Decentralized Control Design with Limited Plant Model Information

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Abstract

Large-scale control systems are often composed of several smaller interconnected units. For these systems, it is common to employ local controllers, which observe and act locally. At the heart of common control design procedures for distributed systems lies the often implicit assumption that the designer has access to the global plant model information when designing a local controller. However, there are several reasons why such plant model information would not be globally known. One reason could be that the designer wants the parameters of each local controller to only depend on local model information, so that the controllers are not modified if the model parameters of a particular subsystem change. It might also be the case that the design of each local controller is done by individual designers with no access to the global plant model, for instance, due to the fact that the designers refuse to share their model information since they consider it private. This class of problems, which we refer to as limited model information control design, is the topic of the thesis.

First, we investigate the achievable closed-loop performance of discrete-time linear time-invariant plants under a separable quadratic cost performance with structured static state-feedback controllers. To do so, we introduce control design strategies as mappings, which construct controllers by accessing the plant model information in a constrained way according to a given design graph. We compare control design strategies using the competitive ratio as a performance metric, that is, we compare the worst case control performance for a given design strategy normalized with the optimal control performance based on full model information. An explicit minimizer of the competitive ratio is sought. As this minimizer might not be unique, we further search for the ones that are undominated, that is, there is no other control design strategy in the set of limited model information design strategies with a better closed-loop performance for all possible plants while maintaining the same worst-case ratio. We study the trade-off between the amount of model information exploited by a control design strategy and the best possible closed-loop performance. We generalize this setup to structured dynamic state-feedback controllers for H_2-performance. Surprisingly, the optimal control design strategy with limited model information is still a static one. This is the case even though the optimal decentralized state-feedback controller with full model information is dynamic. Finally, we discuss the design of dynamic controllers for disturbance accommodation under limited model information. This problem is of special interest because the best limited model information control design in this case is a dynamic control design strategy. The optimal controller can be separated into a static feedback law and a dynamic disturbance observer. For constant disturbances, it is shown that this structure corresponds to proportional-integral control.

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Farhad
Stockholm, January 2012

Contents

Notations

Sets

$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{T}$	The unit circle in $\mathbb{C}$
$\mathcal{L}_\infty$	The set of Lebesgue measurable functions bounded on $\mathbb{T}$
$\mathcal{R}$	The set of proper real rational functions
$\mathcal{RL}_\infty$	The set of proper real rational functions in $\mathcal{L}_\infty$
$\mathcal{S}^n_{++}$ $(\mathcal{S}^n_{+})$	The set of symmetric positive definite (semidefinite) matrices
$\mathcal{A}$	All other sets are denoted by calligraphic letters
$\mathcal{A}^c$	The complement of $\mathcal{A}$

Matrices

A	Matrices are denoted by capital roman letters
A_j	j^{th} row of matrix A
A_{ij}	Submatrix i, j of matrix A with dimension and position defined in the text
a_{ij}	Entry i, j of matrix A
$A > (\geq) 0$	The real symmetric matrix A is positive definite (semidefinite)
$A > (\geq) B$	$A - B > (\geq) 0$
$\underline{\sigma}(Y)$	The smallest singular value of the matrix Y

$\overline{\sigma}(Y)$	The largest singular value of the matrix Y
$\underline{\lambda}(Y)$	The smallest eigenvalue of the matrix Y
$\overline{\lambda}(Y)$	The largest eigenvalue of the matrix Y

Graphs

G	Graphs are denoted by capital roman letters. All considered graphs are directed
$\{1, \ldots, q\}$	Vertex set of G
E	Edge set of G
S	The adjacency matrix of G whose entry $s_{ij} = 1$ if $(j, i) \in E$ and $s_{ij} = 0$ otherwise for all $1 \leq i, j \leq q$
$G \subseteq G'$	G is a subgraph of G'. The edge set of G is a subset of the edge set of G'
$i \to j$	A link between vertices i, j in a graph G such that $(i, j) \in E$
sink	Vertex i such that there does not exist $j \neq i$ with $(i, j) \in E$
loop	A loop of length t in G is a set of distinct vertices $\{i_1, ..., i_t\}$ such that $i_1 \to i_2 \to \cdots \to i_t \to i_1$

Others

e_i	The column vector with all entries zero except the i^{th} entry which is equal to one
$\delta : \mathbb{Z} \to \mathbb{Z}$	The unit-impulse function which is equal to one at origin and zero anywhere else

Part I

Introduction

CHAPTER 1

Introduction

Many modern large-scale systems, such as aircraft and satellite formations [1, 2], automated highways and other shared infrastructures [3, 4], flexible structures [5, 6], and supply chains [7, 8], consist of several subsystems coupled through their dynamics, controllers, or performance objectives. When regulating these systems, it is often advantageous to adopt a distributed control architecture, in which the overall controller is composed of interconnected subcontrollers, each of which accesses a subset of the plant's state measurements. A common but often implicit assumption for distributed control system is that the design can be performed in a centralized fashion, with full knowledge of the plant model. However, this assumption is far from being warranted in practice. Removing this assumption from the control design procedure generates a new class of problems, namely limited model information control design problems. For these problems, we are interested in studying the challenges facing decision-makers (agents) in a dynamical system who must select some control variables in order to optimize a social function using only partial knowledge of the model governing the system (in addition to the partial knowledge of the system state). The described problem is closely related to the classical problem of distributed decision-making using partial information [9–12]. In distributed decision-making using partial information, the aim is to develop algorithms that always produce feasible solutions with reasonable values of the objective function. This problem appears in many areas ranging from computer science problems, such as managing a large-scale communication network [12] and distributed task assignment [12–14], to economical and financial problems, such as inventory models [15–18] and supply chains [19–23].

The rest of the chapter is organized as follows. We begin by giving a motivating application for studying control design with limited model information in Section 1.1. In Section 1.2, we discuss the reasons behind the lack of a global model information in optimal control design and we present two examples to illustrate the problem. We describe the underlying mathematical formulation for control design with limited plant model information in Section 1.3. In Section 1.4, we revisit the examples introduced in Section 1.2 to illustrate the framework. Finally, we conclude this chapter by the thesis outline and contributions in Section 1.5.

1.1 Motivating Application

To illustrate and motivate the importance of control design with limited plant model information, we consider a highly complex large-scale dynamical system, namely, the Baltic sea region electricity transmission grid portrayed in Figure 1.1. The power is generated in several large power generators and transmitted through the network to the power consumers. The power network consists of tens-of-thousands of components (e.g., generators, transmission lines, conversion stations, etc) connected together. These components have local interactions with each other because of the grid, which results in a specific system dynamics. In the thesis, we capture the structured dynamics through a plant graph.

For a power transmission grid, one of the design goals is to optimally regulate the voltage, active and reactive power, and frequency. To do so, the designer employs many sensors (e.g., phasor measurement units) to measure voltage, active or reactive power, and frequency over the network. These units transmit their measurements over a communication network to the control stations. Due to communication limitations and the large scale and complexity of the grid, all sensor information cannot instantaneously be available to any controller in the system. Therefore, the controller cannot use full state measurements of the system, but only access a subset of the states in each local controller. In this thesis, this property is illustrated using a control graph, that is, a directed graph that identify the communication links between subsystems and subcontrollers. The absence of full state measurement in a networked control system brings challenges in designing stabilizing and optimal controllers, which we discuss later.

Power network control systems are highly complex time-varying dynamical systems, which are very hard to completely model for several reasons. First of all, these systems are social-technical systems meaning that they are composed of a technical layer (electrical components and their interconnections) and a social layer working together [24]. The social layer consists of the end users who put physical constraints on the technical layer and the human operators who change the structure of the technical layer and manage the production levels to control the power flow. In the design procedure, the behavior of the social layer is partially unknown (although to some extent predictable by the historical data and the regulations). Second, several different power production companies compete with each other over the production levels. The network manager regulates the power production companies based on their prices and the public demand. As a consequence, a varying set of companies with different generator types (e.g., thermal, wind, hydro, etc) provide

Figure 1.1: Electricity transmission grid in the Baltic sea region (Courtesy of Nordregio http://www.nordregio.se/, Designer: P.G. Lindblom).

the power needed across the network. These competing companies are unwilling to share their (private) information about the network as that might compromise their financial benefits by giving tactical advantages to other companies in power auctions. Third, power networks are typically made of nonlinear components, although it is common, to design linear controllers with acceptable closed-loop performance based on linearized models. These controllers are functions of the linearized subsystems' model (and, in turn, functions of their operating points). These subsystems (e.g., generators) change their operating points in response to the power demand and physical constraints. Finally, safety constraints must be satisfied at all time instances to protect the electrical equipments and end users from harm in faulty conditions or other hazardous situations. Therefore, safety switches automatically connect or disconnect electrical components or transmission lines (to meet these safety requirements). These switches change the topology of the network and the transmission lines impedances. Now, noting that these power networks are typically implemented over a vast geographical area (even across different countries) makes it extremely difficult (perhaps impossible) to gather all the model information (entire network topology, line impedances, operating conditions, etc) at one place. Even if one could gather all these information and implement a new controller based on them, it might take very long and by then the information might be outdated. This delay may even lead to instability of the closed-loop system. This motivates our interest in designing local controllers based on only local model information of the plant to be controlled. The amount of information that is available in each local subsystem when designing its controller, in this thesis, is captured using a design graph, that is, a directed graph which indicates the dependency of each local controller on different parts of the global plant model.

1.2 Model Information Limitations

When regulating a large-scale system composed of several interconnected subsystems, it makes sense to adopt a distributed or decentralized control architecture, in which the controller itself is made of interconnected subcontrollers. At the heart of traditional distributed or decentralized control design problems is the assumption that the control design is done with the global knowledge of the plant model. However, this assumption is seldom warranted, for instance, because of the following three reasons:

- **Maintenance:** To simplify control systems tuning and maintenance, it is desirable that each local controller to be only a function of local subsystem parameters, so that the resulting local controller does not need to be modified if the model parameters of a particular subsystem change over time. Otherwise, the designer might be required to reconfigure and tune every subcontroller any time she observes a change in a local parameter. These local parameter changes might be due to several reasons, including changing operating conditions, material fatigue, weather conditions, and scheduled services.

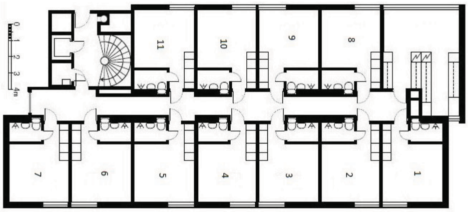

Figure 1.2: The floor plan of a half block of a student house.

- **Availability:** The lack of availability of the complete model of the plant, at the time of the design, restrict the designer to only use local model information in each subsystem control design. This is because often the design of each local controller is done by a different designer (possibly in a different company, organization, or country) with no access to the global plant model at the time of design, as the complete model information is not available yet, or to be changed later. This is becoming more and more common as engineers implement a system as a whole using commercially available pre-designed modules (off-the-shelf components). These modules are designed, in advance, with no prior knowledge of their possible use or future operating condition. Thus, they are required to work with an acceptable performance under almost any circumstances.

- **Privacy:** Privacy constraints, caused by financial incentives or security reason, limit the amount of the model information available in each subsystems when designing its controller. These constraints stem from the fact that, in large-scale control systems, different subsystems typically belong to different individuals, and these individuals might be unwilling to share their model information. Therefore, each subsystem's controller should be designed only based on its own model information.

We capture the amount of plant model information available in the design process to each subcontroller by a design graph. An edge in the design graph from a subsystem to a subcontroller represents that the subcontroller can use the model parameters of that subsystem. Therefore, we deal with a limited model information control design whenever the underlying design graph is not a complete graph.

The three aforementioned reasons (maintenance, availability, and privacy) contribute to the motivation for studying how the amount of the model information available in each subsystems influence the control design performance. Let us illustrate the control design problem through two examples: a temperature control problem in Example 1.1 and a vehicle platooning problem in Example 1.2.

Example 1.1 (Temperature Control): Let us consider the problem of regulating the temperature in $q = 11$ rooms on a floor of a half block of a student house, where each room can be warmed by a single heater (see Figure 1.2). The corridors and stairways are supposed to have ambient temperature. Let us denote the average temperature of room i by $\bar{x}_i$. By applying Euler's constant step discretization scheme to the continuous-time model (both in time and space), we obtain the following difference equation

$$\bar{x}_i(k+1) = \sum_{j \neq i} \alpha_{ij}(\bar{x}_j(k) - \bar{x}_i(k)) + \beta_i(\bar{x}_a - \bar{x}_i(k)) + u_i(k), \tag{1.1}$$

where $\bar{x}_a$ is the ambient temperature, which is assumed to be constant, and β_i and α_{ij} are constants representing the average heat loss rates of room i to the ambient and to room j, respectively. The goal is to regulate the temperature of each room at a prescribed value by minimizing the performance criterion

$$J = \sum_{k=0}^{\infty} \sum_{i=1}^{q} (\bar{x}_i(k) - r_i)^2 + (u_i(k) - u_i^*)^2, \tag{1.2}$$

where r_i, for each i, is the reference temperature of room i, and u_i^*, for each i, is the steady-state control signal of room i. Note that, in the case of the infinite horizon control cost function, the steady-state control signals is nonzero and related to the reference points [25], as otherwise the performance criterion would become infinity.

The characteristics of each room (such as opening doors and windows, places of the furniture, the type and the brightness of the wallpapers or paint, thickness of the walls, etc) affect its model parameters $\{\beta_i\} \cup \{\alpha_{ij} \mid j \neq i\}$. These parameters may not be available to other rooms' thermostat due to several reasons including:

- Maintenance: Consider the case that the land-lady wants each subcontroller to be only a function of the corresponding subsystem parameters to avoid disturbing other tenants whenever something changes in a single room (due to opening or closing windows, redecoration, renovation, etc), as these system parameters would change quite frequently, and the global optimal controller must be updated every time that a single parameter gets updated (e.g., someone opens or closes a window).

- Privacy: It might be the case that these characteristics depend on some private information (like the decoration of the room or opening/closing of the windows) and the tenants might be unwilling to share it with the thermostat of other rooms (e.g., due to a risk of theft).

Besides, the tenants also want to guarantee some reasonable bounds on the closed-loop performance of the system because of environmental factors and the constantly increasing energy prices. Therefore, this problem is a simple illustration of designing optimal controller with limited model information.

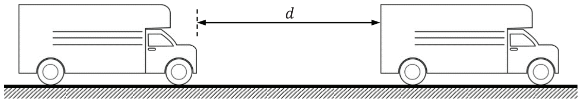

Figure 1.3: Regulating the distance between two trucks.

Example 1.2 (Vehicle Platooning): As the simplest case for vehicle platooning, consider the problem of regulating the distance between two trucks illustrated in Figure 1.3. Applying Euler's constant step discretization scheme to the continuous-time model of each truck, one gets

$$\begin{bmatrix} x_i(k+1) \\ v_i(k+1) \end{bmatrix} = \left(I + \Delta T \begin{bmatrix} 0 & 1 \\ 0 & -\alpha_i/m_i \end{bmatrix}\right) \begin{bmatrix} x_i(k) \\ v_i(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_i/m \end{bmatrix} u_i(k),$$

where $x_i(k)$ is the truck position, $v_i(k)$ the velocity, m_i the mass, α_i the viscous drag coefficient, β_i the power conversion quality coefficient, and ΔT the sampling time. As a natural choice, the designer wants to minimize the cost function

$$J = \sum_{k=0}^{\infty} \left[q_d(x_2(k) - x_1(k) - d^*)^2 + \sum_{i=1,2} q_v(v_i(k) - v^*)^2 + r(u_i(k) - u_i^*)^2 \right],$$

to regulate the distance between the trucks with minimum control effort. Note that u_i^* is a steady-state control signal and it is a function of the reference points $\alpha_i v^*/\beta_i$. We can write the reduced-order system using the distance between trucks and their velocities as state variables

$$z(k+1) = Az(k) + Bu(k),$$

where

$$z(k) = \begin{bmatrix} v_2(k) - v^* \\ x_2(k) - x_1(k) - d^* \\ v_1(k) - v^* \end{bmatrix}, \qquad u(k) = \begin{bmatrix} u_2(k) - \alpha_2 v^*/\beta_2 \\ u_1(k) - \alpha_1 v^*/\beta_1 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 1 - \Delta T\alpha_2/m_2 & 0 & 0 \\ \Delta T & 1 & -\Delta T \\ 0 & 0 & 1 - \Delta T\alpha_1/m_1 \end{bmatrix}, \; B = \begin{bmatrix} \Delta T\beta_2/m_2 & 0 \\ 0 & 0 \\ 0 & \Delta T\beta_1/m_1 \end{bmatrix}.$$

This leads to the simplified performance criterion

$$J = \sum_{k=0}^{\infty} z(k)^T Q z(k) + u(k)^T R u(k),$$

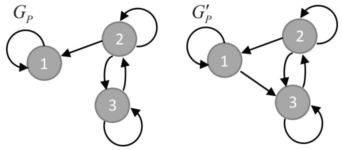

Figure 1.4: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs.

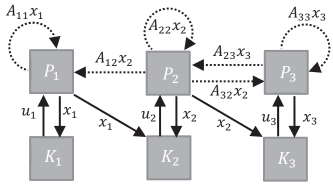

Figure 1.5: Example of the physical interconnection between different subsystems and subcontrollers in a networked control system.

where $Q = \text{diag}(q_v, q_d, q_v)$ and $R = \text{diag}(r, r)$. Note that the characteristics of each truck (e.g., mass, tire quality, break quality, etc) change its model parameters $\{m_i, \alpha_i, \beta_i\}$. Each vehicle control system designer may want its controller to only be a function of its truck parameters because:

- Maintenance: It might be the case that each designer wants the controller to be fixed. The safety constraints might be a motive for this as changing a truck's subcontroller (in an uncontrolled environment) may result in an unpredictable behavior.

- Availability: Each truck's local controller cannot be designed based on the model information of all possible vehicles that it may cooperate with in future traffic scenarios.

- Privacy: The truck parameters (e.g., the truck mass) might not be available to other trucks. For instance, different trucks might belong to the different companies and these companies may wish to honor their costumers privacy.

All truck owners want to guarantee some reasonable bounds on the closed-loop performance of the platoon to reduce the fuel consumption. This problem is hence a viable candidate for optimal control design with limited model information.

1.3 Problem Formulation

In this section, we mathematically formulate the high-level goals of the thesis. Here, we give some of the key definitions. We do not go through the assumptions needed later for validity of the results. These assumptions are highlighted and discussed individually in Papers 1–3.

1.3.1 Plant Model

We start by presenting the most essential problem formulation from Paper 1. Then, we build our way to other cases by sensible extensions of this basic problem.

Let a directed graph $G_{\mathcal{P}} = (\{1, \ldots, q\}, E_{\mathcal{P}})$ with adjacency matrix $S_{\mathcal{P}}$ be given. This directed graph, which we refer to as the plant graph, is common in all the discussed models, and it illustrates the interconnection pattern between subsystems. Let us define the following set of matrices associated with the adjacency matrix $S_{\mathcal{P}}$:

$$\begin{aligned}\mathcal{A}(S_{\mathcal{P}}) = \big\{\bar{A} \in \mathbb{R}^{n \times n} \ | \ &\bar{A}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \\ &\text{for all } \ 1 \leq i, j \leq q \text{ such that } (s_{\mathcal{P}})_{ij} = 0\big\}\,,\end{aligned} \tag{1.3}$$

where, for each $1 \leq i \leq q$, integer number n_i is the dimension of subsystem i. Implicit in these definitions is the fact that $\sum_{i=1}^{q} n_i = n$. Also, we define

$$\mathcal{B} \subseteq \big\{\bar{B} \in \mathbb{R}^{n \times n} \mid \bar{B}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i \neq j \leq q\big\}\,. \tag{1.4}$$

With these definitions, we can introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant dynamical systems of the form

$$x(k+1) = Ax(k) + Bu(k)\ ;\ x(0) = x_0,$$

with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}$, and $x_0 \in \mathbb{R}^n$. Clearly $\mathcal{P}$ is isomorph to $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B} \times \mathbb{R}^n$ and, slightly abusing notation, we will thus identify a plant $P \in \mathcal{P}$ with the corresponding triple $P = (A, B, x_0)$.

Figure 1.4 shows an example of a plant graph $G_{\mathcal{P}}$. Each node represents a subsystem of the system. For instance, the second subsystem in this example may affect the first subsystem and the third subsystem; i.e., submatrices A_{12} and A_{32} can be nonzero. The self-loop for the second subsystem shows that A_{22} may be nonzero. Figure 1.5 illustrates the corresponding physical interconnection between subsystems of the plant in Figure 1.4 by dotted edges. Note that P_1 in Figure 1.5 represents a sink (a node that cannot affect any other node) of $G_{\mathcal{P}}$. The plant graph $G'_{\mathcal{P}}$ in Figure 1.4 has no sink. As we will see later in Papers 1–3, the sinks play a significant role in the nature of the solutions that we present.

