\min_{f \in \mathcal{F}} C(f)$; note that we use inf operator instead of sup operator in this definition in contrast to that of Definition 7.8. See [56] for more explanation regarding the difference between PoS and PoA.

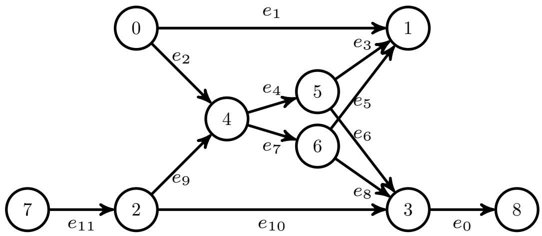

Figure 1: Transportation network in the numerical example.

tion 4.1), the condition of Corollary 7.8 is not satisfied. Therefore, we use the tax $\tilde{\tau}_{e_i}(\phi^{\rm c}_{e_i},\phi^{\rm t}_{e_i}) = (2c_0\alpha b_e a_e)\phi^{\rm t}_{e_i}$ which is developed in Subsection 5.2. This results in

$$\tilde{\ell}^{\rm c}_{e_i}(\phi^{\rm c}_{e_i},\phi^{\rm t}_{e_i}) + \tilde{\tau}_{e_i}(\phi^{\rm c}_{e_i},\phi^{\rm t}_{e_i}) = \alpha^{(i)}_{\rm cc}\phi^{\rm c}_{e_i} + \alpha^{(i)}_{\rm ct}\phi^{\rm t}_{e_i} + \beta^{(i)}_{\rm c},$$
$$\tilde{\ell}^{\rm t}_{e_i}(\phi^{\rm c}_{e_i},\phi^{\rm t}_{e_i}) + \tilde{\tau}_{e_i}(\phi^{\rm c}_{e_i},\phi^{\rm t}_{e_i}) = \alpha^{(i)}_{\rm tc}\phi^{\rm c}_{e_i} + \alpha^{(i)}_{\rm tt}\phi^{\rm t}_{e_i} + \beta^{(i)}_{\rm t},$$

where $\alpha^{(i)}_{\rm ct} = \bar{\alpha}^{(i)}_{\rm ct} + 2c_0\alpha b_e a_e$ and $\alpha^{(i)}_{\rm tt} = \bar{\alpha}^{(i)}_{\rm tt} + 2c_0\alpha b_e a_e$. In this case, we can calculate the potential function as

$$V = \sum_{i=0}^{11}\Big[\frac{1}{2}\alpha^{(i)}_{\rm cc}(\phi^{\rm c}_{e_i})^2 + (\alpha^{(i)}_{\rm ct}\phi^{\rm t}_{e_i} + \beta^{(i)}_{\rm c})\phi^{\rm c}_{e_i} - \alpha^{(i)}_{\rm ct}\phi^{\rm c}_{e_i}\phi^{\rm t}_{e_i} + \frac{1}{2}\alpha^{(i)}_{\rm tt}(\phi^{\rm t}_{e_i})^2 + (\alpha^{(i)}_{\rm tc}\phi^{\rm c}_{e_i} + \beta^{(i)}_{\rm t})\phi^{\rm t}_{e_i}\Big].$$

Noticing that solving a non-convex quadratic programming problem might be numerically intractable in general, we focus on the case in which the potential function is a convex function. Following the argument of [54, p. 71], we know that the potential function is a convex function if and only if

$$\begin{bmatrix} \alpha^{(i)}_{\rm cc} & \frac{1}{2}\alpha^{(i)}_{\rm ct} \\ \frac{1}{2}\alpha^{(i)}_{\rm tc} & \alpha^{(i)}_{\rm tt} \end{bmatrix} \geq 0, \ \ \forall i = \{0,\dots,11\}.$$

Let us pick the parameters for the routing game according to Table 7.1. Furthermore, we choose $(F^{\rm a}_1, F^{\rm b}_1) = (5,1)$, $(F^{\rm a}_2, F^{\rm b}_2) = (3,3)$, and $(F^{\rm a}_3, F^{\rm b}_3) = (2,4)$. After solving the optimization problem in Corollary 7.8, we can extract the path flows and path cost functions shown in Table 7.2 which demonstrate a Nash equilibrium (see Definition 7.6)[13]. In addition, we can calculate

$$\frac{C(f)}{C(f^*)} = 1.0137 \leq 2 = \text{Upper Bound of the PoA},$$

[13]See `http://dl.dropbox.com/u/36867745/HeterogeneousRoutingGame.zip` for the Python code to simulate this numerical example.

Table 7.2: The path flow and path cost function at a Nash equilibrium extracted by minimizing the potential function.

	f_p^c	f_p^t
$p \in \mathcal{P}_1$	4.97	0.79
	0.00	0.00
	0.03	0.21
$p \in \mathcal{P}_2$	1.04	0.06
	0.04	0.00
	1.92	2.94
$p \in \mathcal{P}_3$	0.01	0.00
	1.10	0.00
	0.88	4.00

	$\ell_p^c(f)$	$\ell_p^t(f)$
$p \in \mathcal{P}_1$	12.26	8.35
	13.18	10.29
	12.26	8.35
$p \in \mathcal{P}_2$	13.92	11.22
	13.92	13.98
	13.92	11.22
$p \in \mathcal{P}_3$	28.02	31.73
	28.02	34.48
	28.02	31.72

where f^* denotes the socially optimal flow. This shows that the social cost of the recovered Nash equilibrium is only 1.0137 times the cost of the socially optimal solutions.

8 Conclusions

In this article, we proposed a heterogeneous routing game in which the players may belong to more than one type. The type of each player determines the cost of using an edge as a function of the flow of all types over that edge. We proved that this heterogeneous routing game admits at least one Nash equilibrium. Additionally, we gave a necessary and sufficient condition for the existence of a potential function, which indeed implies that we can transform the problem of finding a Nash equilibrium into an optimization problem. Finally, we developed tolls to guarantee the existence of a potential function. Possible future research will focus on generalizing these results to higher number of types or a continuum of player types.

9 Bibliography

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A Proof of Corollary 7.9

Let us define the Lagrangian as

$$\begin{aligned}L((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) =& V((f_{p'}^{\theta_1})_{p'\in\mathcal{P}},(f_{p'}^{\theta_2})_{p'\in\mathcal{P}}) + \sum_{i=1}^{2}\sum_{e\in E} v_e^i\left(\sum_{p\in\mathcal{P}:e\in p} f_p^{\theta_i} - \phi_e^{\theta_i}\right)\\ &- \sum_{i=1}^{2}\sum_{k=1}^{K} w_k^i\left(\sum_{p\in\mathcal{P}_k} f_p^{\theta_i} - F_k^{\theta_i}\right) - \sum_{i=1}^{2}\sum_{p\in\mathcal{P}} \lambda_p^i f_p^i,\end{aligned}$$

where $(v_e^1)_{e\in\mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}$, $(v_e^2)_{e\in\mathcal{E}} \in \mathbb{R}^{|\mathcal{E}|}$, $(w_k^1)_{k\in[\![K]\!]} \in \mathbb{R}^K$, $(w_k^2)_{k\in[\![K]\!]} \in \mathbb{R}^K$, $(\lambda_p^1)_{p\in\mathcal{P}} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}$, and $(\lambda_p^2)_{p\in\mathcal{P}} \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}$ are Lagrange multipliers. Using Karush–Kuhn–Tucker conditions [54, p. 244], optimality conditions are

$$\begin{aligned}\frac{\partial L}{\partial \phi_e^{\theta_1}} &= \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}} \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u_2)}{\partial \phi_e^{\theta_1}} \mathrm{d}u_2 - \int_0^{\phi_e^{\theta_2}} \frac{\partial}{\partial u}\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u)\mathrm{d}u - v_e^1\\ &= \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - v_e^1 + \int_0^{\phi_e^{\theta_2}}\left(\frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},u)}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},u)}{\partial u}\right)\mathrm{d}u\\ &= \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - v_e^1 = 0, \quad \forall e\in\mathcal{E}, \qquad (18a)\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial \phi_e^{\theta_2}} &= \int_0^{\phi_e^{\theta_1}} \frac{\partial \tilde{\ell}_e^{\theta_1}(u_1,\phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} \mathrm{d}u_1 + \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - \int_0^{\phi_e^{\theta_1}} \frac{\partial}{\partial \phi_e^{\theta_2}}\tilde{\ell}_e^{\theta_1}(t,\phi_e^{\theta_2})\mathrm{d}t - v_e^2\\ &= \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1},\phi_e^{\theta_2}) - v_e^2 = 0, \quad \forall e\in\mathcal{E}, \qquad (18b)\end{aligned}$$

and

$$\frac{\partial}{\partial f_p^{\theta_1}} L = \sum_{e\in p} v_e^1 - w_k^1 - \lambda_p^1 = 0, \quad \forall p\in\mathcal{P}, \qquad (19a)$$

$$\frac{\partial}{\partial f_p^{\theta_2}} L = \sum_{e\in p} v_e^2 - w_k^2 - \lambda_p^2 = 0, \quad \forall p\in\mathcal{P}. \qquad (19b)$$

In addition, the complimentary slackness conditions for inequality constraints result in $\lambda_p^1 f_p^1 = 0$ and $\lambda_p^2 f_p^2 = 0$ for all $p \in \mathcal{P}$. Hence, for all k and $p \in \mathcal{P}_k$, we have

$$\begin{aligned}\ell_p^{\theta_i}(f) &= \sum_{e\in p} \tilde{\ell}_e^{\theta_i}(\phi_e^{\theta_1},\phi_e^{\theta_2})\\ &= \sum_{e\in p} v_e^i && \text{by (18)}\\ &= w_k^i + \lambda_p^i. && \text{by (19)}\end{aligned}$$

Thus, if $f_p^{\theta_i}, f_{p'}^{\theta_i} > 0$, using complimentary slackness, we get $\lambda_p^{\theta_i} = 0$ and $\lambda_{p'}^{\theta_i} = 0$, which results in

$$\ell_p^{\theta_i}(f) = \ell_{p'}^{\theta_i}(f) = w_k^i.$$

Additionally, for all $p'' \in \mathcal{P}_k$, where $f_{p''}^{\theta_i} = 0$, we have $\lambda_p^{\theta_i} \geq 0$ (because of dual feasibility), which results in

$$\ell_{p''}^{\theta_i}(f) = w_k^i + \lambda_{p''}^{\theta_i} \geq w_k^i = \ell_p^{\theta_i}(f).$$

This is the definition of a Nash equilibrium.

B Proof of Proposition 7.10

Note that introducing the tolls $\tilde{\tau}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$ has the same impact on the routing game as replacing the edge cost functions in the original heterogeneous routing game from $\tilde{\ell}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$ to $\tilde{\ell}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta}(\phi_e^{\theta_1}, \phi_e^{\theta_2})$. Thanks to Lemma 7.6, the abstract game based upon this new heterogeneous routing game admits the potential function V if

$$\frac{\partial(\tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2}))}{\partial \phi_e^{\theta_2}} - \frac{\partial(\tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}) + \tilde{\tau}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2}))}{\partial \phi_e^{\theta_1}} = 0.$$

With rearranging the terms in this equality, we can extract the condition in the statement of the proposition.

C Proof of Corollary 7.12

The proof can be seen as a direct application of the result of [57, Ch. 4] to Proposition 7.11. However, let us show this fact following simple algebraic manipulations. Notice that

$$\begin{aligned}\frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} =& \frac{\partial}{\partial \phi_e^{\theta_1}} \left[c_e + \psi_e(\phi_e^{\theta_1} + \phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}} f_e(q, \phi_e^{\theta_1} + \phi_e^{\theta_2} - q) \mathrm{d}q \right] \\ =& \frac{\mathrm{d}\psi_e(u)}{\mathrm{d}u}\Big|_{u=\phi_e^{\theta_1}+\phi_e^{\theta_2}} + \int_0^{\phi_e^{\theta_2}} \frac{\partial f_e(q,u)}{\partial u}\Big|_{u=\phi_e^{\theta_1}+\phi_e^{\theta_2}-q} \mathrm{d}q\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} =& \frac{\partial}{\partial \phi_e^{\theta_2}} \left[c_e + \psi_e(\phi_e^{\theta_1} + \phi_e^{\theta_2}) + \int_0^{\phi_e^{\theta_2}} f_e(q, \phi_e^{\theta_1} + \phi_e^{\theta_2} - q) \mathrm{d}q \right] \\ =& \frac{\mathrm{d}\psi_e(u)}{\mathrm{d}u}\Big|_{u=\phi_e^{\theta_1}+\phi_e^{\theta_2}} + f_e(\phi_e^{\theta_2}, \phi_e^{\theta_1}) + \int_0^{\phi_e^{\theta_2}} \frac{\partial f_e(q,u)}{\partial u}\Big|_{u=\phi_e^{\theta_1}+\phi_e^{\theta_2}-q} \mathrm{d}q.\end{aligned}$$

Therefor, we get

$$\frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}} - \frac{\partial \tilde{\tau}_e(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} = f_e(\phi_e^{\theta_2}, \phi_e^{\theta_1})$$
$$= \frac{\partial \tilde{\ell}_e^{\theta_2}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_1}} - \frac{\partial \tilde{\ell}_e^{\theta_1}(\phi_e^{\theta_1}, \phi_e^{\theta_2})}{\partial \phi_e^{\theta_2}},$$

where the second equality directly follows from the definition of the mapping $f_e : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ in the statement of the corollary. Now, we can use Proposition 7.11 to show that a potential function indeed exists.

PAPER 8

A Study of Truck Platooning Incentives Using a Congestion Game

Farhad Farokhi and Karl H. Johansson

Abstract–We introduce an atomic congestion game with two types of agents, cars and trucks, to model the traffic flow on a road over various time intervals of the day. Cars maximize their utility by finding a trade-off between the time they choose to use the road, the average velocity of the flow at that time, and the dynamic congestion tax that they pay for using the road. In addition to these terms, the trucks have an incentive for using the road at the same time as their peers because they have platooning capabilities, which allow them to save fuel. The dynamics and equilibria of this game-theoretic model for the interaction between car traffic and truck platooning incentives are investigated. We use traffic data from Stockholm to validate parts of the modeling assumptions and extract reasonable parameters for the simulations. We use joint strategy fictitious play and average strategy fictitious play to learn a pure strategy Nash equilibrium of this game. We perform a comprehensive simulation study to understand the influence of various factors, such as the drivers' value of time and the percentage of the trucks that are equipped with platooning devices, on the properties of the Nash equilibrium.

1 Introduction

1.1 Motivation

Urban traffic congestion creates many problems, such as increased transportation delays and fuel consumption, air pollution, and dampened economic growth in heavily congested areas [1–3]. A recent study [3] shows that the transportation has contributed to approximately 15% of the total man-made carbon-dioxide since preindustrial era and suggests that it will be responsible for roughly 16% of the carbon-emission over the next century. To circumvent part of these issues, the local governments in some urban areas introduced congestion taxes to manage the traffic congestion over existing infrastructures. For instance, Stockholm implemented a congestion taxing system in August, 2007 after a seven-month trial period in 2006. A survey of the influence of the congestion taxes over the trial period can be found in [4], which shows significant improvements in travel times as well as favorable economic and environmental effects. Behavioral aspects and other influences of the Stockholm congestion taxing system is discussed in [5–8].

In parallel to reducing the congestion, we can employ other means to improve the fuel efficiency and decrease the carbon emission [1]. One way to improve the fuel efficiency of vehicles is platooning, as vehicles experience a reduced air drag force when they travel in platoons [9–13]. Trucks or heavy-duty vehicles can significantly improve their fuel efficiency by platooning with their peers. In [9], the authors report 4.7%-7.7% reduction in the fuel consumption (depending on the distance between the vehicles among other factors) when two identical trucks move close to each other at 70 km/h. In a futuristic scenario when several trucks are equipped with platooning devices, they are able to save fuel by cooperating with each other. However, implementing truck platooning in a large-scale setup is not easy since a global decision-maker might become complex and the vehicles can belong to competing entities. In addition, it is interesting to study if a desirable behavior can emerge from simple local strategies. In this paper, we consider such a case where the traffic flow can be modeled as a congestion game and the desired behavior corresponds to an equilibrium of this game.

1.2 Related Studies

Modeling the traffic flow using congestion games or routing games is a well-known problem [14–22]. Rosenthal [17] presented a noncooperative game in which a finite number of players compete for using a finite set of resources with application to modeling transport networks. He showed that a class of these games admit at least one pure strategy Nash equilibrium (an action profile in which no agent has an incentive to unilaterally deviate from her action). Later, the authors of [23] showed that atomic congestion games are indeed potential games (i.e., there exists a potential function, such that its variation when only one agent changes her action is equal to the variation of the utility of the corresponding agent) under some conditions and, hence, one can find a Nash equilibrium by minimizing the potential function. For a survey of these and related results, see [24]. Most of these studies modeled the *route selection* using an atomic congestion game. Recently, the authors of [25] utilized a congestion game for modeling instead the time interval in which drivers decide to use a road.

This setup may be extended to weighted congestion games in which every agent is associated with a (splittable or unsplittable) demand (not equal and more than a single unit) that should be routed over the network. In [26], Rosenthal showed that a Nash equilibrium does not necessarily exist in these games if the agents can split their demand. The authors of [27–29] constructed counterexamples to show that a Nash equilibrium does not necessarily exist also for unsplittable demands as well. However, when cost functions (i.e., latencies) of each road are affine functions, an equilibrium certainly exists (and may be found in pseudo-polynomial time) [28]. In [27], it was also proved that an equilibrium may exist for a special class of cost functions (that are only a function of the residual capacity on each edge) on parallel networks. The largest class of latency functions for which the game admits an equilibrium were explored in [30]. It was also shown that a weighted congestion game admits an exact potential function (a weighted potential function) if and only if the set of costs contains only affine functions (affine or exponential functions) [31].

The studies discussed above mainly consider homogeneous congestion games in which all the drivers on a road at any given time interval perceive the same cost function (e.g, the drivers only consider the latency in their decision-making and they all have the same sensitivity to the latency as well). However, in road traffic networks, this assumption might not be realistic. For instance, as we will see in this paper, whenever the drivers include the fuel consumption in their decision making, trucks and cars potentially have different cost functions even if they observe the same latency when using the road. To capture this phenomenon, we extend the model in [25] to an atomic congestion game with two types of agents, namely, cars and trucks. Notice that the problem of heterogeneous congestion and routing games have been studied extensively in the past [32–34]. For instance, in [32], the author formulated a congestion game in which each player has a specific cost function that depends on the congestion. In that study, it was shown that every unweighted congestion game with player-specific cost functions admits at least one equilibrium; however, this results may not be generalized to weighted congestion games with player-specific cost functions in general. In addition, generally, even unweighted congestion games with player-specific cost functions do not admit a potential function. For routing games, in which a continuum of players route an infinitesimal amount of flow, it was proved that a potential function exists if a symmetry condition is satisfied for the cost functions (i.e., various classes of agents bother or delight each other equally) [33, 35]. A class of necessary and sufficient conditions for the existence of potential functions was presented in [36]. Conditions for the (essential) uniqueness of the equilibrium in multi-class routing games were also presented in [37, 38].

Motivated by the fact that the Nash equilibrium is generally inefficient, the price of anarchy (i.e., the worst-case ratio of the social welfare function for a Nash equilibrium over the social welfare function for a socially optimal solution) of atomic congestion games with linear latency functions was studied in [39]. Several studies have proposed congestion taxes (also known as tolls) to improve the social cost function when all the agents are equally sensitive to the proposed taxes [40–43] as well as when they have different sensitivities [44–47]. For instance, in [47], tolls were introduced to minimize the total travel time and the total travel cost (as a bi-objective optimization problem). This setup was generalized in [45] to also admit entities that own several agents (and wish to optimize the combined utility of those agents). The idea of maximizing the reserve capacity of the network was approached in [44]. A scenario in which the network is managed by several decision-makers (with conflicting objectives) across various regions was discussed in [46]. The authors of [25, 40, 41] presented congestion taxes so that the underlying congestion game admits the social welfare as a potential function. This is certainly of interest because it guarantees that the socially optimal decision is also a Nash equilibrium. However, in those studies, the authors needed to introduce a congestion tax for all the agents (and not only a subset of them).

1.3 Contributions

In this paper, we model the traffic flow at non-overlapping intervals of the day using an atomic[1] congestion game with two types of agents. The agents of the first type are cars as well as trucks that do not have platooning equipments. For the sake of brevity, we call all these agents cars. They optimize their utility, which is a sum of the penalty for deviating from their preferred time for using the road, the average velocity of the traffic flow along the road, and the congestion tax that they pay for using the road at that time interval. The agents of the second type are trucks equipped with platooning devices. For the sake of brevity, we call these agents trucks. In addition to the above mentioned terms, they have an incentive for using the road with other trucks (due to an increased chance for platooning and, hence, reducing their fuel consumption).

We model the average velocity of the flow at each time interval as an affine function of the number of the vehicles that are using the road at that time interval. We use real traffic data from the northbound E4 highway from Lilla Essingen to the end of Fredhällstunneln in Stockholm to validate this modeling assumption.

We determine a necessary condition for the existence of a potential function for the introduced atomic congestion game with two types of agents and use this condition to prove that in general the congestion game is not a potential game. Therefore, we devise appropriate congestion taxes (specifically, a congestion taxing policy for cars and a platooning subsidy for trucks) to guarantee the existence of a potential function. Based on this result, we prove that the atomic congestion game admits at least one pure strategy Nash equilibrium under the proposed congestion

[1]We use the term atomic to emphasize the fact that we are not dealing with a continuum of players or fractional flows when modeling the traffic flow as a congestion game [48, 49].

tax–subsidy policy. Equipped with these results, we use joint strategy fictitious play and average strategy fictitious play to learn a Nash equilibrium. Intuitively, we interpret the learning algorithm as the way drivers decide on a daily basis to choose the time interval on which they are using the road by optimizing their utility given the history of their actions. Iterating over days, the drivers' decisions (i.e., the profile of the learning algorithm) converges almost surely to a pure strategy Nash equilibrium. Note that the potential games are certainly not the only classes of games for which variants of the fictitious play (e.g., joint strategy fictitious play) may converge to an equilibrium. To mention a few example, the authors of [23, 50] introduced ordinal potential games and weighted potential games as two families of games for which the fictitious play converges in beliefs to a mixed strategy Nash equilibrium. For (generalized) ordinal potential games, one may also deduce the convergence of the joint strategy fictitious play to a pure strategy Nash equilibrium with probability one [40]. These families of games are certainly more general than (exact) potential games. In this paper, as a starting point, we present necessary conditions for the existence of (exact) potential functions as well as imposing congestion taxes for guaranteeing the existence of such functions. Although conservative, this approach perhaps can be justified in the introduced problem due to the existence of intuitive taxing and subsidy policies (see Subsection 3.2). A viable direction for future work is to investigate necessary and sufficient conditions so that a congestion game belongs to the category of ordinal or weighted potential games. In addition, as also mentioned earlier, congestion taxes were presented in [25, 40, 41] so that the congestion game admits the social welfare as a potential function. However, in contrast to the results of this paper, the authors of [40, 41] introduced a congestion tax for all the agents (and not only a subset of them) to improve the efficiency and considered homogeneous congestion games (with only one type of agents).