1.3.2 Controller Model

Let a control graph $G_{\mathcal{K}}$ with adjacency matrix $S_{\mathcal{K}}$ be given. The control laws of interest are static linear state-feedback control laws of the form

$$u(k) = Kx(k),$$

where

$$\begin{aligned}K \in \mathcal{K}(S_{\mathcal{K}}) = \big\{\bar{K} \in \mathbb{R}^{n \times n} \ | \ &\bar{K}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \\ &\text{for all } \ 1 \leq i, j \leq q \text{ such that } (s_{\mathcal{K}})_{ij} = 0\big\}\,.\end{aligned} \tag{1.5}$$

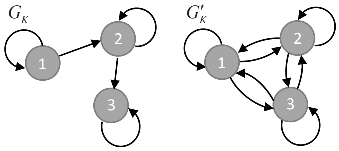

Figure 1.6: $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs.

An example of a control graph $G_{\mathcal{K}}$ is given in Figure 1.6. Each node represents a subsystem–controller pair of the overall system. For instance, $G_{\mathcal{K}}$ shows that the second subsystem's controller can use state measurements of the first subsystem besides its own state measurements. Solid edges in Figure 1.5 correspond to the edges of the control graph $G_{\mathcal{K}}$. Figure 1.6 shows $G'_{\mathcal{K}}$ which is a complete graph. This control graph indicates that each subcontroller has access to full state measurements of all subsystems.

1.3.3 Control Design Method

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}(S_{\mathcal{K}})$. Just like plants and controllers, a control design method can exhibit structure which, in turn, can be captured by a directed graph which we call the design graph as it illustrates the amount of the information available to each subsystem in control design procedure. Let a control design method Γ be partitioned according to subsystems dimensions as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{bmatrix}$$

and the design graph $G_{\mathcal{C}} = (\{1,\ldots,q\}, E_{\mathcal{C}})$ with adjacency matrix $S_{\mathcal{C}}$ be given. Each block Γ_{ij} represents a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B} \to \mathbb{R}^{n_i \times n_j}$. We say that a control design strategy Γ has structure $G_{\mathcal{C}}$ if and only if, for all i, the map $[\Gamma_{i1} \ \cdots \ \Gamma_{iq}]$ is only a function of

$$\{[A_{j1} \ \cdots \ A_{jq}], B_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}. \tag{1.6}$$

The set of all control design methods with structure $G_{\mathcal{C}}$ is denoted by $\mathcal{C}$. When $G_{\mathcal{C}}$ is not a complete graph, we refer to $\Gamma \in \mathcal{C}$ as being a limited model information control design method.

An example of a design graph $G_{\mathcal{C}}$ is given in Figure 1.7. Each node represents a subsystem–controller pair of the overall system. For instance, $G_{\mathcal{C}}$ shows that the third subsystem's model is available to the designer of the second subsystem's controller but not the first subsystem's model. Figure 1.7 shows a fully disconnected design graph with self-loops in $G'_{\mathcal{C}}$. A local designer in this case can only rely on the model of its corresponding subsystem. Note that the conventional networked control system block diagram in Figure 1.5 does not feature the design graph.

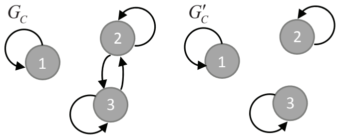

Figure 1.7: $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

1.3.4 Performance Metric

The goal of this thesis is to investigate the influence of the plant, control, and design graphs on the quality of controllers constructed by limited model information control design methods. To each plant $P \in \mathcal{P}$ and controller $K \in \mathcal{K}$, we associate a closed-loop performance criterion

$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k), \tag{1.7}$$

where $Q, R \in \mathcal{S}_{++}^{n}$ are block diagonal matrices, with each diagonal block entry belonging to $\mathcal{S}_{++}^{n_i}$. The closed-loop performance criterion could be changed later according to the application in-hand (as we do in Papers 2 and 3). Now, assume that a plant graph $G_{\mathcal{P}}$ and a control graph $G_{\mathcal{K}}$ are given. Furthermore, assume that, for every plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that

$$J_P(K^*(P)) \leq J_P(K), \ \forall K \in \mathcal{K}.$$

The mapping $K^* : P \to K^*(P)$ is not itself required to lie in the set $\mathcal{C}$, as every component of the optimal controller may depend on all entries of the plant model. The competitive ratio of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$" equals one. Now, we formulate the main question of this thesis regarding the connection between closed-loop performance, plant structure, controller structure, and limited model information control design as follows. For given plant, control, and design graphs, we would like to determine

$$\Gamma^* \in \arg\min_{\Gamma \in \mathcal{C}} r_{\mathcal{P}}(\Gamma). \tag{1.8}$$

Since this minimizer might not be unique, we define a partial order (domination) on the set $\mathcal{C}$. A control design method Γ is said to dominate another control design method Γ' if

$$J_P(\Gamma(P)) \leq J_P(\Gamma'(P)), \ \forall \ P \in \mathcal{P}, \tag{1.9}$$

with strict inequality holding for at least one plant in $\mathcal{P}$. When $\Gamma' \in \mathcal{C}$ and no control design method $\Gamma \in \mathcal{C}$ exists that dominates Γ', we say that Γ' is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$. In the thesis, we are interested in determining the control design strategies in (1.8) that are undominated.

1.3.5 Problem Formulation Extensions

In Paper 2, we introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant dynamical systems of the form

$$x(k+1) = Ax(k) + Bu(k) + Hw(k)\ ;\ x(0) = 0,$$

where $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}$, and

$$H \in \mathcal{H} \subseteq \left\{\bar{H} \in \mathbb{R}^{n\times n} \mid \bar{H}_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \le i \ne j \le q\right\}.$$

Thus, we identify a plant $P \in \mathcal{P}$ in Paper 2, with the corresponding triple $P = (A, B, H) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B} \times \mathcal{H}$. We generalize the set of control laws of interest to dynamic linear state-feedback control

$$\begin{aligned}\mathcal{K}(S_{\mathcal{K}}) = \{\bar{K} \in (\mathcal{RL}_\infty)^{n\times n} \mid \bar{K}_{ij} = 0 \in (\mathcal{RL}_\infty)^{n_i\times n_j} \\ \text{for all } 1 \le i, j \le q \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}.\end{aligned}$$

We also use the H_2-norm of the closed-loop system from the exogenous input $w(k)$ to the output

$$y(k) = \begin{bmatrix} C^T & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & D^T \end{bmatrix} u(k)$$

where $C, D \in \mathbb{R}^{n\times n}$ are block diagonal matrices, with each diagonal block entry belonging to $\mathbb{R}^{n_i\times n_i}$.

According to the specific structure of $\mathcal{B}$ given in (1.4), each subsystem is fully-actuated, with as many input as states, and controllable in just one time step. Possible generalization of the results to a (restricted) family of under-actuated systems is also discussed in Paper 2.

In Paper 3, we fix $n_i = m_i = 1$ for all $1 \le i \le n$ in (1.3)–(1.4), and introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant dynamical systems of the form

$$\begin{aligned}x(k+1) &= Ax(k) + B(u(k) + w(k))\ ;\ x(0) = x_0,\\ w(k+1) &= Dw(k)\ ;\ w(0) = w_0,\end{aligned}$$

with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}$, $x_0 \in \mathbb{R}^n$, $w_0 \in \mathbb{R}^n$, and

$$D \in \mathcal{D} = \left\{\bar{D} \in \mathbb{R}^{n\times n} \mid \bar{d}_{ij} = 0 \in \mathbb{R} \text{ for all } 1 \le i \ne j \le n\right\}.$$

We identify a plant $P \in \mathcal{P}$ with the corresponding tuple $P = (A, B, D, x_0, w_0) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B} \times \mathcal{D} \times \mathbb{R}^n \times \mathbb{R}^n$. We also generalize the set of control laws of interest to the set of dynamic linear state-feedback controllers

$$\mathcal{K}(S_{\mathcal{K}}) = \{\bar{K} \in \mathcal{R}^{n\times n} \mid \bar{k}_{ij} = 0 \in \mathcal{R} \text{ for all } 1 \le i, j \le n \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}.$$

We associate the closed-loop performance criterion

$$J_P(K) = \sum_{k=0}^{\infty} x(k)^T Q x(k) + (u(k) + w(k))^T R (u(k) + w(k)),$$

where $Q, R \in \mathcal{S}^n_{++}$ are diagonal matrices.

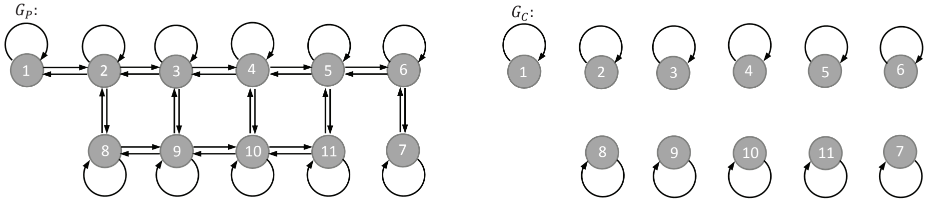

Figure 1.8: The plant graph $G_{\mathcal{P}}$ and design graph $G_{\mathcal{C}}$ of the temperature control system described in Example 1.3.

1.4 Examples Revisited

In this subsection, we revisit the temperature control and the vehicle platooning examples presented in Section 1.2.

Example 1.3 (Temperature Control, continued): Consider the temperature control problem introduced in Example 1.1. Augmenting all the average temperature difference equations in (1.1), and using a simple change of variable $x(k) = \bar{x}(k) - r$ with $r = [r_1 \ \cdots \ r_q]^T \in \mathbb{R}^q$ as the vector of desired temperature, results in a discrete-time linear time-invariant dynamical system of the form

$$x(k+1) = Ax(k) + u(k) + w(k),$$

where $w(k) \in \mathbb{R}^q$ is a constant-disturbance vector given by

$$w(k) = [\beta_1 \ \cdots \ \beta_q]^T \bar{x}_a + Ar - r,$$

and $A \in \mathbb{R}^{q\times q}$ is a model matrix whose entries are defined as

$$a_{ij} = \begin{cases} \alpha_{ij}, & i \neq j, \\ -\beta_i - \sum_{\ell\neq i} \alpha_{i\ell}, & \text{otherwise.} \end{cases}$$

Note that we can consider $Ar - r$ as a part of the disturbance vector whenever subsystems do not know each other set-points. Now, the performance criterion in (1.2) can be written as

$$J = \sum_{k=0}^{\infty} x(k)^T x(k) + (u(k) + w(k))^T (u(k) + w(k)).$$

If two rooms are not adjacent, their temperatures do not affect each other significantly, which we can use to generate the corresponding plant graph. In this particular problem, we have $q = 11$ rooms/subsystems, and each room's dynamics is of dimension one. The plant graph for this family of plants is shown in Figure 1.8 (left). Let the control graph $G_{\mathcal{K}}$ be a supergraph of the plant graph $G_{\mathcal{P}}$, and the design graph $G_{\mathcal{C}}$ be the one in Figure 1.8 (right). The design graph $G_{\mathcal{C}}$ shows that each local controller is designed based on a local subsystem model. Now, one can use the results given in Paper 3 to show that the undominated minimizer of the competitive ratio is the deadbeat proportional-integral control design strategy Γ^{Δ} which, for a fixed plant $P = (A, B, I, x_0, w_0)$, gives the proportional-integral control law

$$u(k) = -B^{-1}Ax(k) - B^{-1}\sum_{i=0}^{k} x(i).$$

This is the case as the plant graph $G_{\mathcal{P}}$ contains no sink. In the case that the plant graph contains one or more sinks, one can take advantage of the knowledge of the location of the sinks to achieve a better closed-loop performance.

Example 1.4 (Vehicle Platooning, continued): Consider the platooning problem in Example 1.2. Let us define the first subsystem as $\underline{z}_1(k) = z_1(k)$ and $\underline{z}_2(k) = [z_2(k)\ z_3(k)]^T$. Unfortunately, the dynamical system introduced in this example does not satisfy one of the assumptions required in Paper 1 (i.e., B is not a square invertible matrix). To use the results given in Paper 1, one can use either (i) the restriction of the platooning problem to velocity regulation, or (ii) the simplified version of the platooning problem under the assumption that the trucks can be modeled as first-order subsystems with the velocity as the control input. For instance, assume that we restrict the platooning problem to the velocity regulation. In this case, we have

$$\Delta v(k+1) = A\Delta v(k) + B\Delta u(k)$$

where

$$\Delta v(k) = \begin{bmatrix} v_1(k) - v^* \\ v_2(k) - v^* \end{bmatrix}, \qquad \Delta u(k) = \begin{bmatrix} u_1(k) - \alpha_1 v^*/\beta_1 \\ u_2(k) - \alpha_1 v^*/\beta_1 \end{bmatrix},$$

with v^* as the reference velocity, and

$$A = \begin{bmatrix} 1 - \Delta T\alpha_1/m_1 & 0 \\ 0 & 1 - \Delta T\alpha_2/m_2 \end{bmatrix}, \qquad B = \begin{bmatrix} \Delta T\beta_1/m_1 & 0 \\ 0 & \Delta T\beta_2/m_2 \end{bmatrix}.$$

Furthermore, let us consider the performance measure

$$J = \sum_{k=0}^{\infty} \Delta v(k)^T \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \Delta v(k) + \Delta u(k)^T \Delta u(k).$$

Unfortunately, the performance measure does not obey the assumptions of Paper 1 as it is nonseparable (i.e., Q is not diagonal). To fix this, we use the change of variable

$$z(k) = Q^{1/2}\Delta v(k) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \Delta v(k),$$

which gives

$$J = \sum_{k=0}^{\infty} z(k)^T z(k) + \Delta u(k)^T \Delta u(k),$$

and

$$z(k+1) = \bar{A} z(k) + \bar{B} \Delta u(k),$$

where $\bar{A} = Q^{1/2} A Q^{-1/2}$ and $\bar{B} = Q^{1/2} B$. For any fixed $(\bar{A}, \bar{B})$, the control law for the deadbeat control design strategy Γ^{Δ} is

$$\begin{aligned}
\Delta u(k) &= \Gamma^{\Delta}(\bar{A}, \bar{B}) z(k) \\
&= \Gamma^{\Delta}(Q^{1/2} A Q^{-1/2}, Q^{1/2} B) Q^{1/2} \Delta v(k) \\
&= -(Q^{1/2} B)^{-1} (Q^{1/2} A Q^{-1/2}) Q^{1/2} \Delta v(k) \\
&= -B^{-1} A \Delta v(k).
\end{aligned}$$

Therefore, subsystems i control law becomes only a function of its own parameters $\{\alpha_i, \beta_i, m_i\}$ (i.e., local model information), and consequently, $\Gamma^{\Delta} \in \mathcal{C}$. Thus, although the assumptions of Paper 1 are not completely fulfilled, the conclusions are still valid.

1.5 Thesis Outline and Contributions

The rest of this thesis is organized as follows.

Chapter 2: Background

A review of the pre-existing literature on generic properties of structured systems, distributed and decentralized control design, decision-making (optimization) with partial information, and limited model information control design is given in this chapter.

Chapter 3: Conclusions and Future Work

A summary of the results of the thesis and possible directions for future research are presented in this chapter.

Paper 1: Optimal Control Design with Limited Model Information

In this paper, we introduce the family of limited model information control design methods, which construct controllers by accessing the plant's model in a constrained way, according to a given design graph. We investigate the achievable closed-loop performance of discrete-time linear time-invariant plants under a separable quadratic cost performance measure with structured static state-feedback controllers. We find the optimal control design strategy (in terms of the competitive ratio and domination) when the control designer has access to the local model information and the global interconnection structure of the plant. At last, we study the trade-off between the amount of model information exploited by a control design method and the best closed-loop performance (in terms of the competitive ratio) of controllers it can produce. This paper is under review for journal publication as:

> F. Farokhi, C. Langbort, K. H. Johansson, "Optimal Control Design with Limited Model Information," 2011. Submitted.

A preliminary version of the paper was presented as:

> F. Farokhi, C. Langbort, K. H. Johansson, "Control Design with Limited Model Information," in *American Control Conference, Proceedings of the*, pp. 4697–4704, 2011.

Paper 2: Dynamic Control Design Based on Limited Model Information

The design of optimal H_2 dynamic controllers for interconnected linear systems under limited plant model information is considered in this paper. An explicit minimizer of the competitive ratio is found. It is shown that this control design strategy is not dominated by any other strategy with the same amount of model information. The result applies to a wide class of system interconnections, controller structures, and design information. This paper was recently presented as:

> F. Farokhi, K. H. Johansson, "Dynamic Control Design Based on Limited Model Information," in *Communication, Control, and Computing, Proceedings of the 49th Annual Allerton Conference on*, pp. 1576–1583, 2011.

Paper 3: Decentralized Disturbance Accommodation with Limited Plant Model Information

The optimal control design for disturbance accommodation with limited model information is considered in this paper. As it is shown in Papers 1 and 2, when it comes to designing optimal centralized or partially structured decentralized state-feedback controllers with limited model information, the best control design strategy (in terms of competitive ratio and domination) is static. This is true even though the optimal partially structured decentralized state-feedback controller with full model information is dynamic. In this paper, we show that, in contrast, the best limited model information control design strategy for the disturbance accommodation problem gives a dynamic controller. We find an explicit minimizer of the competitive ratio and we show that it is undominated. This optimal controller can be separated into a static feedback law and a dynamic disturbance observer. For constant disturbances, it is shown that this structure corresponds to proportional-integral control. This paper was recently submitted for journal publication as:

> F. Farokhi, C. Langbort, K. H. Johansson, "Decentralized Disturbance Accommodation with Limited Plant Model Information," 2011. Submitted.

A preliminary version of this paper was submitted for a conference presentation as:

> F. Farokhi, C. Langbort, K. H. Johansson, "Optimal Disturbance-Accommodation with Limited Model Information," Submitted to the American Control Conference 2012.

CHAPTER 2

Background

In this chapter, we review the available literature on decentralized and distributed control and decision-making with partial information. The primary goals of these reviews are to show the lack of a mathematical framework for studying the optimal control design with limited model information and to present the necessary background for the main results of the thesis.

According to [26], a networked control system is “a spatially distributed systems in which sensors, actuators, and controllers are connected to each other through a band-limited digital communication network”. Figure 2.1 illustrates an example of a networked control system which is composed of several subcontrollers C_i and subsystems P_i connected to each other through a communication network, such as wireless communication network, high-speed connection bus, etc. The network topology shows how different sensors can communicate with different subcontrollers and how these subcontrollers relay back their commands to the corresponding actuators.

Networked control systems have several characteristics. First, these systems are typically distributed geographically over a vast area like the motivating power grid application in Chapter 1. It is natural to assume that a given subsystem can only influence a strict subset of neighboring subsystems (due to the geographical constraints). Therefore, the geographical profile of the system and its underlying physical characteristics dictate the interconnection pattern between subsystems. In many situations, the interconnections of the subsystems are fixed (and given) in advance. This property of large-scale control system has attracted a lot of attention through the time and many have studied the generic properties of structured systems. We take a deeper look into structured systems in Section 2.1.

Second, any communication medium brings limitations, such as band-limited channels, sampling and quantization issues, variable delays, packet drop-outs, etc.

A realistic communication network has band-limited channels, that is, it can only relay a limited amount of data per unit of time. Therefore, it might not make sense to assume in designing each subcontroller that the subcontroller has access to the full state measurements of the plant. Note that even if each channel has high bandwidth, the point-to-point capacity of a large multi-hop network can still be very limited [27]. The absence of full state information gives rise to several challenges in designing stabilizing and optimal controllers which we discuss in Section 2.2.

Finally, in large-scale dynamical systems, it may be extremely difficult (perhaps impossible) to identify all system parameters and update them globally. One can only hope that the designer knows the local parameter variations and update the corresponding subcontroller based on them. This fact motivates optimal control design with limited model information. We briefly review the literature on this problem in Section 2.4.

The rest of the chapter is organized as follows. We begin by introducing the generic properties of structured systems in Section 2.1. In Section 2.2, we present an overview of the literature on decentralized and distributed control design. In Section 2.3, we briefly review decision-making with partial information. We summarize some of the recent attempts in control design with limited model information in Section 2.4.

2.1 Generic Properties of Structured Systems

The study of structured systems dates back almost four decades [28–32]. In [28], the author first introduced the definition that a pair of matrices (A, B) is structurally controllable if there exists a controllable pair of matrices (A', B') with the same structure as (A, B). A structurally controllable system can be shown to be controllable for almost all parameter combinations, except for some cases with zero measure that might occur when the system parameters satisfy certain equality constraints [28–30]. Thus, the structural controllability helps the designer to overcome the inherently incomplete knowledge of the system parameters. There exist graph theoretic conditions for verifying structured controllability [28]. A set of algebraic conditions has been presented in [29, 31] to check structured controllability. It is interesting to note that, as structured controllability gives controllability of a continuum of linearized systems, the aforementioned results may also provide a sufficient condition for controllability of many nonlinear systems [33–35].

Many classical control results were generalized to structured systems. For instance, the problem of input–output decoupling of structured systems has been discussed in [36–38]. The problem of disturbance rejection and disturbance decoupling was addressed initially in [39–41]. Decentralized control of structured systems was considered in [42–45]. For instance, the authors in [42] presented necessary and sufficient conditions for controllability under a decentralized information structure. In [43], the authors studied geometric properties of structured systems using graph-theoretic tools. They also obtained graph-theoretic conditions used to determine

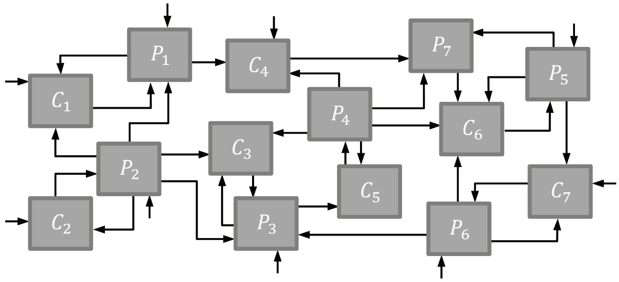

Figure 2.1: Illustrative example of a networked control system.

stabilizability of structured interconnected systems via decentralized feedback control. The decentralized stabilization and pole placement of structured system has been discussed in [46]. Parts of these results were also generalized to descriptor systems in [47]. More related studies can be found in a recent survey of structured systems and their generic properties [48]. There has been also some work in fault detection and isolation for structured systems. For instance, in [49], the authors provided necessary and sufficient graph-theoretic conditions under which the fault detection and isolation problem has a solution. Later, the sensor location problem for fault diagnosis in structured systems was discussed in [50]. Recently, a necessary and sufficient graph-theoretic condition for the existence of vulnerabilities that are inherent to the power network interconnection structure has been developed in [51].

2.2 Distributed and Decentralized Control Design

Band-limited channels in a networked control system force us to design distributed and decentralized controller as subcontrollers in the overall system might have access only to a strict subset of the state measurements. Distributed and decentralized control and estimation in large-scale and networked systems is a well-studied problem [52–55].

There is a huge body of literature on stabilizing decentralized systems. For instance, the authors of [56–59] showed that the absence of so-called fixed modes is a necessary and sufficient condition for stabilizability of a linear time-invariant dynamical system with a time-invariant decentralized controller. Later, this result was extended to show that a time-varying controller might be able to eliminate the fixed modes that are not structurally fixed modes and as a result, a linear time-invariant dynamical system could be stabilized with a decentralized controller even when fixed modes are present [60, 61]. Fixed modes can also be eliminated with vibrational control or sampling techniques [62–64]. It has also been shown that if a fixed mode cannot be eliminated by a decentralized periodically time-varying controller, then it cannot be eliminated by any decentralized controller [65, 66].

There are contributions in multi-agent systems related to distributed control, such as a Nyquist-like condition for stability of a formation using the individual plant transfer function and the Laplacian of the graph describing the network topology [67]. This work has been generalized to the stability of multi-input multi-output dynamical systems with arbitrary dynamical interconnection between the subsystems with fixed interaction topology [68]. The coordination of a group of autonomous agents when the graph topology changes over time has been considered in [69, 70]. These works were generalized to a framework for stability analysis of interconnected systems where the topology can potentially be time-varying [71]. The authors in [72] presented an algorithm for designing controllers that preserve the stability of the closed-loop system under any interconnection and communication typology. Recently, a simple method for the design of decentralized stabilizing controllers for large-scale interconnected systems has been proposed in [73].

There has been a great effort in designing optimal distributed and decentralized controller. Witsenhausen showed that a linear controller is not optimal for a quadratic performance criterion with a linear time-invariant system subject to Gaussian noise under the distributed information constraint in general and the cost function is not necessarily convex in the controller variables [74]. The authors in [75, 76] established that the discrete-time version of the Witsenhausen counter-example is NP-complete. There has been some effort also to identify the cases where a linear solution is optimal. For instance, Witsenhausen identified some cases where the resulting optimal controller were linear [77]. The authors in [78] showed that under a partially nested information pattern the optimal controller is a linear controller. It has been shown in [79] that the optimal controller is linear, if each subcontroller has access to all the previously implemented control values and observations made by any other subsystem in the system before the current time and its own observations including the current time. There were some studies under the spatial invariance assumption [80, 81]. Some other control structures were shown to result in optimal linear controllers [82, 83]. In [84], the author presented a solution to the optimal decentralized state-feedback control design problem for partially nested information structure. Recently, it has been shown that under the quadratic invariance and internal quadratic invariance information patterns, one can formulate structured H_∞- and H_2-optimal control design as convex optimization problems [85–88]. This formulation results in an explicit solution for the problem of designing decentralized H_2-optimal controllers for a spacial class of systems [89–91]. The authors in [92–94] also using the partially ordered sets introduced an explicit solution to the decentralized state-feedback H_2-optimal control design problem for some classes of plant interconnection and information structure. The problem of designing optimal distributed controllers was recently approached using team decision theory in [95, 96]. This work was further generalized to solve the stochastic linear quadratic control problem under power constraints [97]. In this work, the output-feedback problem is also considered. Later, the team decision theory was used to develop optimal distributed H_∞-optimal controllers when each subsystems has access to the state measurements and control signals of those subsystems that can affect it [98].

There has been studies on designing optimal controller for positive systems with more general structures. For instance, the authors in [99, 100] gave a necessary and sufficient condition for existence of a diagonal Lyapunov function for positive systems. They also showed that, in this case, an H_∞-optimal control design problem can be written as a convex optimization problem (and therefore it is computationally tractable). Later, the author in [101] proved that for positive discrete-time linear time-invariant systems the H_∞- and ℓ_1-norms are equal to each other. It was also shown that the problem of designing an optimal controller for these systems can be written as a convex optimization problem under some conditions on the controller structure.

There has been studies on sub-optimal distributed and decentralized control design because, as it was mentioned earlier, the problem of synthesizing these controllers for arbitrary information patterns is NP-complete. The authors in [102] considered the problem of designing sub-optimal static and fixed-order dynamic structured compensators. Some approaches were based on gradient descent, Newton, and quasi-Newton algorithms [103–108]. A set of sufficient Linear Matrix Inequalities for finding distributed controllers was presented in [109]. In [110], the authors presented an algorithm for designing a near-optimal decentralized controller that replicates the behavior of the optimal centralized controller. The problem of near-optimal decentralized output regulation of hierarchical systems subject to disturbances has been studied in [111]. In [112, 113], the problem of designing an optimal decentralized state-feedback controller has been solved on a finite-horizon using dynamic programming. In these papers, the authors provided both a computationally intensive optimal solution and a sub-optimal solution that is computationally more tractable. A receding horizon approach to develop a sub-optimal controller was considered in [8, 114]. A recent result was introduced in [115] using decomposition methods in distributed optimization accompanied with a special stopping criteria to synthesize a sub-optimal controller with closed-loop performance guarantees.

2.3 Decision-Making with Partial Information

The problem of decision-making with incomplete information and the value of information is a well-studied problem in economics [9, 116]. For instance, in [116], the author studied the degradation in economic decisions caused by the lack of information and communication between both competing and cooperating agents. He also gave an estimate of the value of information in a network using this degradation factors. In [117, 118], the value of information in distributed algorithmic decision-making has been studied. The value of information was captured using the competitive ratio [11, 119] which was defined based on the so-called regret ratio in economics [9]. The authors in [12] studied the standard linear programming problem when each agent just knows a restricted subset of constraint coefficients. They motivated this problem using distributed decision-making in network management (see also [120, 121]), distributed task assignment problem, and organization theory. The problem has been generalized to dynamic cases in [122, 123]. These studies are the origin of the definitions in this thesis on competitive ratio and domination for decentralized control design with limited plant model information. The theory of competitive analysis of distributed algorithms was used later to compare the cost of a distributed on-line algorithm to the cost of an optimal distributed algorithm to study the performance of online distributed algorithms in [124]. The authors in [125, 126] discussed a settings where several agents jointly solve a coordination game and studied the value of information in these games. The linear programming problem with privacy constraints was later discussed in [127].

2.4 Limited Model Information Control Design

The problem of designing controllers using uncertain plant model information is a classical topic in control theory [128–133]. In robust control design, the goal is to design a controller such that some level of performance of the controlled system is guaranteed irrespective of changes in the plant dynamics within a predefined bound around a given nominal global model. This is different from designing an optimal controller without a global model since in optimal control design with limited model information, subsystems do not have any prior information about the other subsystems' model (i.e., there are no nominal model for the subsystems in the design procedure) and there are no, apriori known, bound on the model uncertainties.

There has been some interesting approaches for tackling limited model information control design problem, although not specifically tailored for it. For instance, references [134–137] introduced methods for designing sub-optimal decentralized controllers without a global dynamical model of the system. In these papers, the authors assume that the plant consists of an interconnection of weakly coupled subsystems. They design an optimal controller for each subsystem using only the corresponding local model, and connect the obtained subcontrollers to construct a global controller. They show that, when the coupling is negligible, this latter controller is satisfactory in terms of closed-loop stability and performance. However, as coupling strength increases, even closed-loop stability guarantees are lost. The motivation behind their studies was to design fully-decentralized near-optimal controllers for large-scale dynamical systems and to avoid numerical complications, stemming from the high dimension of the system, by splitting the original problem into several smaller ones. Other approaches such as [4, 8] are based on receding horizon control and use decomposition methods to solve each step's optimization problem in a decentralized manner with only limited information exchange between subsystems.

As one can see, what is missing from the literature is a rigorous characterization of the best closed-loop performance that can be attained through limited model information design and, a study of the trade-off between the closed-loop performance and the amount of exchanged information. In this thesis, our goal is to introduce a mathematical framework to try to partially fill this gap and also to study the described trade-off.

The problem of designing an optimal controller with limited model information, in the current setup, was first approached in [122, 123]. In these papers, the authors introduced control design strategies as mappings from the set of plants of interest to the set of eligible controllers. They investigated the quality of the controllers that these control design strategies construct. This quality was measured by a quadratic closed-loop performance criterion. To do so, they introduced competitive ratio as a performance metric and the domination as a partial order on the set of limited model information control design strategies, to study the intrinsic limitations of limited model information control design strategies. Previously, there were no other metrics specifically proposed for control design strategies. The authors defined the competitive ratio as the worst case ratio of the cost of a control design strategy to the cost of the optimal control design with full model information, similar to Section 1.3. They worked with communication-less control design strategies as an extreme family of limited model information control design strategies that only rely on each subsystem model for designing the corresponding subcontroller. They used the term communication-less to illustrate the fact that different parts of these control design strategies would not exchange model information (and equivalently would not communicate) with each other. The subsystems were assumed to be scalar. Under these assumptions, it was proved that, when dealing with continuous-time linear time-invariant dynamical systems, the competitive ratio of any control design strategy is always unbounded. Thus, they focused on discrete-time linear time-invariant systems and found an explicit minimizer of the competitive ratio over the set of limited model information control design strategies. Since this minimizer might not be unique, they also proved that it is undominated, that is, there is no other control design method that acts always better while having the same worst-case ratio. This undominated minimizer of the competitive ratio was the deadbeat control design strategy. Towards the end, they briefly studied the amount of information needed to find a control design strategy with a lower competitive ratio than the deadbeat control design strategy or to dominate it.

The results presented in the papers of this thesis considerably extend the contributions of [122, 123], but build on the same framework. We consider limited model information control design for interconnections of fully-actuated discrete-time linear time-invariant subsystems (of arbitrary order) with a quadratic separable cost function [138, 139]. We investigate the best closed-loop performance achievable by structured static state-feedback controllers constructed by limited model information design strategies. We show that the result depends crucially on the subsystems interconnection pattern and state measurement availability (i.e., the plant graph and control graph). We extend the fact proven in [122] that the deadbeat strategy is the best limited model information control design method when there is no subsystem that cannot affect any other subsystem and each subcontroller has access to at least the state measurements of those subsystems that affect it. However, the deadbeat control design strategy is dominated when there is a subsystem that could not affect any other subsystem. We find a better, undominated, limited model information control design method, which, although having the same competitive ratio as the deadbeat control design strategy, can achieve a better closed-loop performance in average. We also characterize the amount of model information needed to achieve a better competitive ratio than the deadbeat control design strategy. In [140], we generalize these results to structured dynamic state-feedback controllers when the closed-loop performance criterion is H_2-norm of the closed-loop transfer function. Surprisingly, the optimal control design strategy (in the sense of competitive ratio) with limited model information is a static one. This is case even though the optimal decentralized state-feedback controller with full model information is dynamic itself [90, 91]. We also partially remove the assumption that all the subsystems are fully-actuated and generalize the result for a class of under-actuated systems where the sinks (in the plant graph) are not required to be fully-actuated. Later, we also discuss the design of dynamic controllers for disturbance accommodation in [141, 142]. This problem is of special interest because of the fact that the best limited model information control design is a dynamic control design strategy contrary to all previous results where the best limited model information control design strategy was a static one. This dynamic control design strategy can be divided into two parts; a static part which was previously introduced in [138–140] and an observer for canceling the disturbances. For constant disturbances, it is shown that this structure corresponds to proportional-integral control [142].

CHAPTER 3

Conclusions and Future Work

In this chapter, we present a short summary of the contributions of the thesis, and the directions for future work.

In Paper 1, we presented a framework for the study of optimal control design under limited model information, and investigated the connection between the quality of controllers produced by a design method and the amount of plant model information available to it. This is mathematically done for a set of discrete-time linear time-invariant plants under a separable quadratic performance measure with structured static state-feedback controllers. We showed that the best performance achievable by a limited model information control design method crucially depends on the structure of the plant graph and, thus, that giving the designer access to this graph, even without a detailed model of all plant subsystems, results in superior design, in the sense of domination.

In Paper 2, we considered optimal H_2 dynamic control design for interconnected linear systems under limited plant model information. We found an explicit undominated minimizer of the competitive ratio for a large class of system interconnections, controller structure, and design information. It was also shown when it comes to designing optimal centralized or partially structured decentralized state-feedback controllers with limited model information, the best control design strategy (in terms of competitive ratio) is a static one. This is true even though the optimal structured decentralized state-feedback controller with full model information is dynamic. We were also able to relax the assumption that all the subsystems are fully-actuated for sinks in the plant graph.

In Paper 3, we studied the design of optimal disturbance accommodation controllers with limited model information. We adapted the notion of limited model information control design strategies to handle disturbance accommodation to study the cases where the best limited model information control design is a dynamic control design strategy. We found an explicit minimizer of the competitive ratio and we showed that it is undominated. We split this optimal control design strategy into a static part for regulating the state of the systems and a deadbeat observer for canceling the disturbance effect. There are several directions to further expand the work presented in the thesis. We list some of these directions below.

In most of the current results, we have the assumption that each subsystem is fully-actuated, with as many inputs as states, and controllable in one time-step; i.e., all matrices $B \in \mathcal{B}$ are square invertible. This seems to be the most restrictive assumption on the results. We are able to relax this assumption for the sinks in the plant graph, that is, the sinks of the plant graph can be under-actuated. In future, we can study the possibility of relaxing this assumption for more cases. This can be done with several approaches. For instance, one approach is to define the competitive ratio based on only a compact set of plants with possibly pre-defined bounds on the value of the parameters of each subsystem. As some of the issues concerning the under-actuated subsystems arise when the parameters becomes large enough, this might help us to relax the assumption that all matrices B are square invertible. Another approach is to focus on under-actuated subsystems that are capable of decoupling themselves from other subsystems in one time-step. For instance, if subsystem i, in addition to its current model information, have access to $\{A_{ji} | j \neq i\}$, it can always hide inside the unobservable subspace of these matrices; i.e., we can design and implement a subcontroller that makes this unobservable subspace both an invariant subspace and a reachable subspace within one time-step. Thus, from other subsystems perspective, such a subsystem would behave like a fully-actuated subsystem in feedback with the deadbeat control design strategy. Although mathematically interesting, unfortunately, as the number of other subsystems increases the probability that such a subspace is non-trivial is very slim. This is troublesome as the only controller that can make the origin both an invariant subspace and a reachable subspace within one time-step is the deadbeat control design strategy (and it requires B to be a square invertible matrix).

The current results only hold for separable closed-loop performance measures. It might be interesting to see what sort of assumptions are required to generalize these results to the case that behavior of the subsystems are linked to each other through the cost function. In case that the matrix R is block diagonal but not the matrix Q, similar to the platooning problem in Example 1.4, one might be able to transform the problem into an optimal control design problem with separable closed-loop performance measures, and keep the local controllers as a function of only local parameters. In this case, the results of this thesis might be extended, but, in general, the solution to his problem might not be that straight-forward. We can also look at the general problem of designing optimal controller with limited model information for other closed-loop performance measures like H_∞- or ℓ_1-norm of the closed-loop transfer function.

Another extension is to let designers communicate with each other and explore the question what is a good way of signaling to other subsystems without revealing all the parameters, for instance, by finding the minimum amount of information needed for designing the optimal controller (e.g., see [143–145] for such bounds on distributed computation).

In some cases, it is beneficial to define an average competitive ratio and to guarantee the performance of the closed-loop system with a given probability. The current definition of the competitive ratio using the worst-case behavior might be restrictive as these extreme cases might not happen in reality. For instance, average competitive ratio might be useful in studying the effect of packet drop-outs in a communication network since, in a large-scale system, there are many subsystems and all these subsystems cannot keep track of all the acknowledgement signals. The subsystems could model the information flow pattern based on historical data. Assuming that each subsystem can only gather the acknowledgement signals related to itself and model the rest of the network as a stochastic process, one might search for finding the optimal controller under such a limited information regime, and to guarantee the closed-loop performance with a high-probability (or in average).

Other approaches like using mechanism design, by giving different subsystems financial incentives for sharing their private information (e.g., [146]) in a selfish scenario, might be interesting to investigate as well.

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Part II

Papers

PAPER 1

Optimal Control Design with Limited Model Information

Farhad Farokhi, Cédric Langbort, and Karl H. Johansson

Abstract–We introduce the family of limited model information control design methods, which construct controllers by accessing the plant's model in a constrained way, according to a given design graph. We investigate the achievable closed-loop performance of discrete-time linear time-invariant plants under a separable quadratic cost performance measure with structured static state-feedback controllers. We find the optimal control design strategy (in terms of the competitive ratio and domination metrics) when the control designer has access to the local model information and the global interconnection structure of the plant-to-be-controlled. At last, we study the trade-off between the amount of model information exploited by a control design method and the best closed-loop performance (in terms of the competitive ratio) of controllers it can produce.

An early version of this paper was presented at the American Control Conference, 2011 [1].

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1 Introduction

Many modern control systems, such as aircraft and satellite formation [2, 3], automated highways and other shared infrastructure [4, 5], flexible structures [6], and supply chains [7], consist of a large number of subsystems coupled through their performance goals or dynamics. When regulating this kind of plant, it is often advantageous to adopt a distributed control architecture, in which the controller itself is composed of interconnected subcontrollers, each of which accesses a strict subset of the plant's output. Several control synthesis methods have been proposed over the past decades that result in distributed controllers of this form, with various types of closed-loop stability and performance guarantees (e.g., [8–16]). Most recently, the tools presented in [17] and [18] revealed how to exploit the specific interconnection of classes of plants (the so-called quadratically invariant systems) to formulate convex optimization problems for the design of structured H_∞- and H_2- optimal controllers. A common thread in this part of the literature is the assumption that, even though the controller is structured, its design can be performed in a centralized fashion, with full knowledge of the plant model. However, especially in the case of supply chain and shared infrastructure, this assumption is not always warranted, as the design of each subcontroller may need to be carried out by a different control designer, with no access to the global model of the plant, although its interconnection structure and the common closed-loop cost function to be minimized are public knowledge. This class of problems, which we refer to as "limited model information control design problems", is the main object of interest in the present paper.