Finally, using the parameters extracted from the real congestion data, we construct a simulation setup to study the performance of the learning algorithms as well as the properties of the Nash equilibrium. For instance, we study the robustness to perturbations of the learning algorithm, e.g., accidents along the road, sudden weather changes, or temporary road constructions. We also consider the case when the drivers value their time differently, where the values are motivated by survey data from Stockholm area [51].

1.4 Paper Organization

The rest of the paper is organized as follows. In Section 2, we formulate the considered congestion game. We find a necessary condition of the existence of a potential function in Section 3. In Sections 4 and 5, we respectively introduce the joint strategy fictitious play and the average strategy fictitious play to learn a Nash equilibrium of the congestion game. Finally, we present the simulations in Section 6 and conclude the paper in Section 7.

1.5 Notation

Let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of real, integer, and natural numbers, respectively. Furthermore, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define $[\![N]\!] = \{1, \dots, N\}$ for any $N \in \mathbb{N}$. In this paper, all other sets are denoted by calligraphic letters such as $\mathcal{R}$. We use $|\mathcal{R}|$ to denote the cardinality of $\mathcal{R}$. Finally, we define the characteristic function $\mathbf{1}_{x=y}$ ($\mathbf{1}_{x\geq y}$) to be equal one whenever $x = y$ ($x \geq y$) holds true and to be equal to zero otherwise.

2 Game-Theoretic Model

We model the traffic flow at certain time intervals of the day on a given road using an atomic congestion game. The agents in this congestion game are the vehicles (or, rather the drivers of these vehicles) and their actions are the time intervals that they choose to use the road at each day. Let us divide the time of the day into $R \in \mathbb{N}$ non-overlapping intervals and denote each interval by r_i for $i \in [\![R]\!]$. The set of all these intervals (i.e., agents' actions) is denoted by $\mathcal{R} = \{r_1, r_2, \dots, r_R\}$. We consider the case where the underlying congestion game is composed of two types of agents. As specified in the introduction, we name the agents of the first type cars and the agents of the second type trucks throughout the paper. We assume N cars and M trucks are playing in this congestion game and denote the actions of the cars and the trucks by $z = \{z_i\}_{i=1}^N$ and $x = \{x_i\}_{i=1}^M$, respectively. Let us describe the utilities of the cars and the trucks in the following subsections.

2.1 Car Utility

Car $i \in [\![N]\!]$ maximizes its utility given by

$$U_i(z_i, z_{-i}, x) = \xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) + v_{z_i}(z, x) + p_i^{\mathrm{c}}(z, x), \tag{1}$$

where the mapping $\xi_i^{\mathrm{c}} : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ describes the penalty for deviating from the preferred time interval for using the road denoted by $T_i^{\mathrm{c}} \in \mathcal{R}$ (e.g., due to being late for work or delivering goods), $v_{z_i}(z, x)$ is the average velocity of the traffic flow at time interval z_i, and $p_i^{\mathrm{c}}(z, x)$ is a potential congestion tax for using the road on a specific time interval.

Following [25, 52, 53], we assume that $v_r(z, x)$ (i.e., the average velocity at time interval $r \in \mathcal{R}$) is linearly dependent on the road congestion

$$n_r(z, x) = \sum_{\ell=1}^{N} \mathbf{1}_{\{z_\ell = r\}} + \sum_{\ell=1}^{M} \mathbf{1}_{\{x_\ell = r\}}, \tag{2}$$

which is the total number of vehicles (both cars and trucks) that are using the road at $r \in \mathcal{R}$. Let us use real traffic data from sensors on the northbound E4 highway in Stockholm from Lilla Essingen to the end of Fredhällstunneln (see Figure 1) to validate this assumption. The measurements are extracted during October 1–15,

Figure 1: The dashed black curve shows the segment of northbound E4 highway between Lilla Essingen and Fredhällstunneln in Stockholm where we are using to validate the model and extract reasonable parameters.

2012. Figure 2 illustrates the average velocity of the flow as a function of the number of vehicles. As we can see, for up to 1000 vehicles, a linear relationship

$$v_r(z, x) = a n_r(z, x) + b \tag{3}$$

with $a = -0.0110$ and $b = 84.9696$ describes the data well. However, for higher numbers of the vehicles, it fails to capture the behavior of around 20% of the data (shown by the red dots in Figure 2). Note that some of these outlier measurements can be caused by traffic accidents, sudden weather changes during the day, or temporary road constructions. A viable direction for future work is to introduce more complex velocity models in which the average velocity of the traffic flow may depend on the number of vehicles in the neighboring time intervals in addition to the current one. We may also need to separate the effect of cars and trucks as one may expect heavier and larger vehicles to contribute more to the traffic congestion. However, in this paper, we use the simple model presented in (3) and instead focus on platooning incentives.

The choice of the penalty mappings ξ_i^{c}, $i \in [\![N]\!]$, does not change the theoretical results presented in the paper, but it can capture various models of the drivers. For instance, following [25], we can use $\xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) = \alpha_i^{\mathrm{c}} |z_i - T_i^{\mathrm{c}}|$, with scalar $\alpha_i^{\mathrm{c}} < 0$, to describe the case where the driver of car i is penalized by deviating from the preferred time interval. With this function, the driver get penalized symmetrically no matter if she uses the road sooner or later than T_i^{c}. By increasing $|\alpha_i^{\mathrm{c}}|$, she becomes less flexible. Another penalty function is $\xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) = \alpha_i^{\mathrm{c}} \max(z_i - T_i^{\mathrm{c}}, 0)$,

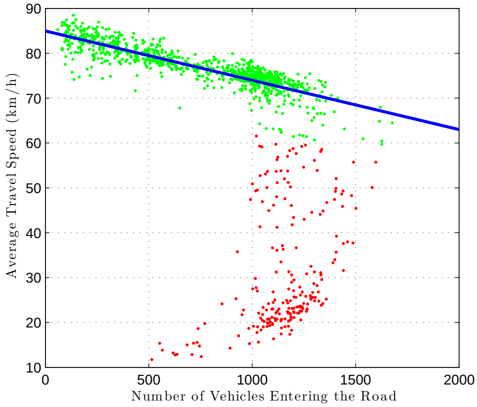

Figure 2: Average velocity of the traffic flow as a function of the number of vehicles that are entering the segment of northbound E4 highway between Lilla Essingen and Fredhällstunneln for 15 min time intervals.

which penalizes the driver of car i only for being late. For the simulations in the paper, we assume that all vehicles use the first penalty mapping.

2.2 Truck Utility

Truck $j \in [\![M]\!]$ maximizes its utility given by

$$V_j(x_j, x_{-j}, z) = \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) + v_{x_j}(z, x) + p_i^{\mathrm{t}}(z, x) + \beta v_{x_j}(z, x) g(m_{x_j}(x)), \quad (4)$$

where, similar to the utilities of the cars, $\xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}})$ is the penalty for deviating from the preferred time T_j^{t} for using the road, $v_{x_j}(z, x)$ is the average velocity of the traffic flow, and $p_i^{\mathrm{t}}(z, x)$ is a potential congestion tax for using the road at time interval x_j. Trucks have an extra term $\beta v_{x_j}(z, x) g(m_{x_j}(x))$ in their utility because of their benefit in using the road at the same time as the other trucks. Here, $g : [\![M]\!] \to \mathbb{R}$ is a nondecreasing function and $m_r(x) = \sum_{\ell=1}^{M} \mathbf{1}_{\{x_\ell = r\}}$ is the number of trucks that are using the road at time interval $r \in \mathcal{R}$. The increased utility can be justified by the fact that whenever there are many trucks on the road at the same time interval, they can potentially collaborate to form platoons and thereby increase the fuel efficiency. It should be noted that this extra utility is a function of the average velocity of the flow since trucks cannot save a significant amount of fuel through platooning whenever traveling at low velocities [9, 54]. The function $g : [\![M]\!] \to \mathbb{R}$ describes the dependency of the platooning incentive on the number of trucks that are using the road at that time interval. Again, the choice of this function does not change the mathematical results presented in this paper, but it can help us to capture the relationship between the fuel saving and the number of the trucks on the road. For instance, $g(m_{x_j}(x)) = m_{x_j}(x)$ shows that the vehicles can even benefit from a low number of trucks but $g(m_{x_j}(x)) = m_{x_j}(x)\mathbf{1}_{m_{x_j}(x)\geq\tau}$ describes the case where the trucks do not benefit until they reach a critical number $\tau \in \mathbb{N}$. For the simulations, we use the first mapping.

Notice that in the utilities U_i in (1) and V_j in (4), we introduced congestion taxes for cars and trucks. Later, they are used to ensure that the described game is a potential game. Such a game admits at least one pure strategy Nash equilibrium and we can use joint strategy fictitious play and average strategy fictitious play to learn that equilibrium. A viable direction for future research could be to design taxing policies so as to enforce a socially optimal behavior, such as an optimal carbon emission profile, using mechanism design theory [55].

2.3 Congestion Game

Now, we are ready to define a congestion game with two types of players using normal-form representation of strategic games [56, 57].

Definition 8.1 (Car–Truck Congestion Game)*: A car–truck congestion game is defined as a tuple* $\mathcal{G} = ((\mathcal{R})_{i=1}^{N+M}; ((U_i)_{i=1}^{N}, (V_j)_{j=1}^{M}))$*, that is, a combination of* $N + M$ *players with action space* $(\mathcal{R})_{i=1}^{N+M}$ *and utilities* $((U_i)_{i=1}^{N}, (V_j)_{j=1}^{M}))$.

A pure strategy Nash equilibrium for a car–truck congestion game is a pair $(z, x) \in \mathcal{R}^N \times \mathcal{R}^M$ such that

$$\begin{aligned} U_i(z_i, z_{-i}, x) &\geq U_i(z_i', z_{-i}, x), \qquad \forall z_i' \in \mathcal{R}, \qquad i \in [\![N]\!], \\ V_j(x_j, x_{-j}, z) &\geq V_j(x_j', x_{-j}, z), \qquad \forall x_j' \in \mathcal{R}, \qquad j \in [\![M]\!]. \end{aligned}$$

To prove the existence of a pure strategy Nash equilibrium or to use various learning algorithms for finding an equilibrium, we focus on a subclass of games, namely, potential games [23]. A car–truck congestion game is a potential game with potential function $\Phi : \mathcal{R}^N \times \mathcal{R}^M \to \mathbb{R}$ if

$$\begin{aligned} \Phi(x, z_i, z_{-i}) - \Phi(x, z_i', z_{-i}) &= U_i(z_i, z_{-i}, x) - U_i(z_i', z_{-i}, x), \qquad \forall i \in [\![N]\!], \\ \Phi(x_j, x_{-j}, z) - \Phi(x_j', x_{-j}, z) &= V_j(x_j, x_{-j}, z) - V_j(x_j', x_{-j}, z), \qquad \forall j \in [\![M]\!]. \end{aligned}$$

With these definitions in hand, we are ready to present the results of the paper.

3 Existence of Potential Function

Atomic congestion games with one type of agents (corresponding to the case where $M = 0$ or $N = 0$) are known to admit a potential function even without congestion taxes [23, 25, 43]. In this section, we show that this property does not hold for car–truck congestion games unless we devise an appropriate taxing scheme.

3.1 Necessary Condition for the Existence of a Potential Function

Let $\Phi : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ be a given mapping. Define

$$\Delta_{x_j \to x'_j} \Phi(x, z) = \Phi(x, z) - \Phi(x', z)$$
$$\Delta_{z_i \to z'_i} \Phi(x, z) = \Phi(x, z) - \Phi(x, z'),$$

where $x' = (x'_j, x_{-j})$ and $z' = (z'_i, z_{-i})$. Using simple algebra, we can show that the operators commute, i.e.,

$$\Delta_{z_i \to z'_i} \Delta_{x_j \to x'_j} \Phi(x, z) = \Delta_{x_j \to x'_j} \Delta_{z_i \to z'_i} \Phi(x, z).$$

Now, we are ready to prove the following useful result.

Proposition 8.1 *A car–truck congestion game admits a potential function only if*

$$\Delta_{x_i \to x'_j} \Delta_{z_i \to z'_i} V_j(z, x) = \Delta_{z_i \to z'_i} \Delta_{x_i \to x'_j} U_i(z, x),$$

for all $i \in [\![N]\!]$ *and* $j \in [\![M]\!]$.

Proof: Let $\Phi(x, z)$ be a potential function for the congestion game. Define $x' = (x'_j, x_{-j})$ and $z' = (z'_i, z_{-i})$. Then, it must satisfy

$$\Delta_{x_j \to x'_j} V_j(x, z) = \Delta_{x_j \to x'_j} \Phi(x, z), \tag{5}$$

for all $z \in \mathcal{R}^N$, $x \in \mathcal{R}^M$, and $x'_j \in \mathcal{R}$. Again, when noting that $\Phi(x, z)$ is a potential function, we get

$$\Phi(x, z) = \Phi(x, z') + \Delta_{z_i \to z'_i} U_i(z, x) \tag{6a}$$
$$\Phi(x', z) = \Phi(x', z') + \Delta_{z_i \to z'_i} U_i(z, x') \tag{6b}$$

for all $z \in \mathcal{R}^N$, $x \in \mathcal{R}^M$, $z'_i \in \mathcal{R}$, and $x'_j \in \mathcal{R}$. Substituting (6) into (5) results in

$$\begin{aligned}
\Delta_{x_j \to x'_j} V_j(x, z) =& \Phi(x, z) - \Phi(x', z) \\
=& \Delta_{x_j \to x'_j} \Phi(x, z') + \Delta_{z_i \to z'_i} U_i(z, x) - \Delta_{z_i \to z'_i} U_i(z, x') \\
=& \Delta_{x_j \to x'_j} \Phi(x, z') + \Delta_{z_i \to z'_i} \Delta_{x_i \to x'_j} U_i(z, x) \\
=& \Delta_{x_j \to x'_j} V_j(x, z') + \Delta_{z_i \to z'_i} \Delta_{x_i \to x'_j} U_i(z, x),
\end{aligned}$$

where the last equality follows from the definition of the potential function. Therefore, we get the identity in the statement of the theorem. ■

This shows that it might not be possible to find a potential functions for the congestion game with two types of players.

Corollary 8.2 *Let $p_i^{\mathrm{c}}(z,x) = 0$ for $i \in [\![N]\!]$ and $p_j^{\mathrm{t}}(z,x) = 0$ for $j \in [\![M]\!]$. A car–truck congestion game admits a potential function only if $\beta = 0$ or g is equal to zero everywhere.*

Proof: First, by simple algebraic manipulations, we prove the identity in

$$\begin{aligned}
&\Delta_{x_i \to x'_j}\Delta_{z_i \to z'_i} V_j(z,x) \\
&= \Delta_{x_i \to x'_j}\Delta_{z_i \to z'_i}\big(\xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) + v_{x_j}(z,x) + \beta v_{x_j}(z,x) g(m_{x_j}(x))\big) \\
&= \Delta_{x_i \to x'_j}\Delta_{z_i \to z'_i}\big(v_{x_j}(z,x) + \beta v_{x_j}(z,x) g(m_{x_j}(x))\big) \\
&= \Delta_{x_i \to x'_j}\big(v_{x_j}(z,x) - v_{x_j}(z',x) + \beta v_{x_j}(z,x) g(m_{x_j}(x)) - \beta v_{x_j}(z',x) g(m_{x_j}(x))\big) \\
&= \Delta_{x_i \to x'_j}\big(a[\mathbf{1}_{x_j = z_i} - \mathbf{1}_{x_j = z'_i}][1 - \beta g(m_{x_j}(x))]\big) \\
&= a[\mathbf{1}_{x_j = z_i} - \mathbf{1}_{x_j = z'_i}][1 - \beta g(m_{x_j}(x))] - a[\mathbf{1}_{x'_j = z_i} - \mathbf{1}_{x'_j = z'_i}][1 - \beta g(m_{x'_j}(x'))] \\
&= a[\mathbf{1}_{x_j = z_i} + \mathbf{1}_{x'_j = z'_i} - \mathbf{1}_{x_j = z'_i} - \mathbf{1}_{x'_j = z_i}] \\
&\quad - a\beta[\mathbf{1}_{x_j = z_i} - \mathbf{1}_{x_j = z'_i}] g(m_{x_j}(x)) + a\beta[\mathbf{1}_{x'_j = z_i} - \mathbf{1}_{x'_j = z'_i}] g(m_{x'_j}(x')) \\
&= a[\mathbf{1}_{x_j = z_i} + \mathbf{1}_{x'_j = z'_i} - \mathbf{1}_{x_j = z'_i} - \mathbf{1}_{x'_j = z_i}] \\
&\quad + a\beta[\mathbf{1}_{x_j = z'_i}\mathbf{1}_{x'_j = z_i} - \mathbf{1}_{x_j = z_i}\mathbf{1}_{x'_j = z'_i}][1 - \mathbf{1}_{z_j = z'_i}][g(m_{x_j}(x)) + g(m_{x'_j}(x'))]
\end{aligned} \tag{7}$$

Similarly, we can show that

$$\Delta_{z_i \to z'_i}\Delta_{x_i \to x'_j} U_i(z,x) = a[\mathbf{1}_{x_j = z_i} + \mathbf{1}_{x'_j = z'_i} - \mathbf{1}_{x_j = z'_i} - \mathbf{1}_{x'_j = z_i}].$$

Therefore, following Proposition 8.1, the introduced congestion game admits a potential function only if

$$\beta[\mathbf{1}_{x_j = z'_i}\mathbf{1}_{x'_j = z_i} - \mathbf{1}_{x_j = z_i}\mathbf{1}_{x'_j = z'_i}][1 - \mathbf{1}_{z_j = z'_i}][g(m_{x_j}(x)) + g(m_{x'_j}(x'))] = 0$$

for all x, z and x'_j, z'_i. This is indeed only possible if $\beta = 0$ or if g is equal to zero everywhere. ∎

Potential games have many desirable attributes. For instance, these games always admit at least one pure strategy Nash equilibrium. In addition, many learning algorithms, such as, joint strategy fictitious play, are known to extract a pure strategy Nash equilibrium for potential games. Given these important properties, a natural question that comes to mind is that whether it is possible to guarantee the existence of a potential function by imposing appropriate congestion taxes. We answer this question in the next subsection.

3.2 Imposing Taxes to Guarantee the Existence of a Potential Function

In this subsection, we propose a taxing and a subsidy policy that guarantee the existence of a potential function for the car–truck congestion game.

Theorem 8.3 *Let each car $i \in [\![N]\!]$ pay the congestion tax*

$$p_i^{\mathrm{c}}(z,x) = a\beta \sum_{\ell=1}^{m_{z_i}(x)} g(\ell), \tag{8}$$

for using the road at time interval $z_i \in \mathcal{R}$. Then, the car–truck congestion game is a potential game with the potential function

$$\begin{aligned}\Phi(x,z) = &\sum_{i=1}^{N} \xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) + \sum_{j=1}^{M} \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) + \sum_{r=1}^{R} \sum_{k=1}^{n_r(x,z)} (ak+b) \\ &+ \sum_{r=1}^{R} \beta(an_r(x,z)+b) \sum_{\ell=1}^{m_r(x)} g(\ell) - a\beta \sum_{r=1}^{R} \sum_{\ell=1}^{m_r(x)} \sum_{k=1}^{\ell-1} g(k).\end{aligned}$$

Furthermore, this game admits at least one pure strategy Nash equilibrium.

Proof: The proof of this lemma follows the same line of reasoning as in the proof of Proposition 4.1 in [25]. First, we need to define the following notations

$$\begin{aligned}\Phi_1(x,z) &= \sum_{i=1}^{N} \xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) + \sum_{j=1}^{M} \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}), \\ \Phi_2(x,z) &= \sum_{r=1}^{R} \sum_{k=1}^{n_r(x,z)} (ak+b), \\ \Phi_3(x,z) &= \sum_{r=1}^{R} \beta(an_r(x,z)+b) \sum_{\ell=1}^{m_r(x)} g(\ell), \\ \Phi_4(x,z) &= -a\beta \sum_{r=1}^{R} \sum_{\ell=1}^{m_r(x)} \sum_{k=1}^{\ell-1} g(k).\end{aligned}$$

Let us start by analyzing the trucks. If $x_j = x_j'$, the result trivially holds. Consequently, we consider the case where $x_j \neq x_j'$, which results in

$$\Phi(x_j, x_{-j}, z) - \Phi(x_j', x_{-j}, z) = \sum_{k=1}^{4} \Phi_k(x_j, x_{-j}, z) - \Phi_k(x_j', x_{-j}, z).$$

We continue the proof by considering each term of this summation separately. For the first term, clearly, we have

$$\Phi_1(x_j, x_{-j}, z) - \Phi_1(x_j', x_{-j}, z) = \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) - \xi_j^{\mathrm{t}}(x_j', T_j^{\mathrm{t}}).$$

Let us define $x' = (x'_j, x_{-j})$. For the second term, we have

$$\begin{aligned}\Phi_2(x_j, x_{-j}, z) - \Phi_2(x'_j, x_{-j}, z) &= \sum_{r=1}^{R} \sum_{k=1}^{n_r(x,z)} (ak+b) - \sum_{r=1}^{R} \sum_{k=1}^{n_r(x',z)} (ak+b) \\ &= \sum_{k=1}^{n_{x_j}(x,z)} (ak+b) + \sum_{k=1}^{n_{x'_j}(x,z)} (ak+b) \\ &\quad - \sum_{k=1}^{n_{x_j}(x',z)} (ak+b) - \sum_{k=1}^{n_{x'_j}(x',z)} (ak+b),\end{aligned}$$

where the second equality holds because of the fact that $n_r(x, z) = n_r(x', z)$ for all $r \neq x_j, x'_j$. Note that

$$n_{x_j}(x', z) = n_{x_j}(x, z)–1, \; n_{x'_j}(x, z) = n_{x'_j}(x', z)–1, \tag{9}$$

and as a result,

$$\Phi_2(x_j, x_{-j}, z) - \Phi_2(x'_j, x_{-j}, z) = (an_{x_j}(z, x) + b) - (an_{x'_j}(z, x') + b).$$

For the third term, we get the identity in

$$\begin{aligned}&\Phi_3(x_j, x_{-j}, z) - \Phi_3(x'_j, x_{-j}, z) \\ &\quad= \sum_{r=1}^{R} \beta(an_r(x,z)+b) \sum_{\ell=1}^{m_r(x)} g(\ell) - \sum_{r=1}^{R} \beta(an_r(x',z)+b) \sum_{\ell=1}^{m_r(x')} g(\ell) \\ &\quad= \beta(an_{x_j}(x,z)+b) \sum_{\ell=1}^{m_{x_j}(x)} g(\ell) + \beta(an_{x'_j}(x,z)+b) \sum_{\ell=1}^{m_{x'_j}(x)} g(\ell) \\ &\qquad - \beta(an_{x_j}(x',z)+b) \sum_{\ell=1}^{m_{x_j}(x')} g(\ell) - \beta(an_{x'_j}(x',z)+b) \sum_{\ell=1}^{m_{x'_j}(x')} g(\ell) \\ &\quad= \beta(an_{x_j}(x,z)+b) g(m_{x_j}(x)) - \beta(an_{x'_j}(x',z)+b) g(m_{x'_j}(x')) \\ &\qquad + a\beta \sum_{\ell=1}^{m_{x_j}(x)-1} g(\ell) - a\beta \sum_{\ell=1}^{m_{x'_j}(x')-1} g(\ell),\end{aligned} \tag{10}$$

where the last equality follows from using (9) and the fact that $m_{x_j}(x') = m_{x_j}(x) - 1$ and $m_{x'_j}(x) = m_{x'_j}(x') - 1$. Finally, using the same argument as in the case of the second term and the third term, we get

$$\Phi_4(x_j, x_{-j}, z) - \Phi_4(x'_j, x_{-j}, z) = -a\beta \sum_{\ell=1}^{m_{x_j}(x)-1} g(\ell) + a\beta \sum_{\ell=1}^{m_{x'_j}(x')-1} g(\ell).$$