Control design based on uncertain plant model information is a classical topic in the robust control literature [19–22]. However, designing an optimal controller without a global model is different from a robust control problem. In optimal control design with limited model information, subsystems do not have any prior information about the other subsystems' model; i.e., there is no nominal model for design procedure and there is no bound on the model uncertainties. There has been some interesting approaches for tackling this problem. For instance, references [23–26] introduced methods for designing sub-optimal decentralized controllers without a global dynamical model of the system. In these papers, the authors assume that the large-scale system to be controlled consists of an interconnection of weakly coupled subsystems. They design an optimal controller for each subsystem using only the corresponding local model, and connect the obtained subcontrollers to construct a global controller. They show that, when coupling is negligible, this latter controller is satisfactory in terms of closed-loop stability and performance. However, as coupling strength increases, even closed-loop stability guarantees are lost. Other approaches such as [5, 7] are based on receding horizon control and use decomposition methods to solve each step's optimization problem in a decentralized manner with only limited information exchange between subsystems. What is missing from the literature, however, is a rigorous characterization of the best closed-loop performance that can be attained through limited model information design and, a study of the trade off between the closed-loop performance and the amount of exchanged information. We tackle this question in the present paper. We are particularly interested in the same applications as [5, 7], namely supply chains and shared infrastructure, which have been shown to be well-modeled by dynamically-coupled but possibly cost-decoupled interconnected systems.

Limited model information control design occurs naturally in this context, since the subsystems often belong to different entities, which may consider their model information private and may thus be reluctant to share it with the others. In this case, the designers may have to resort to "communication-less" strategies in which subcontroller K_i depends solely on the description of subsystem i's model. Another reason for using communication-less strategies in more general design situations, even when the circulation of plant information is not restricted a priori, is that the resulting sub-controller K_i does not need to be modified if the characteristics of a particular subsystem, which is not directly connected to subsystem i, vary.

In this paper, we study the properties of general limited model information control design methods. We investigate the relationship between the amount of plant information available to the designers, the nature of the plant interconnection graph, and the quality (measured by the closed-loop control goal) of controllers that can be constructed using their knowledge. To do so, we look at limited model information and communication-less control design methods as belonging to a special class of maps between the plant and controller sets, and make use of the competitive ratio and domination metrics introduced in [27] to characterize their intrinsic limitations. To the best of our knowledge, there are no other metrics specifically tuned to control design methods. We address much more general classes of subsystems and of limitations on the model information available to the designer than is done in [27].

Specifically, we consider limited model information control design for interconnections of fully-actuated (i.e., with invertible B-matrix) discrete-time linear time-invariant subsystems with quadratic separable cost function. Our choice of such a cost is motivated by our interest in dynamically-coupled but cost-decoupled plants, while our assumption on the B-matrix is a technical assumption which, as we show in the last section of the paper, can be partially removed in some cases. We investigate the best closed-loop performance achievable by *structured* static state feedback controllers constructed by limited model information design strategies. We show that the result depends crucially on the plant graph and the control graph. In the case where the plant graph contains no sink and the control graph is a supergraph of the plant graph, we extend the fact proven in [27] that the deadbeat strategy is the best communication-less control design method. However, the deadbeat control design strategy is dominated when the plant graph has sinks, and we exhibit a better, undominated, communication-less control design method, which, although having the same competitive ratio as the deadbeat control design strategy, takes advantage of the knowledge of the sinks' location to achieve a better closed-loop performance in average. We characterize the amount of model information needed to achieve better competitive ratio than the deadbeat control design strategy. This amount of information is expressed in terms of properties of the design graph; a directed graph which indicates the dependency of each subsystem's controller on different parts of the global dynamical model.

This paper is organized as follows. After formulating the problem of interest and defining the performance metrics in Section 2, we characterize the best communication-less control design method according to both competitive ratio and domination metrics in Section 3. In Section 4, we show that achieving a strictly better competitive ratio than these control design methods requires a complete design graph when the plant graph is itself complete. Finally, we end with a discussion on extensions in Section 5 and the conclusions in Section 6.

1.1 Notation

Sets will be denoted by calligraphic letters, such as $\mathcal{P}$ and $\mathcal{A}$. If $\mathcal{A}$ is a subset of $\mathcal{M}$ then $\mathcal{A}^c$ is the complement of $\mathcal{A}$ in $\mathcal{M}$, i.e., $\mathcal{M} \setminus \mathcal{A}$.

Matrices are denoted by capital roman letters such as A. A_j will denote the j^{th} row of A. A_{ij} denotes a sub-matrix of matrix A, the dimension and the position of which will be defined in the text. The entry in the i^{th} row and the j^{th} column of the matrix A is a_{ij}.

Let S^n_{++} (S^n_{+}) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$. $A > (\geq) 0$ means that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq) B$ means that $A - B > (\geq) 0$.

$\underline{\lambda}(Y)$ and $\bar{\lambda}(Y)$ denote the smallest and the largest eigenvalues of the matrix Y, respectively. Similarly, $\underline{\sigma}(Y)$ and $\bar{\sigma}(Y)$ denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i denotes the column-vector with all entries zero except the i^{th} entry, which is equal to one.

All graphs considered in this paper are directed, possibly with self-loops, with vertex set $\{1, ..., q\}$ for some positive integer q. If $G = (\{1, ..., q\}, E)$ is a directed graph, we say that i is a sink if there does not exist $j \neq i$ such that $(i, j) \in E$. A loop of length t in G is a set of distinct vertices $\{i_1, ..., i_t\}$ such that $(i_t, i_1) \in E$ and $(i_p, i_{p+1}) \in E$ for all $1 \leq p \leq t-1$. We will sometimes refer to this loop as $(i_1 \to i_2 \to \cdots \to i_t \to i_1)$. The adjacency matrix S of graph G is the $q \times q$ matrix whose entries satisfy

$$s_{ij} = \begin{cases} 1 & \text{if } (j, i) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Since the set of vertices is fixed here, a subgraph of G is a graph whose edge set is a subset of the edge set of G and a supergraph of G is a graph of which G is a subgraph. We use the notation $G' \supseteq G$ to indicate that G' is a supergraph of G.

2 Control Design with Limited Model Information

2.1 Plant Model

Let a graph $G_{\mathcal{P}} = (\{1, ..., q\}, E_{\mathcal{P}})$ be given, with adjacency matrix $S_{\mathcal{P}} \in \{0, 1\}^{q \times q}$. We define the following set of matrices associated with $S_{\mathcal{P}}$:

$$\mathcal{A}(S_{\mathcal{P}}) = \{A \in \mathbb{R}^{n \times n} | A_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_{\mathcal{P}})_{ij} = 0\}, \tag{1}$$

where for each $1 \leq i \leq q$, integer number n_i is the dimension of subsystem i. Implicit in these definitions is the fact that $\sum_{i=1}^{q} n_i = n$. Also, for a given scalar $\epsilon > 0$, we let

$$\mathcal{B}(\epsilon) = \{B \in \mathbb{R}^{n \times n} \mid \underline{\sigma}(B) \geq \epsilon, B_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i \neq j \leq q\}. \tag{2}$$

The set $\mathcal{B}(\epsilon)$ defined in (2) is made of invertible block-diagonal square matrices since $\underline{\sigma}(B) \geq \epsilon > 0$ for each matrix $B \in \mathcal{B}(\epsilon) \subseteq \mathbb{R}^{n \times n}$. With these definitions, we can introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant dynamical systems of the form

$$x(k+1) = Ax(k) + Bu(k) \; ; \; x(0) = x_0, \tag{3}$$

with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $x_0 \in \mathbb{R}^n$. Clearly $\mathcal{P}$ is isomorph to $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathbb{R}^n$ and, slightly abusing notation, we will thus identify a plant $P \in \mathcal{P}$ with the corresponding triple (A, B, x_0).

A plant $P \in \mathcal{P}$ can be thought of as the interconnection of q subsystems, with the structure of the interconnection specified by the graph $G_{\mathcal{P}}$ (i.e., subsystem j's output feeds into subsystem i only if $(j, i) \in E_{\mathcal{P}}$). As a consequence, we refer to $G_{\mathcal{P}}$ as the "plant graph". We will denote the ordered set of state indices pertaining to subsystem i as $\mathcal{I}_i$, i.e., $\mathcal{I}_i := (1 + \sum_{j=1}^{i-1} n_j, \ldots, n_i + \sum_{j=1}^{i-1} n_j)$. For subsystem i, state vector and input vector are defined as

$$\underline{x}_i = \begin{bmatrix} x_{\ell_1} & \cdots & x_{\ell_{n_i}} \end{bmatrix}^T, \qquad \underline{u}_i = \begin{bmatrix} u_{\ell_1} & \cdots & u_{\ell_{n_i}} \end{bmatrix}^T$$

where the ordered set of indices $(\ell_1, \ldots, \ell_{n_i}) \equiv \mathcal{I}_i$, and its dynamics is specified by

$$\underline{x}_i(k+1) = \sum_{j=1}^{q} A_{ij} \underline{x}_j(k) + B_{ii} \underline{u}_i(k).$$

According to the specific structure of $\mathcal{B}(\epsilon)$ given in (2), each subsystem is fully-actuated, with as many input as states, and controllable in one time-step. Possible generalization of the results to a (restricted) family of under-actuated systems is discussed in Section 5.

Figure 1(a) shows an example of a plant graph $G_{\mathcal{P}}$. Each node represents a subsystem of the system. For instance, the second subsystem in this example may

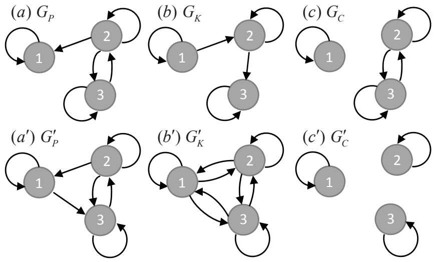

Figure 1: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs, $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs, and $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

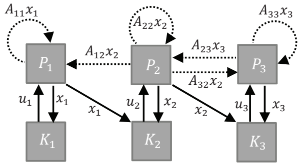

Figure 2: Physical interconnection between different subsystems and controllers corresponding to $G_{\mathcal{P}}$ and $G_{\mathcal{K}}$ in Figures 1(a) and 1(b), respectively.

affect the first subsystem and the third subsystem; i.e., sub-matrices A_{12} and A_{32} can be non-zero. The self-loop for the second subsystem shows that A_{22} may be non-zero. Figure 2 illustrates the corresponding physical interconnection between subsystems of the plant in Figure 1(a) by dotted edges. Note that P_1 in Figure 2 represents a sink of $G_{\mathcal{P}}$ in Figure 1(a). The plant graph $G'_{\mathcal{P}}$ in Figure 1(a') has no sink. The control graph $G_{\mathcal{K}}$ is introduced in the next subsection.

2.2 Controller Model

Let a control graph $G_{\mathcal{K}}$ be given, with adjacency matrix $S_{\mathcal{K}}$. The control laws of interest in this paper are linear static state-feedback control laws of the form

$$u(k) = Kx(k),$$

where

$$K \in \mathcal{K}(S_{\mathcal{K}}) = \{K \in \mathbb{R}^{n \times n} | K_{ij} = 0 \in \mathbb{R}^{n_i \times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}. \tag{4}$$

In particular, when $G_{\mathcal{K}}$ is a complete graph, $\mathcal{K}(S_{\mathcal{K}}) = \mathbb{R}^{n \times n}$, while, if $G_{\mathcal{K}}$ is totally disconnected with self-loops, $\mathcal{K}(S_{\mathcal{K}})$ represents the set of fully-decentralized controllers. When adjacency matrix $S_{\mathcal{K}}$ is not relevant or can be deduced from context, we refer to the set of controllers as $\mathcal{K}$.

An example of a control graph $G_{\mathcal{K}}$ is given in Figure 1(b). Each node represents a subsystem-controller pair of the overall system. For instance, $G_{\mathcal{K}}$ shows that the second subsystem's controller can use state measurements of the first subsystem besides its own state measurements. Solid edges in Figure 2 correspond to the edges of the control graph $G_{\mathcal{K}}$. Figure 1(b') shows $G'_{\mathcal{K}}$ which is a complete graph. This control graph indicates that each subsystem has access to full state measurements of all other subsystems; i.e., $\mathcal{K}(S_{\mathcal{K}}) = \mathbb{R}^{n\times n}$.

2.3 Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}$. Just like plants and controllers, a control design method can exhibit structure which, in turn, can be captured by a design graph. Let a control design method Γ be partitioned according to subsystems dimensions as

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{bmatrix} \tag{5}$$

and a graph $G_{\mathcal{C}} = (\{1, ..., q\}, E_{\mathcal{C}})$ be given, with adjacency matrix $S_{\mathcal{C}}$. Each block Γ_{ij} represents a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathbb{R}^{n_i \times n_j}$. Control design method Γ can be further partitioned in the form

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix},$$

where each γ_{ij} is a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathbb{R}$. We say that Γ has structure $G_{\mathcal{C}}$ if, for all i, the map $[\Gamma_{i1} \ \cdots \ \Gamma_{iq}]$ is only a function of

$$\{[A_{j1} \ \cdots \ A_{jq}], B_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}. \tag{6}$$

In words, a control design method has structure $G_{\mathcal{C}}$ if and only if, for all i, the subcontroller of subsystem i is constructed with knowledge of the plant model of only those subsystems j such that $(j, i) \in E_{\mathcal{C}}$. The set of all control design methods with structure $G_{\mathcal{C}}$ will be denoted by $\mathcal{C}$. In the particular case where $G_{\mathcal{C}}$ is the totally disconnected graph with self-loops (meaning that every node in the graph has a self-loop; i.e, $S_{\mathcal{C}} = I_q$), we say that a control design method in $\mathcal{C}$ is "communication-less", so as to capture the fact that subsystem i's subcontroller is constructed with no information coming from (and, hence, no communication with) any other subsystem j, $j \neq i$. Therefore, the design graph indicates knowledge (or lack thereof) of entire block rows in the aggregate system matrix. When $G_{\mathcal{C}}$ is not a complete graph, we refer to $\Gamma \in \mathcal{C}$ as being "a limited model information control design method".

Note that $\mathcal{C}$ can be considered as a subset of the set of functions from $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon)$ to $\mathcal{K}(S_{\mathcal{K}})$, since a design method with structure $G_{\mathcal{C}}$ is not a function of initial state x_0. Hence, when $\Gamma \in \mathcal{C}$ we will write $\Gamma(A, B)$ instead of $\Gamma(P)$ for plant $P = (A, B, x_0) \in \mathcal{P}$.

An example of a design graph $G_{\mathcal{C}}$ is given in Figure 1(c). Each node represents a subsystem-controller pair of the overall system. For instance, $G_{\mathcal{C}}$ shows that the third subsystem's model is available to the designer of the second subsystem's controller but not the first subsystem's model. Figure 1(c') shows a fully disconnected design graph with self-loops $G'_{\mathcal{C}}$. A local designer in this case can only rely on the model of its corresponding subsystem; i.e., the design strategy is communication-less. Note that Figure 2 does not feature the design graph.

2.4 Performance Metrics

The goal of this paper is to investigate the influence of the plant and design graph on the properties of controllers constructed by limited model information control design methods. To this end, we will use two performance metrics for control design methods. These performance metrics are adapted from the notions of competitive ratio and domination introduced in [27], so as to take plant, controller, and control design structures into account. Following the approach in [27], we start by associating a closed-loop performance criterion to each plant $P = (A, B, x_0) \in \mathcal{P}$ and controller $K \in \mathcal{K}$. As explained in the introduction, we are particularly interested in dynamically-coupled but cost-decoupled systems in this paper, hence, we use a cost of the form

$$J_P(K) = \sum_{k=1}^{\infty} x(k)^T Q x(k) + \sum_{k=0}^{\infty} u(k)^T R u(k), \tag{7}$$

where $Q \in S_{++}^n$ and $R \in S_{++}^n$ are block diagonal matrices, with each diagonal block entry belonging to $S_{++}^{n_i}$. We make the following two standing assumptions:

Assumption 1 $Q = R = I$.

This is without loss of generality because the change of variables $(\bar{x}, \bar{u}) = (Q^{1/2}x, R^{1/2}u)$ transforms the performance criterion and state space representation into

$$J_P(K) = \sum_{k=1}^{\infty} \bar{x}(k)^T \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T \bar{u}(k), \tag{8}$$

and

$$\begin{aligned} \bar{x}(k+1) &= Q^{\frac{1}{2}} A Q^{-\frac{1}{2}} \bar{x}(k) + Q^{\frac{1}{2}} B R^{-\frac{1}{2}} \bar{u}(k) \\ &= \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k), \end{aligned}$$

respectively, without affecting the plant, control, or design graph (due to the block diagonal structure of Q and R).

Assumption 2 *The set of matrices $\mathcal{B}(\epsilon)$ is replaced with the set of diagonal matrices with diagonal entries greater than or equal to ϵ.*

This assumption is without loss of generality. Indeed, consider a plant $P = (A, B, x_0) \in \mathcal{P}$. Every sub-system's B_{ii} matrix has a singular value decomposition $B_{ii} = U_{ii}\Sigma_{ii}V_{ii}^T$ with $\Sigma_{ii} \geq \epsilon I_{n_i \times n_i}$. Combining these singular value decompositions together results in a singular value decomposition for matrix $B = U\Sigma V^T$ where $U = \operatorname{diag}(U_{11}, U_{22}, \cdots, U_{qq})$, $\Sigma = \operatorname{diag}(\Sigma_{11}, \Sigma_{22}, \cdots, \Sigma_{qq})$, and $V = \operatorname{diag}(V_{11}, V_{22}, \cdots, V_{qq})$. Defining $\bar{x}(k) = U^T x(k)$ and $\bar{u}(k) = V^T u(k)$ results in

$$\bar{x}(k+1) = U^T A U \bar{x}(k) + U^T B V \bar{u}(k),$$

where $U^T BV$ is diagonal. Because of the block diagonal structure of matrices U and V, the change of variables $(A, B, x_0) \mapsto (U^T AU, U^T BV, U^T x_0)$ does not affect the plant, control, or design graph. In addition, the cost function becomes

$$\begin{aligned} J_P(K) &= \sum_{k=1}^{\infty} \bar{x}(k)^T U^T U \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T V^T V \bar{u}(k) \\ &= \sum_{k=1}^{\infty} \bar{x}(k)^T \bar{x}(k) + \sum_{k=0}^{\infty} \bar{u}(k)^T \bar{u}(k), \end{aligned}$$

which is of the form (8), because both U and V are unitary matrices. We are now ready to define the performance metrics of interest in this paper.

Definition 1 (Competitive Ratio) *Let a plant graph $G_{\mathcal{P}}$, control graph $G_{\mathcal{K}}$ and constant $\epsilon > 0$ be given. Assume that, for every plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that*

$$J_P(K^*(P)) \leq J_P(K), \ \forall K \in \mathcal{K}.$$

The competitive ratio of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,x_0)\in\mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$" equals one.

Note that the mapping $K^* : P \to K^*(P)$ is not itself required to lie in the set $\mathcal{C}$, as every component of the optimal controller may depend on all entries of the model matrices A and B.

Definition 2 (Domination) *A control design method Γ is said to dominate another control design method Γ' if*

$$J_P(\Gamma(A,B)) \leq J_P(\Gamma'(A,B)), \quad \forall\, P = (A, B, x_0) \in \mathcal{P}, \tag{9}$$

with strict inequality holding for at least one plant in $\mathcal{P}$. When $\Gamma' \in \mathcal{C}$ and no control design method $\Gamma \in \mathcal{C}$ exists that satisfies (9), we say that Γ' is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$.

2.5 Problem Formulation

With the definitions of the previous subsections in hand, we can reformulate the main question of this paper regarding the connection between closed-loop performance, plant structure, and limited model information control design as follows. For a given plant graph, control graph, and design graph, we would like to determine

$$\arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma). \tag{10}$$

Since several design methods may achieve this minimum, we are interested in determining which ones of these strategies are *undominated*.

In [27], this problem was solved in the case when $G_{\mathcal{P}}$ and $G_{\mathcal{K}}$ are complete graphs, $G_{\mathcal{C}}$ is a totally disconnected graph with self-loops (i.e., $S_{\mathcal{C}} = I_q$), and $\mathcal{B}(\epsilon)$ is replaced with singleton $\{I_n\}$. In this paper, we investigate the role of more general plant and design graphs. We also extend the results in [27] for scalar subsystems to subsystems of arbitrary order $n_i \geq 1$, $1 \leq i \leq q$.

3 Plant Graph Influence on Achievable Performance

In this section, we study the relationship between the plant graph and the achievable closed-loop performance in terms of the competitive ratio and domination.

Definition 3 *The deadbeat control design method* $\Gamma^{\Delta} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \rightarrow \mathcal{K}$ *is defined as*

$$\Gamma^{\Delta}(A,B) = -B^{-1}A, \text{ for all } P = (A,B,x_0) \in \mathcal{P}.$$

This control design method is communication-less; i.e., the control design for the subsystem i is a function of the model of the subsystem i only, because subsystem i's controller gain $\left[\Gamma^{\Delta}_{i1}(A,B) \ \cdots \ \Gamma^{\Delta}_{iq}(A,B)\right]$ equals to $B_{ii}^{-1}\left[A_{i1} \ \cdots \ A_{iq}\right]$. The name "deadbeat" comes from the fact that the closed-loop system obtained by applying controller $\Gamma^{\Delta}(A,B)$ to plant $P = (A,B,x_0)$ reaches the origin in just one time-step [28].

Remark 1 *Note that for the case that the control graph $G_{\mathcal{K}}$ is a complete graph; i.e., $\mathcal{K} = \mathbb{R}^{n\times n}$, there exists a controller $K^*(P)$ satisfying the assumptions of Definition 1 for all $P \in \mathcal{P}$, namely, the optimal linear quadratic regulator which is independent of the initial condition of the plant. For incomplete control graphs, the optimal control design strategy $K^*(P)$ (if exists) might become a function of the initial condition [29]. Hence, we will use $K^*(A,B)$ instead of $K^*(P)$ when the control graph $G_{\mathcal{K}}$ is a complete graph for each plant $P = (A,B,x_0) \in \mathcal{P}$ to emphasize this fact.*

Form Definition 1, the notation $K^*(P)$ is reserved for the optimal control design strategy for any given control graph $G_{\mathcal{K}}$. In the particular case, we use the notation $K^*_C(A,B)$ to denote the *unstructured* optimal control design strategy (i.e., $G_{\mathcal{K}}$ is a complete graph).

Lemma 1 *Let the control graph $G_{\mathcal{K}}$ be a complete graph. The cost of the optimal control design strategy K^* is lower-bounded by*

$$J_P(K^*(A,B)) \geq \left(\frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B)+1}\right) J_P(\Gamma^\Delta(A,B)),$$

for all plants $P = (A, B, x_0) \in \mathcal{P}$.

Proof: See Appendix A. ■

Theorem 2 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then the competitive ratio of the deadbeat control design method Γ^Δ is*

$$r_{\mathcal{P}}(\Gamma^\Delta) = 1 + 1/\epsilon^2.$$

Proof: Irrespective of the control graph $G_{\mathcal{K}}$ and for all plants $P \in \mathcal{P}$, it is true that $J_P(K_C^*(A,B)) \leq J_P(K^*(P))$. Therefore, we get

$$\frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(P))} \leq \frac{J_P(\Gamma^\Delta(A,B))}{J_P(K_C^*(A,B))}. \tag{11}$$

Now, using Lemma 1, we know that

$$\frac{J_P(\Gamma^\Delta(A,B))}{J_P(K_C^*(A,B))} \leq 1 + \frac{1}{\underline{\sigma}^2(B)}, \tag{12}$$

for all $P = (A, B, x_0) \in \mathcal{P}$. Combining (12) and (11) results in

$$r_{\mathcal{P}}(\Gamma^\Delta) = \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma^\Delta(A,B))}{J_P(K^*(P))} \leq 1 + \frac{1}{\epsilon^2}.$$

To show that this upper bound is attained, let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ where $1 \leq i \neq j \leq q$ and $(s_{\mathcal{P}})_{ij} \neq 0$ (such indices i and j exist because plant graph $G_{\mathcal{P}}$ has no isolated node by assumption). Consider the system $A = e_{i_1} e_{j_1}^T$ and $B = \epsilon I$. The unique positive definite solution of the discrete algebraic Riccati equation

$$A^T X A - A^T X B (I + B^T X B)^{-1} B^T X A = X - I, \tag{13}$$

is $X = I + [1/(1+\epsilon^2)] e_{j_1} e_{j_1}^T$. Consequently, the centralized controller $K_C^*(A,B) = -\epsilon/(1+\epsilon^2) e_{i_1} e_{j_1}^T$ belongs to the set $\mathcal{K}(S_{\mathcal{K}})$ because $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Thus, we get

$$J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) \leq J_{(A,B,e_{j_1})}(K_C^*(A,B)) \tag{14}$$

since $K^*(P)$ has a lower cost than any other controller in $\mathcal{K}(S_{\mathcal{K}})$. On the other hand, it is evident that

$$J_{(A,B,e_{j_1})}(K_C^*(A,B)) \leq J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) \tag{15}$$

because the centralized controller has access to more state measurements. Using (14) and (15) simultaneously results in

$$\begin{aligned} J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) &= J_{(A,B,e_{j_1})}(K_C^*(A,B)) \\ &= 1/(1+\epsilon^2). \end{aligned}$$

On the other hand $\Gamma^\Delta(A,B) = -[1/\epsilon]e_{i_1}e_{j_1}^T$ and $J_{(A,B,e_{j_1})}(\Gamma^\Delta(A,B)) = 1/\epsilon^2$. Therefore, $r_{\mathcal{P}}(\Gamma^\Delta) = 1 + 1/\epsilon^2$. ■

Remark 2 *Consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 1(a) and the control graph $G'_{\mathcal{K}}$ in Figure 1(b′). Theorem 2 shows that, if we apply the deadbeat control design strategy to this particular problem, the performance of the deadbeat control design strategy, at most, can be $1 + 1/\epsilon^2$ times the cost of the optimal control design strategy K^*.*

Remark 3 *There is no loss of generality in assuming that there is no isolated node in the plant graph $G_{\mathcal{P}}$, since it is always possible to design a controller for an isolated subsystem without any model information about the other subsystems and without impacting cost (7). In particular, this implies that there are $q \geq 2$ vertices in the graph because for $q = 1$ the only subsystem that exists is an isolated node in the plant graph.*

Remark 4 *For implementation of the deadbeat control design strategy in each node, we only need the state measurements of the neighbors of that node. For the implementation of the optimal control design strategy K^* when the control graph has many more links than the plant graph, the controller gain $K^*(P)$ is not necessarily a sparse matrix.*

With this characterization of Γ^Δ in hand, we are now ready to tackle problem (10).

3.1 First case: plant graph $G_{\mathcal{P}}$ with no sink

In this subsection, we show that the deadbeat control method Γ^Δ is undominated by communication-less control design methods for plants in $\mathcal{P}$, when $G_{\mathcal{P}}$ contains no sink. We also show that Γ^Δ exhibits the smallest possible competitive ratio among such control design methods.

First, we state the following two lemmas.

Lemma 3 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. A control design method $\Gamma \in \mathcal{C}$ has bounded competitive ratio only if the following implication holds for all $1 \leq i \leq q$ and all j:*

$$a_{\ell j} = 0 \text{ for all } \ell \in \mathcal{I}_i \Rightarrow \gamma_{\ell j}(A,B) = 0 \text{ for all } \ell \in \mathcal{I}_i,$$

where $\mathcal{I}_i$ is the set of indices related to subsystem i; i.e., $\mathcal{I}_i = (1+\sum_{z=1}^{i-1} n_z, \dots, n_i + \sum_{z=1}^{i-1} n_z)$.

Proof: See Appendix B. ■

Lemma 4 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Assume the plant graph $G_{\mathcal{P}}$ has at least one loop. Then,*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2 \tag{16}$$

for all limited model information control design method Γ in $\mathcal{C}$.

Proof: See Appendix C. ■

Using these two lemmas, we are ready to state and prove one of the main theorems in this paper and, as a result, find the solution to problem (10) when the plant graph $G_{\mathcal{P}}$ contains no sink.

Theorem 5 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and no sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2.$$

Proof: From Lemma 1.4.23 in [30], we know that a directed graph with no sink must have at least one loop. Hence $G_{\mathcal{P}}$ must contain a loop. The result then follows from Lemma 4. ■

Remark 5 *Theorem 5 shows that $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Delta})$ for any control design strategy $\Gamma \in \mathcal{C}$, and as a result the deadbeat control design method Γ^{Δ} becomes a minimizer of the competitive ratio function $r_{\mathcal{P}}$ over the set of communication-less design methods.*

We now turn our attention to domination properties of the deadbeat control design strategy.

Lemma 6 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. The deadbeat control design strategy Γ^{Δ} is undominated, if there is no sink in the plant graph $G_{\mathcal{P}}$.*

Proof: See Appendix D. ■

The following theorem shows that the deadbeat control design strategy is undominated by communication-less design methods if and only if the plant graph $G_{\mathcal{P}}$ has no sink. It thus provides a good trade-off between worst-case and average performance.

Theorem 7 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then the deadbeat control design method Γ^{Δ} is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$ if and only if the plant graph $G_{\mathcal{P}}$ has no sink.*

Proof: Proof of the "if" part of the theorem, is given by Lemma 6.

For ease of notation in this proof, we use $[\Gamma]_i = [\Gamma_{i1} \ \cdots \ \Gamma_{iq}]$ and $[A]_i = [A_{i1} \ \cdots \ A_{iq}]$.

In order to prove the "only if" part of the theorem, we need to show that if the plant graph has a sink (i.e., if there exists j such that $(s_{\mathcal{P}})_{ij} = 0$ for every $i \neq j$), then there exists a control design method Γ which dominates the deadbeat control design method. We exhibit such a strategy.

Without loss of generality, we can assume that $(s_{\mathcal{P}})_{iq} = 0$ for all $i \neq q$, in which case every matrix A in $\mathcal{A}(S_{\mathcal{P}})$ has the structure

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1,q-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{q-1,1} & \cdots & A_{q-1,q-1} & 0 \\ A_{q1} & \cdots & A_{q,q-1} & A_{qq} \end{bmatrix}.$$

Define $\bar{x}_0 = [\ x_1(0) \quad \cdots \quad x_{q-1}(0)\ ]^T$, and let control design strategy Γ be defined by

$$\begin{bmatrix} -B_{11}^{-1}A_{11} & \cdots & -B_{11}^{-1}A_{1,q-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -B_{q-1,q-1}^{-1}A_{q-1,1} & \cdots & -B_{q-1,q-1}^{-1}A_{q-1,q-1} & 0 \\ K_{q1}(A,B) & \cdots & K_{q,q-1}(A,B) & K_{qq}(A,B) \end{bmatrix}$$

for all $P = (A, B, x_0) \in \mathcal{P}$, with

$$\begin{aligned} \bar{K}(A,B) : &= [\ K_{q1}(A,B) \quad \cdots \quad K_{q,q-1}(A,B) \quad K_{qq}(A,B)\ ] \\ &= -(I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq} [A]_q, \end{aligned}$$

where X_{qq} is the unique positive definite solution to the discrete algebraic Riccati equation

$$A_{qq}^T X_{qq} B_{qq} (I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq} A_{qq} - A_{qq}^T X_{qq} A_{qq} + X_{qq} - I = 0. \qquad (17)$$

In words, control design strategy Γ applies the deadbeat strategy to subsystems 1 to $q-1$ while, on subsystem q, it uses the same sub-controller as in the optimal controller for the plant

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}\hat{u}(k), \qquad (18)$$

with cost function

$$J^{(2)}_{(A,B,x_0)}(\bar{K}) = \sum_{k=1}^{\infty} \hat{x}(k)^T Q \hat{x}(k) + \sum_{k=0}^{\infty} \hat{u}(k)^T \hat{u}(k),$$

where $Q = \mathrm{diag}(0, \ldots, 0, I_{n_q \times n_q})$, the matrix $\hat{A}$ is defined as $[\hat{A}]_q = [A]_q$ and $[\hat{A}]_z = 0$ for all $z \neq q$, and furthermore, the matrix $\hat{B}$ is defined as $\hat{B} = \mathrm{diag}(0, \ldots, 0, B_{qq})$. Note that Γ is indeed communication-less since $\bar{K}(A, B)$ defined above can be computed with the sole knowledge of the q^{th} lower block of A and B. Because of the structure of matrices in $\mathcal{A}(S_{\mathcal{P}})$ and this characterization of Γ, we have

$$J_{(A,B,x_0)}(\Gamma(A, B)) = J^{(1)}_{(A,B,x_0)} + J^{(2)}_{(A,B,x_0)}(\bar{K}(A, B)),$$

where $J^{(1)}_{(A,B,x_0)} = \bar{x}_0^T \bar{A}^T \bar{B}^{-T} \bar{B}^{-1} \bar{A} \bar{x}_0$, with

$$\bar{A} = \begin{bmatrix} A_{11} & \cdots & A_{1,q-1} \\ \vdots & \ddots & \vdots \\ A_{q-1,1} & \cdots & A_{q-1,q-1} \end{bmatrix},$$

and $\bar{B} = \mathrm{diag}(B_{11}, \ldots, B_{q-1,q-1})$ and $J^{(2)}_{(A,B,x_0)}(\bar{K}(A, B))$ is the closed-loop cost for system (18). Since $\bar{K}(A, B)$ is the optimal controller for this cost, $J^{(2)}_{(A,B,x_0)}(\bar{K}(A, B)) = x_0^T \hat{A}^T W \hat{A} x_0$, where

$$W = \mathrm{diag}(0, \ldots, 0, X_{qq} - X_{qq} B_{qq} (I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq}).$$

Using part 2 of Subsection 3.5.2 in [31], we have the matrix inversion identity

$$X - XY(I + ZXY)^{-1} ZX = (X^{-1} + YZ)^{-1},$$

which results in

$$\begin{aligned} W_{qq} &= X_{qq} - X_{qq} B_{qq} (I + B_{qq}^T X_{qq} B_{qq})^{-1} B_{qq}^T X_{qq} \\ &= (X_{qq}^{-1} + B_{qq} B_{qq}^T)^{-1} \\ &< B_{qq}^{-T} B_{qq}^{-1}. \end{aligned}$$

Note that X_{qq}^{-1} exists because $X_{qq} \geq I$ which follows from the discrete algebraic Riccati equation in (17). This inequality implies that

$$\hat{A}^T W \hat{A} < \hat{A}^T (\hat{B}^\dagger)^T \hat{B}^\dagger \hat{A}$$

where $\hat{B}^\dagger = \mathrm{diag}(0, \ldots, 0, B_{qq}^{-1})$. Thus

$$\begin{aligned} J_{(A,B,x_0)}(\Gamma(A, B)) &= J^{(1)}_{(A,B,x_0)} + J^{(2)}_{(A,B,x_0)}(\bar{K}(A, B)) \\ &< J_{(A,B,x_0)}(\Gamma^\Delta(A, B)), \end{aligned}$$

for all $P = (A, B, x_0) \in \mathcal{P}$ such that the q^{th} lower block of A is not zero, unless the $J_{(A,B,x_0)}(\Gamma(A, B)) = J_{(A,B,x_0)}(\Gamma^\Delta(A, B))$. Thus, control design method Γ dominates the deadbeat control design method Γ^Δ. ∎

Remark 6 *Consider the limited model information design problem given by the plant graph $G'_{\mathcal{P}}$ in Figure 1(a′), the control graph $G'_{\mathcal{K}}$ in Figure 1(b′), and the design graph $G'_{\mathcal{C}}$ in Figure 1(c′). Theorems 5 and 7 show that the deadbeat control design strategy Γ^{Δ} is the best control design strategy that one can propose based on the local model of subsystems and the plant graph, because the deadbeat control design strategy is the minimizer of the competitive ratio and it is undominated.*

Remark 7 *It should be noted that, the proof of the "only if" part of the Theorem 7 is constructive. We use this construction to build a control design strategy for the plant graphs with sinks in next subsection.*

3.2 Second case: plant graph $G_{\mathcal{P}}$ with at least one sink

In this section, we consider the case where plant graph $G_{\mathcal{P}}$ has $c \geq 1$ sinks. Accordingly, its adjacency matrix $S_{\mathcal{P}}$ is of the form

$$S_{\mathcal{P}} = \left[\begin{array}{c|c} (S_{\mathcal{P}})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_{\mathcal{P}})_{21} & (S_{\mathcal{P}})_{22} \end{array}\right], \tag{19}$$

where

$$(S_{\mathcal{P}})_{11} = \begin{bmatrix} (s_{\mathcal{P}})_{11} & \cdots & (s_{\mathcal{P}})_{1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q-c,1} & \cdots & (s_{\mathcal{P}})_{q-c,q-c} \end{bmatrix},$$

$$(S_{\mathcal{P}})_{21} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,1} & \cdots & (s_{\mathcal{P}})_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q,1} & \cdots & (s_{\mathcal{P}})_{q,q-c} \end{bmatrix},$$

and

$$(S_{\mathcal{P}})_{22} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,q-c+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (s_{\mathcal{P}})_{qq} \end{bmatrix},$$

where we assume, without loss of generality, that the vertices are numbered such that the sinks are labeled $q-c+1, \ldots, q$. With this notation, let us now introduce the control design method Γ^{Θ} defined by

$$\Gamma^{\Theta}(A,B) = -\mathrm{diag}(B_{11}^{-1}, \ldots, B_{q-c,q-c}^{-1}, W_{q-c+1}(A,B), \ldots, W_q(A,B))A \tag{20}$$

for all $(A,B) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon)$, where

$$W_i(A,B) = (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} \tag{21}$$

for all $q-c+1 \leq i \leq q$ and X_{ii} is the unique positive definite solution of the discrete algebraic Riccati equation

$$A_{ii}^T X_{ii} B_{ii} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} A_{ii} - A_{ii}^T X_{ii} A_{ii} + X_{ii} - I = 0. \tag{22}$$

The control design method Γ^{Θ} applies the deadbeat strategy to every subsystem that is not a sink and, for every sink, applies the same optimal control law as if the node were decoupled from the rest of the graph. We will show that when the plant graph contains sinks, Γ^{Θ} has, in worst case, the same competitive ratio as the deadbeat strategy. Unlike the deadbeat strategy, it has the additional property of being undominated by communication-less methods for plants in $\mathcal{P}$ when the plant graph $G_{\mathcal{P}}$ has sinks.

Lemma 8 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Let Γ be a control design strategy in $\mathcal{C}$. Suppose that there exist i and $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$ and that node i is not a sink. The competitive ratio of Γ is bounded only if*

$$A_{ij} + B_{ii}\Gamma_{ij}(A,B) = 0, \quad \textit{for all } P = (A, B, x_0) \in \mathcal{P}.$$

Proof: See Appendix E. ∎

Remark 8 *Lemma 8 shows that a necessary condition for a bounded competitive ratio is to decouple the nodes that are not sinks from the rest of the network.*

Theorem 9 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink, and the control graph $G_{\mathcal{K}}$ be a complete graph. Then the competitive ratio of the communication-less design method Γ^{Θ} introduced in (20) is*

$$r_{\mathcal{P}}(\Gamma^{\Theta}) = \begin{cases} 1, & \textit{if } (S_{\mathcal{P}})_{11} = 0 \textit{ and } (S_{\mathcal{P}})_{22} = 0, \\ 1 + 1/\epsilon^2, & \textit{otherwise.} \end{cases}$$

Proof: Based on Theorem 2 we know that, for every plant $P = (A, B, x_0) \in \mathcal{P}$

$$J_{(A,B,x_0)}(K^*(A,B)) \geq \frac{\epsilon^2}{1+\epsilon^2} x_0^T A^T B^{-T} B^{-1} A x_0, \tag{23}$$

In addition, proceeding as in the proof of the “only if” part of the Theorem 7, we know that

$$J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B)) \geq J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B)). \tag{24}$$

Plugging equation (24) into equation (23) results in

$$\frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} \leq 1 + \frac{1}{\epsilon^2} \text{ for all } P = (A, B, x_0) \in \mathcal{P}.$$

As a result, $r_{\mathcal{P}}(\Gamma^{\Theta}) \leq 1 + 1/\epsilon^2$. To show that this upper-bound is tight, we now exhibit plants for which it is attained. We use a different construction depending on matrices $(S_{\mathcal{P}})_{11}$ and $(S_{\mathcal{P}})_{22}$. If $(S_{\mathcal{P}})_{11} \neq 0$, two situations can occur.
Case 1: $(S_{\mathcal{P}})_{11} \neq 0$ and it is not diagonal. There exist $1 \leq i \neq j \leq q - c$ such that

$(s_\mathcal{P})_{ij} \neq 0$. In this case, choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A = e_{i_1} e_{j_1}^T$ and $B = \epsilon I$. Then, for $x_0 = e_{j_1}$, we find that

$$\frac{J_{(A,B,x_0)}(\Gamma^\Theta(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} = \frac{1/\epsilon^2}{1/(1+\epsilon^2)} = 1 + \frac{1}{\epsilon^2}$$

because the control design Γ^Θ acts like the deadbeat control design method on this plant.