Combining all these differences, we get

$$\begin{aligned}\Phi(x_j, x_{-j}, z) - \Phi(x'_j, x_{-j}, z) =& \beta(an_{x_j}(x, z) + b)g(m_{x_j}(x)) \\ &- \beta(an_{x'_j}(x', z)+b)g(m_{x'_j}(x')) \\ &+ \xi^{\mathrm{t}}_j(x_j, T^{\mathrm{t}}_j) - \xi^{\mathrm{t}}_j(x'_j, T^{\mathrm{t}}_j) \\ &+ (an_{x_j}(z, x) + b) - (an_{x'_j}(z, x') + b) \\ =& V_j(x_j, x_{-j}, z) - V_j(x'_j, x_{-j}, z).\end{aligned}$$

Now, let us prove this fact for the cars as well. If $z_i = z'_i$, the result trivially holds. Thus, we investigate the case where $z_i \neq z'_i$. Similarly, we consider each term of the summation separately. For the first term, we have

$$\Phi_1(x, z_i, z_{-i}) - \Phi_1(x, z'_i, z_{-i}) = \xi^{\mathrm{c}}_i(z_i, T^{\mathrm{c}}_i) - \xi^{\mathrm{c}}_i(z'_i, T^{\mathrm{c}}_i).$$

We define the notation $z' = (z'_i, z_{-i})$. Following a similar reasoning as in the case of the trucks, for the second and the third terms, we get

$$\Phi_2(x, z_i, z_{-i}) - \Phi_2(x, z'_i, z_{-i}) = (an_{z_i}(z, x) + b) - (an_{z'_i}(z', x) + b),$$

and

$$\Phi_3(x, z_i, z_{-i}) - \Phi_3(x, z'_i, z_{-i}) = a\beta \sum_{\ell=1}^{m_{z_i}(x)} g(\ell) - a\beta \sum_{\ell=1}^{m_{z'_i}(x)} g(\ell).$$

For the forth term, we get $\Phi_4(x, z_i, z_{-i}) - \Phi_4(x, z'_i, z_{-i}) = 0$ since this term is only a function of x which is not changed. Again, combining all these differences, we get

$$\begin{aligned}\Phi(x, z_i, z_{-i}) - \Phi(x, z'_i, z_{-i}) =& (an_{z_i}(z, x) + b) - (an_{z'_i}(z', x) + b) \\ &+ \xi^{\mathrm{c}}_i(z_i, T^{\mathrm{c}}_i) - \xi^{\mathrm{c}}_i(z'_i, T^{\mathrm{c}}_i) \\ &+ a\beta \sum_{\ell=1}^{m_{z_i}(x)} g(\ell) - a\beta \sum_{\ell=1}^{m_{z'_i}(x)} g(\ell) \\ =& U_i(z_i, z_{-i}, x) - U_i(z'_i, z_{-i}, x).\end{aligned}$$

Finally, note that every potential game admits at least one pure strategy Nash equilibrium [23]. ■

Remark 8.1 *Note the tax $p^{\mathrm{c}}_i(z, x)$ grows quadratically with the number of the trucks that are using the road at that time interval if the mapping $g : [\![M]\!] \to \mathbb{R}$ is an affine function. Therefore, the congestion tax policy $p^{\mathrm{c}}_i(z, x)$ in Theorem 8.3 forces the cars to avoid the time intervals that the trucks use to travel together.*

Instead of taxing the cars, we can also introduce a platooning subsidy for the trucks to get a potential game.

Theorem 8.4 *Let each truck $j \in [\![M]\!]$ receive the subsidy*

$$p_j^{\mathrm{t}}(x,z) = \beta(v_0 - (an_{x_j}(z,x) + b))m_{x_j}(x), \tag{11}$$

for a given $v_0 \in \mathbb{R}$. Then, the car–truck congestion game is a potential game with the potential function

$$\Psi(x,z) = \sum_{i=1}^{N} \xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) + \sum_{j=1}^{M} \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) + \sum_{r=1}^{R} \sum_{k=1}^{n_r(x,z)} (ak+b) + \beta v_0 \sum_{r=1}^{R} \sum_{\ell=1}^{m_r(x)} g(\ell).$$

Furthermore, this game admits at least one pure strategy Nash equilibrium.

Proof: Let us start with trucks. Note that with the introduced policy, the utility of truck j is equal

$$V_j(x_j, x_{-j}, z) = \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) + v_{x_j}(z,x) + \beta v_0 g(m_{x_j}(x)).$$

Let us define $x' = (x_j', x_{-j})$. If $x_j = x_j'$, the result trivially holds. Therefore, without loss of generality, we consider the case where $x_j \neq x_j'$. In what follows, we examine each term in the cost function separately. First, we define $\Psi_1(x,z) = \sum_{i=1}^{N} \xi_i^{\mathrm{c}}(z_i, T_i^{\mathrm{c}}) + \sum_{j=1}^{M} \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}})$. Now, it is easy to see that

$$\Psi_1(x,z) - \Psi_1(x',z) = \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) - \xi_j^{\mathrm{t}}(x_j', T_j^{\mathrm{t}}).$$

Second, we define $\Psi_2(x,z) = \sum_{r=1}^{R} \sum_{k=1}^{n_r(x,z)} (ak+b)$. For this term, we can show that

$$\begin{aligned}\Psi_2(x,z) - \Psi_2(x',z) &= \sum_{r=1}^{R} \sum_{k=1}^{n_r(x,z)} (ak+b) - \sum_{r=1}^{R} \sum_{k=1}^{n_r(x',z)} (ak+b) \\ &= \sum_{k=1}^{n_{x_j}(x,z)} (ak+b) + \sum_{k=1}^{n_{x_j'}(x,z)} (ak+b) - \sum_{k=1}^{n_{x_j}(x',z)} (ak+b) - \sum_{k=1}^{n_{x_j'}(x',z)} (ak+b),\end{aligned}$$

where the second equality holds because of the fact that $n_r(x,z) = n_r(x',z)$ for all $r \neq x_j, x_j'$. Noticing that $n_{x_j}(x',z) = n_{x_j}(x,z) - 1$ and $n_{x_j'}(x,z) = n_{x_j'}(x',z) - 1$, we know that

$$\Psi_2(x,z) - \Psi_2(x',z) = (an_{x_j}(z,x) + b) - (an_{x_j'}(z,x') + b).$$

Finally, we define $\Psi_3(x,z) = \sum_{r=1}^{R}\sum_{\ell=1}^{m_r(x)} g(\ell)$. In this case, we can show that

$$\begin{aligned}\Psi_3(x,z) - \Psi_3(x',z) =& \sum_{r=1}^{R}\sum_{\ell=1}^{m_r(x)} g(\ell) - \sum_{r=1}^{R}\sum_{\ell=1}^{m_r(x')} g(\ell)\\ =& \sum_{\ell=1}^{m_{x_j}(x)} g(\ell) + \sum_{\ell=1}^{m_{x'_j}(x)} g(\ell) - \sum_{\ell=1}^{m_{x_j}(x')} g(\ell) - \sum_{\ell=1}^{m_{x'_j}(x')} g(\ell)\\ =& g(m_{x_j}(x)) - g(m_{x'_j}(x')).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\Psi(x,z) - \Psi(x',z) =& \Psi_1(x,z) - \Psi_1(x',z) + \Psi_2(x,z) - \Psi_2(x',z)\\ &+ \beta v_0(\Psi_3(x,z) - \Psi_3(x',z))\\ =& \xi_j^{\mathrm{t}}(x_j, T_j^{\mathrm{t}}) - \xi_j^{\mathrm{t}}(x'_j, T_j^{\mathrm{t}}) + v_{x_j}(x,z) - v_{x'_j}(x',z)\\ &+ \beta v_0(g(m_{x_j}(x)) - g(m_{x'_j}(x')))\\ =& V_j(x_j, x_{-j}, z) - V_j(x'_j, x_{-j}, z).\end{aligned}$$

The proof for cars follows the same line of reasoning. ■

Remark 8.2 *Note that if v_0 is greater than the average velocity of the flow, the trucks get paid to use the road at the same time as their peers. This way the government incentivizes the trucks to form platoons. This subsidy is technically the difference between the amount of the fuel that the trucks would have saved if they formed a platoon at velocity v_0 instead of the actual average velocity of the traffic flow $an_r(z,x)+b$. Therefore, the trucks would benefit from traveling together even at low velocities (which is a scenario where the trucks do not increase their fuel efficiency significantly through platooning). However, if v_0 is smaller than the average velocity of the flow, we reduce the extra utility that the trucks would receive from traveling together (and technically $p_j^{\mathrm{t}}(x,z)$ becomes a tax rather than a subsidy). Therefore, it becomes less likely for the trucks to stick together. To emphasize the fact that we are willing to pay the trucks rather than taxing them (and hence, dealing with the first scenario), we call $p_j^{\mathrm{t}}(x,z)$ a subsidy.*

4 Joint Strategy Fictitious Play

We start by briefly introducing the learning algorithm and, then, analyzing its convergence.

4.1 Learning Algorithm

Algorithm 3 Joint strategy fictitious play for learning a Nash equilibrium.

Require: $p \in (0, 1)$
Ensure: (x^*, z^*)
for $t = 0, 1, \ldots$ **do**
  for $i = 1, \ldots, N$ **do**
    Calculate $z_i' \in \arg\max_{r \in \mathcal{R}} \hat{U}_i(r; t-1)$
    if $U_i(z_i', z_{-i}(t-1), x(t-1)) \leq U_i(z_i(t-1), z_{-i}(t-1), x(t-1))$ **then**
      $z_i(t) \leftarrow z_i(t-1)$
    else
      With probability $1-p$, $z_i(t) \leftarrow z_i(t-1)$, otherwise $z_i(t) \leftarrow z_i'$
    end if
    for $j = 1, \ldots, M$ **do**
      Calculate $x_j' \in \arg\max_{r \in \mathcal{R}} \hat{V}_j(r; t-1)$
      if $V_j(z(t-1), x_j', x_{-j}(t-1)) \leq V_j(z(t-1), x_j(t-1), x_{-j}(t-1))$ **then**
        $x_j(t) \leftarrow x_j(t-1)$
      else
        With probability $1-p$, $x_j(t) \leftarrow x_j(t-1)$, otherwise $x_j(t) \leftarrow x_j'$
      end if
    end for
  end for
end for

Assume that the agents follow the joint strategy fictitious play algorithm [40]. To do so, the agents calculate an average utility given the history of the actions. At time step $t \in \mathbb{N}_0$, car $i \in [\![N]\!]$ computes $\hat{U}_i(r; t)$ using the recursive equation

$$\hat{U}_i(r; t) = (1 - \lambda_t)\hat{U}_i(r; t-1) + \lambda_t U_i(r, z_{-i}(t), x(t)), \tag{12}$$

with the initial condition $\hat{U}_i(r; -1) = \xi_i^{\mathrm{c}}(r, T_i^{\mathrm{c}})$ for all $r \in \mathcal{R}$. In (12), $\lambda_t \in (0, 1]$ is a forgetting factor which captures the extent that the agents forget the actions from the past. If $\lambda_t = 1$, the agents are myopic (i.e., only consider the actions from the previous time step) while if $\lambda_t = 1/t$, the agents value the whole history at the same level. Following the same approach, truck $j \in [\![M]\!]$ calculates $\hat{V}_j(r; t)$ using the recursive equation

$$\hat{V}_j(r; t) = (1 - \lambda_t)\hat{V}_j(r; t-1) + \lambda_t V_j(r, x_{-j}(t), z(t)),$$

with $\hat{V}_j(r; -1) = \xi_j^{\mathrm{t}}(r, T_j^{\mathrm{t}})$ for all $r \in \mathcal{R}$. Algorithm 3 shows the joint strategy fictitious play for the car–truck congestion game.

4.2 Convergence Analysis

Noting that with appropriate taxes the introduced congestion game is a potential game, we can use the result of [40] to conclude the convergence of the learning algorithm.

Theorem 8.5 *Let the action profile of the agents be generated by the joint strategy fictitious play in Algorithm 3. Assume that* $\lambda_t = \lambda \in (0, 1)$ *or* $\lambda_t = 1/t$ *for all* $t \in \mathbb{N}$. *Then, this action profile almost surely converges to a pure strategy Nash equilibrium of the car–truck congestion game, if either the cars pay the congestion tax* $p_i^{\mathrm{c}}(z, x)$ *in* (8) *or the trucks receive the platooning subsidy* $p_j^{\mathrm{t}}(x, z)$ *in* (11).

Proof: The proof is a consequence of combining Theorems 2.1 and 3.1 in [40] with Theorems 8.3 and 8.4. ■

Note that the joint strategy fictitious play might be restrictive in some aspects. For instance, all the agents must have access to all the individual decisions taken by the other agents to calculate the average cost function. In the next section, we adapt the average strategy fictitious play introduced in [25] as an alternative. This learning algorithm requires instead a central node to broadcast the congestion prediction (i.e., an average of all the players actions) for all time intervals per day.

5 Average Strategy Fictitious Play

First, we introduce the average strategy fictitious play and study its convergence by extending parts of the proofs in [25].

5.1 Learning Algorithm

Before introducing the learning algorithm, we have to make the following standing assumptions:

Assumption 8.1 *The congestion tax policies satisfy*

- $p_i^{\mathrm{c}}(z, x)$, $i \in [\![N]\!]$, *is only a function of* $n_{z_i}(x, z), m_{z_i}(x)$;
- $p_j^{\mathrm{t}}(x, z)$, $j \in [\![M]\!]$, *is only a function of* $n_{x_j}(x, z), m_{x_j}(x)$.

This assumption means that the congestion tax can only be function of the traffic flow rather than the individual actions of the agents. The congestion taxing policy that we introduced in the previous section satisfies this assumption. To emphasize this fact, from now on, we write $p_i^{\mathrm{c}}(n_{z_i}(x, z), m_{z_i}(x))$ and $p_j^{\mathrm{t}}(n_{x_j}(x, z), m_{x_j}(x))$ with some abuse of notation.

Now, we can introduce the average strategy fictitious play. To initialize the algorithm, we let the agents pick an arbitrary action from the set $\mathcal{R}$ at the first time step. We assume that there exists a central node[2] that can observe the traffic

[2]This central node is assumed to be a not-for-profit organization. Therefore, it is not trying to optimize its income or loss (i.e., the summation of the received taxes or the distributed subsidies) and, hence, it would not strategically deviate from the intended algorithm. Certainly, introducing a mechanism with profitable organizations as a central node can be a viable avenue for future research (to attract the private sector for implementing this part).

flow at each time interval. This central node uses the following recursive update laws to calculate the average number of the cars and trucks in each time interval

$$\bar{n}_r^{\mathrm{c}}(t) = (1-\lambda)\bar{n}_r^{\mathrm{c}}(t-1) + \lambda \sum_{\ell=1}^{N} \mathbf{1}_{\{z_\ell(t)=r\}},$$
$$\bar{n}_r^{\mathrm{t}}(t) = (1-\lambda)\bar{n}_r^{\mathrm{t}}(t-1) + \lambda \sum_{\ell=1}^{M} \mathbf{1}_{\{x_\ell(t)=r\}},$$

with $\bar{n}_r^{\mathrm{c}}(0) = \sum_{\ell=1}^{N} \mathbf{1}_{\{z_\ell(0)=r\}}$ and $\bar{n}_r^{\mathrm{t}}(0) = \sum_{\ell=1}^{M} \mathbf{1}_{\{x_\ell(0)=r\}}$ for all $r \in \mathcal{R}$. The superscripts c and t show that the aforementioned property is related to the cars or the trucks, respectively. In these recursive update laws, we should choose the forgetting factor $\lambda \in (0,1)$ to capture the extent with which we value the congestion information from the past. We can think of the numbers $\bar{n}_r^{\mathrm{c}}(t)$ and $\bar{n}_r^{\mathrm{t}}(t)$ as the forecasts that the central node (e.g., the department of transportation, the radio station, etc) announces on a day-to-day basis about the traffic flow for each time interval of the day. These values have a memory to remember the congestion in earlier days and get updated based on the actual observation of the traffic flow every midnight.

Additionally, car $i \in [\![N]\!]$ and truck $j \in [\![M]\!]$ keep track of the average number of times that they have chosen $r \in \mathcal{R}$ following the recursive update laws

$$\bar{w}_{r,i}^{\mathrm{c}}(t) = (1-\lambda)\bar{w}_{r,i}^{\mathrm{c}}(t-1) + \lambda \mathbf{1}_{\{z_i(t)=r\}},$$
$$\bar{w}_{r,j}^{\mathrm{t}}(t) = (1-\lambda)\bar{w}_{r,j}^{\mathrm{t}}(t-1) + \lambda \mathbf{1}_{\{x_j(t)=r\}},$$

with $\bar{w}_{r,i}^{\mathrm{c}}(0) = \mathbf{1}_{\{z_i(0)=r\}}$ and $\bar{w}_{r,j}^{\mathrm{t}}(0) = \mathbf{1}_{\{x_j(0)=r\}}$ for all $r \in \mathcal{R}$. Finally, for all $i \in [\![N]\!]$ and $j \in [\![M]\!]$, we define the new "average" cost functions in

$$\begin{aligned}
\tilde{V}_j(r;t) =& [a(\bar{n}_r^{\mathrm{c}}(t) + \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,j}^{\mathrm{t}}(t) + 1) + b] \\
&+ \beta[a(\bar{n}_r^{\mathrm{c}}(t) + \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,j}^{\mathrm{t}}(t) + 1) + b] g(\bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,j}^{\mathrm{t}}(t) + 1) \\
&+ \xi_j^{\mathrm{t}}(r, T_j^{\mathrm{t}}) + p_j^{\mathrm{t}}(\bar{n}_r^{\mathrm{c}}(t) + \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,j}^{\mathrm{t}}(t) + 1, \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,j}^{\mathrm{t}}(t) + 1), \quad (13a) \\
\tilde{U}_i(r;t) =& \xi_i^{\mathrm{c}}(r, T_i^{\mathrm{c}}) + [a(\bar{n}_r^{\mathrm{c}}(t) + \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,i}^{\mathrm{c}}(t) + 1) + b] \\
&+ p_i^{\mathrm{c}}(\bar{n}_r^{\mathrm{c}}(t) + \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,i}^{\mathrm{c}}(t) + 1, \bar{n}_r^{\mathrm{t}}(t)). \quad (13b)
\end{aligned}$$

Now, if we follow Algorithm 4, we expect to converge to a Nash equilibrium.

5.2 Convergence Analysis

First, we need to prove an intermediate lemma which shows that if Algorithm 4 reaches a Nash equilibrium, it stays there forever.

Lemma 8.6 *Let each truck $j \in [\![M]\!]$ receive the subsidy*

$$p_j^{\mathrm{t}}(x,z) = \beta(v_0 - (a n_{x_j}(z,x) + b)) m_{x_j}(x),$$

for a given $v_0 \in \mathbb{R}$. If $x(t)$ and $z(t)$, generated by Algorithm 4, is a pure strategy Nash equilibrium, and $z_i(t) \in \arg\max_{r \in \mathcal{R}} \tilde{U}_i(r; t-1)$ for all $i \in [\![N]\!]$ and $x_j(t) \in$

Algorithm 4 Average strategy fictitious play for learning a Nash equilibrium.

Require: $p \in (0, 1)$
Ensure: (x^*, z^*)
for $t = 1, 2, \ldots$ **do**
  for $i = 1, \ldots, N$ **do**
    Calculate $z_i' \in \arg\max_{r\in\mathcal{R}} \tilde{U}_i(r; t-1)$
    if $U_i(z_i', z_{-i}(t-1), x(t-1)) \leq U_i(z_i(t-1), z_{-i}(t-1), x(t-1))$ **then**
      $z_i(t) \leftarrow z_i(t-1)$
    else
      With probability $1-p$, $z_i(t) \leftarrow z_i(t-1)$, otherwise $z_i(t) \leftarrow z_i'$
    end if
    for $j = 1, \ldots, M$ **do**
      Calculate $x_j' \in \arg\max_{r\in\mathcal{R}} \tilde{V}_j(r; t-1)$
      if $V_j(z(t-1), x_j', x_{-j}(t-1)) \leq V_j(z(t-1), x_j(t-1), x_{-j}(t-1))$ **then**
        $x_j(t) \leftarrow x_j(t-1)$
      else
        With probability $1-p$, $x_j(t) \leftarrow x_j(t-1)$, otherwise $x_j(t) \leftarrow x_j'$
      end if
    end for
  end for
end for

$\arg\max_{r\in\mathcal{R}} \tilde{V}_j(r; t-1)$ *for all* $j \in [\![M]\!]$, *then* $x(t') = x(t)$ *and* $z(t') = z(t)$ *for all* $t' \geq t$.