Case 2: $(S_\mathcal{P})_{11} \neq 0$ and it is diagonal. There exists $1 \leq i \leq q-c$ such that $(s_\mathcal{P})_{ii} \neq 0$. Pick an index $i_1 \in \mathcal{I}_i$. In that case, consider $A(r) = r e_{i_1} e_{i_1}^T$ and $B = \epsilon I$. For $x_0 = e_{i_1}$, the optimal cost is

$$J_{(A(r),B,x_0)}(K^*(A(r),B)) = \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon^2},$$

which results in

$$\lim_{r\to 0} \frac{J_{(A,B,x_0)}(\Gamma^\Theta(A,B))}{J_{(A,B,x_0)}(K^*(A,B))} = 1 + \frac{1}{\epsilon^2}.$$

Now suppose that $(S_\mathcal{P})_{11} = 0$. Again, two different situations can occur.

Case 3: $(S_\mathcal{P})_{11} = 0$ and $(S_\mathcal{P})_{22} \neq 0$. There exists $q-c+1 \leq i \leq q$ such that $(s_\mathcal{P})_{ii} \neq 0$. From the assumption that the plant graph contains no isolated node, we know that there must exist $1 \leq j \leq q-c$ such that $(s_\mathcal{P})_{ij} \neq 0$. Accordingly, let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and consider the 2-parameter family of matrices $A(r,s)$ in $\mathcal{A}(S_\mathcal{P})$ with all entries equal to zero except $a_{i_1 i_1}$, which is equal to r, and $a_{i_1 j_1}$, which is equal to s. Let $B = \epsilon I$. For any initial condition x_0, the corresponding closed-loop performance is

$$J_{(A(r,s),B,x_0)}(\Gamma^\Theta(A(r,s),B)) = \beta_\Theta x_0^T a(r,s) a(r,s)^T x_0,$$

where we have let $a(r,s) = A(r,s)_{i_1}^T$ and β_Θ is

$$\beta_\Theta = \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2ar^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon^2 r^2}.$$

Besides, the optimal closed-loop performance can be computed as

$$J_{(A(r,s),B,x_0)}(K^*(A(r,s),B)) = \beta_{K^*} x_0^T a(r,s) a(r,s)^T x_0,$$

where β_{K^*} is

$$\beta_{K^*} = \frac{\epsilon^2 s^2 + r^2(1+\epsilon^2) - (\epsilon^2+1)^2 + \sqrt{c_+ c_-}}{2\epsilon^2(\epsilon^2+1)(s^2+r^2)},$$

$$c_\pm = (\epsilon^2 s^2 + (r^2 \pm 2r)(\epsilon^2+1) + (\epsilon^2+1)^2).$$

Then,

$$
\begin{aligned}
r_{\mathcal{P}}(\Gamma^{\Theta}) &\geq \lim_{r\to\infty,\frac{s}{r}\to\infty} \frac{J_{(A(r,s),B,x_0)}(\Gamma^{\Theta}(A(r,s),B))}{J_{(A(r,s),B,x_0)}(K^*(A(r,s),B))} \\
&= 1 + \frac{1}{\epsilon^2}
\end{aligned}
$$

Case 4: $(S_{\mathcal{P}})_{11} = 0$ *and* $(S_{\mathcal{P}})_{22} = 0$. Then, every matrix $A \in \mathcal{A}(S_{\mathcal{P}})$ has the form $\left[\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right]$ and, in particular, is nilpotent of degree 2; i.e., $A^2 = 0$. In this case, the Riccati equation yielding the optimal control gain $K^*(A,B)$ can be readily solved, and we find that $K^*(A,B) = -(I + B^T B)^{-1} B^T A$ for all (A,B). As a result, $K^*(A,B) = \Gamma^{\Theta}(A,B)$ for all plant $P = (A,B,x_0) \in \mathcal{P}$ (since $W_i(A,B) = (I + B_{ii}^T B_{ii})^{-1} B_{ii}^T$ for all $q - c + 1 \leq i \leq q$), which implies that the competitive ratio of Γ^{Θ} against plants in $\mathcal{P}$ is equal to one. ∎

In Theorem 9, the control graph $G_{\mathcal{K}}$ is assumed to be a complete graph. We needed this assumption to calculate the cost of the optimal control design strategy $K^*(P)$ when $(S_{\mathcal{P}})_{11} = 0$ and $(S_{\mathcal{P}})_{22} \neq 0$ which is not an easy task when the control graph $G_{\mathcal{K}}$ is incomplete. However, more can be said if $(S_{\mathcal{P}})_{11} \neq 0$.

Corollary 10 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then*

$$
r_{\mathcal{P}}(\Gamma^{\Theta}) = \begin{cases} 1, & \text{if } (S_{\mathcal{P}})_{11} = 0 \text{ and } (S_{\mathcal{P}})_{22} = 0, \\ 1 + 1/\epsilon^2, & \text{if } (S_{\mathcal{P}})_{11} \neq 0. \end{cases}
$$

Proof: According to Theorem 9, for $(S_{\mathcal{P}})_{11} \neq 0$, we get

$$
\begin{aligned}
r_{\mathcal{P}}(\Gamma^{\Theta}) &= \sup_{P\in\mathcal{P}} \frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K^*(P))} \\
&\leq \sup_{P\in\mathcal{P}} \frac{J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B))}{J_{(A,B,x_0)}(K_C^*(A,B))} = 1 + \frac{1}{\epsilon^2}.
\end{aligned}
$$

Case 1: $(S_{\mathcal{P}})_{11} \neq 0$ *and it is not diagonal.* For the special plant introduced in Case 1 in the proof of Theorem 9, we have $J_{(A,B,e_{j_1})}(K_C^*(A,B)) = J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1}))$ since $A = e_{i_1} e_{j_1}^T$ is a nilpotent matrix. The rest of the proof is similar to Case 1 in the proof of Theorem 9.
Case 2: $(S_{\mathcal{P}})_{11} \neq 0$ *and it is diagonal.* Note that, for the special plant introduced Case 2 in the proof of Theorem 9, we have

$$
K_C^*(A,B) = -\frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon r^2} A
$$

which shows $K_C^*(A,B) \in \mathcal{K}(S_{\mathcal{K}})$ and similar to the proof of Theorem 2, we get $J_{(A,B,e_{i_1})}(K_C^*(A,B)) = J_{(A,B,e_{i_1})}(K^*(A,B,e_{i_1}))$. The rest of the proof is similar

to Case 2 in the proof of Theorem 9.
Case 3: $(S_{\mathcal{P}})_{11} = 0$ *and* $(S_{\mathcal{P}})_{22} = 0$. Then, every $A \in \mathcal{A}(S_{\mathcal{P}})$ is nilpotent matrix which results in $J_P(K^*(P)) = J_P(K^*_C(A,B))$. The rest of the proof is similar to Case 4 in the proof of Theorem 9. ■

Theorem 11 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink, the control graph $G_{\mathcal{K}}$ be a complete graph, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops. Then the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2,$$

if either $(S_{\mathcal{P}})_{11}$ is not diagonal or $(S_{\mathcal{P}})_{22} \neq 0$.

Proof: *Case 1: $(S_{\mathcal{P}})_{11} \neq 0$ and it is not diagonal.* Then, there exist $1 \leq i, j \leq q-c$ and $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and consider the matrix A defined by $A = e_{i_1} e_{j_1}^T$ and $B = \epsilon I$. From Lemma 8, we know that a communication-less method Γ has a bounded competitive ratio only if $\Gamma(A,B) = -B^{-1}A$ (because node i is a part of $(S_{\mathcal{P}})_{11}$ and it is not a sink). Therefore

$$r_{\mathcal{P}}(\Gamma) \geq \frac{J_{(A,B,e_{j_1})}(\Gamma(A,B))}{J_{(A,B,e_{j_1})}(K^*(A,B))} = 1 + \frac{1}{\epsilon^2}$$

for any such method.
Case 2: $(S_{\mathcal{P}})_{22} \neq 0$. There thus exists $q-c+1 \leq i \leq q$ such that $(s_{\mathcal{P}})_{ii} \neq 0$. Note that, there exists $1 \leq j \leq q-c$ such that $(s_{\mathcal{P}})_{ij} \neq 0$, since there is no isolated node in the plant graph. Choose indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Consider A defined as $A = r e_{i_1} e_{j_1}^T + s e_{i_1} e_{i_1}^T$ and $B = \epsilon I$. As indicated in the proof of Theorem 9, control design strategy Γ^{Θ} yields the globally optimal controller with limited model information for plants in this family. Hence, we know that $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$ for every communication-less strategy Γ. ■

In Theorem 11, the control graph $G_{\mathcal{K}}$ is assumed to be a complete graph since we used the proof of Theorem 9 for the case that $(S_{\mathcal{P}})_{22} \neq 0$.

Corollary 12 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node and at least one sink, the control graph $G_{\mathcal{K}}$ be a complete graph, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph with self-loops. Then the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2,$$

if $(S_{\mathcal{P}})_{11}$ is not diagonal.

Proof: Considering that for the nilpotent matrix $A = e_{i_1} e_{j_1}^T$, we get $J_{(A,B,e_{j_1})}(K^*(A,B,e_{j_1})) = J_{(A,B,e_{j_1})}(K^*_C(A,B))$, the rest of the proof is similar to Case 1 in the proof of Theorem 11. ■

Remark 9 *Combining Theorems 9 and 11 implies that if either* $(S_{\mathcal{P}})_{11}$ *is not diagonal or* $(S_{\mathcal{P}})_{22} \neq 0$*, control design method* Γ^{Θ} *exhibits the same competitive ratio as the deadbeat control strategy, which is the smallest ratio achievable by a communication-less control method. Therefore, it is a solution to problem (10). Furthermore, if* $(S_{\mathcal{P}})_{11}$ *and* $(S_{\mathcal{P}})_{22}$ *are both zero, then* Γ^{Θ} *is equal to* K^**, which shows that* Γ^{Θ} *is a solution to problem (10), in this case too.*

Remark 10 *The case where* $(S_{\mathcal{P}})_{11}$ *is diagonal and* $(S_{\mathcal{P}})_{22} = 0$ *is still open.*

The next theorem shows that Γ^{Θ} is a more desirable control design method than the deadbeat control design strategy when the plant graph $G_{\mathcal{P}}$ has sinks, since it is then undominated by communication-less design methods.

Theorem 13 *Let the plant graph* $G_{\mathcal{P}}$ *contain no isolated node and at least one sink, the design graph* $G_{\mathcal{C}}$ *be a totally disconnected graph with self-loops, and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$*. The control design method* Γ^{Θ} *is undominated by any control design method* $\Gamma \in \mathcal{C}$*.*

Proof: See Appendix F. ■

Remark 11 *Consider the limited model information design problem given by the plant graph* $G_{\mathcal{P}}$ *in Figure 1(a), the control graph* $G'_{\mathcal{K}}$ *in Figure 1(b′), and the design graph* $G'_{\mathcal{C}}$ *in Figure 1(c′). Theorems 9, 11, and 13 together show that, the control design strategy* Γ^{Θ} *is the best control design strategy that one can propose based on the local model information and the plant graph, because the control design strategy* Γ^{Θ} *is a minimizer of the competitive ratio and it is undominated.*

Remark 12 *For general weight matrices* Q *and* R *appearing in the performance cost, the competitive ratio of both the deadbeat control design strategy* Γ^{Δ} *and the control design strategy* Γ^{Θ} *is* $1 + \bar{\sigma}(R)/(\underline{\sigma}(Q)\epsilon^2)$*. In particular, the competitive ratio has a limit equal to one as* $\bar{\sigma}(R)/\underline{\sigma}(Q)$ *goes to zero. We thus recover the well-known observation (e.g., [32]) that, for discrete-time linear time-invariant systems, the optimal linear quadratic regulator approaches the deadbeat controller in the limit of "cheap control".*

4 Design Graph Influence on Achievable Performance

In the previous section, we have shown that communicat-ion-less control design methods (i.e., $G_{\mathcal{C}}$ is totally disconnected with self-loops) have intrinsic performance limitations, and we have characterized minimal elements for both the competitive ratio and domination metrics. A natural question is "given plant graph $G_{\mathcal{P}}$, which design graph $G_{\mathcal{C}}$ is necessary to ensure the existence of $\Gamma \in \mathcal{C}$ with better competitive ratio than Γ^{Δ} and Γ^{Θ} ?". We tackle this question in this section.

Theorem 14 *Let the plant graph $G_{\mathcal{P}}$ and the design graph $G_{\mathcal{C}}$ be given and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. If one of the following conditions is satisfied then $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$ for all $\Gamma \in \mathcal{C}$:*

1. *$G_{\mathcal{P}}$ contains the path $k \to i \to j$ with distinct nodes i, j, and k while $(j,i) \notin E_{\mathcal{C}}$.*

2. *There exist $i \neq j$ such that $n_i \geq 2$ and $(i,j) \in E_{\mathcal{P}}$ while $(j,i) \notin E_{\mathcal{C}}$.*

Proof: We prove the case when condition (1) holds. The proof for condition (2) is similar.

Let i, j, and k be three distinct nodes such that $(s_{\mathcal{P}})_{ik} \neq 0$ and $(s_{\mathcal{P}})_{ji} \neq 0$ (i.e., the path $k \to i \to j$ is contained in the plant graph $G_{\mathcal{P}}$). Let us pick $i_1 \in \mathcal{I}_i$, $j_1 \in \mathcal{I}_j$ and $k_1 \in \mathcal{I}_k$ and consider the 2-parameter family of matrices $A(r,s)$ in $\mathcal{A}(S_{\mathcal{P}})$ with all entries equal to zero except $a_{i_1k_1}$, which is equal to r, and $a_{j_1i_1}$, which is equal to s. Let $B = \epsilon I$ and let $\Gamma \in \mathcal{C}$ be a limited model information with design graph $G_{\mathcal{C}}$. For $x_0 = e_{k_1}$, we have

$$J_{(A(r,s),B,e_{k_1})}(\Gamma(A(r,s),B)) \geq (r + \epsilon\gamma_{i_1k_1}(A,B))^2[\gamma_{j_1i_1}^2 + (s + \epsilon\gamma_{j_1i_1}(A,B))^2]$$

where $\gamma_{i_1k_1}$ cannot be a function of s because $(j,i) \notin E_{\mathcal{C}}$. Note that, irrespective of the choice of $\gamma_{j_1i_1}(A,B)$, we have

$$J_{(A(r,s),B,e_{k_1})}(\Gamma(A(r,s),B)) \geq \frac{(r + \epsilon\gamma_{i_1k_1}(A,B))^2 s^2}{1+\epsilon^2}.$$

The cost of the deadbeat control design on this plant satisfies

$$J_{(A(r,s),B,e_{k_1})}(\Gamma^{\Delta}(A(r,s),B)) = r^2/\epsilon^2,$$

and thus

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &= \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(K^*(P))} \\ &= \sup_{P \in \mathcal{P}} \left[\frac{J_P(\Gamma(A,B))}{J_P(\Gamma^{\Delta}(A,B))} \frac{J_P(\Gamma^{\Delta}(A,B))}{J_P(K^*(P))} \right] \\ &\geq \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B))}{J_P(\Gamma^{\Delta}(A,B))}, \\ &\geq \lim_{s \to \infty} \frac{\epsilon^2 (r + \epsilon\gamma_{i_1k_1}(A,B))^2 s^2}{(1+\epsilon^2) r^2}. \end{aligned} \tag{25}$$

This shows that $r_{\mathcal{P}}(\Gamma)$ is unbounded unless $r + \epsilon\gamma_{i_1k_1}(A(r,s),B) = 0$ for all r, s. Now consider the 1-parameter family of matrices $\bar{A}(r)$ with all entries equal to zero except $a_{i_1k_1}$, which is equal to r. Because of $(j,i) \notin E_{\mathcal{C}}$, we know that $\Gamma_z(\bar{A}(r),B) = \Gamma_z(A(r,s),B)$ for all $z \in \mathcal{I}_i$. Thus

$$J_{(\bar{A}(r),B,e_{k_1})}(\Gamma(\bar{A}(r),B)) \geq r^2/\epsilon^2.$$

On the other hand, similar to the proof of Theorem 2, we can compute the optimal controller for systems in this 1−parameter family and find

$$\begin{aligned} J_{(\bar{A}(r),B,e_{k_1})}(K^*(\bar{A}(r),B,e_{k_1})) &= J_{(\bar{A}(r),B,e_{k_1})}(K^*_{\mathcal{C}}(\bar{A}(r),B)) \\ &= r^2/(1+\epsilon^2), \end{aligned}$$

As a result, we get

$$r_{\mathcal{P}}(\Gamma) \geq \frac{r^2/\epsilon^2}{r^2/(1+\epsilon^2)} = 1 + \frac{1}{\epsilon^2},$$

which concludes the proof for this case. ■

Remark 13 *Consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 1(a), the control graph $G'_{\mathcal{K}}$ in Figure 1(b′), and the design graph $G_{\mathcal{C}}$ in Figure 1(c). Theorem 14 shows that, because the plant graph $G_{\mathcal{P}}$ contains the path $3 \to 2 \to 1$ but the design graph $G_{\mathcal{C}}$ does not contain $1 \to 2$, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ would be greater than or equal to $1 + 1/\epsilon^2$.*

Corollary 15 *Let both the plant graph $G_{\mathcal{P}}$ and the control graph $G_{\mathcal{K}}$ be complete graphs. If the design graph $G_{\mathcal{C}}$ is not equal to $G_{\mathcal{P}}$, then $r_{\mathcal{P}}(\Gamma) \geq 1 + 1/\epsilon^2$ for all $\Gamma \in \mathcal{C}$.*

Proof: The proof is a direct application of Theorem 14 with condition (1) fulfilled. ■

Remark 14 *Corollary 15 shows that, when $G_{\mathcal{P}}$ is a complete graph, achieving a better competitive ratio than the deadbeat design strategy requires each subsystem to have full knowledge of the plant model when constructing each subcontroller.*

5 Extensions to Under-Actuated Subsystems

In the previous sections, we gave an explicit solution to the problem in (10) under the assumption that all the subsystems are fully-actuated; i.e., all the matrices $B \in \mathcal{B}(\epsilon)$ are square invertible matrices. In this section, we briefly discuss an extension to more general (but still restricted) under-actuated systems.

Consider the limited model information control design problem given with the plant graph $G_{\mathcal{P}}$, the control graph $G_{\mathcal{K}}$, and the design graph $G_{\mathcal{C}}$ given in Figure 3. The state space representation of the system is given as

$$\begin{bmatrix} \underline{x}_1(k+1) \\ \underline{x}_2(k+1) \end{bmatrix} = A \begin{bmatrix} \underline{x}_1(k) \\ \underline{x}_2(k) \end{bmatrix} + B \begin{bmatrix} \underline{u}_1(k) \\ \underline{u}_2(k) \end{bmatrix},$$

where

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \; B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix},$$

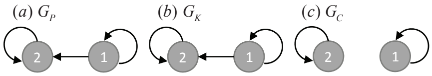

Figure 3: Plant graph $G_{\mathcal{P}}$, control graph $G_{\mathcal{K}}$, and design graph $G_{\mathcal{C}}$ used to illustrate an extension to under-actuated systems.

with $\underline{x}_1(k) \in \mathbb{R}^{n_1}$, $\underline{x}_2(k) \in \mathbb{R}^{n_2}$, $\underline{u}_1(k) \in \mathbb{R}^{n_1}$, and $\underline{u}_2(k) \in \mathbb{R}^{m_2}$ for some given integers $n_1 \geq 1$, $n_2 > m_2 \geq 1$. Thus, for the second subsystem the matrix $B_{22} \in \mathbb{R}^{n_2 \times m_2}$ is a non-square matrix, and as a result the second subsystem is an under-actuated subsystem. Let us assume that the matrices A_{21}, A_{22}, B_{22} satisfy the "matching condition"; i.e., the pair (A_{22}, B_{22}) is controllable and $\mathrm{span}(A_{21}) \subseteq \mathrm{span}(B_{22})$ [33]. Besides, assume that for all matrices B, we have $\underline{\sigma}(B) \geq \epsilon$ for some $\epsilon > 0$. For this case, we have

$$\Gamma^{\Theta}(A, B) = -\operatorname{diag}(B_{11}^{-1}, W_2(A_{22}, B_{22}))A,$$

where $W_2(A_{22}, B_{22})$ is defined in (21). Note that we do not require the matrix B_{22} to be square invertible. Under some additional conditions and following a similar approach as above, it can be shown that the control design strategy Γ^{Θ} becomes an undominated minimizer of the competitive ratio over the set of limited model information control design strategies. This result can be generalized to cases with higher number of subsystems as long as the sinks in the plant graph $G_{\mathcal{P}}$ are the only under-actuated subsystems [34].

6 Conclusion

We presented a framework for the study of control design under limited model information, and investigated the connection between the quality of controllers produced by a design method and the amount of plant model information available to it. We showed that the best performance achievable by a limited model information control design method crucially depends on the structure of the plant graph and, thus, that giving the designer access to this graph, even without a detailed model of all plant subsystems, results in superior design, in the sense of domination. Possible future work will focus on extending the present framework to dynamic controllers and/or where disturbances are present.

7 Bibliography

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A Proof of Lemma 1

For any plant $P = (A, B, x_0) \in \mathcal{P}$, the optimal controller $K^*(P)$ exists (because the plant is controllable since B is invertible by assumption) and can be computed using the unique positive definite solution to the discrete algebraic Riccati equation

$$X = A^T XA - A^T XB(I + B^T XB)^{-1} B^T XA + I. \tag{26}$$

The corresponding cost is $J_P(K^*(A, B)) = x_0^T (X - I)x_0$. Inserting the product BB^{-1} before every matrix A and $B^{-T}B^T$ after every matrix A^T in (26) results in

$$\begin{aligned} X - I = {} & A^T B^{-T} B^T XBB^{-1} A \\ & - A^T B^{-T} B^T XB(I + B^T XB)^{-1} B^T XBB^{-1} A. \end{aligned} \tag{27}$$

Naming $B^T XB$ as Y simplifies (27) into

$$X - I = A^T B^{-T} [Y - Y(I + Y)^{-1} Y]B^{-1} A. \tag{28}$$

Note that Y is a positive definite matrix because X is positive definite and B is full rank. Let us denote the right-hand side of (28) by $A^T B^{-T} g(Y)B^{-1} A$. Then we can make the following two claims regarding the rational function $g(\cdot)$.

Claim 1: The function $y \mapsto g(y) = y/(1+y)$ is a monotonically increasing over $\mathbb{R}^+$.

Claim 2: Let $Y \in S^n_{++}$ and D, T be diagonal and unitary matrices, respectively, such that $Y = T^T D T$. Then $g(Y) = T^T \operatorname{diag}(g(d_{ii}))T$, where d_{ii} are the diagonal elements of D (and the eigenvalues of Y).

Claim 1 is proved by computing the derivative of g over $\mathbb{R}^+$, while Claim 2 follows from the fact that all matrices involved in the computation of $g(Y)$ can be diagonalized in the same basis. Using these two claims, we find that, for all Y with eigenvalues denoted by $\lambda_1(Y), \ldots, \lambda_n(Y)$

$$\begin{aligned} X - I &= A^T B^{-T} g(Y) B^{-1} A \\ &= A^T B^{-T} T^T \operatorname{diag}(g(\lambda_i(Y))) T B^{-1} A \\ &\geq (g(\underline{\lambda}(Y))) A^T B^{-T} B^{-1} A, \end{aligned} \tag{29}$$

where $\underline{\lambda}(Y)$ is a positive number because matrix Y is a positive definite matrix. Now, according to [35],

$$\underline{\lambda}(X) \geq \underline{\lambda}(A^T(I + BB^T)^{-1}A + I) \geq \frac{\underline{\sigma}^2(A)}{1 + \bar{\sigma}^2(B)} + 1. \tag{30}$$

Using (30) in inequality $\underline{\lambda}(Y) \geq \underline{\sigma}^2(B)\underline{\lambda}(X)$ gives

$$\underline{\lambda}(Y) \geq \frac{\underline{\sigma}^2(B)\underline{\sigma}^2(A)}{1 + \bar{\sigma}^2(B)} + \underline{\sigma}^2(B), \tag{31}$$

and, because of the claim 1 and the inequality in (31), we will have

$$\begin{aligned} g(\underline{\lambda}(Y)) &\geq \frac{\underline{\sigma}^2(B)[\underline{\sigma}^2(A) + \bar{\sigma}^2(B) + 1]}{1 + \bar{\sigma}^2(B) + \underline{\sigma}^2(B)[\underline{\sigma}^2(A) + \bar{\sigma}^2(B) + 1]} \\ &\geq \frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B) + 1}. \end{aligned} \tag{32}$$

Combining (29) and (32) results in

$$X - I \geq \frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B) + 1} A^T B^{-T} B^{-1} A,$$

and, therefore

$$\begin{aligned} J_P(K^*(A,B)) &= x_0^T (X - I) x_0 \\ &\geq \left(\frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B) + 1} \right) x_0^T (A^T B^{-T} B^{-1} A) x_0 \\ &= \left(\frac{\underline{\sigma}^2(B)}{\underline{\sigma}^2(B) + 1} \right) J_P(\Gamma^{\Delta}(A,B)). \end{aligned}$$

B Proof of Lemma 3

Let $\Gamma \in \mathcal{C}$ and assume that the implication does not hold, i.e., that there exists a matrix A and indices i, j with $\ell_0 \in \mathcal{I}_i$ such that $a_{\ell j} = 0$ for all $\ell \in \mathcal{I}_i$ but $\gamma_{\ell_0 j}(A, B) \neq 0$. Consider matrix $\bar{A}$ such that $\bar{A}_\ell = A_\ell$ for all $\ell \in \mathcal{I}_i$ and $\bar{A}_z = 0$ for all $z \notin \mathcal{I}_i$. Based on the definition of limited-model-information control design methods, we know that $\Gamma_\ell(\bar{A}, B) = \Gamma_\ell(A, B)$ for all $\ell \in \mathcal{I}_i$ and $\Gamma_z(\bar{A}, B) = 0$ for all $z \notin \mathcal{I}_i$ (because $\Gamma_z(A, B) = \Gamma_z(0, B)$ for all $z \notin \mathcal{I}_i$ and, as shown in [27], it is necessary that $\Gamma(0, B) = 0$ for a finite competitive ratio). For $x = e_j$, we have

$$\begin{aligned} J_{(\bar{A},B,e_j)}(\Gamma(\bar{A}, B)) &\geq \sum_{\ell \in \mathcal{I}_i} \gamma_{\ell j}(\bar{A}, B)^2 \\ &= \sum_{\ell \in \mathcal{I}_i} \gamma_{\ell j}(A, B)^2 \\ &\geq \gamma_{\ell_0 j}(A, B)^2 > 0. \end{aligned}$$

Using (25), we get

$$r_{\mathcal{P}}(\Gamma) \geq \frac{J_{(\bar{A},B,e_j)}(\Gamma(\bar{A}, B))}{J_{(\bar{A},B,e_j)}(\Gamma^{\Delta}(\bar{A}, B))} = \infty,$$

since $J_{(\bar{A},B,e_j)}(\Gamma^{\Delta}(\bar{A}, B)) = 0$. This proves the claim by contrapositive.

C Proof of Lemma 4

Clearly, it is enough to prove inequality (16) for control design methods with a finite competitive ratio.

Let $G_{\mathcal{P}}$ have a loop and $\Gamma \in \mathcal{C}$ have finite competitive ratio. Without loss of generality, let us assume that the nodes of graph $G_{\mathcal{P}}$ are numbered such that it admits the following loop of length ℓ: $1 \to 2 \to \cdots \to \ell \to 1$. Let us choose indices $i_1 \in \mathcal{I}_1$, $i_2 \in \mathcal{I}_2$, …, $i_\ell \in \mathcal{I}_\ell$ and consider the one-parameter family of matrices $\{A(r)\}$ defined by $a_{i_2 i_1}(r) = r$, $a_{i_3 i_2}(r) = r$, …, $a_{i_\ell i_{\ell-1}}(r) = r$, $a_{i_1 i_\ell}(r) = r$, and all other entries equal to zero, for all r. Let $B = \epsilon I$. Because of Lemma 3, the controller gain entries $\gamma_{j_2 i_1}(A(r), B)$ for all $j_2 \in \mathcal{I}_2$, $\gamma_{j_3 i_2}(A(r), B)$ for all $j_3 \in \mathcal{I}_3$, …, $\gamma_{j_\ell i_{\ell-1}}(A(r), B)$ for all $j_\ell \in \mathcal{I}_\ell$, $\gamma_{j_1 i_\ell}(A(r), B)$ for all $j_1 \in \mathcal{I}_1$ can be non-zero, but all other entries of the controller gain $\Gamma(A(r), B)$ are zero for all r. As a result, the characteristic polynomial of matrix $A(r) + B\Gamma(A(r), B)$ can be computed as:

$$\begin{aligned} \lambda^{n-\ell}[\lambda^\ell - (-1)^\ell &(r + \epsilon\gamma_{i_2 i_1}(A(r), B))(r + \epsilon\gamma_{i_3 i_2}(A(r), B)) \\ &\times \cdots \times (r + \epsilon\gamma_{i_\ell i_{\ell-1}}(A(r), B))(r + \epsilon\gamma_{i_1 i_\ell}(A(r), B))]. \end{aligned}$$

Now, note that because Γ has a bounded competitive ratio against $\mathcal{P}$ by assumption, this polynomial should be stable for all r. (Indeed, Γ can have a finite competitive ratio only if $A + B\Gamma(A, B)$ is stable for all matrices A, otherwise it would yield an infinite cost for some plants while the corresponding optimal cost remains bounded since the pair (A, B) is controllable for all plant in $\mathcal{P}$). As a result, we must have

$$\begin{aligned} &|(r + \epsilon\gamma_{i_2 i_1}(A(r), B)) \cdots (r + \epsilon\gamma_{i_1 i_\ell}(A(r), B))| \\ &\qquad = |r + \epsilon\gamma_{i_2 i_1}(A(r), B)| \cdots |r + \epsilon\gamma_{i_1 i_\ell}(A(r), B)| < 1 \end{aligned} \tag{33}$$

for all r. Let $\{r_z\}_{z=1}^{\infty}$ be a sequence of real numbers with the property that r_z goes to infinity as z goes to infinity. From (33), we know that there exists an index $\bar{m}$ such that

$$\forall N,\ \exists z > N \text{ such that } |r_z + \epsilon\gamma_{i_{\bar{m}\oplus 1} i_{\bar{m}}}(A(r_z), B)| < 1, \tag{34}$$

where "$\oplus$" designated addition modulo ℓ; i.e., $i \oplus j = (i + j) - \lfloor (i + j)/\ell \rfloor \ell$ where $\lfloor x \rfloor = \max\{y \in \mathbb{Z} | y \le x\}$ for all $x \in \mathbb{R}$. Indeed, if this is not the case, it is true that

$$\forall m,\ \exists N_m \text{ such that } |r_z + \epsilon\gamma_{i_{m\oplus 1} i_m}(A(r_z), B)| \ge 1, \forall z > N_m.$$

Then, for all $z > \max_m N_m$ and all m,

$$|r_z + \epsilon\gamma_{i_{m\oplus 1} i_m}(A(r_z), B)| \ge 1$$

which contradicts (33). Without loss of generality (since this just amounts to renumbering the nodes in the plant graph), we assume that $\bar{m} = 1$. Using (34), we can then construct a subsequence $\{r_{\phi(z)}\}$ of $\{r_z\}$ with the property that

$$|r_{\phi(z)} + \epsilon\gamma_{i_2 i_1}(A(r_{\phi(z)}), B)| < 1 \text{ for all } z.$$

Now introduce the sequence of matrices $\{\bar{A}(z)\}_{z=1}^{\infty}$ defined by $\bar{A}_{i_2 i_1}(z) = r_{\phi(z)}$ for all z and every other row equal to zero. For large enough z (and hence, large enough $r_{\phi(z)}$), we get

$$\begin{aligned} J_{(\bar{A}(z), B, e_{i_1})}(\Gamma(\bar{A}(z), B)) &\ge \gamma_{i_2 i_1}(\bar{A}(z), B)^2 \\ &= \gamma_{i_2 i_1}(A(r_{\phi(z)}), B)^2 \\ &\ge \frac{(|r_{\phi(z)}| - 1)^2}{\epsilon^2}, \end{aligned}$$

and thus

$$\frac{J_{(\bar{A}(z), B, e_{i_1})}(\Gamma(\bar{A}(z), B))}{J_{(\bar{A}(z), B, e_{i_1})}(K^*(\bar{A}(z), B, e_{i_1}))} \ge \frac{(|r_{\phi(z)}| - 1)^2/\epsilon^2}{r_{\phi(z)}^2/(1 + \epsilon^2)}.$$

This, in particular, implies that

$$r_{\mathcal{P}}(\Gamma) \ge \lim_{z\to\infty} \frac{J_{(\bar{A}(z), B, e_{i_1})}(\Gamma(\bar{A}(z), B))}{J_{(\bar{A}(z), B, e_{i_1})}(K^*(\bar{A}(z), B, e_{i_1}))} \ge 1 + 1/\epsilon^2.$$

Note that $\bar{A}(z)$ is a nilpotent matrix for all z, and thus

$$J_{(\bar{A}(z),B,e_{i_1})}(K^*(\bar{A}(z),B,e_{i_1})) = J_{(\bar{A}(z),B,e_{i_1})}(K_C^*(\bar{A}(z),B))$$

similar to the proof of Theorem 2, and therefore

$$J_{(\bar{A}(z),B,e_{i_1})}(K_C^*(\bar{A}(z),B)) = r_{\phi(z)}^2/(1+\epsilon^2)$$

using the unique positive-definite solution of discrete algebraic Riccati equation in (13).

D Proof of Lemma 6

We prove that if there is no sink in the plant graph (i.e., according to [30], if $\forall j \exists k,\ k \neq j$, such that $(s_{\mathcal{P}})_{kj} \neq 0$) then the deadbeat control design method is undominated. For proving this claim, we are going to prove that for any control design $\Gamma \in \mathcal{C} \backslash \{\ \Gamma^\Delta\}$, there exits a plant $P = (A, B, x_0) \in \mathcal{P}$ such that $J_P(\Gamma(A,B)) > J_P(\Gamma^\Delta(A,B)) = x_0^T[A^T B^{-T} B^{-1} A]x_0$. We will proceed in several steps, which require us to partition the set of limited model information control design methods $\mathcal{C}$ as follows

$$\mathcal{C} = \mathcal{L}^c \cup \mathcal{W}_1 \cup \mathcal{W}_2 \cup \{\Gamma^\Delta\},$$

where

$$\begin{aligned}\mathcal{L} := \{\Gamma \in \mathcal{C} | \exists \Lambda_j : \mathbb{R}^{n_j \times n} \times \mathbb{R}^{n_j \times n_j} \rightarrow \mathbb{R}^{n_j \times n_j},\\ [\Gamma(A,B)]_j = \Lambda_j([A]_j, B_{jj})[A]_j,\ \text{for all } j = 1, \cdots, q\},\end{aligned}$$

$$\begin{aligned}\mathcal{W}_1 := \{\Gamma \in \mathcal{L} | \exists j, i \neq j \text{ and } A_{ij} \in \mathbb{R}^{n_i \times n_j} \text{ nonzero s.t.}\\ I + B_{ii}\Lambda_i([0\ \cdots\ 0\ A_{ij}\ 0\ \cdots\ 0], B_{ii}) \neq 0\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}_2 := \{\Gamma \in \mathcal{L} \backslash \mathcal{W}_1 | \exists i \in \{1, \cdots, q\}, [A]_i \in \mathbb{R}^{n_i \times n},\ \text{with}\\ \text{appropriate structure such that } I + B_{ii}\Lambda_i([A]_i, B_{ii}) \neq 0\}.\end{aligned}$$

First, we prove that the deadbeat control design method is undominated by control design strategies in $\mathcal{L}^c$. Let $\Gamma \in \mathcal{L}^c$ and let j be such that $\exists j_1 \in \mathcal{I}_j$ which $\Gamma_{j_1}(\bar{A}, B)^T$ cannot be written as a linear combination of vectors in the set $\{\bar{A}_i^T, \forall i \in \mathcal{I}_j\}$ for some matrix $\bar{A}$ and matrix B. Let $a_i^T = \bar{A}_i$ for all $i \in \mathcal{I}_j$ and consider matrix A such that the row $A_i = a_i^T$ for all $i \in \mathcal{I}_j$ and $A_i = 0$ for all $i \in \mathcal{I}_j^c$. If $\Gamma(0,B) \neq 0$, then Γ cannot dominate Γ^Δ (since $\Gamma^\Delta(0,B) = 0$ for all x_0) and, thus, there is no loss of generality in assuming that $\Gamma(0,B) = 0$ for all x_0, and, in turn that $\Gamma_i(A,B) = 0$ for all $i \in \mathcal{I}_j^c$. Let us also denote $\Gamma(A,B)$ by K and $\Gamma_i(A,B) = \Gamma_i(\bar{A},B)$ by K_i^T for all $i \in \mathcal{I}_j$. For all x_0,

$$J_{(A,B,x_0)}(\Gamma(A,B)) \geq x_0^T[K^T K + (A+BK)^T(A+BK)]x_0,$$

and

$$\begin{aligned}&J_{(A,B,x_0)}(\Gamma(A,B)) - J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B))\\&\quad\geq x_0^T[A^T(I-B^{-T}B^{-1})A + A^TBK + K^TB^TA + K^T(I+B^TB)K]x_0.\end{aligned} \tag{35}$$

We know that $\mathrm{null}(A) = \mathrm{span}\{A_i^T, \forall i \in \mathcal{I}_j\}^{\perp} \neq \{0\}$, because $n_j < n$. On the other hand, we know that there exists an $j_1 \in \mathcal{I}_j$ such that $K_{j_1} \notin \mathrm{span}\{A_i^T, \forall i \in \mathcal{I}_j\}$ which shows that

$$\mathrm{span}\{A_i^T, \forall i \in \mathcal{I}_j\} \subsetneq \mathrm{span}\{A_i^T, \forall i \in \mathcal{I}_j\} + \mathrm{span}\{K_i^T, \forall i \in \mathcal{I}_j\},$$

Thus, we can choose an initial condition $x_0 \in \mathrm{null}(A)$ such that $Kx_0 \neq 0$. Using this x_0 in (35) results in

$$J_{(A,B,x_0)}(\Gamma(A,B)) - J_{(A,B,x_0)}(\Gamma^{\Delta}(A,B)) \geq x_0^T[K^T(I+B^TB)K]x_0 > 0. \tag{36}$$

Therefore, the control design strategies in $\mathcal{L}^c$ cannot dominate the deadbeat control design strategy Γ^{Δ}.

Second, we prove that the deadbeat control design strategy is undominated by control design methods in $\mathcal{W}_1$. Let $\Gamma \in \mathcal{W}_1$ and let j be such that $(I + B_{ii}\Lambda_i(\begin{bmatrix}0 & \cdots & 0 & \bar{A}_{ij} & 0 & \cdots & 0\end{bmatrix}, B_{ii})) \neq 0$ for some $i \neq j$. It means that there exists at least $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ such that $\bar{a}_{i_1j_1} \neq 0$ and $\bar{a}_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(\bar{A}, B) \neq 0$. Using the structure matrix, we know that there exits a $\ell \neq i$ such that $(s_{\mathcal{P}})_{\ell i} \neq 0$. Choose an index $\ell_1 \in \mathcal{I}_\ell$. Consider the matrix A defined by $[A]_i = [\bar{A}]_i$, $a_{\ell_1 i_1} = r$ and all other entries equal to zero. Then, $[\Gamma(A,B)]_i = \Lambda_i([A]_i, B_{ii})[A]_i$, $[\Gamma(A,B)]_\ell = \Lambda_\ell([A]_\ell, B_{\ell\ell})[A]_\ell$ (because $\Gamma \in \mathcal{L}$), and $[\Gamma(A,B)]_z = 0$ for all $z \neq i, \ell$. Denote $\Gamma(A,B)$ by K. We have

$$\begin{aligned}J_{(A,B,x_0)}(\Gamma(A,B)) \geq x_0^T[&(A+BK)^TK^TK(A+BK)\\&+((A+BK)^2)^T(A+BK)^2]x_0.\end{aligned}$$

Using $x_0 = e_{j_1}$ results in

$$\begin{aligned}&J_{(A,B,e_{j_1})}(\Gamma(A,B)) - J_{(A,B,e_{j_1})}(\Gamma^{\Delta}(A,B)) \geq\\&\qquad[k_{\ell_1i_1}^2 + (r + b_{\ell_1\ell_1}k_{\ell_1i_1})^2](a_{i_1j_1} + b_{i_1i_1}k_{i_1j_1})^2 - \sum_{z\in\mathcal{I}_i} a_{zj_1}^2/b_{zz}^2.\end{aligned} \tag{37}$$

Note that, irrespective of the choice of the controller gain $k_{\ell_1i_1}$,

$$k_{\ell_1i_1}^2 + (r + b_{\ell_1\ell_1}k_{\ell_1i_1})^2 \geq r^2/(1 + b_{\ell_1\ell_1}^2),$$

and as a result,

$$\lim_{r\to+\infty}[k_{\ell_1i_1}^2 + (r + b_{\ell_1\ell_1}k_{\ell_1i_1})^2](a_{i_1j_1} + b_{i_1i_1}k_{i_1j_1})^2 = \infty,$$

because $a_{i_1j_1} + b_{i_1i_1}k_{i_1j_1} \neq 0$. Hence, we can always construct A with appropriate choice of index ℓ and a scalar r large enough to make the right hand side of the expression (37) positive. As a result, $\Gamma \in \mathcal{W}_1$ cannot dominate Γ^Δ.