Proof: The proof of this lemma follows the same line of reasoning as in the proof of Proposition 4.2 in [25]. Here, we only prove the results for the trucks as the proof for the cars is technically the same. First, note that for all $r \in \mathcal{R}$, we get

$$\begin{aligned}
\bar{n}_r^{\mathrm{c}}(t) + \bar{n}_r^{\mathrm{t}}(t) - \bar{w}_r^{\mathrm{t}}(t) &= (1-\lambda)\bar{n}_r^{\mathrm{c}}(t-1) + \lambda\sum_{\ell=1}^{N}\mathbf{1}_{\{z_\ell(t)=r\}} \\
&\quad + (1-\lambda)\bar{n}_r^{\mathrm{t}}(t-1) + \lambda\sum_{\ell=1}^{M}\mathbf{1}_{\{x_\ell(t)=r\}} \\
&\quad - (1-\lambda)\bar{w}_{r,j}^{\mathrm{t}}(t-1) - \lambda\mathbf{1}_{\{x_j(t)=r\}} \\
&= (1-\lambda)(\bar{n}_r^{\mathrm{c}}(t-1) + \bar{n}_r^{\mathrm{t}}(t-1) - \bar{w}_r^{\mathrm{t}}(t-1)) \\
&\quad + \lambda(n_r(x(t), z(t)) - \mathbf{1}_{\{x_j(t)=r\}}), \qquad (14a)
\end{aligned}$$

$$\begin{aligned}
\bar{n}_r^{\mathrm{t}}(t) - \bar{w}_{r,j}^{\mathrm{t}}(t) &= (1-\lambda)\bar{n}_r^{\mathrm{t}}(t-1) + \lambda\sum_{\ell=1}^{M}\mathbf{1}_{\{x_\ell(t)=r\}} \\
&\quad - (1-\lambda)\bar{w}_{r,j}^{\mathrm{t}}(t-1) - \lambda\mathbf{1}_{\{x_j(t)=r\}} \\
&= (1-\lambda)(\bar{n}_r^{\mathrm{t}}(t-1) - \bar{w}_{r,j}^{\mathrm{t}}(t-1)) \\
&\quad + \lambda(m_r(x(t)) - \mathbf{1}_{\{x_j(t)=r\}}). \qquad (14b)
\end{aligned}$$

Now, using these update laws and the proposed subsidy policy in (11), we get

$$\begin{aligned}\tilde{V}_j(r;t) &= \xi_j^{\mathrm{t}}(r,T_j^{\mathrm{t}}) + a(\bar{n}_r^{\mathrm{c}}(t)+\bar{n}_r^{\mathrm{t}}(t)-\bar{w}_r^{\mathrm{t}}(t)+1)\\ &\quad + b + \beta v_0(\bar{n}_r^{\mathrm{t}}(t)-\bar{w}_{r,j}^{\mathrm{t}}(t)+1)\\ &= \xi_j^{\mathrm{t}}(r,T_j^{\mathrm{t}}) + a(1-\lambda)(\bar{n}_r^{\mathrm{c}}(t-1)+\bar{n}_r^{\mathrm{t}}(t-1)-\bar{w}_r^{\mathrm{t}}(t-1))\\ &\quad + a(\lambda(n_r(x(t),z(t))-\mathbf{1}_{\{x_j(t)=r\}})+1)\\ &\quad + b + \beta v_0(1-\lambda)(\bar{n}_r^{\mathrm{t}}(t-1)-\bar{w}_{r,j}^{\mathrm{t}}(t-1))\\ &\quad + \beta v_0(\lambda(m_r(x(t))-\mathbf{1}_{\{x_j(t)=r\}})+1)\\ &= (1-\lambda)\tilde{V}_j(r;t-1)+\lambda V_j(r,x_{-j}(t),z(t)).\end{aligned}$$

Therefore, we can prove that

$$\begin{aligned}\tilde{V}_j(x_j(t);t) &= (1-\lambda)\tilde{V}_j(x_j(t);t-1)+\lambda V_j(x_j(t),x_{-j}(t),z(t))\\ &\geq (1-\lambda)\tilde{V}_j(r;t-1)+\lambda V_j(r,x_{-j}(t),z(t))\\ &= \tilde{V}_j(r;t)\end{aligned}$$

for any $r \in \mathcal{R}$, where the inequality is direct consequence of the fact that the pair $x(t)$ and $z(t)$ is a pure strategy Nash equilibrium and $x_j(t) \in \arg\max_{r\in\mathcal{R}} \tilde{V}_j(r;t-1)$ for all $j \in [\![M]\!]$. Thus, $x_j(t) \in \arg\max_{r\in\mathcal{R}} \tilde{V}_j(r;t)$ and as a result, we get $x_j(t+1) = x_j(t)$ (following Algorithm 4). Now, using a simple mathematical induction, we can show $x_j(t+k) = x_j(t)$ for all $k \in \mathbb{N}$. ∎

Theorem 8.7 *Let the action profile of the agents be generated by the average strategy fictitious play in Algorithm 4. Then, this action profile almost surely converges to a pure strategy Nash equilibrium of the car–truck congestion game, if the trucks receive the platooning subsidy* $p_j^{\mathrm{t}}(x,z)$ *in* (11).

Proof: The proof follows from using Theorem 8.4 and Lemma 8.6 in the proof of Theorem 4.1 in [25]. ∎

6 Numerical Example

Let us assume that $N = 10000$ cars and $M = 100$ trucks are using the segment of the highway illustrated in Figure 1 from 7:00am to 9:00am on a daily basis. We divide the time horizon into eight equal non-overlapping intervals. Hence, we fix the action set as $\mathcal{R} = \{1,\dots,8\}$, where each number represents an interval of 15 min. Let T_i^{c}, $i \in [\![N]\!]$, be randomly chosen from the set $\mathcal{R}$ using the discrete distribution

$$\mathbb{P}\{T_i^{\mathrm{c}} = n\} = \begin{cases} 1/6, & n = 2,4,\\ 1/4, & n = 3,\\ 1/12, & \text{otherwise.}\end{cases}$$

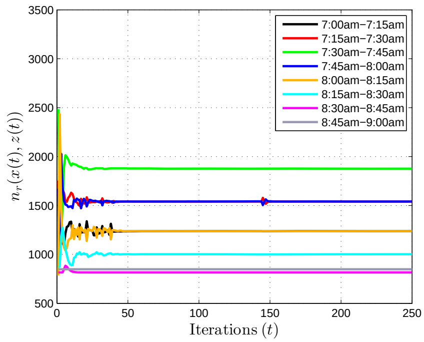

Figure 3: $n_r(x(t), z(t))$, $r \in \mathcal{R}$, versus the iteration number for $\beta = 10^{-3}$ when using the joint strategy fictitious play in Algorithm 3 with $p = 0.4$ and $\lambda_t = 3 \times 10^{-2}$ for all $t \in \mathbb{N}_0$.

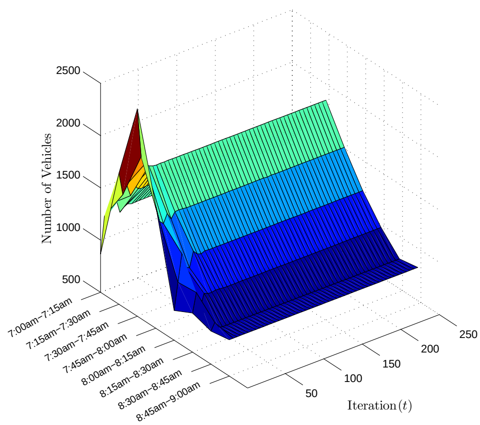

Figure 4: Number of the vehicles in each time interval for $\beta = 10^{-3}$ when using the joint strategy fictitious play in Algorithm 3 with $p = 0.4$ and $\lambda_t = 3 \times 10^{-2}$ for all $t \in \mathbb{N}_0$.

Let us also use a similar probability distribution to extract T_j^{t}, $j \in [\![M]\!]$. Hence, we consider the case where the drivers statistically prefer to use the road at $r = 3$ which corresponds to 7:30am to 7:45am. Let α_i^{c}, $i \in [\![N]\!]$, and α_j^{t}, $j \in [\![M]\!]$, be randomly generated following a uniform distribution within the interval $[-7.5, -2.5]$. Finally, let $a = -0.0110$ and $b = 84.9696$ as discussed in Section 2.

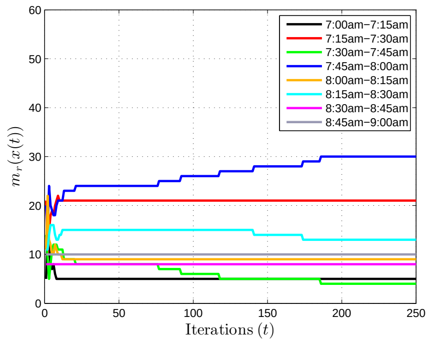

Figure 5: $m_r(x(t))$, $r \in \mathcal{R}$, versus the iteration number for $\beta = 10^{-3}$ when using the joint strategy fictitious play in Algorithm 3 with $p = 0.4$ and $\lambda_t = 3 \times 10^{-2}$ for all $t \in \mathbb{N}_0$.

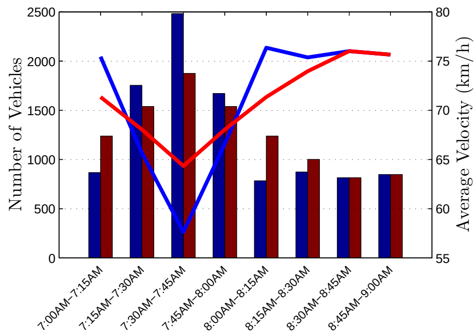

Figure 6: Number of the vehicles and the average velocity of the traffic flow in each time interval for the case where the drivers neglect the congestion in their decision making (blue) and for the learned pure strategy Nash equilibrium (red).

6.1 Learning Algorithm Performance

In this subsection, we start by simulating the joint strategy fictitious play in Algorithm 3. Let us fix $\beta = 10^{-3}$, $p = 0.4$, and $\lambda_t = 3 \times 10^{-2}$ for all $t \in \mathbb{N}_0$. Figure 3 illustrates the number of the vehicles (both cars and trucks) that are using a specific time interval to commute $n_r(x(t), z(t))$, $r \in \mathcal{R}$, as a function of the iteration number. As can be seen in this figure, the learning algorithm converges to a pure strategy Nash equilibrium in this example relatively fast[3]. Figure 4 shows the evo-

[3]Recall that there are $|\mathcal{R}|^{M+N}$ possible action combinations in a car–truck congestion game. Therefore, in this example, we have $8^{10100} \simeq 10^{9100}$ possible action combinations. To put this number into perspective, recall that there are around 10^{80} atoms in the visible universe.

lution of the traffic distribution. Figure 5 shows the number of trucks $m_r(x(t))$, $r \in \mathcal{R}$, that are using the road on various time intervals. For instance, at the learned Nash equilibrium, thirty trucks use the time interval 7:45am to 8:00am while at the same time, most of them avoid using 7:15am to 7:30am because it is highly congested (and they would not save much fuel if they commute at this time).

6.2 Nash Equilibrium Efficiency

Figure 6 shows the number of the vehicles in each time interval and the corresponding average velocity in that time interval. The blue color denotes the case where the drivers do not consider the congestion in their decision making; i.e., they commute whenever pleases them, $z_i = T_i^{\mathrm{c}}$ for all $i \in [\![N]\!]$ and $x_j = T_j^{\mathrm{t}}$ for all $j \in [\![M]\!]$. The red color denotes the case where the drivers implement the pure strategy Nash equilibrium that they have learned using Algorithm 3. As we can see in this figure, the proposed congestion game reduces the average commuting time (increases the average velocity). Following [58], we can define the social cost

$$\begin{aligned} S(x,z) &= \min_{r\in\mathcal{R}} v_r(z,x) \\ &= \min_{r\in\mathcal{R}} a n_r(x,z) + b \\ &= a(\max_{r\in\mathcal{R}} n_r(x,z)) + b, \end{aligned}$$

where the last equality holds because of the fact that $a < 0$. This social cost is the worst-case average velocity of the traffic flow[4]. Another definition of social cost could be the total fuel consumption or the overall carbon emission. In a utopia, the government should be able to implement a global solution of the optimization problem

$$(x^\bullet, z^\bullet) \in \underset{(z,x)\in\mathcal{R}^N\times\mathcal{R}^M}{\arg\max} \; S(x,z),$$

to achieve the lowest congestion at all time intervals. However, this solution cannot be implemented in a society with strategic (selfish) agents since they have no incentive for following a socially optimal decision $(x^\bullet, z^\bullet)$. Note that since $a < 0$, we have

$$\begin{aligned} (x^\bullet, z^\bullet) &\in \underset{(z,x)\in\mathcal{R}^N\times\mathcal{R}^M}{\arg\max} \; \min_{r\in\mathcal{R}} \; a n_r(x,z) + b \\ &\in \underset{(z,x)\in\mathcal{R}^N\times\mathcal{R}^M}{\arg\min} \; \max_{r\in\mathcal{R}} \; n_r(x,z), \end{aligned}$$

[4]This cost function is an example of a Rawlsian social cost function (i.e., the worst-case cost function of the players). Another possible choice of social cost function is a utilitarian social cost function (i.e., summation of the individual cost functions of all the players); see [59, p. 413] for more information regarding the difference between these two categories of social cost functions.

and as a result, we get

$$\begin{aligned} S(x^\bullet, z^\bullet) &= a \left\lceil \frac{N+M}{|\mathcal{R}|} \right\rceil + b \\ &= 71.0766\,\text{km/h}. \end{aligned}$$

Therefore, we have

$$\frac{S(x^\bullet, z^\bullet)}{S(x^*, z^*)} = 1.1048,$$

which shows that the acquired pure strategy Nash equilibrium (x^*, z^*) is not efficient with respect to the introduced welfare function[5]. However, it is somewhat better than the case where the drivers do not consider the congestion in their decision making (i.e. they travel whenever pleases them) as

$$\frac{S(x^\bullet, z^\bullet)}{S(\{T_j^{\text{t}}\}_{j=1}^M, \{T_i^{\text{c}}\}_{i=1}^N)} = 1.2330.$$

6.3 Robustness of the Learning Algorithm

Let us now consider the case where on the fiftieth day of learning (i.e., iteration $t = 50$) an unexpected behavior (e.g., a traffic accident) significantly decreases the average velocity of the traffic flow during 7:15am and 8:00am (i.e., for $r = 2, 3, 4$). To reflect this matter in the simulations, we assume that on the fiftieth iteration, the average velocity for $r = 2, 3, 4$ is given by $(an_r(x(t), z(t)) + b)/10$. Figure 7 illustrates the number of vehicles that are using a specific time interval to commute $n_r(x(t), z(t))$, $r \in \mathcal{R}$, as a function of the iteration numbers. Note that there is a sudden drop in the number of the vehicles that are using the time intervals corresponding to $r = 2, 3, 4$ for a while (around twenty iterations) after the accident. However, the learning process recovers the Nash equilibrium after another fifty iterations.

6.4 Effect of the Fuel-Saving Coefficient

In this subsection, we aim at illustrating the effect of the fuel-saving coefficient β on the behavior of the trucks. We perform all the simulations using the joint strategy fictitious play introduced in Algorithm 3 with $p = 0.4$ and $\lambda_t = 3 \times 10^{-2}$ for all $t \in \mathbb{N}_0$. Figure 8 illustrates the number of trucks for the learned Nash equilibrium at different time intervals for various choices of the coefficient β. As we expect, when $\beta = 0$, the trucks are reluctant to platoon (but instead stick to the time that favors them the most). However, as we increase the coefficient β, a higher number of trucks drive at the same time interval. Note that for $\beta = 4 \times 10^{-3}$, all hundred trucks use the road during exactly one time interval (i.e, 8:00am to 8:15am).

[5]It is worth mentioning that if we choose the potential function Φ in Theorem 8.3 as the social welfare function, the learned Nash equilibrium is indeed efficient since Algorith, 3 results in a

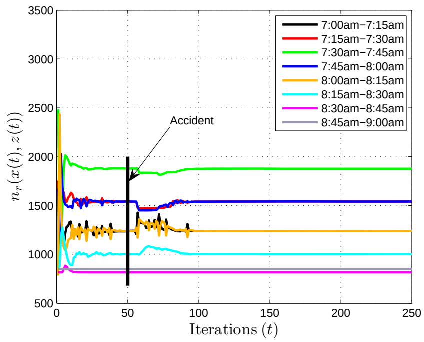

Figure 7: $n_r(x(t), z(t))$, $r \in \mathcal{R}$, versus the iteration number when an unexpected behavior (e.g., an accident) disrupt the traffic flow on the fiftieth day of learning.

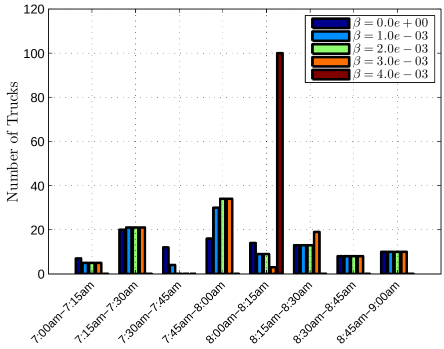

Figure 8: Number of the trucks in each time interval for various choices of the coefficient β.

6.5 Drivers Having Different Time Values

In 2001, the consulting firm Inregia in Sweden, by the request of Swedish Institute for Transport and Communications Analysis, performed a survey to estimate the value of time for the road users in Stockholm [35, 51]. This study showed that various groups of people value their time differently. According to the study, drivers valued time as 0.98, 3.30, and 0.19 SEK/min for work and school commuting trips, business trips, and other trips, respectively [35, 51]. Let us include this effect in the introduced congestion game setup. Assume that in the utility of car $i \in [\![N]\!]$,

local maximizer of this potential function. However, such a choice does not have any practical implications.

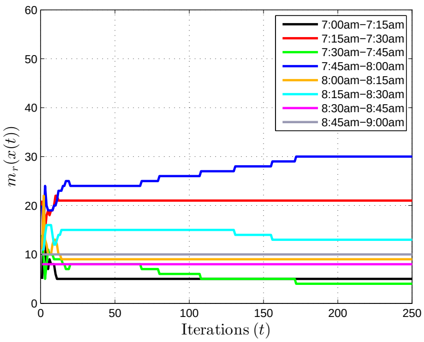

Figure 9: $m_r(x(t))$, $r \in \mathcal{R}$, versus the iteration number for the case where the drivers value their time differently.

we set the term

$$p_i^{\mathrm{c}}(z, x) = \delta_i^{-1}\left(a\beta \sum_{\ell=1}^{m_{z_i}(x)} g(\ell)\right),$$

where $\delta_i > 0$ is the value of time for the driver of car i. For work and school commuting trips, we scale the value of time to $\delta_i = 1.00$. Therefore, we get $\delta_i = 3.37$ and $\delta_i = 0.19$ for business trips and other trips, respectively. Now, allow us to randomly distribute the cars into three groups of work and school trips, business trips, and other trips with probabilities 0.754, 0.036, 0.210, respectively, as suggested in [35]. Figure 9 shows the number of trucks in each time interval as a function of the iteration number in this case. Comparing with Figure 5, we can clearly see that in this example, the difference in the value of time has not changed the behavior of trucks (certainly in the Nash equilibrium, but the transient response is different). Figure 10 shows the number of the cars in each time interval for the case where the drivers value their time differently subtracted by number of the cars in each time interval for the case where the drivers value their time equally. Clearly, the cars that value their time the most, or equivalently, the ones that are willing to pay higher congestion taxes (i.e., $\delta_i = 1.00, 3.37$), can move to the time interval where thirty trucks are traveling. However, the cars that do not value their time much (i.e., $\delta_i = 0.19$) switch to a less expensive alternative.

6.6 Trucks with and Without Platooning Equipment

Few trucks are currently fitted with platooning equipments. In this subsection, we try to understand the influence of this matter on the properties of the learned Nash equilibrium. To illustrate the effect of trucks without platooning equipment, let us consider two types of trucks where the first type can indeed participate in platoons and the second type does not have the necessary equipments for doing

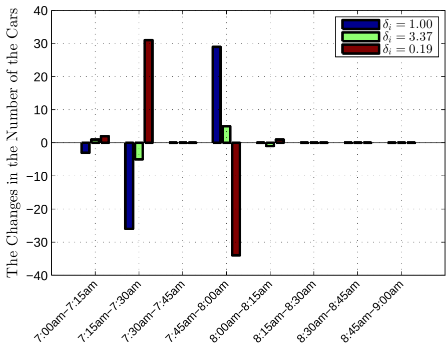

Figure 10: Number of the cars in each time interval for the case where the drivers value their time differently subtracted by number of the cars in each time interval for the case where their drivers value the time equally.

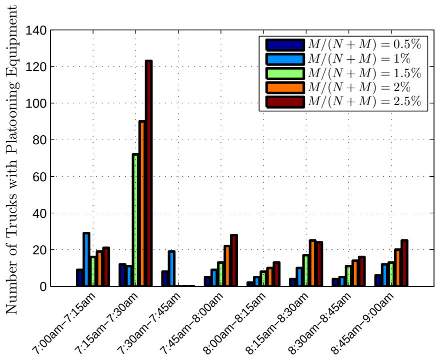

Figure 11: Number of the vehicles in each time interval for the learned pure strategy Nash equilibrium for various choices of $M/(M+N)$.

so. We count the second type of trucks as ordinary cars since they do not benefit from traveling at the same time interval as the other trucks. Hence, N shows the number of ordinary cars together with the trucks without platooning equipment and M denotes the number of trucks that can potentially participate in forming the platoons. We fix $N + M = 10000$. Figure 11 illustrates the number of the trucks that have platooning equipment in each time interval for various ratios of $M/(M + N)$. Evidently, the number of the trucks (with platooning equipment) in most of the time intervals grows linearly with $M/(M + N)$ (as we expect since there are more trucks). However, some of the intervals, such as, 7:30am to 7:45am become less favorable (as they are highly congested) and the trucks in these intervals completely move to their neighboring intervals as $M/(M + N)$ increases.

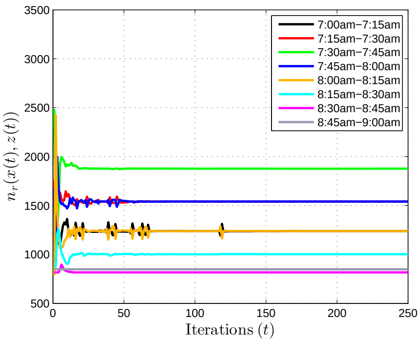

Figure 12: $n_r(x(t), z(t))$, $r \in \mathcal{R}$, versus the iteration number when the congestion tax is updated with a delay of $D = 30$ days.

6.7 Announcing Congestion Taxes in Advance

A drawback of the presented formulation is that the congestion taxes are dynamic and must be calculated (and enforced) instantly based on the number of the vehicles in each time interval. Although dynamic congestion taxing has been implemented on several occasions (e.g., San Diego I-15 High-Occupancy Toll Lanes in which the tolls vary dynamically with the level of congestion [60]), they proved to be controversial (or, cumbersome to understand for the drivers at the least). Therefore, one might consider the case in which the tolls for day $t + D$ are announced at the end of day t for all $t \in \mathbb{N}_0$ (so that the drivers have time to digest this information and act accordingly). To simulate such a scenario, we note that the congestion tax $p_i^{\mathrm{c}}(t)$ that car $i \in [\![N]\!]$ must pay for using the road at time interval $z_i(t) \in \mathcal{R}$ on iteration $t \in \mathbb{N}_0$ is equal

$$p_i^{\mathrm{c}}(t) = \begin{cases} a\beta \sum_{\ell=1}^{m_{z_i(t)}(x(t-D))} g(\ell), & t > D, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 12 illustrates the number of the vehicles for each time interval $n_r(x(t), z(t))$, $r \in \mathcal{R}$, versus the iteration number when the congestion tax is updated with a delay of $D = 30$ days. Evidently, there are more oscillations in comparison to Figure 3, however, the algorithm converges rapidly to a pure strategy Nash equilibrium.

6.8 Average Strategy Fictitious Play

In this subsection, we use the average strategy fictitious play with $\beta = 10^{-3}$, $\lambda = 3 \times 10^{-2}$, and $p = 0.4$. We also implement the platooning subsidy in Theorem 8.4 with $v_0 = 85$. Figure 13 illustrates $n_r(x(t), z(t))$, $r \in \mathcal{R}$, versus the iteration number. The proposed algorithm clearly converges to a Nash equilibrium relatively fast.

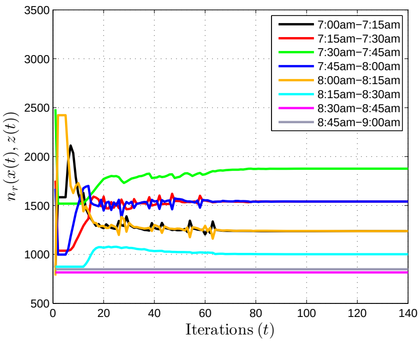

Figure 13: $n_r(x(t), z(t))$, $r \in \mathcal{R}$, versus the iteration number for $\beta = 10^{-3}$ and $v_0 = 85$ when using the average strategy fictitious play in Algorithm 4.

7 Conclusions and Future Work

We introduced a model for traffic flow on a specific road at various time intervals per day using an atomic congestion game with two types of agents (namely, cars and trucks). Cars only optimize their trade-off between using the road at the time they prefer, the average velocity of the traffic flow, and the congestion tax they are paying. However, trucks benefit from using the road at the same time as the other trucks. We motivated this extra utility using an increased possibility of platooning with the other trucks and as a result, saving fuel. We used congestion data from Stockholm to validate the linear relationship between the average velocity of commuting and the number of the vehicles that are using the road at that time. We devised appropriate tax or subsidy policies to create a potential game. Then, we used the joint strategy fictitious play and the average strategy fictitious play to learn a pure strategy Nash equilibrium of this game. We conducted a comprehensive simulation study to analyze the effect of different factors on the properties of the learned Nash equilibrium. As a future work, we can consider using mechanism design tools to enforce a socially optimal solution, such as, an optimal carbon emission profile, through appropriate congestion tax policy. Finally, in this paper, we did not consider the routing aspects of the problem. It would be of great interest in future research to combine the departure-time selection and the route selection problems in the context of understanding the platooning incentives.

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Part 3:

Stochastic Sensor Scheduling

PAPER 9

Stochastic Sensor Scheduling for Networked Control Systems

Farhad Farokhi and Karl H. Johansson

Abstract–Optimal sensor scheduling with applications to networked estimation and control systems is considered. We model sensor measurement and transmission instances using jumps between states of a continuous-time Markov chain. We introduce a cost function for this Markov chain as the summation of terms depending on the average sampling frequencies of the subsystems and the effort needed for changing the parameters of the underlying Markov chain. By minimizing this cost function through extending Brockett's recent approach to optimal control of Markov chains, we extract an optimal scheduling policy to fairly allocate the network resources among the control loops. We study the statistical properties of this scheduling policy in order to compute upper bounds for the closed-loop performance of the networked system, where several decoupled scalar subsystems are connected to their corresponding estimator or controller through a shared communication medium. We generalize the estimation results to observable subsystems of arbitrary order. Finally, we illustrate the developed results numerically on a networked system composed of several decoupled water tanks.

1 Introduction

1.1 Motivation

Emerging large-scale control applications in smart infrastructures [1], intelligent transportation systems [2], aerospace systems [3], and power grids [4], are typically implemented over a shared communication medium. Figure 1 illustrates an example of such a networked system, where L decoupled subsystems are connected to their subcontrollers over a wireless communication network. A set of sensors in each subsystem sample its state and transmit the measurements over the wireless network to the corresponding subcontroller. Then, the subcontroller calculates an actuation signal (based on the transmitted observation history) and directly applies it to the subsystem. Unfortunately, traditional digital control theory mostly results in conservative networked controllers because the available methods often assume that the sampling is done periodically with a fixed rate [5, 6]. When utilizing these periodic sampling methods, the network manager should allocate communication instances (according to the fixed sampling rates) to each control loop considering the worst-case possible scenario, that is, the maximum number of active control loops. In a large control system with thousands of control loops, fixed scheduling of communication instances imposes major constraints because network resources are allocated even if a particular control loop is not active at the moment. This restriction is more evident in ad-hoc networked control systems where many control loops may join or leave the network or switch between active and inactive states. Therefore, we need a scheduling method to set the sampling rates of the individual control loops adaptively according to their requirements and the overall network resources. We address this problem in this paper by introducing an optimal stochastic sensor scheduling scheme.

1.2 Related Studies

In networked control systems, communication resources need to be efficiently shared between multiple control loops in order to guarantee a good closed-loop performance. Despite that communication resources in large networks almost always are varying over time due to the need from the individual users and physical communication constraints, the early networked control system literature focused on situations with fixed communication constraints; e.g., bit-rate constraints [7–10] and packet loss [11–14]. Only recently, some studies have targeted the problem of integrated resource allocation and feedback control; e.g., [15–20].

The problem of sharing a common communication medium or processing unit between several users is a well-known problem in computer science, wireless communication, and networked control [21–24]. For instance, the authors in [25] proposed a scheduler to allocate time slots between several users over a long horizon. In that scheduler, the designer must first manually assign shares (of a communication medium or processing unit) that an individual user should receive. Then, each user achieves its pre-assigned share by means of probabilistic or deterministic algorithms [25, 26]. The authors in [27, 28] proved that implementing the task with the earliest deadline achieves the optimum latency in case of both synchronous and asynchronous job arrivals. In [29], a scheduling policy based on static priority assignment to the tasks was introduced. Many studies in communication literature have also considered the problem of developing protocols in order to avoid the interference between several information sources when using a common communication medium. Examples of such protocols are both time-division and frequency-division multiple access [30, 31]. Contrary to all these studies, in this paper, we automatically determine the communication instances (and, equivalently, the sampling rates) of the subsystems in a networked system based on the number of active control loops at any given moment. We use a continuous-time Markov chain to model the optimal scheduling policy.

Markov chains are very convenient tools in control and communication [32, 33]. Markov jump linear systems with underlying parameters switching according to a given Markov chain has been studied in the control literature [34–37]. The problem of controlled Markov chains has always been actively pursued [38–41]. In a recent study by Brockett [42], an explicit solution to the problem of optimal control of observable continuous-time Markov chains for a class of quadratic cost functions was presented. In that paper, the underlying continuous-time Markov chain was described using the so-called unit vector representation [42, 43]. Then, the finite horizon problem and its generalization to infinite horizon cost functions were considered. We extend that result to derive the optimal scheduling policy in this paper.

In the study [44], the authors developed a stochastic sensor scheduling policy using Markov chains. Contrary to this paper, they considered a discrete-time Markov chain to get a numerically tractable algorithm for optimal sensor scheduling. The algorithm in [44] uses one of the sensors at each time step while here, the continuous-time Markov chain can rest in one of its states to avoid sampling any of the sensors. Furthermore, the cost function in [44] was not written explicitly in terms of the Markov chain parameters, but instead it was based on the networked system performance when using a Markov chain for sampling the sensors. However, our proposed scheduling policy results in a separation between designing the Markov chain parameters and networked system, which enables us to describe the cost function needed for deriving our optimal sensor scheduling policy only in terms of the Markov chain parameters.

1.3 Main Contributions

The objective of the paper is to find a dynamic scheduling policy to fairly allocate the network resources between the subsystems in a networked system such as the one in Figure 1. Specifically, we employ a continuous-time Markov chain for scheduling the sensor measurement and transmission instances. We use time instances of the jumps between states of this continuous-time Markov chain to model

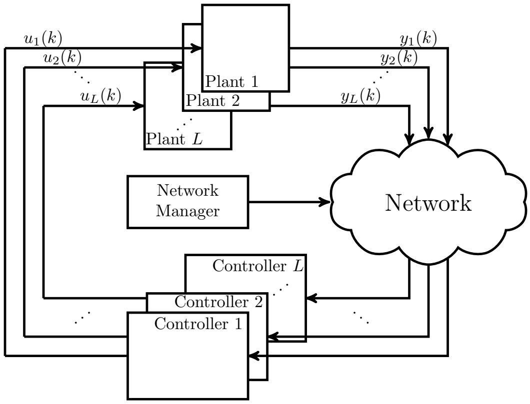

Figure 1: An example of a networked control system.

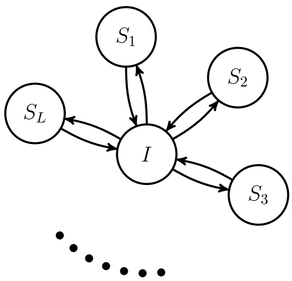

Figure 2: Flow diagram of the continuous-time Markov chain used for modeling the proposed stochastic scheduling policy.

the sampling instances; i.e., whenever there is a jump from an idle state in the Markov chain to a state that represent a subsystem in the networked system, we sample that particular subsystem and transmit its state measurement across the shared communication network to the corresponding subcontroller. Figure 2 illustrates the flow diagram of the proposed Markov chain. Every time that a jump from the idle node I to node S_ℓ, $1 \leq \ell \leq L$, occurs in this continuous-time Markov chain, we sample subsystem ℓ and send its state measurement to subcontroller ℓ. The idle state I helps to tune the sampling rates of the subsystems independently. As an approximation of the wireless communication network, we assume that the sampling and communication are instantaneous; i.e., the sampling and transmission delays are negligible in comparison to the subsystems response time. We still want to limit the amount of communication per time unit to reduce the energy consumption and network resources.

We mathematically model the described continuous-time Markov chain using unit vector representation [42, 43]. We introduce a cost function that is a combination of the average sampling frequencies of the subsystems (i.e., the average frequency of the jumps between the idle state and the rest of the states in the Markov chain) and the effort needed for changing the scheduling policy (i.e., changing the underlying Markov chain parameters). We expand the results presented in [42] to minimize the cost function over both finite and infinite horizons. Doing so, we find an explicit minimizer of the cost function and develop the optimal scheduling policy accordingly. This policy fairly allocates sampling instances among the sensors in the networked system. The proposed optimal scheduling policy works particularly well for ad-hoc sensor networks since we can easily accommodate for the changes in the network configuration by adding an extra state to the Markov chain (and, in turn, by adding an extra term to the cost function) whenever a new sensor becomes active and by removing a state from the Markov chain (and, in turn, by removing the corresponding term from the cost function) whenever a sensor becomes inactive. The idea of dynamic peer participation (or churn) in peer-to-peer networks have been extensively studied in the communication literature [45, 46]. However, not much attention has been paid to this problem for networked control and estimation.

Later, we focus on networked estimation as an application of the proposed stochastic sensor scheduling policy. We start by studying a networked system composed of several scalar subsystems and calculate an explicit upper bound for the estimation error variance as a function of the statistics of the measurement noise and the scheduling policy. The statistics of the scheduling policy are implicitly dependent on the cost function. Hence, we can achieve the required level of performance by finely tuning the cost function design parameters. We generalize these estimation results to higher-order subsystems when noisy state measurements of the subsystems are available. In the case where noisy output measurements of the subsystems are available, we derive an estimator based on the discrete-time Kalman filter and calculate an upper bound for the variance of its error given a specific sequence of sampling instances. Lastly, we consider networked control as an application of the proposed sensor scheduling policy. We assume that the networked control system is composed of scalar subsystems that are in feedback interconnection with impulsive controllers (i.e., controllers that ideally reset the state of the system whenever a new measurement arrives). We find an upper bound for the closed-loop performance of the subsystems as a function of the statistics of the measurement noise and the scheduling policy. We generalize this result to pulse and exponential controllers.

1.4 Paper Outline

The rest of the paper is organized as follows. In Section 2, we introduce the optimal stochastic scheduling policy and calculate its statistics. We apply the proposed stochastic scheduling policy to networked estimation and control systems in Sections 3 and 4, respectively. In Section 5, we illustrate the developed results numerically on a networked system composed of several decoupled water tanks. Finally, we present the conclusions and directions for future research in Section 6.

1.5 Notation

The sets of integer and real numbers are denoted by $\mathbb{Z}$ and $\mathbb{R}$, respectively. We use $\mathbb{O}$ and $\mathbb{E}$ to denote the sets of odd and even numbers. For any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we define $\mathbb{Z}_{>(\geq)n} = \{m \in \mathbb{Z} \mid m > (\geq) n\}$ and $\mathbb{R}_{>(\geq)x} = \{y \in \mathbb{R} \mid y > (\geq) x\}$, respectively. We use calligraphic letters, such as $\mathcal{A}$ and $\mathcal{X}$, to denote any other set.

We use capital roman letters, such as A and C, to denote matrices. For any matrix A, a_{ij} denotes its entry in the i-th row and the j-th column.

Vector e_i denotes a column vector (where its size will be defined in the text) with all entries equal zero except its i-th entry which is equal to one. For any vector $x \in \mathbb{R}^n$, we define the entry-wise operator $x^{\cdot 2} = [x_1^2 \ \ \cdots \ \ x_n^2]^\top$.

2 Stochastic Sensor Scheduling

In this section, we develop an optimal stochastic scheduling policy for networked systems, where several sensors are connected to the corresponding controllers or estimators over a shared communication medium. Let us start by modeling the stochastic scheduling policy using continuous-time Markov chains.

2.1 Sensor Scheduling Using Continuous-Time Markov Chains

We employ continuous-time Markov chains to model the sampling instances of the subsystems. To be specific, every time that a jump from the idle node I to node S_ℓ, $1 \leq \ell \leq L$, occurs in the continuous-time Markov chain described by the schematic flow diagram in Figure 2, we sample subsystem ℓ. We use unit vector representation to mathematically model this continuous-time Markov chain [42, 43].

We define the set $\mathcal{X} = \{e_1, e_2, \ldots, e_n\} \subset \mathbb{R}^n$ where $n = L + 1$. The Markov chain state $x(t) \in \mathbb{R}^n$ takes value from $\mathcal{X}$, which is the reason behind naming this representation as the unit vector representation. We associate nodes S_1, S_2, ..., S_L, and I in the Markov chain flow diagram with unit vectors e_1, e_2, ..., e_L, and e_n, respectively. Following the same approach as in [43], we can model the Markov chain in Figure 2 by the Itô differential equation

$$\mathrm{d}x(t) = \sum_{\ell=1}^{L} \Big(G'_{\ell n} x(t) \ \mathrm{d}N'_{\ell n}(t) + G'_{n\ell} x(t) \ \mathrm{d}N'_{n\ell}(t) \Big), \tag{1}$$

where $\{N'_{n\ell}(t)\}_{t \in \mathbb{R}_{\geq 0}}$ and $\{N'_{\ell n}(t)\}_{t \in \mathbb{R}_{\geq 0}}$, $1 \leq \ell \leq L$, are Poisson counter processes[1] with rates $\lambda_{n\ell}(t)$ and $\lambda_{\ell n}(t)$, respectively. These Poisson counters determine the

[1] Recall that a Poisson counter $N(t)$ is a stochastic process with independent and stationary increments that starts from zero $N(0) = 0$. Additionally, $\mathbb{P}\{N(t + \Delta t) - N(t) = k\} = (\int_t^{t+\Delta t} \lambda(t)\mathrm{d}t)^k \exp(\int_t^{t+\Delta t} \lambda(t)\mathrm{d}t)/k!$ for any $t, \Delta t \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. In the limit, when replacing Δt with $\mathrm{d}t$ and $\Delta N(t) = N(t+\Delta t) - N(t)$ with $\mathrm{d}N(t)$, we get $\mathbb{P}\{\mathrm{d}N(t) = 0\} = 1 - \lambda(t)\mathrm{d}t$, $\mathbb{P}\{\mathrm{d}N(t) = 1\} = \lambda(t)\mathrm{d}t$, and $\mathbb{P}\{\mathrm{d}N(t) = k\} = 0$ for $k \in \mathbb{Z}_{\geq 2}$. For a detailed discussion on Poisson counters, see [43, 47].

rates of jump from S_ℓ to I, and vice versa. In addition, we have $G'_{\ell n} = (e_\ell - e_n)e_n^\top$ and $G'_{n\ell} = (e_n - e_\ell)e_\ell^\top$, $1 \leq \ell \leq L$. Let us define $m = 2L$. Now, we can rearrange the Itô differential equation in (1) as

$$\mathrm{d}x(t) = \sum_{i=1}^{m} G_i x(t)\, \mathrm{d}N_i(t), \tag{2}$$

where $\{N_i(t)\}_{t\in\mathbb{R}_{\geq 0}}$, $1 \leq i \leq m$, is a Poisson counter process with rate denoted as

$$\mu_i(t) = \begin{cases} \lambda_{n,\lfloor (i-1)/2\rfloor+1}(t), & i \in \mathbb{O}, \\ \lambda_{\lfloor (i-1)/2\rfloor+1,n}(t), & i \in \mathbb{E}, \end{cases} \tag{3}$$

and

$$G_i = \begin{cases} G'_{n,\lfloor (i-1)/2\rfloor+1}, & i \in \mathbb{O}, \\ G'_{\lfloor (i-1)/2\rfloor+1,n}, & i \in \mathbb{E}. \end{cases} \tag{4}$$

The Poisson counters $\{N_i(t)\}_{t\in\mathbb{R}_{\geq 0}}$, $1 \leq i \leq m$, determine the rates of jump between the states of the Markov chain in (2). Now, noting that this Markov chain models the sampling instances $\{T_i^\ell\}_{i=0}^\infty$, $1 \leq \ell \leq L$, using the jumps that occur in its state $x(t)$, we can control the average sampling frequencies of the sensors through the rates $\mu_i(t)$, $1 \leq i \leq m$. Similar to [42], we assume that we can control the rates as

$$\mu_i(t) = \mu_{i,0} + \sum_{j=1}^{m} \alpha_{ij} u_j(t), \tag{5}$$

and thereby control the average sampling frequencies[2]. In (5), $\alpha_{ij} \in \mathbb{R}$, $1 \leq i, j \leq m$, are constant parameters that determine the sensitivity of Poisson counters' jump rates with respect to control inputs u_j for $1 \leq j \leq m$. Control signals $u_j(t)$, $1 \leq j \leq m$, are chosen in order to minimize the cost function

$$J = \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T \sum_{\ell=1}^{L} \xi_\ell\, e_n^\top x(t)\, \mathrm{d}N_{2\ell}(t) + u(t)^\top u(t)\, \mathrm{d}t\right\}, \tag{6}$$

where $\xi_\ell \in \mathbb{R}_{\geq 0}$, $1 \leq \ell \leq L$, are design parameters. Note that the cost function (6) consists of two types of terms: $\frac{1}{T}\int_0^T e_n^\top x(t)\mathrm{d}N_{2\ell}(t)$ denotes the average frequency of the jumps from I to S_ℓ in the Markov chain (i.e., the average sampling frequency of sensor ℓ) and $\frac{1}{T}\int_0^T u(t)^\top u(t)\mathrm{d}t$ penalizes the control effort in regulating this frequency. If the latter term is removed, the problem would become ill-posed as the optimal rates $\mu_i(t)$ then is zero and $\mathbb{E}\{\mathrm{d}N_i(t)\} = 0$. Consequently, the average sampling frequencies of the sensors vanish.

[2]Notice that the number of control inputs in (5) does not have to be the same as the number of Poisson counters for the proofs in Subsection 2.2 to hold. However, we decided to follow the convention of [42] because we use these results in Sections 3 and 4 to optimally schedule sensors in a networked system in which we can control all the rates $(\mu_i(t))_{i=1}^m$ directly.

Considering the identity $\mathbb{E}\{\mathrm{d}N_{2\ell}(t)\} = (\mu_{2\ell,0} + \sum_{j=1}^{m} \alpha_{2\ell,j} u_j(t))\mathrm{d}t$, we can rewrite the cost function in (6) as

$$J = \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T c^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\ \mathrm{d}t\right\}, \tag{7}$$

where $c = e_n \sum_{\ell=1}^{L} \xi_\ell \mu_{2\ell,0}$ and $S \in \mathbb{R}^{m\times n}$ is a matrix whose entries are defined as $s_{ji} = \sum_{\ell=1}^{L} \xi_\ell \alpha_{2\ell,j}$ if $i = n$ and $s_{ji} = 0$ otherwise. In the rest of this paper, we use the notation S_i, $1 \le i \le m$, to denote i-th row of matrix S. In the next subsection, we find a policy that minimizes (7) with respect to the rate control law (5) and subject to the Markov chain dynamics (2). Doing so, we develop an optimal scheduling policy which fairly allocates the network resources (i.e., the sampling instances) between the devices in a sensor network.

2.2 Optimal Sensor Scheduling

We start by minimizing the finite horizon version of the cost function in (6). The proof of the following theorem is a slight generalization of Brockett's result in [42] but follows the same line of reasoning[3].

Theorem 9.1 *Consider a continuous-time Markov chain evolving on* $\mathcal{X} = \{e_1, \ldots, e_n\} \subset \mathbb{R}^n$, *generated by (2). Let us define matrices* $A = \sum_{i=1}^{m} \mu_{i,0} G_i$ *and* $B_i = \sum_{j=1}^{m} \alpha_{ji} G_j$, *where for all* $1 \le i, j \le m$, G_i *and* α_{ij} *are introduced in (4) and (5), respectively. Assume that, for given* $T \in \mathbb{R}_{>0}$ *and* $c : [0, T] \to \mathbb{R}^n$, *the differential equation*

$$\dot{k}(t) = -c(t) - A^\top k(t) + \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top k(t))^{\cdot 2};\ k(T) = k_f, \tag{8}$$

has a solution on $[0, T]$ *such that, for each* $(t, x) \in [0, T] \times \mathcal{X}$, *the operator* $A - \sum_{i=1}^{m} \frac{1}{2}(k(t)^\top B_i + S_i) x B_i$ *is an infinitesimal generator. Then, the control law*

$$u_i(t, x) = -\frac{1}{2}\left(k(t)^\top B_i + S_i\right) x(t), \quad 1 \le i \le m, \tag{9}$$

minimizes

$$J = \mathbb{E}\left\{\frac{1}{T}\int_0^T c(t)^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t + \frac{1}{T} k_f^\top x(T)\right\}.$$

Furthermore, $J = \frac{1}{T} k(0)^\top \mathbb{E}\{x(0)\}$.

[3]The statement makes use of the concept of infinitesimal generators. See [48, pp. 124] for definition and discussion.

Proof: See Appendix A. ∎

Notice that for some parameter settings of the cost function, the operator $A - \sum_{i=1}^{m} \frac{1}{2}(k(t)^\top B_i + S_i)xB_i$ may not be an infinitesimal generator. A future avenue of research could be to characterize these cases and to present conditions for avoiding them.

Based on Theorem 9.1, we are able to solve the following infinite-horizon version of the optimal scheduling policy. In the infinite-horizon case, we need to assume that the parameters of the Markov chain and the cost function are time invariant.

Corollary 9.2 *Consider a continuous-time Markov chain evolving on* $\mathcal{X} = \{e_1, \ldots, e_n\} \subset \mathbb{R}^n$*, generated by (2). Let us define matrices* $A = \sum_{i=1}^{m} \mu_{i,0} G_i$ *and* $B_i = \sum_{j=1}^{m} \alpha_{ji} G_j$*, where for all* $1 \leq i, j \leq m$*,* G_i *and* α_{ij} *are introduced in (4) and (5), respectively. Assume that, for a given* $c \in \mathbb{R}^n$*, the nonlinear equation*

$$\begin{bmatrix} A^\top & -\mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix} \begin{bmatrix} k_0 \\ \varrho \end{bmatrix} - \frac{1}{4} \begin{bmatrix} \sum_{i=1}^{m} (S_i^\top + B_i^\top k_0)^{\cdot 2} \\ 0 \end{bmatrix} = \begin{bmatrix} -c \\ 0 \end{bmatrix}, \tag{10}$$

has a solution $(k_0, \varrho) \in \mathbb{R}^n \times \mathbb{R}$ *such that, for all* $x \in \mathcal{X}$*, the operator* $A - \sum_{i=1}^{m} \frac{1}{2}(k_0^\top B_i + S_i)xB_i$ *is an infinitesimal generator. Then, the control law*

$$u_i(t, x) = -\frac{1}{2}(k_0^\top B_i + S_i)x(t), \quad 1 \leq i \leq m, \tag{11}$$

minimizes

$$J = \lim_{T \to \infty} \mathbb{E} \left\{ \frac{1}{T} \int_0^T c^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t) \mathrm{d}t \right\}.$$

Furthermore, we have $J = \varrho$*.*

Proof: See Appendix B. ∎

Corollary 9.2 introduces an optimal scheduling policy to fairly allocate measurement transmissions among sensors according to the cost function in (6). By changing the design parameters ξ_ℓ, $1 \leq \ell \leq L$, we can tune the average sampling frequencies of the subsystems according to their performance requirements. In addition, by adding an extra term to the cost function whenever a new subsystem is introduced or by removing a term whenever a subsystem is detached, we can easily accommodate for dynamic changes in an ad-hoc network. In the remainder of this section, we analyze the asymptotic properties of the optimal scheduling policy in Corollary 9.2.

2.3 Average Sampling Frequencies

In this subsection, we study the relationship between the Markov chain parameters and the effective sampling frequencies of the subsystems. Recalling from the problem formulation, $\{T_i^\ell\}_{i=0}^\infty$ denotes the sequence of time instances that the state of the Markov chain in (1) jumps from the idle node I to S_ℓ and hence, subsystem ℓ is sampled. Mathematically, we define these time instances as

$$T_0^\ell = \inf\{t \geq 0 \mid \exists\, \epsilon > 0 : x(t-\epsilon) = e_n \wedge x(t) = e_\ell\},$$

and

$$T_{i+1}^\ell = \inf\{t \geq T_i^\ell \mid \exists\, \epsilon > 0 : x(t-\epsilon) = e_n \wedge x(t) = e_\ell\},$$

for all $i \in \mathbb{Z}_{\geq 0}$. Furthermore, we define the sequence of random variables $\{\Delta_i^\ell\}_{i=0}^\infty$ such that $\Delta_i^\ell = T_{i+1}^\ell - T_i^\ell$ for all $i \in \mathbb{Z}_{\geq 0}$. These random variables denote the time interval between any two successive sampling instances of sensor ℓ. We make the assumption that the first and second samples happen within finite time almost surely:

Assumption 9.1 $\mathbb{P}\{T_0^\ell < \infty\} = 1$ *and* $\mathbb{P}\{T_1^\ell < \infty\} = 1$.

This assumption is not restrictive. Note that it is trivially satisfied if the number of subsystems is finite, the Markov chain is irreducible, and the rates of Poisson processes are finite and uniformly bounded away from zero. Let us present the following simple lemma.

Lemma 9.3 $\{\Delta_i^\ell\}_{i=0}^\infty$ *are identically and independently distributed random variables.*

Proof: See [49]. ■

Lemma 9.3 implies that $\mathbb{E}\{\Delta_i^\ell\}$ is not a function of i (and, therefore, it ensures several of the expressions, that are presented later, are indeed well-defined). Now, we are ready to state state our main result concerning the average sampling frequency of the sensors denoted by

$$f_\ell = \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T e_n^\top x(t)\,\mathrm{d}N_{2\ell}(t)\right\}, \quad 1 \leq \ell \leq L.$$

Theorem 9.4 *Let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Define $p(t) = \mathbb{E}\{x(t)\}$ using*

$$\dot{p}(t) = \left(A - \frac{1}{2}\sum_{i=1}^{m} B_i \Lambda(k_0^\top B_i + S_i)\right) p(t), \; p(0) = \mathbb{E}\{x(0)\}, \tag{12}$$

where

$$\Lambda(k_0^\top B_i + S_i) = \operatorname{diag}((k_0^\top B_i + S_i)e_1, \ldots, (k_0^\top B_i + S_i)e_n).$$

If $\lim_{t\to\infty} p(t)$ exists, the average sampling frequency of sensor ℓ is equal to

$$\begin{aligned} f_\ell &= \frac{1}{\mathbb{E}\{\Delta_i^\ell\}} \\ &= \left(\mu_{2\ell,0} - \frac{1}{2}\sum_{j=1}^{m} \alpha_{2\ell,j}(k_0^\top B_j + S_j)e_n\right) e_n^\top \lim_{t\to\infty} p(t). \end{aligned}$$

Proof: See Appendix C. ■

Theorem 9.4 allows us to calculate the average sampling frequencies of the subsystems. We use these average sampling frequencies to bound the closed-loop performance of the networked system when the proposed optimal scheduling policy is implemented.

3 Applications to Networked Estimation

In this section, we study networked estimation based on the proposed stochastic scheduling policy. Let us start by presenting the system model and the estimator. As a starting point, we introduce a networked system that is composed of scalar decoupled subsystems. In Subsections 3.3 and 3.4, we generalize some of the results to decoupled higher-order subsystems.

3.1 System Model and Estimator

Consider the networked system illustrated in Figure 1, where subsystem ℓ, $1 \leq \ell \leq L$, is a scalar stochastic system described by

$$\mathrm{d}z_\ell(t) = -\gamma_\ell z_\ell(t)\,\mathrm{d}t + \sigma_\ell\,\mathrm{d}w_\ell(t);\; z_\ell(0) = 0, \tag{13}$$

with given model parameters $\gamma_\ell, \sigma_\ell \in \mathbb{R}_{\geq 0}$. Note that all subsystems are stable. The stochastic processes $\{w_\ell(t)\}_{t\in\mathbb{R}_{\geq 0}}$, $1 \leq \ell \leq L$, are statistically independent Wiener processes with zero mean. Estimator ℓ receives state measurements $\{y_i^\ell\}_{i=0}^\infty$ at time instances $\{T_i^\ell\}_{i=0}^\infty$, such that

$$y_i^\ell = z_\ell(T_i^\ell) + n_i^\ell; \qquad \forall i \in \mathbb{Z}_{\geq 0}, \tag{14}$$

where $\{n_i^\ell\}_{i=0}^\infty$ denotes measurement noise sequence, which is composed of independently and identically distributed Gaussian random variables with zero mean and specified standard deviation η_ℓ. Let each subsystem adopt a simple estimator of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_\ell(t) = -\gamma_\ell \hat{z}_\ell(t); \qquad \hat{z}_\ell(T_i^\ell) = y_i^\ell, \tag{15}$$

for $t \in [T_i^\ell, T_{i+1}^\ell)$. We define the estimation error $e_\ell(t) = z_\ell(t) - \hat{z}_\ell(t)$. Estimator ℓ only has access to the state measurements of subsystem ℓ at specific time instances $\{T_i^\ell\}_{i=0}^\infty$ but is supposed to reconstruct the signal at any time $t \in \mathbb{R}_{\geq 0}$. Notice that this estimator is not optimal. In Subsection 3.4, we will consider estimators based on Kalman filtering instead.

3.2 Performance Analysis: Scalar Subsystems

In this subsection, we present an upper bound for the performance of the introduced networked estimator. The following theorem presents upper bounds for the estimation error variance for the cases where the measurement noise is small or large, respectively.

Theorem 9.5 *Assume that subsystem ℓ, $1 \leq \ell \leq L$, is described by (13) and let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Then, if $\eta_\ell \leq \sqrt{1/(2\gamma_\ell)}\sigma_\ell$, the estimation error variance is bounded by*

$$\mathbb{E}\{e_\ell^2(t)\} \leq \eta_\ell^2 e^{-2\gamma_\ell/f_\ell} + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell/f_\ell}\right), \tag{16}$$

otherwise, if $\eta_\ell > \sqrt{1/(2\gamma_\ell)}\sigma_\ell$,

$$\mathbb{E}\{e_\ell^2(t)\} \leq \eta_\ell^2 + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell/f_\ell}\right). \tag{17}$$

Proof: See Appendix D. ■

Note that the upper bound (16) is tighter than (17) when the equality $\eta_\ell = \sqrt{1/(2\gamma_\ell)}\sigma_\ell$ holds. In the next two subsections, we generalize these results to higher-order subsystems.

3.3 Performance Analysis: Higher-Order Subsystems with Noisy State Measurement

Let us assume that subsystem ℓ, $1 \leq \ell \leq L$, is described by

$$\mathrm{d}z_\ell(t) = A_\ell z_\ell(t)\mathrm{d}t + H_\ell \mathrm{d}w_\ell(t); \qquad z_\ell(0) = 0, \tag{18}$$

where $z_\ell(t) \in \mathbb{R}^{d_\ell}$ is its state with $d_\ell \in \mathbb{Z}_{\geq 1}$ and A_ℓ is its model matrix satisfying $\overline{\lambda}(A_\ell + A_\ell^\top) < 0$ where $\overline{\lambda}(\cdot)$ denotes the largest eigenvalue of a matrix. In addition, $\{w_\ell(t)\}_{t\in\mathbb{R}_{\geq 0}}$, $1 \leq \ell \leq L$, is a tuple of statistically independent Wiener processes with zero mean. Estimator ℓ receives noisy state-measurements $\{y_i^\ell\}_{i=0}^\infty$ at time instances $\{T_i^\ell\}_{i=0}^\infty$, such that

$$y_i^\ell = z_\ell(T_i^\ell) + n_i^\ell; \quad \forall i \in \mathbb{Z}_{\geq 0}, \tag{19}$$

where $\{n_i^\ell\}_{i=0}^\infty$ denotes the measurement noise and is composed of independently and identically distributed Gaussian random variables with $\mathbb{E}\{n_i^\ell\} = 0$ and $\mathbb{E}\{n_i^\ell n_i^{\ell\top}\} = R_\ell$. We define the estimation error as $e_\ell(t) = z_\ell(t) - \hat{z}_\ell(t)$, where for all $t \in [T_i^\ell, T_{i+1}^\ell)$, the state estimate $\hat{z}_\ell(t)$ is derived by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_\ell(t) = A_\ell \hat{z}_\ell(t);\ \hat{z}_\ell(T_i^\ell) = y_i^\ell.$$

The next theorem presents an upper bound for the variance of this estimation error. For scalar subsystems, the introduced upper bound in (20) is equivalent to the upper bound in (17).

Theorem 9.6 *Assume that subsystem ℓ, $1 \leq \ell \leq L$, is described by (18) and let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Then, the estimation error variance is bounded by*

$$\mathbb{E}\{\|e_\ell(t)\|^2\} \leq \operatorname{tr}(R_\ell) + \frac{\operatorname{tr}(H^\top H)}{|\overline{\lambda}(A_\ell + A_\ell^\top)|}\left(1 - e^{\overline{\lambda}(A_\ell + A_\ell^\top)/f_\ell}\right). \tag{20}$$

Proof: See Appendix E. ■

It is possible to refine the upper bound (20) for the case where $\operatorname{tr}(R_\ell) \leq 1/(|\overline{\lambda}(A_\ell + A_\ell^\top)|) \operatorname{tr}(H^\top H)$, following a similar argument as in the proof of Theorem 9.5.

3.4 Performance Analysis: Higher-Order Subsystems with Noisy Output Measurement

In this subsection, we assume that estimator ℓ, $1 \leq \ell \leq L$, receives noisy output measurements $\{y_i^\ell\}_{i=0}^\infty$ at time instances $\{T_i^\ell\}_{i=0}^\infty$, such that

$$y_i^\ell = C_\ell z_\ell(T_i^\ell) + n_i^\ell; \quad \forall i \in \mathbb{Z}_{\geq 0}, \tag{21}$$

where $C_\ell \in \mathbb{R}^{p_\ell \times d_\ell}$ (for a given output vector dimension $p_\ell \in \mathbb{Z}_{\geq 1}$ such that $p_\ell \leq d_\ell$) and the measurement noise $\{n_i^\ell\}_{i=0}^\infty$ is a sequence of independently and identically distributed Gaussian random variables with $\mathbb{E}\{n_i^\ell\} = 0$ and $\mathbb{E}\{n_i^\ell n_i^{\ell\top}\} = R_\ell$. For any sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$, we can discretize the stochastic continuous-time system in (18) as

$$z_\ell[i+1] = F_\ell[i] z_\ell[i] + G_\ell[i] w_\ell[i],$$

where $z_\ell[i] = z(T_i^\ell)$, $F_\ell[i] = e^{A(T_{i+1}^\ell - T_i^\ell)}$, and the sequence $\{G_\ell[i]\}_{i=0}^\infty$ is chosen such that

$$G_\ell[i] G_\ell[i]^\top = \int_0^{T_{i+1}^\ell - T_i^\ell} e^{A\tau} H H^\top e^{A^\top \tau} \mathrm{d}\tau, \qquad \forall i \in \mathbb{Z}_{\geq 0}.$$

In addition, $\{w_\ell[i]\}_{i=0}^\infty$ is a sequence of independently and identically distributed Gaussian random variables with zero mean and unity variance. It is evident that $y_\ell[i] = C_\ell z_\ell[i] + n_i^\ell$. We run a discrete-time Kalman filter over these output measurements to calculate the state estimates $\{\hat{z}_\ell[i]\}_{i=0}^\infty$ with error covariance matrix $P_\ell[i] = \mathbb{E}\{(z_\ell[i] - \hat{z}_\ell[i])(z_\ell[i] - \hat{z}_\ell[i])^\top\}$. For inter-sample times $t \in [T_i^\ell, T_{i+1}^\ell)$, we use a simple prediction filter

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_\ell(t) = A_\ell \hat{z}_\ell(t); \ \hat{z}_\ell(T_i^\ell) = \hat{z}_\ell[i]. \tag{22}$$

Let us define the estimation error as $e_\ell(t) = z_\ell(t) - \hat{z}_\ell(t)$. The next theorem present an upper bound for the estimation error variance.

Theorem 9.7 *Assume that subsystem ℓ, $1 \leq \ell \leq L$, is described by (18). Then, the estimator given by (22) is an optimal mean square error estimator and for any fixed sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$, the estimation error is upper-bounded by*

$$\mathbb{E}\{\|e_\ell(t)\|^2 \,|\, \Delta_i^\ell\} \leq \mathrm{trace}(P_\ell[i]) + \frac{\mathrm{tr}(H^\top H)}{|\overline{\lambda}(A_\ell + A_\ell^\top)|}\left(1 - e^{\overline{\lambda}(A_\ell + A_\ell^\top)\Delta_i^\ell}\right). \tag{23}$$

Proof: See Appendix F. ∎

Note that the upper bound (23) is conditioned on the sampling intervals. Unfortunately, it is difficult to calculate $\mathbb{E}\{\mathrm{trace}(P_\ell[i])\}$ as a function of average sampling frequencies, which makes it hard to eliminate the conditional expectation. However, for the case where $p_\ell = n_\ell$, the upper bound (20) would also hold for the estimator in (22). This is indeed true because (22) is an optimal mean square error estimator.

4 Applications to Networked Control

In this section, we study networked control as an application of the proposed stochastic scheduling policy. Let us start by presenting the system model and the control law. We first present the results for impulsive controllers in Subsection 4.2. However, in Subsections 4.3 and 4.4, we generalize these results to pulse and exponential controllers.

4.1 System Model and Controller

Consider the stochastic control system

$$\mathrm{d}z_\ell(t) = (-\gamma_\ell z_\ell(t) + v_\ell(t))\,\mathrm{d}t + \sigma_\ell\,\mathrm{d}w_\ell(t); \quad z_\ell(0) = z_\ell^0, \tag{24}$$

where $z_\ell(t) \in \mathbb{R}$ and $v_\ell(t) \in \mathbb{R}$, $1 \leq \ell \leq L$, are the state and control input of subsystem ℓ. We assume that each subsystem is in feedback interconnection with a subcontroller governed by the control law

$$v_\ell(t) = -\sum_{i=0}^{\infty} y_i^\ell f(t - T_i^\ell), \tag{25}$$

where $y_i^\ell = z(T_i^\ell) + n_i$ for all $i \in \mathbb{Z}_{\geq 0}$ and $f : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is chosen appropriately to yield a causal controller (i.e., $f(t) = 0$ for all $t < 0$). For instance, using $f(\cdot) = \delta(\cdot)$, where $\delta(\cdot)$ is the impulse function (see [50, p. 1]), results in an impulsive controller, which simply resets the state of its corresponding subsystem to a neighborhood of the origin characterized by the amplitude of the measurement noise whenever a new measurement is received. Without loss of generality, we assume that $z_\ell^0 = 0$ because the influence of the initial condition is only visible until the first sampling instance T_0^ℓ, which is guaranteed to happen in a finite time thanks to Assumption 9.1.

4.2 Performance Analysis: Impulsive Controllers

In this subsection, we present an upper bound for the closed-loop performance of subsystems described in (24) and controlled by an impulsive controller. In this case, for all $t \in [T_i^\ell, T_{i+1}^\ell)$, the closed-loop subsystem ℓ is governed by

$$\mathrm{d}z_\ell(t) = -\gamma_\ell z_\ell(t)\,\mathrm{d}t + \sigma_\ell\,\mathrm{d}w_\ell(t);\; z_\ell(T_i^\ell) = -n_i^\ell.$$

The next theorem presents an upper bound for the performance of this closed-loop system which corresponds to the estimation error upper bound presented in Theorem 9.5.

Theorem 9.8 *Assume that subsystem ℓ, $1 \le \ell \le L$, is described by (24) and let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Then, if $\eta_\ell \le \sqrt{1/(2\gamma_\ell)}\sigma_\ell$, the closed-loop performance of subsystem ℓ is bounded by*

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \le \eta_\ell^2 e^{-2\gamma_\ell/f_\ell} + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell/f_\ell}\right). \tag{26}$$

otherwise,

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \le \eta_\ell^2 + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell/f_\ell}\right). \tag{27}$$

Proof: Similar to the proof of Theorem 9.5. See [49] for details. ∎

Note that the closed-loop performance, measured as the variance of the plant state, is upper bounded by the plant and measurement noise variance. In the next two subsections, we generalize this result to pulse and exponential controllers.

4.3 Performance Analysis: Pulse Controllers

In this subsection, we use a narrow pulse function to approximate the behavior of the impulse function. Let us pick a constant $\rho \in \mathbb{R}_{>0}$. For $t \in [T_i^\ell, T_{i+1}^\ell)$, we use the control law

$$v_\ell(t) = \begin{cases} -y_i^\ell \gamma_\ell e^{-\gamma_\ell \rho}/(1 - e^{-\gamma_\ell \rho}), & T_i^\ell \le t \le T_i^\ell + \rho, \\ 0, & T_i^\ell + \rho < t \le T_{i+1}^\ell, \end{cases} \tag{28}$$

whenever $T_i^\ell + \rho \le T_{i+1}^\ell$, and

$$v_\ell(t) = -y_i^\ell \gamma_\ell e^{-\gamma_\ell \rho}/(1 - e^{-\gamma_\ell \rho}), \qquad T_i^\ell \le t \le T_{i+1}^\ell,$$

otherwise. This controller converges to the impulsive controller as ρ tends to zero.

Theorem 9.9 *Assume that subsystem ℓ, $1 \le \ell \le L$, is described by (24) and let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Then, the closed-loop performance of subsystem ℓ is bounded by*

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \le \left[\eta_\ell^2 + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell/f_\ell}\right)\right] \frac{1}{1 - \mathbb{P}\{\Delta_i^\ell < \rho\}}. \tag{29}$$

Proof: See Appendix G. ■

Note that if ρ tends to zero in (29), we would recover the same upper bound as in the case of the impulsive controller (27). This is true since $\lim_{\rho\to 0}\mathbb{P}\{\Delta_i^\ell < \rho\} = 0$ assuming that the probability distribution of hitting-times of the underlying Markov chain is atom-less at the origin, which is a reasonable assumption when the Poisson jump rates are finite.

4.4 Performance Analysis: Exponential Controllers

In this subsection, we use an exponential function to approximate the impulse function. Let us pick a constant $\theta \in \mathbb{R}_{>0} \setminus \{\gamma_\ell\}$. For all $t \in [T_i^\ell, T_{i+1}^\ell)$, we use the control law

$$v_\ell(t) = (\gamma_\ell - \theta) y_i^\ell e^{-\theta(t-T_i^\ell)}. \tag{30}$$

This controller converges to the impulsive controller as θ goes to infinity.

Theorem 9.10 *Assume that subsystem ℓ, $1 \le \ell \le L$, is described by (24) and let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Then, the closed-loop performance of subsystem ℓ is bounded by*

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \le \left[\eta_\ell^2 + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell/f_\ell}\right)\right]\frac{1}{1-\mathbb{E}\{e^{-2\theta\Delta_i^\ell}\}}. \tag{31}$$

Proof: See Appendix H. ■

Note that if θ goes to infinity, we would recover the same upper bound as in the case of the impulsive controller since $\lim_{\theta\to+\infty}\mathbb{E}\{e^{-2\theta\Delta_i^\ell}\} = 0$ assuming that the probability distribution of hitting-times of the Markov chain is atom-less at the origin. Exponential shape of the control signal is common in biological systems such as in neurological control system [51].

5 Numerical Example

In this section, we demonstrate the developed results on a networked system composed of L decoupled water tanks illustrated in Figure 3 (left), where each tank is linearized about its stationary water level h_ℓ as

$$\mathrm{d}z_\ell(t) = -\frac{a_\ell}{a'_\ell}\sqrt{\frac{g}{2h_\ell}}z_\ell(t)\mathrm{d}t + \mathrm{d}w_\ell(t);\ z_\ell(0) = z_\ell^0.$$

In this model, a'_ℓ is the cross-section of water tank ℓ, a_ℓ is the cross-section of its outlet hole, and g is the acceleration of gravity. Furthermore, $z_\ell(t)$ and $v_\ell(t)$ denote the deviation of the tank's water level from its stationary point and the control input, respectively. Let the initial condition $z_\ell^0 = 0$ as we assume that the tank's water level start at its stationary level. However, due to factors such as input flow fluctuations, the water level drifts away from zero. In the next subsection, we start by numerically demonstrating the estimation results for $L = 2$ water tanks.

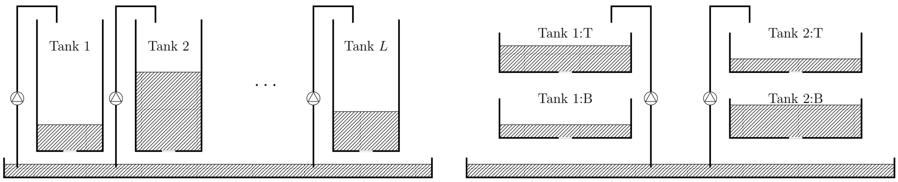

Figure 3: An example of a networked system composed of decoupled scalar subsystems (left) and multivariable subsystems (right).

Table 9.1: Example of average sampling frequencies.

ξ_1	ξ_2	f_1	f_2
0.1	0.1	0.8040	0.8040
0.5	0.1	0.6577	0.8279
1.0	0.1	0.4656	0.8559
2.0	0.1	0.0451	0.9045

5.1 Estimation: Scalar Subsystem

Let us fix the parameters $a_1' = a_2' = 1.00\,\mathrm{m}^2$, $a_1 = 0.20\,\mathrm{m}^2$, $a_2 = 0.10\,\mathrm{m}^2$, $g = 9.80\,\mathrm{m/s}^2$, $h_1 = 0.40\,\mathrm{m}$, and $h_2 = 0.54\,\mathrm{m}$. For these physical parameters, the water tanks can be described by (13) with $\gamma_1 = 0.7$, $\gamma_2 = 0.3$, and $\sigma_1 = \sigma_2 = 1.0$. We sample these subsystems using the Markov chain in (2) with $m = 2L = 4$. We assume that $\mu_i(t) = \mu_{i,0} + u_i(t)$ for all $1 \le i \le 4$, where $\mu_{2\ell,0} = 1$ and $\mu_{2\ell-1,0} = 10$ for $\ell = 1, 2$. We are interested in finding $u_i(t)$, $1 \le i \le 4$, in order to minimize the cost function

$$J = \lim_{T\to\infty} \mathbb{E}\left\{ \frac{1}{T} \int_0^T \Big(0.5 e_3^\top x(t) \mathrm{d}N_2 + 0.1 e_3^\top x(t) \mathrm{d}N_4 + u(t)^\top u(t) \Big) \mathrm{d}t \right\}.$$

Using Corollary 9.2, we get

$$\begin{bmatrix} u_1(t,x) \\ u_2(t,x) \\ u_3(t,x) \\ u_4(t,x) \end{bmatrix} = \begin{bmatrix} -0.0228 & 0 & 0 \\ 0 & 0 & -0.2272 \\ 0 & -0.0228 & 0 \\ 0 & 0 & -0.0272 \end{bmatrix} x(t).$$

Figure 4 (upper-left) illustrates an example of the continuous-time Markov chain state $x(t)$ and the sampling instances $\{T_i^\ell\}_{i=0}^\infty$ of subsystems $\ell = 1, 2$. Using (42), we get the average sampling frequencies $f_1 = 0.66$ and $f_2 = 0.83$. Figure 4 (upper-right) shows the sampling instances when using a periodic scheduling policy with the same sampling frequencies as the average sampling frequencies of the optimal scheduling policy. Note that the optimal scheduling policy allocates the sampling instances according to the jumps between the states of the Markov chain.

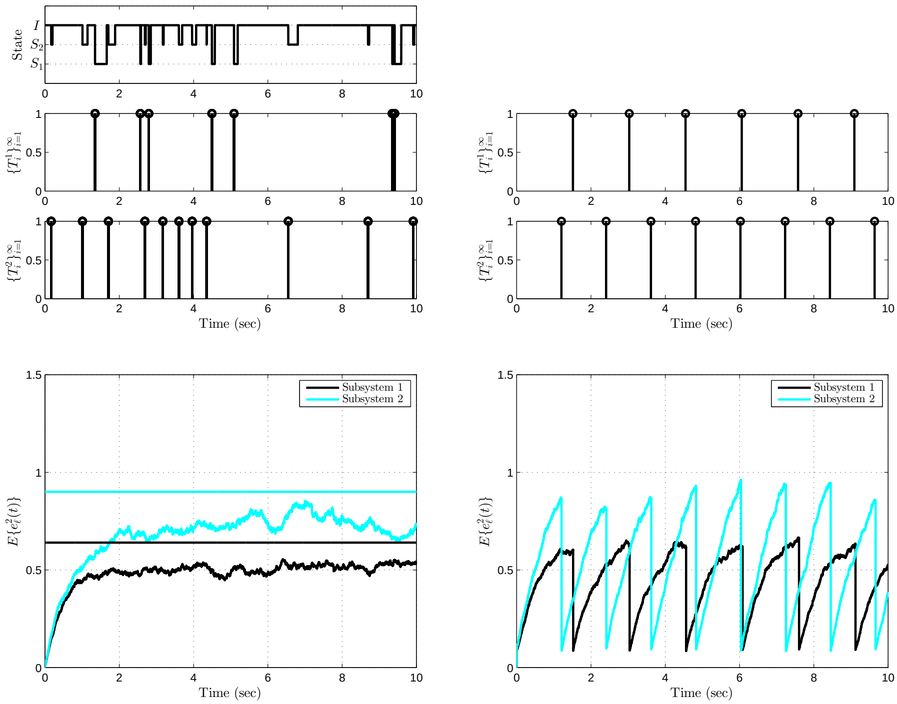

Figure 4: An example of the state of the continuous-time Markov chain used in the optimal scheduling policy and its corresponding sampling instances for both subsystems (upper-left). Sampling instances for both subsystems when using a periodic scheduling policy (upper-right). Estimation error $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations when using the optimal sampling policy (lower-left) and the periodic sampling policy (lower-right).

We can tune the average sampling frequencies of the subsystems by changing the design parameters ξ_ℓ, $1 \leq \ell \leq L$. Table 9.1 illustrates the average sampling frequencies of the subsystems versus different choices of the design parameters ξ_ℓ, $1 \leq \ell \leq L$. It is evident that when increasing (decreasing) ξ_ℓ for a given ℓ, the average sampling frequency of subsystem ℓ decreases (increases).

Let us assume that estimator ℓ has access to state measurements of subsystem ℓ according to (14) with measurement noise variance $\eta_\ell = 0.3$ for $\ell = 1, 2$. Figure 4 (lower-left) illustrates the estimation error variance $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations when using the optimal scheduling policy. The horizontal lines represent the theoretical upper bounds derived in Theorem 9.5; i.e., $\mathbb{E}\{e_1^2(t)\} \leq 0.64$ and $\mathbb{E}\{e_2^2(t)\} \leq 0.90$. Note that the approximations of the estimation error variances would eventually converge to the exact expectation value as the number of

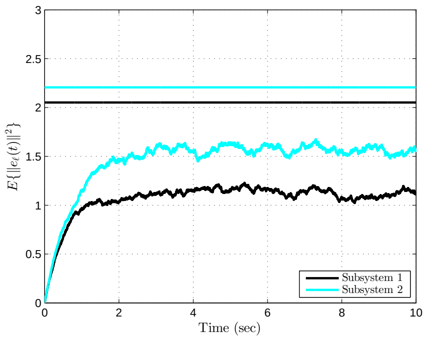

Figure 5: Estimation error $\mathbb{E}\{\|e_\ell(t)\|^2\}$ for 1000 Monte Carlo simulations and its comparison to the theoretical results when $d_\ell = 2$ for $\ell = 1, 2$.

simulations goes to infinity, and that the theoretical bounds are relatively close. Figure 4 (lower-right) illustrates the estimation error variance $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations when using the periodic scheduling policy that is portrayed in Figure 4 (upper-right). Note that the saw-tooth behavior is due to the fact that the sampling instances are fixed in advance and they are identical for each Monte Carlo simulation.

5.2 Estimation: Higher-Order Subsystems with Noisy State Measurement

Let us focus on a networked system composed of only two subsystems where each subsystem is a serial interconnection of two water tanks. Figure 3 (right) illustrates such a networked system. In this case, subsystem ℓ can be described by (18) with

$$A_\ell = \begin{bmatrix} -(a_{\ell,\mathrm{T}}/a'_{\ell,\mathrm{T}})\sqrt{g/(2h_{\ell,\mathrm{T}})} & 0 \\ +(a_{\ell,\mathrm{T}}/a'_{\ell,\mathrm{T}})\sqrt{g/(2h_{\ell,\mathrm{T}})} & -(a_{\ell,\mathrm{B}}/a'_{\ell,\mathrm{B}})\sqrt{g/(2h_{\ell,\mathrm{B}})} \end{bmatrix},$$

where the parameters marked with T and B belong to the top and the bottom tanks, respectively. Let us fix parameters $a'_{1,\mathrm{T}} = a'_{1,\mathrm{B}} = a'_{2,\mathrm{T}} = a'_{2,\mathrm{B}} = 1.00\,\mathrm{m}^2$, $a_{1,\mathrm{T}} = a_{1,\mathrm{B}} = 0.20\,\mathrm{m}^2$, $a_{2,\mathrm{T}} = a_{2,\mathrm{B}} = 0.10\,\mathrm{m}^2$, $h_{1,\mathrm{T}} = h_{1,\mathrm{B}} = 0.40\,\mathrm{m}$, and $h_{2,\mathrm{T}} = h_{2,\mathrm{B}} = 0.54\,\mathrm{m}$.

Let us assume that estimator ℓ has access to the noisy state measurements of subsystem ℓ (with noise variance $\mathbb{E}\{n_i^\ell n_i^{\ell\top}\} = 0.09 I_{2\times 2}$) at sampling instances $\{T_i^\ell\}_{i=0}^\infty$ enforced by the optimal scheduling policy described in Subsection 5.1. Figure 5 shows the estimation error variance $\mathbb{E}\{\|e_\ell(t)\|^2\}$. The horizontal lines in this figure show the theoretical bounds calculated in Theorem 9.6; i.e., $\mathbb{E}\{\|e_1(t)\|^2\} \le 2.05$ and $\mathbb{E}\{\|e_2(t)\|^2\} \le 2.21$. In comparison with the scalar case in Figure 4 (lower-

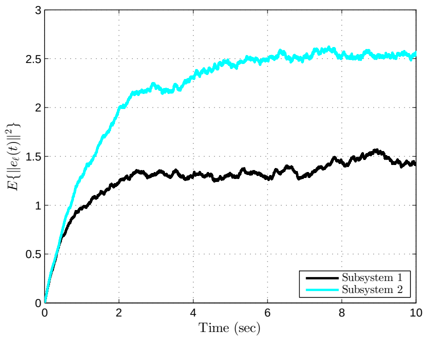

Figure 6: Estimation error $\mathbb{E}\{\|e_\ell(t)\|^2\}$ for 1000 Monte Carlo simulations when using Kalman-filter based estimator.

left), note that the bounds in Figure 5 are less tight. The reason for this is that the dimension of the subsystems are now twice the previous case.

5.3 Estimation: Higher-Order Subsystems with Noisy Output Measurement

In this subsection, we use output measurements $y_i^\ell = [0 \quad 1] z_\ell(T_i^\ell) + n_i^\ell$ for all $i \in \mathbb{Z}_{\geq 0}$, where $\mathbb{E}\{(n_i^\ell)^2\} = 0.09$ for $\ell = 1, 2$. We use the Kalman filter based scheme introduced in Subsection 3.4 for estimating the state of each subsystem. Figure 6 illustrates the estimation error variance $\mathbb{E}\{\|e_\ell(t)\|^2\}$. As mentioned earlier, it is difficult to calculate $\mathbb{E}\{\text{trace}(P_\ell[i])\}$ as a function of the average sampling frequencies and hence, we do not have any theoretical results for comparison. Note that the upper bound presented in Theorem 9.7 is only valid for a fixed sequence of sampling instances. This problem can be an interesting direction for future research.

5.4 Estimation: Ad-hoc Sensor Network

As discussed earlier, an advantage of using the introduced optimal scheduling policy is that we can accommodate for changes in ad-hoc networked systems. To portray this property, let us consider a networked system that can admit up to $L = 70$ identical subsystems described by (13) with $\gamma_\ell = 0.3$ and $\sigma_\ell = 1.0$ for $1 \leq \ell \leq 70$. When all the subsystems are active, we sample them using the Markov chain in (2) with $m = 2L = 140$. We assume that $\mu_i(t) = \mu_{i,0} + u_i(t)$ for $1 \leq i \leq 140$, where $\mu_{2\ell,0} = 10$ and $\mu_{2\ell-1,0} = 50$ for $1 \leq \ell \leq 70$. In this case, we are also interested in

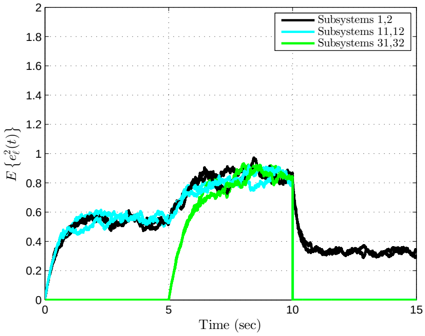

Figure 7: Estimation error $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations over an ad-hoc networked system with the optimal sampling policy (left) and the periodic sampling policy (right).

calculating an optimal scheduling policy that minimizes

$$J = \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T \sum_{\ell=1}^{70} 0.1 e_{71}^\top x(t)\mathrm{d}N_{2\ell} + u(t)^\top u(t)\mathrm{d}t\right\}. \tag{32}$$

However, when some of the subsystems are inactive, we simply remove their corresponding nodes from the Markov chain flow diagram in Figure 2 and set their corresponding terms in (32) equal to zero. Let us assume that for $t \in [0,5)$, only 30 subsystems are active, for $t \in [5,10)$, all 70 subsystems are active, and finally, for $t \in [10,15]$, only 10 subsystems are active. Figure 7 (left) and (right) illustrate the estimation error variance $\mathbb{E}\{e_\ell^2(t)\}$ for 1000 Monte Carlo simulations when using the optimal scheduling policy and the periodic scheduling policy, respectively. Since in the periodic scheduling policy, we have to fix the sampling instances in advance, we must determine the sampling periods according to the worst-case scenario (i.e., when the networked system is composed of 70 subsystems). Therefore, when using the periodic sampling, the networked system is not using its true potential for $t \in [0,5)$ and $t \in [10,15]$, but the estimation error is fluctuating substantially over the whole time range. The proposed optimal scheduling policy adapts to the demand of the system, see Figure 7 (right). For instance, as shown in Figure 7 (left), when subsystems 31 and 32 become active for $t \in [5,10)$, the overall sampling frequencies of the subsystems decreases (and, in turn, the estimation error variance increases), but when they become inactive again for $t \in [10,15]$, the average sampling frequencies increase (and, in turn, the estimation error variance decreases). Hence, this example illustrates the dynamic benefits of our proposed stochastic scheduling approach.

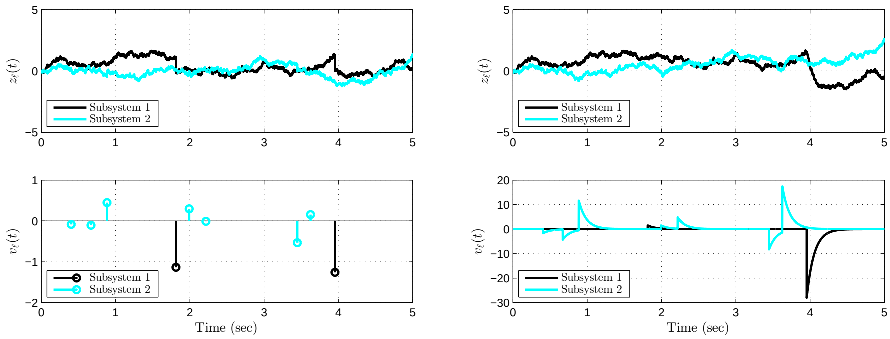

Figure 8: An example of state and control of the closed-loop subsystems when using the impulsive controller (left) and the exponential controller with $\theta = 10$ (right).

5.5 Controller: Decoupled Scalar Subsystems

In this subsection, we briefly illustrate the networked control results for $L = 2$ subsystems. Let the subsystems be described by (24) with $\gamma_1 = 0.7$, $\gamma_2 = 0.3$, and $\sigma_1 = \sigma_2 = 1.0$. Let us assume that controller ℓ has access to state measurements of subsystem ℓ according to (14) with measurement noise variance $\eta_\ell = 0.3$ for $\ell = 1, 2$. We use the optimal scheduling policy derived in Subsection 5.1 for assigning sampling instances. Figure 8 (left) and (right) illustrate an example of the state and the control signal for both subsystems when using the impulsive and exponential controllers (with $\theta = 10$), respectively. Note that in Figure 8 (left), the control signal of the impulsive controller only portrays the energy that is injected to the subsystem (i.e., the integral of the impulse function) and not its exact value. Figure 9 (left) and (right) show the closed-loop performance $\mathbb{E}\{z_\ell^2(t)\}$ when using the impulsive and exponential controllers, respectively. The horizontal lines illustrate the theoretical upper bounds derived in Theorem 9.8. Note that the exponential controller gives a worse performance than the impulse controller. This is normal as we design the exponentials only as an approximation of the impulse train.

5.6 Controller: Coupled Scalar Subsystems

Consider a networked system composed of $L = 70$ interconnected subsystems, where subsystem ℓ, $1 \leq \ell \leq 70$, can be described by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} z_\ell(t) =& 0.1(z_{\mathrm{mod}(\ell-1,L)}(t) - z_\ell(t)) + 0.1(z_{\mathrm{mod}(\ell+1,L)}(t) - z_\ell(t)) \\ &+ v_\ell(t) + w_\ell(t); \qquad\qquad z_\ell(0) = 0,\end{aligned}$$

with notation $\mathrm{mod}(i, j) = i - \lfloor i/j \rfloor j$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{>0}$. In this model, $z_\ell(t)$, $v_\ell(t)$, and $w_\ell(t)$ respectively denote the state, the control input, and the exogenous input. Each subsystem transmits its state measurement over the wireless

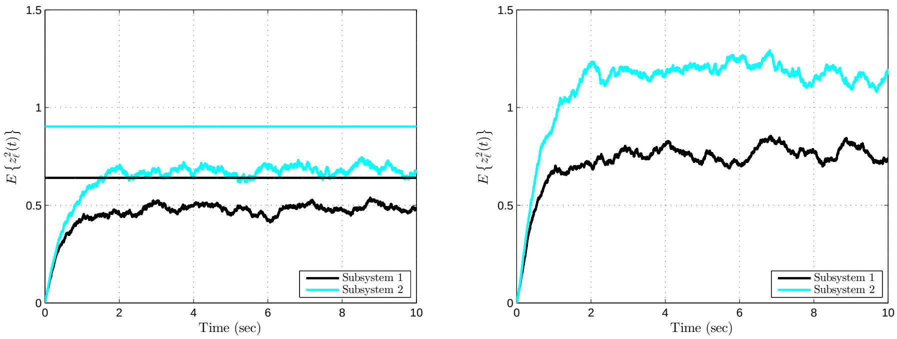

Figure 9: Closed-loop performance measure $\mathbb{E}\{z_\ell^2(t)\}$ for 1000 Monte Carlo simulations when using the impulsive controller (left) and the exponential controller (right).

network at instances $\{T_i^\ell\}_{i=0}^\infty$ to its subcontroller. Hence, at any time $t \in \mathbb{R}_{\geq 0}$, subcontroller ℓ has access to the state measurements $z_\ell(T^\ell_{M_t^\ell})$ where recalling from the earlier definitions $M_t^\ell = \max\left\{i \geq 1 \,|\, T_i^\ell \leq t\right\}$. Each subcontroller simply implement the following decentralized proportional-integral control law

$$v_\ell(t) = -1.2 z_\ell(T^\ell_{M_t^\ell}) - 0.3 \int_0^t z_\ell(T^\ell_{M_\tau^\ell})\mathrm{d}\tau.$$

We sample the subsystems using the Markov chain in (2) with $m = 2L = 140$. We assume that $\mu_i(t) = \mu_{i,0} + u_i(t)$ for $1 \leq i \leq 140$, where $\mu_{2\ell,0} = 10$ and $\mu_{2\ell-1,0} = 70$ for $1 \leq \ell \leq 70$. Let us consider the following disturbance rejection scenario. We assume that $w_\ell(t) \equiv 0$ for $\ell \neq 4, 26$, $w_4(t) = \mathrm{step}(t)$, and $w_{26}(t) = -0.4\,\mathrm{step}(t-15)$, where $\mathrm{step} : \mathbb{R} \to \{0, 1\}$ is the heaviside step function (i.e., $\mathrm{step}(t) = 1$ for $t \in \mathbb{R}_{\geq 0}$ and $\mathrm{step}(t) = 0$, otherwise). Let us denote $t \in [0, 15)$ and $t \in [15, 30]$ as the first phase and the second phase, respectively. During each phase, we find the infinite horizon optimal scheduling policy which minimizes

$$J = \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T \sum_{\ell=1}^{70} \xi_\ell\, e_{71}^\top x(t)\mathrm{d}N_{2\ell} + u(t)^\top u(t)\mathrm{d}t\right\}.$$

We fix $\xi_\ell = 10$ for $\ell = 4$ over the first phase and for $\ell = 26$ over the second phase. In addition, we fix $\xi_\ell = 20$ for $\ell = 3, 5$ over the first phase and for $\ell = 25, 27$ over the second phase. Finally, we set $\xi_\ell = 30$ for the rest of the subsystems. This way, we can ensure that we more frequently sample the subsystems that are most recently disturbed by a nonzero exogenous input signal (and the ones that are directly interacting with them). Figure 10 (left) and (right) illustrate an example of the system state and control input when using the described optimal scheduling policy and the periodic scheduling policy, respectively. For the periodic scheduling

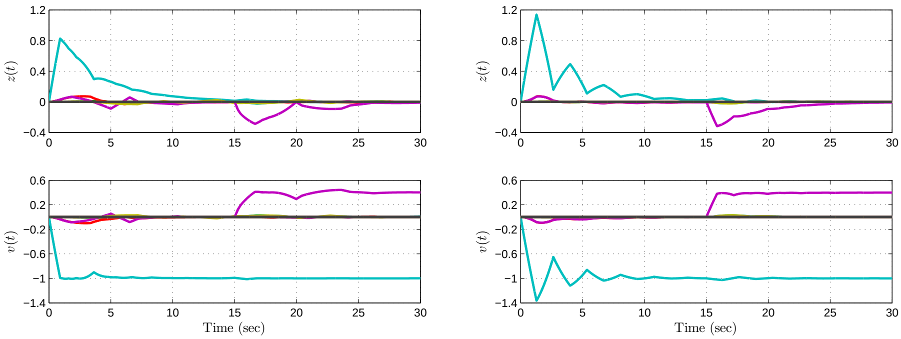

Figure 10: An example of state and control signal using the optimal sampling policy (left) and the periodic sampling policy (right).

policy, we have fixed the sampling frequencies according to the worst-case scenario (i.e., the average frequencies of the optimal scheduling policy when $\xi_\ell = 10$ for all $1 \leq \ell \leq 70$ corresponding to the case where all the subsystems are disturbed). As we expect, for this particular example, the closed-loop performance is better with the optimal scheduling policy than with the periodic scheduling policy. This is indeed the case because the optimal scheduling policy adapts the sampling rates of the subsystems according to their performance requirements.

6 Conclusions

In this paper, we used a continuous-time Markov chain to optimally schedule the measurement and transmission time instances in a sensor network. As applications of this optimal scheduling policy, we studied networked estimation and control of large-scale system that are composed of several decoupled scalar stochastic subsystems. We studied the statistical properties of this scheduling policy to compute bounds on the closed-loop performance of the networked system. Extensions of the estimation results to observable subsystems of arbitrary dimension were also presented. As a future work, we could focus on obtaining better performance bounds for estimation and control in networked system as well as combining the estimation and control results for achieving a reasonable closed-loop performance when dealing with observable and controllable subsystems of arbitrary dimension. An interesting extension is also to consider zero-order hold and other control function for higher-order subsystems.

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A Proof of Theorem 9.1

We follow a similar reasoning as in [42] to calculate the optimal Poisson rates. By adding and subtracting the term $k(t)^\top \mathbb{E}\{x(t)\}\big|_0^T$ from the right hand-side of the scaled cost function $TJ - k_f^\top \mathbb{E}\{x(T)\}$, we get

$$\begin{aligned} TJ - k_f^\top \mathbb{E}\{x(T)\} &= \mathbb{E}\left\{\int_0^T c(t)^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t\right\} \\ &= -\,k(t)^\top \mathbb{E}\{x(t)\}\big|_0^T + k(t)^\top \mathbb{E}\{x(t)\}\big|_0^T \\ &\quad + \mathbb{E}\left\{\int_0^T c(t)^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t\right\}. \end{aligned} \tag{33}$$

Using the identity

$$k(t)^\top \mathbb{E}\left\{x(t)\right\}\Big|_0^T = \mathbb{E}\left\{\int_0^T \mathrm{d}\langle k(t), x(t)\rangle\right\}$$

inside (33), we get

$$\begin{aligned} TJ - k_f^\top \mathbb{E}\left\{x(T)\right\} =& - k(t)^\top \mathbb{E}\left\{x(t)\right\}\Big|_0^T + \mathbb{E}\left\{\int_0^T \mathrm{d}\langle k(t), x(t)\rangle\right\} \\ &+ \mathbb{E}\left\{\int_0^T c(t)^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t\right\}. \end{aligned} \tag{34}$$

Using Itô's Lemma [48, p. 49], we know that

$$\mathrm{d}\langle k(t), x(t)\rangle = \langle \dot{k}(t), x(t)\rangle \mathrm{d}t + \sum_{i=1}^{m} \langle k(t), G_i x(t)\ \mathrm{d}N_i(t)\rangle,$$

which transforms (34) into

$$\begin{aligned} TJ - k_f^\top \mathbb{E}\left\{x(T)\right\} =& - k(t)^\top \mathbb{E}\left\{x(t)\right\}\Big|_0^T \\ &+ \mathbb{E}\left\{\int_0^T \langle \dot{k}(t), x(t)\rangle \mathrm{d}t + \sum_{i=1}^{m} \langle k(t), G_i x(t)\ \mathrm{d}N_i(t)\rangle\right\} \\ &+ \mathbb{E}\left\{\int_0^T c(t)^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t\right\}. \end{aligned}$$

Taking expectation over $x(t)$ and the Poisson processes $\{N_i(t)\}_{t\in\mathbb{R}_{\geq 0}}$, $1 \leq i \leq m$, we get

$$\begin{aligned} TJ - k_f^\top \mathbb{E}\left\{x(T)\right\} =& - k(t)^\top \mathbb{E}\left\{x(t)\right\}\Big|_0^T + \int_0^T \langle \dot{k}(t) + c(t) + A^\top k(t), p(t)\rangle \mathrm{d}t \\ &+ \mathbb{E}\left\{\int_0^T u(t)^\top u(t) + \sum_{i=1}^{m} u_i(t)(S_i x(t) + \langle k(t), B_i x(t)\rangle)\mathrm{d}t\right\}, \end{aligned} \tag{35}$$

where, for $1 \leq i \leq m$, S_i is i-th row of matrix S and $p(t) = \mathbb{E}\{x(t)\}$. We can rewrite (35) as

$$\begin{aligned} TJ - k_f^\top \mathbb{E}\{x(T)\} =& \int_0^T \langle \dot{k}(t) + c(t) + A^\top k(t) - \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top k(t))^{\cdot 2}, p(t)\rangle \mathrm{d}t \\ &- k(t)^\top \mathbb{E}\left\{x(t)\right\}\Big|_0^T \\ &+ \mathbb{E}\left\{\int_0^T \sum_{i=1}^{m} \left\| u_i(t) + \frac{1}{2}(k(t)^\top B_i + S_i)x(t)\right\|^2 \mathrm{d}t\right\}, \end{aligned} \tag{36}$$

using completion of squares. As there exists a well-defined solution to the differential equation (8), the first integral in (36) vanishes. Hence, the optimal control law is given by (9) since this control law minimizes the last term of (36). Consequently, equation (36) gives

$$TJ = k_f^\top \mathbb{E}\{x(T)\} - k(t)^\top \mathbb{E}\{x(t)\}\big|_0^T = k(0)^\top \mathbb{E}\{x(0)\}.$$

This completes the proof.

B Proof of Corollary 9.2

Since $x(t) \in \mathcal{X}$ is bounded (because $\|x(t)\|_2 \equiv 1$ for $t \in \mathbb{R}_{\geq 0}$), we get the identity in

$$\begin{aligned} &\lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T c^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t\right\} \\ &= \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T c^\top x(t) + u(t)^\top S x(t) + u(t)^\top u(t)\mathrm{d}t + \frac{1}{T}k_0^\top x(T)\right\}. \end{aligned} \tag{37}$$

According to Theorem 9.1, in order to minimize (37) for any fixed $T \in \mathbb{R}_{>0}$, we have

$$\dot{k}(t) = -c(t) - A^\top k(t) + \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top k(t))^{\cdot 2}, \tag{38}$$

with the final condition $k(T) = k_0$. Defining $q(t) = k(T-t) - k_0 - \varrho\mathbf{1}t$, we get

$$\begin{aligned} \dot{q}(t) = &-\dot{k}(T-t) - \varrho\mathbf{1} \\ &= A^\top k(T-t) + c - \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top k(T-t))^{\cdot 2} - \varrho\mathbf{1} \\ &= A^\top(q(t) + k_0 + \varrho\mathbf{1}t) + c - \varrho\mathbf{1} \\ &\quad - \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top(q(t) + k_0 + \varrho\mathbf{1}t))^{\cdot 2}. \end{aligned}$$

Note that $A^\top\mathbf{1} = 0$ and $B_i^\top\mathbf{1} = 0$, $1 \leq i \leq m$, as A and B_i are infinitesimal generators. Hence,

$$\begin{aligned} \dot{q}(t) &= A^\top(q(t) + k_0) + c - \varrho\mathbf{1} - \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top(q(t) + k_0))^{\cdot 2} \\ &= A^\top q(t) - \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top(q(t) + k_0))^{\cdot 2} + \frac{1}{4}\sum_{i=1}^{m}(S_i^\top + B_i^\top k_0)^{\cdot 2}. \end{aligned} \tag{39}$$

Notice that $q^* = 0$ is an equilibrium of (39), so $q(t) = 0$ for all $t \in [0, T]$ since $q(0) = k(T) - k_0 = 0$. Therefore, we get $k(t) = k_0 + \varrho\mathbf{1}(T-t)$, which results in

$\frac{1}{2}(k(t)^\top B_i + S_i) = \frac{1}{2}(k_0^\top B_i + S_i)$, since $\mathbf{1}^\top B_i = 0$, $1 \leq i \leq m$. As a result, when T goes to infinity, the controller which minimizes (37) is given by (11). Furthermore, we have

$$\begin{aligned} J &= \lim_{T\to\infty} \frac{1}{T} k(0)^\top \mathbb{E}\{x(0)\} \\ &= \lim_{T\to\infty} \frac{1}{T} \big(k_0 + \varrho \mathbf{1}(T-0)\big)^\top \mathbb{E}\{x(0)\} \\ &= \varrho \mathbf{1}^\top \mathbb{E}\{x(0)\} \\ &= \varrho. \end{aligned}$$

Finally, notice that the condition $\mathbf{1}^T k_0 = 0$ in the second row of (10) reduces the number of solutions k_0 that satisfy the nonlinear equation in the first row of (10). Removing this condition, $k_0 + \vartheta\mathbf{1}$ for any $\vartheta \in \mathbb{R}$ is a solution. Notice that all these parallel solutions result in the same control law because $((k_0 + \vartheta\mathbf{1})^\top B_i + S_i) = (k_0^\top B_i + S_i)$ following the fact that $\mathbf{1}^\top B_i = 0$ for all $1 \leq i \leq N$.

C Proof of Theorem 9.4

Before stating the proof of Theorem 9.4, we need to state the following useful lemma.

Lemma 9.11 *Let the sequence of sampling instances $\{T_i^\ell\}_{i=0}^\infty$ satisfy Assumption 9.1. Then,*

$$\lim_{t\to\infty} \frac{M_t^\ell}{t} \overset{as}{=} \frac{1}{\mathbb{E}\{\Delta_i^\ell\}}, \tag{40}$$

where $M_t^\ell = \max\left\{i \geq 1 \mid T_i^\ell \leq t\right\}$ counts the number of jumps prior to any given time $t \in \mathbb{R}_{\geq 0}$ and $x \overset{as}{=} y$ means that $\mathbb{P}\{x = y\} = 1$.

Proof: See [49]. ∎

Now, we are ready state the proof of Theorem 9.4. The proof of equality $f_\ell = 1/\mathbb{E}\{\Delta_i^\ell\}$ directly follows from applying Lemma 9.11 in conjunction with that $M_T^\ell = \int_0^T e_n^\top x(t)\mathrm{d}N_{2\ell}(t)$. Now, we can compute $p(t)$ using

$$\dot{p}(t) = Ap(t) + \mathbb{E}\left\{\sum_{i=1}^m u_i(t, x(t))B_i x(t)\right\},\ p(0) = \mathbb{E}\{x(0)\} \tag{41}$$

Substituting (11) inside (41), we get

$$\begin{aligned}
\dot{p}(t) &= Ap(t) - \frac{1}{2}\mathbb{E}\left\{\sum_{i=1}^{m}(k_0^\top B_i + S_i)x(t)B_i x(t)\right\} \\
&= Ap(t) - \frac{1}{2}\mathbb{E}\left\{\sum_{i=1}^{m}(k_0^\top B_i + S_i)\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{bmatrix} B_i \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t)\end{bmatrix}\right\} \\
&= Ap(t) - \frac{1}{2}\mathbb{E}\left\{\sum_{i=1}^{m} B_i \begin{bmatrix} x_1(t)\sum_{j=1}^{n}(k_0^\top B_i + S_i)e_j x_j(t) \\ \vdots \\ x_n(t)\sum_{j=1}^{n}(k_0^\top B_i + S_i)e_j x_j(t)\end{bmatrix}\right\}.
\end{aligned}$$

Note that $x_\zeta(t)\sum_{j=1}^{n}(k_0^\top B_i + S_i)e_j x_j(t) = (k_0^\top B_i + S_i)e_\zeta x_\zeta(t)$ for $1 \le \zeta \le n$, since $x(t) \in \mathcal{X}$ is a unit vector in $\mathbb{R}^n$. Therefore, we get (12). Now, noticing that $p(t)$ converges exponentially to a nonzero steady-state value as time goes to infinity (because otherwise $\lim_{t\to\infty} p(t)$ does not exist), we can expand the expression for the average sampling frequencies of the sensors as

$$\begin{aligned}
f_\ell &= \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T e_n^\top x(t)\left(\mu_{2\ell,0} + \sum_{j=1}^{m}\alpha_{2\ell,j}u_j\right)\mathrm{d}t\right\} \\
&= \lim_{T\to\infty} \mathbb{E}\left\{\frac{1}{T}\int_0^T e_n^\top x(t)\left(\mu_{2\ell,0} - \frac{1}{2}\sum_{j=1}^{m}\alpha_{2\ell,j}(k_0^\top B_j + S_j)x(t)\right)\mathrm{d}t\right\} \\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T e_n^\top p(t)\left(\mu_{2\ell,0} - \frac{1}{2}\sum_{j=1}^{m}\alpha_{2\ell,j}(k_0^\top B_j + S_j)e_n\right)\mathrm{d}t \\
&= \left(\mu_{2\ell,0} - \frac{1}{2}\sum_{j=1}^{m}\alpha_{2\ell,j}(k_0^\top B_j + S_j)e_n\right)e_n^\top \lim_{t\to\infty} p(t),
\end{aligned} \tag{42}$$

where the third equality follows again from the fact that $x(t) \in \mathcal{X}$ is a unit vector.

D Proof of Theorem 9.5

Prior to proving this theorem, we should state the following simple lemma.

Lemma 9.12 *Let the function $g : \mathbb{R}_{\ge 0} \to \mathbb{R}$ be defined as $g(t) = c_1 e^{-2\gamma t} + \frac{c_2}{2\gamma}(1 - e^{-2\gamma t})$ with given scalars $c_1, c_2 \in \mathbb{R}$ and $\gamma \in \mathbb{R}_{>0}$ such that $2\gamma c_1 \le c_2$. Then,*
(a) g is a non-decreasing function on its domain;
(b) g is a concave function on its domain.

Proof: See [49]. ■

Now, we can prove Theorem 9.5. Using Itô's Lemma [48, p. 49], for all $t \in [T_i^\ell, T_{i+1}^\ell)$, we get

$$\begin{aligned} \mathrm{d}e_\ell(t) &= \left(-\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_\ell(t) - \gamma_\ell e_\ell(t) - \gamma_\ell \hat{z}_\ell(t)\right)\mathrm{d}t + \sigma_\ell \mathrm{d}w_\ell(t) \\ &= -\gamma_\ell e_\ell(t)\mathrm{d}t + \sigma_\ell \mathrm{d}w_\ell(t), \end{aligned}$$

with the initial condition $e_\ell(T_i^\ell) = -n_i^\ell$. First, let us consider the case where $\eta_\ell \le \sqrt{1/(2\gamma_\ell)}\sigma_\ell$. Again, using Itô's Lemma, we get

$$\mathrm{d}(e_\ell^2(t)) = (-2\gamma_\ell\, e_\ell^2(t) + \sigma_\ell^2)\mathrm{d}t + 2e_\ell(t)\sigma_\ell \mathrm{d}w_\ell(t),$$

and as a result

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{e_\ell^2(t) \,|\, \Delta_i^\ell\} = -2\gamma_\ell\, \mathbb{E}\{e_\ell^2(t) \,|\, \Delta_i^\ell\} + \sigma_\ell^2,$$

where $\mathbb{E}\{e_\ell^2(T_i^\ell) \,|\, \Delta_i^\ell\} = \eta_\ell^2$. Hence, for all $t \in [T_i^\ell, T_{i+1}^\ell)$, we have

$$\mathbb{E}\{e_\ell^2(t) \,|\, \Delta_i^\ell\} = \eta_\ell^2 e^{-2\gamma_\ell(t-T_i^\ell)} + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell(t-T_i^\ell)}\right).$$

Now, using Lemma 9.12 (a), it is easy to see that

$$\mathbb{E}\{e_\ell^2(t) \,|\, \Delta_i^\ell\} \le \eta_\ell^2 e^{-2\gamma_\ell \Delta_i^\ell} + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right).$$

Note that

$$\begin{aligned} \mathbb{E}\{e_\ell^2(t)\} &= \mathbb{E}\{\mathbb{E}\{e_\ell^2(t) \,|\, \Delta_i^\ell\}\} \\ &\le \mathbb{E}\left\{\eta_\ell^2 e^{-2\gamma_\ell \Delta_i^\ell} + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right)\right\}. \end{aligned} \tag{43}$$

By using Lemma 9.12 (b) along with Jensen's Inequality [48, p. 320], we can transform (43) into (16). For the case where $\eta_\ell > \sqrt{1/(2\gamma_\ell)}\sigma_\ell$, we can similarly derive the upper bound

$$\mathbb{E}\{e_\ell^2(t) \,|\, \Delta_i^\ell\} \le \eta_\ell^2 + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right),$$

which results in (17), again using Jensen's Inequality.

E Proof of Theorem 9.6

Using Itô's Lemma, for all $t \in [T_i^\ell, T_{i+1}^\ell)$, we get

$$\begin{aligned} \mathrm{d}\|e_\ell(t)\|^2 &= e_\ell(t)^\top (A_\ell + A_\ell^\top) e_\ell(t)\mathrm{d}t + \mathrm{tr}(H^\top H)\mathrm{d}t \\ &\quad + e_\ell(t)^\top H \mathrm{d}w_\ell(t) + \mathrm{d}w_\ell(t)^\top H^\top e_\ell(t), \end{aligned}$$

and as a result

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\|e_\ell(t)\|^2 \mid \Delta_i^\ell\} = \operatorname{tr}(H^\top H) + \mathbb{E}\{e_\ell(t)^\top (A_\ell + A_\ell^\top) e_\ell(t) \mid \Delta_i^\ell\}$$
$$\leq \operatorname{tr}(H^\top H) + \overline{\lambda}(A_\ell + A_\ell^\top)\mathbb{E}\{\|e_\ell(t)\|^2 \mid \Delta_i^\ell\},$$

with the initial condition $\mathbb{E}\{\|e_\ell(T_i^\ell)\|^2\} = \operatorname{tr}(R_\ell)$. Now, using the Comparison Lemma [52, p.102], we get

$$\mathbb{E}\{\|e_\ell(t)\|^2 \mid \Delta_i^\ell\} \leq \operatorname{tr}(R_\ell) e^{\overline{\lambda}(A_\ell + A_\ell^\top)t} + \frac{\operatorname{tr}(H^\top H)}{|\overline{\lambda}(A_\ell + A_\ell^\top)|}\left(1 - e^{\overline{\lambda}(A_\ell + A_\ell^\top)t}\right),$$

for $t \in [T_i^\ell, T_{i+1}^\ell)$. Using Lemma 9.12 (presented in Appendix D) and Jensen's Inequality, we get (20).

F Proof of Theorem 9.7

First, note that for $t \in [T_i^\ell, T_{i+1}^\ell)$, the estimator

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{z}_\ell(t) = A_\ell \hat{z}_\ell(t);\ \hat{z}_\ell(T_i^\ell) = \{z_\ell(T_i^\ell) \mid y_1^\ell, \ldots, y_i^\ell\},$$

is an optimal mean square error estimator. This is in fact true since the estimator ℓ has not received any new information over $[T_i^\ell, t]$ and it should simply predict the state using the best available estimation $\{z_\ell(T_i^\ell) \mid y_1^\ell, \ldots, y_i^\ell\}$. Now, recalling from [53], we know that $\{z_\ell(T_i^\ell) \mid y_1^\ell, \ldots, y_i^\ell\} = \{z_\ell[i] \mid y_1^\ell, \ldots, y_i^\ell\} = \hat{z}_\ell[i]$. This completes the first part of the proof. For the rest, note that following a similar reasoning as in the proof of Theorem 9.6, for all $t \in [T_i^\ell, T_{i+1}^\ell)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\{\|e_\ell(t)\|^2 \mid \Delta_i^\ell\} \leq \operatorname{tr}(H^\top H) + \overline{\lambda}(A_\ell + A_\ell^\top)\mathbb{E}\{\|e_\ell(t)\|^2 \mid \Delta_i^\ell\},$$

with the initial condition

$$\mathbb{E}\{\|e_\ell(T_i^\ell)\|^2\} = \mathbb{E}\{(z_\ell[i] - \hat{z}_\ell[i])^\top (z_\ell[i] - \hat{z}_\ell[i])\} = \operatorname{trace}(P_\ell[i]),$$

which results in (23) again using the Comparison Lemma.

G Proof of Theorem 9.9

To simplify the calculations, we introduce the change of variable $z'_\ell(t) = z_\ell(t) + \zeta_\ell(t)$ for all $t \in [T_i^\ell, T_{i+1}^\ell)$, where

$$\zeta_\ell(t) = \begin{cases} n_i^\ell \frac{e^{-\gamma_\ell \rho}}{1 - e^{-\gamma_\ell \rho}}\left(1 - e^{-\gamma_\ell (t - T_i^\ell)}\right) - z_\ell(T_i^\ell)\frac{e^{-\gamma_\ell (t - T_i^\ell)} - e^{-\gamma_\ell \rho}}{1 - e^{-\gamma_\ell \rho}}, & t \in [T_i^\ell, T_i^\ell + \rho), \\ n_i^\ell e^{-\gamma_\ell (t - T_i^\ell)}, & t \in [T_i^\ell + \rho, T_{i+1}^\ell). \end{cases}$$

Now, using Itô's Lemma [48, p. 49], we get

$$\mathrm{d}z'_\ell(t) = -\gamma_\ell z'_\ell(t)\mathrm{d}t + \sigma_\ell \mathrm{d}w_\ell(t);\; z'_\ell(T_i^\ell) = 0.$$

Hence, for all $t \in [T_i^\ell, T_{i+1}^\ell)$, we get

$$\begin{aligned}\mathbb{E}\left\{z_\ell^2(t) \,|\, \Delta_i^\ell\right\} &= \mathbb{E}\left\{z_\ell'^2(t) + \zeta_\ell^2(t) \,|\, \Delta_i^\ell\right\} \\ &\leq \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right) + \zeta_\ell^2(t),\end{aligned}$$

where the first equality is due to the fact that $\mathbb{E}\{\zeta_\ell(t) z'_\ell(t)\} = 0$ because the random process $\{w_\ell(t)\}_{t \in (T_i^\ell, T_{i+1}^\ell)}$ is independent of $\zeta_\ell(t)$ and $\mathbb{E}\{z'_\ell(t)\} = 0$. As a result

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \leq \mathbb{E}\left\{\frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right)\right\} + \eta_\ell^2 + \mathbb{E}\{z_\ell^2(T_i^\ell)\}\mathbb{P}\{\Delta_i^\ell < \rho\}. \tag{44}$$

Using Lemma 9.12 (b), presented in Appendix D, and Jensen's Inequality, we can simplify (44) as

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \leq \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell / f_\ell}\right) + \eta_\ell^2 + \mathbb{E}\{z_\ell^2(T_i^\ell)\}\mathbb{P}\{\Delta_i^\ell < \rho\}. \tag{45}$$

Note that by evaluating (45) as t goes to T_{i+1}^ℓ, we can extract a difference equations for the closed-loop performance (i.e., an algebraic equation that relates $\mathbb{E}\{z_\ell^2(T_{i+1}^\ell)\}$ to $\mathbb{E}\{z_\ell^2(T_i^\ell)\}$ for all i). By solving this difference equation and substituting the solution into (45), we get

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \leq \sum_{k=0}^{i}\left[\frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell / f_\ell}\right) + \eta_\ell^2\right]\left(\mathbb{P}\{\Delta_i^\ell < \rho\}\right)^k,$$

for all $t \in [T_i^\ell, T_{i+1}^\ell)$, and as a result

$$\begin{aligned}\mathbb{E}\left\{z_\ell^2(t)\right\} &\leq \sum_{k=0}^{\infty}\left[\frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell / f_\ell}\right) + \eta_\ell^2\right]\left(\mathbb{P}\{\Delta_i^\ell < \rho\}\right)^k \\ &= \left[\frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell / f_\ell}\right) + \eta_\ell^2\right]\frac{1}{1 - \mathbb{P}\{\Delta_i^\ell < \rho\}}.\end{aligned}$$

This concludes the proof.

H Proof of Theorem 9.10

Using the same argument as in the proof of Theorem 9.9, we obtain

$$\mathbb{E}\left\{z_\ell^2(t) \,|\, \Delta_i^\ell\right\} \leq \eta_\ell^2 + \frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right) + \mathbb{E}\{z_\ell^2(T_i^\ell)\}e^{-2\theta\Delta_i^\ell},$$

for all $t \in [T_i^\ell, T_{i+1}^\ell)$, and as a result

$$\mathbb{E}\left\{z_\ell^2(t)\right\} \leq \eta_\ell^2 + \mathbb{E}\left\{\frac{\sigma_\ell^2}{2\gamma_\ell}\left(1 - e^{-2\gamma_\ell \Delta_i^\ell}\right)\right\} + \mathbb{E}\{z_\ell^2(T_i^\ell)\}\mathbb{E}\{e^{-2\theta\Delta_i^\ell}\}.$$

Similar to the proof of Theorem 9.9, we can simplify this expression into (31) using Lemma 9.12 (b), presented in Appendix D, and Jensen's Inequality.