Third, we prove that the deadbeat control design strategy is undominated by control design methods in $\mathcal{W}_2$. Let $\Gamma \in \mathcal{W}_2$ and index i and vector $[\bar{A}]_i$ be such that $I + \Lambda_i([\bar{A}]_i, B_{ii}) \neq 0$. Thus we know that there exists at least $i_1 \in \mathcal{I}_i$ such that $\bar{A}_{i_1} \neq 0$ and $\bar{A}_{i_1} + b_{i_1i_1}\Gamma_{i_1}(\bar{A}, B) \neq 0$. Based on the structure matrix we know that there exits $\ell \neq i$ such that $(s_{\mathcal{P}})_{\ell i} \neq 0$. Choose an index $\ell_1 \in \mathcal{I}_\ell$. Consider the matrix A defined by $[A]_i = [\bar{A}]_i$ and $a_{\ell_1 i_1} = r$ and all other entries of A equal to zero. Then $[A]_i + B_{ii}[\Gamma(A,B)]_i = (I + B_{ii}\Lambda_i([A]_i, B_{ii}))[A]_i$ and $[A]_j + B_{jj}[\Gamma(A,B)]_j = 0$ for all $j \neq i$ (and, in particular, $j = \ell$ since Γ does not belong to $\mathcal{W}_1$). Again, K will stand for $\Gamma(A,B)$. We have

$$\begin{aligned} &K^TK + (A+BK)^TK^TK(A+BK) - A^TB^{-T}B^{-1}A \\ &\quad\geq (A_{i_1} + b_{i_1i_1}\Gamma_{i_1}(A,B))^T(A_{i_1} + b_{i_1i_1}\Gamma_{i_1}(A,B))r^2/b_{\ell_1\ell_1}^2 - \sum_{z\in\mathcal{I}_i} A_z^TA_z/b_{zz}^2, \end{aligned}$$

and hence, since $A_{i_1} + b_{i_1i_1}\Gamma_{i_1}(A,B) \neq 0$, we can choose r large enough to ensure that this matrix has a strictly positive eigenvalue. Thus, the control design strategy $\Gamma \in \mathcal{W}_2$ cannot dominate Γ^Δ.

E Proof of Lemma 8

The proof is by contrapositive. Let Γ be communication-less and assume that there exist matrices A and B and indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ such that $a_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(A,B) \neq 0$. Choose an index $k_1 \in \mathcal{I}_k$. Consider the one-parameter family of matrices $\bar{A}(r)$ defined by $[\bar{A}(r)]_i = [A]_i$, $\bar{a}_{k_1i_1} = r$, and all other entries of $\bar{A}(r)$ being equal to zero for all r. We know that $[\Gamma(\bar{A}(r),B)]_i = [\Gamma(A,B)]_i$ and $\Gamma_{\bar{k}}(\bar{A}(r),B) = \gamma_{\bar{k}i_1}(r)e_{i_1}^T$ for all $\bar{k} \in \mathcal{I}_k$ (because of Lemma 3), $[\Gamma(\bar{A}(r),B)]_z = 0$ for all $z \neq i,k$. For $x_0 = e_{j_1}$, we have

$$\begin{aligned} J_{(\bar{A}(r),B,e_{j_1})}(\Gamma(\bar{A}(r),B)) &\geq (a_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(A,B))^2 \\ &\qquad \times [\gamma_{k_1i_1}(r)^2 + (r + b_{k_1k_1}\gamma_{k_1i_1}(r))^2]. \end{aligned}$$

The minimum value of function $y \mapsto [y^2 + (r + b_{k_1k_1}y)^2]$ is $r^2/(1 + b_{k_1k_1}^2)$. Hence, irrespective of function $\gamma_{k_1i_1}$,

$$J_{(\bar{A}(r),B,e_{j_1})}(\Gamma(\bar{A}(r),B)) \geq (a_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(A,B))^2 r^2/(1 + b_{k_1k_1}^2).$$

Note that the term $(a_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(A,B))^2$ is independent from r because Γ is communication-less. In addition,

$$J_{(\bar{A}(r),B,e_{j_1})}(\Gamma^\Delta(\bar{A}(r),B)) = \sum_{z\in\mathcal{I}_i} \frac{\bar{a}_{zj_1}^2}{b_{zz}^2} = \sum_{z\in\mathcal{I}_i} \frac{a_{zj_1}^2}{b_{zz}^2}$$

for all r and, thus, $J_{(\bar{A}(r),B,e_{j_1})}(\Gamma^{\Delta}(\bar{A}(r),B))$ is also independent from r. Then, proceeding as in (25), we deduce that

$$r_{\mathcal{P}}(\Gamma) \geq \frac{(a_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(A,B))^2}{(1+b_{k_1k_1}^2)J_{(\bar{A}(r),B,e_{j_1})}(\Gamma^{\Delta}(\bar{A}(r),B))} \lim_{r\to\infty} r^2.$$

Since $(a_{i_1j_1} + b_{i_1i_1}\gamma_{i_1j_1}(A,B)) \neq 0$ by assumption, we then deduce that Γ has an unbounded competitive ratio, which proves the lemma by contrapositive.

F Proof of Theorem 13

We prove that for any control design method $\Gamma \in \mathcal{C}\backslash\{\Gamma^{\Theta}\}$, there exists a plant $P = (A, B, x_0) \in \mathcal{P}$ such that $J_P(\Gamma(A,B)) > J_P(\Gamma^{\Theta}(A,B))$. Like in the proof of Theorem 3.6, we partition the set of limited model information control design methods $\mathcal{C}$ as follows

$$\mathcal{C} = \mathcal{L}^c \cup \mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2 \cup \{\Gamma^{\Theta}\},$$

where

$$\begin{aligned}\mathcal{L} := \{\Gamma \in \mathcal{C} | \exists \Lambda_i : \mathbb{R}^{n_i\times n} \times \mathbb{R}^{n_i\times n_i} \to \mathbb{R}^{n_i\times n_i},\\ [\Gamma(A,B)]_i = \Lambda_i([A]_i, B_{ii})[A]_i, \text{ for all } i = 1,\cdots,q\},\end{aligned}$$

$$\mathcal{W}_0 := \{\Gamma \in \mathcal{L}, \exists i \in \{q-c+1,\ldots,q\} \text{ such that } \Lambda_i([A]_i, B_{ii}) \neq W_i([A]_i, B_{ii})\},$$

with W_i defined as in equation (21),

$$\begin{aligned}\mathcal{W}_1 := \{\Gamma \in \mathcal{L} \setminus \mathcal{W}_0 | \exists i \in \{1,\cdots,q-c\}, \exists j \neq i \text{ and } A_{ij} \in \mathbb{R}^{n_i\times n_j}\\ \text{nonzero such that } I + B_{ii}\Lambda_i([0 \ \cdots \ 0 \ A_{ij} \ 0 \ \cdots \ 0], B_{ii}) \neq 0\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}_2 := \{\Gamma \in \mathcal{L}\backslash\mathcal{W}_0 \cup \mathcal{W}_1 | \exists i \in \{1,\cdots,q-c\}, [A]_i \in \mathbb{R}^{n_i\times n},\\ \text{with appropriate structure such that } I + B_{ii}\Lambda_i([A]_i, B_{ii}) \neq 0\}.\end{aligned}$$

First, we prove that Γ^{Θ} is undominated by control design methods in $\mathcal{L}^c$. Let $\Gamma \in \mathcal{L}^c$ and let i be such that there exists a plant with matrix $\bar{A}$ with the property that sub-controller $[\Gamma]_i([\bar{A}]_i, B_{ii})^T$ does not belong to the linear subspace spanned by the columns of $[\bar{A}]_i^T$. If $1 \leq i \leq q-c$ then, proceeding as in the proof of Theorem 7, we can find matrices A, B and initial condition x_0 such that $J_P(\Gamma(P)) > J_P(\Gamma^{\Delta}(P)) = J_P(\Gamma^{\Theta}(P))$ for $P = (A, B, x_0)$ (with the last equality following from the structure of matrix A). Hence, without loss of generality, we assume that $q - c + 1 \leq i \leq q$. Consider matrix A defined as $[A]_i = [\bar{A}]_i$ and $[A]_j = 0$ for all $j \neq i$. For this particular matrix A and any B, x_0 we know from the proof of the "only if" part of the Theorem 7 that $\Gamma^{\Theta}(A, B, x_0)$ is the globally optimal controller. Hence, every other control design method in $\mathcal{C}$ leads to a controller with greater performance criterion than Γ^{Θ} for this particular type of plants. Therefore, the control design Γ^{Θ} is undominated by control design methods in $\mathcal{L}^c$.

The same reasoning shows that Γ^{Θ} is also undominated by control design methods in $\mathcal{W}_0$.

We now prove that Γ^{Θ} is undominated by control design strategies in $\mathcal{W}_1$. Let $\Gamma \in \mathcal{W}_1$ and let $1 \leq i \leq q-c$ be such that $(I + B_{ii}\Lambda_i([\bar{A}]_i, B_{ii})) \neq 0$ where $[\bar{A}]_i = \begin{bmatrix} 0 & \cdots & 0 & \bar{A}_{ij} & 0 & \cdots & 0 \end{bmatrix}$ for some $j \neq i$. This means that there exists at least one $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ such that $\bar{a}_{i_1 j_1} \neq 0$ and $\bar{a}_{i_1 j_1} + b_{i_1 i_1}\gamma_{i_1 j_1}(A, B) \neq 0$. Because subsystem i is not a sink (since $1 \leq i \leq q-c$), we know that there exists a $z \neq i$ such that $(s_{\mathcal{P}})_{zi} \neq 0$. If $1 \leq z \leq q-c$ we can again proceed as in the proof of Theorem 7 to construct a plant P for which $J_P(\Gamma(P)) > J_P(\Gamma^{\Theta}(P))$. Thus, without loss of generality, we assume that $q-c+1 \leq z \leq q$. Choose an index $z_1 \in \mathcal{I}_z$ and consider the matrix A defined by $[A]_i = [\bar{A}]_i$, $a_{z_1 i_1} = r$ and all other entries equal to zero. Then, $[\Gamma(A, B)]_i = \Lambda_i([A]_i, B_{ii})[A]_i$, $[\Gamma(A, B)]_z = -b_{z_1 z_1}/(1 + b_{z_1 z_1}^2)[A]_z$ (because $\Gamma \notin \mathcal{W}_0 \cup \mathcal{L}^c$), and $[\Gamma(A, B)]_t = 0$ for all $t \neq i, z$. Denoting $\Gamma(A, B)$ by K, we see that

$$\begin{aligned} J_{(A,B,x_0)}(\Gamma(A,B)) \geq x_0^T [&(A+BK)^T K^T K (A+BK) \\ &+((A+BK)^2)^T (A+BK)^2] x_0 \end{aligned}$$

for all $B \in \mathcal{B}(\epsilon)$ and x_0. Taking $x_0 = e_{j_1}$ then results in

$$\begin{aligned} J_{(A,B,e_{j_1})}(\Gamma(A,B)) - J_{(A,B,e_{j_1})}(\Gamma^{\Theta}(A,B)) \geq \\ [k_{z_1 i_1}^2 + (r + b_{z_1 z_1} k_{z_1 i_1})^2](a_{i_1 j_1} + b_{i_1 i_1} k_{i_1 j_1})^2 - \sum_{t \in \mathcal{I}_i} a_{t j_1}^2 / b_{tt}^2. \end{aligned} \tag{38}$$

Note that, irrespective of the choice of the controller gain $k_{z_1 i_1}$,

$$k_{z_1 i_1}^2 + (r + b_{z_1 z_1} k_{z_1 i_1})^2 \geq r^2/(1 + b_{z_1 z_1}^2),$$

and as a result,

$$\lim_{r \to +\infty} [k_{z_1 i_1}^2 + (r + b_{z_1 z_1} k_{z_1 i_1})^2](a_{i_1 j_1} + b_{i_1 i_1} k_{i_1 j_1})^2 = +\infty,$$

because $a_{i_1 j_1} + b_{i_1 i_1} k_{i_1 j_1} \neq 0$. Hence, we can always construct A with appropriate choice of index z and a scalar r large enough to make the cost difference positive. As a result, Γ cannot dominate Γ^{Θ}.

Finally, we prove that Γ^{Θ} is undominated by control design methods in $\mathcal{W}_2$. Let $\Gamma \in \mathcal{W}_2$ and index $1 \leq i \leq q-c$ and model sub-matrices $[\bar{A}]_i$ and B_{ii} such that $I + \Lambda_i([\bar{A}]_i, B_{ii}) \neq 0$. Therefore, we know that there exists at least one index $i_1 \in \mathcal{I}_i$ such that $\bar{A}_{i_1} \neq 0$ and $\bar{A}_{i_1} + b_{i_1 i_1}\Gamma_{i_1}(\bar{A}, B) \neq 0$. Based on the fact that node i is not a sink, we know that there exists $z \neq i$ such that $(s_{\mathcal{P}})_{zi} \neq 0$. For the same reasons as before we again restrict ourselves, without loss of generality, to the case where $q-c+1 \leq z \leq q$. Consider the matrix A defined by $[A]_i = [\bar{A}]_i$ and $a_{z_1 i_1} = r$ and all other entries of A equal to zero. Then, $[A]_i + [\Gamma(A,B)]_i = (I + \Lambda_i([A]_i, B_{ii}))[A]_i$ and $[\Gamma(A,B)]_z = -b_{z_1 z_1}/(1 + b_{z_1 z_1}^2)[A]_z$ (because $\Gamma \notin \mathcal{W}_0 \cup \mathcal{L}^c$). Again, K will stand for $\Gamma(A,B)$. Then, for all $B \in \mathcal{B}(\epsilon)$ and x_0

$$\begin{aligned} J_{(A,B,x_0)}(\Gamma(A,B)) - J_{(A,B,x_0)}(\Gamma^{\Theta}(A,B)) \\ \geq x_0^T (A_{i_1} + b_{i_1 i_1} \Gamma_{i_1}(A,B))^T (A_{i_1} + b_{i_1 i_1} \Gamma_{i_1}(A,B)) x_0 \times \\ r^2 b_{z_1 z_1}^2 / (1 + b_{z_1 z_1}^2)^2 - \sum_{t \in \mathcal{I}_i} x_0^T A_t^T A_t x_0 / b_{tt}^2, \end{aligned}$$

and hence, since $A_{i_1} + b_{i_1 i_1} \Gamma_{i_1}(A,B) \neq 0$, we can choose r large enough to ensure that this difference is strictly positive for some $x_0 \in \mathbb{R}^n$ since the inner matrix will have a strictly positive eigenvalue for large values of r. Thus, the control design strategy $\Gamma \in \mathcal{W}_2$ cannot dominate the control design Γ^{Θ}.

PAPER 2

Dynamic Control Design Based on Limited Model Information

Farhad Farokhi and Karl H. Johansson

Abstract–The design of optimal H_2 dynamic controllers for interconnected linear systems using limited plant model information is considered. Control design strategies based on various degrees of model information are compared using the competitive ratio as a performance metric, that is, the worst case control performance for a given design strategy normalized with the optimal control performance based on full model information. An explicit minimizer of the competitive ratio is found. It is shown that this control design strategy is not dominated by any other strategy with the same amount of model information. The result applies to a class of system interconnections and design information characterized through given plant, control, and design graphs.

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1 Introduction

Many large-scale physical systems are composed of several smaller interconnected units. For these interconnected systems, it seems natural to employ local controllers which observe local states and control local inputs. The problem of designing such subcontrollers is usually addressed in the decentralized and distributed control literature [1–3]. Lately, there has been some efforts in formulating the problem of designing optimal decentralized controllers as a convex optimization problem for some specific classes of subsystem interconnection [4–8]. At the heart of all these decentralized and distributed control problems is the assumption that the control design is done with complete knowledge of the plant model. This is however not always possible in large-scale systems. It might be the case that (a) different subsystems belong to different individuals and they might be unwilling to share their model information since they may consider these information private, (b) the design of each subcontroller is done by a different designer with no access to the global plant model since in the time of design the complete model information is not available, or (c) the designer is interested in designing each subcontroller using only local model information, so that the resulting subcontrollers do not need to be modified if the model parameters of a particular subsystem change over time. We call this special class of control design problems limited model information control design problems [9, 10]. In these problems, we assume that only some part of the plant model information is available to each subcontroller designer, but that the system interconnection structure and the common closed-loop cost function to be minimized are global knowledge.

The main contribution of this paper is to study the influence of the subsystem interconnection, the controller structure, and the amount of model information available to each subdesign on the closed-loop performance that a limited model information control design method can produce. We compare the control design methods using a performance metric called the competitive ratio, that is, the worst case control performance for a given design strategy normalized with the optimal control performance based on full model information. We find an explicit minimizer of the competitive ratio for a wide range of problems. Since this minimizer might not be unique, we show that it is also undominated, that is, there is no other control design method that acts always better while having the same worst-case ratio.

This paper is organized as follows. We formulate the problem of interest in Section 2. We define a control design strategy and find its competitive ratio in Section 3. In Section 4, we study the influence of interconnection pattern between different subsystems on the best limited model information control design method. We further study the achievable performance of limited model information design strategies when the controllers that they can produce are structured in Section 5. The trade-off between the amount of plant information available to different parts of a control design strategy and the quality of controllers it can produce is considered in Section 6. Finally, we give the discussions on extensions in Section 7 and end with the conclusions in Section 8.

1.1 Notation

The sets of integer numbers, natural numbers, real numbers, and complex numbers are denoted respectively by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$. The boundary of the unit circle in $\mathbb{C}$ is shown by $\mathbb{T}$. The space of Lebesgue measurable functions that are bounded on $\mathbb{T}$ is presented by $\mathcal{L}_\infty$ and $\mathcal{RL}_\infty$ is the set of real proper rational transfer functions in $\mathcal{L}_\infty$. Additionally, all other sets are denoted by calligraphic letters such as $\mathcal{P}$ and $\mathcal{A}$.

Matrices are denoted by capital roman letters such as A. The entry in the i^{th} row and the j^{th} column of matrix A is a_{ij}. A_j will denote the j^{th} row of A. A_{ij} denotes a submatrix of matrix A, the dimension and the position of which will be defined in the text.

$A > (\geq)0$ means that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq)B$ means $A - B > (\geq)0$. Let $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$.

All graphs considered in this paper are directed with vertex set $\{1, \ldots, q\}$ for a given $q \in \mathbb{N}$. All self-loops are present in the graphs that we consider in this paper, that is, $(i, i) \in E$ for all $1 \leq i \leq q$. We say that a vertex i is a sink if there does not exist $j \neq i$ such that $(i, j) \in E$. The adjacency matrix $S \in \{0, 1\}^{q\times q}$ of graph G is a matrix whose entry $s_{ij} = 1$ if $(j, i) \in E$ and $s_{ij} = 0$ otherwise for all $1 \leq i, j \leq q$. In this paper, since the set of vertices is fixed for all the graphs, a subgraph of a graph G is a graph whose edge set is a subset of the edge set of G and a supergraph of a graph G is a graph of which G is a subgraph. We use the notation $G' \supseteq G$ to indicate that G' is a supergraph of G.

$\underline{\sigma}(Y)$ and $\bar{\sigma}(Y)$ denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i denotes the column vector with all entries zero except the i^{th} entry which is equal to one. The function $\delta : \mathbb{Z} \to \{0, 1\}$ is the unit-impulse function which is equal to one at origin and zero anywhere else.

2 Problem Formulation

2.1 Plant Model

Let a plant graph $G_\mathcal{P}$ with adjacency matrix $S_\mathcal{P}$ be given. Based on the adjacency matrix $S_\mathcal{P}$, we define the following set of matrices

$$\mathcal{A}(S_\mathcal{P}) = \{\bar{A} \in \mathbb{R}^{n\times n} \mid \bar{A}_{ij} = 0 \in \mathbb{R}^{n_i\times n_j} \text{ for all } 1 \leq i, j \leq q \text{ such that } (s_\mathcal{P})_{ij} = 0\},$$

where for each $1 \leq i \leq q$, $n_i \in \mathbb{N}$ is the order of subsystem i and consequently $\sum_{i=1}^q n_i = n$. Besides, we define

$$\mathcal{B}(\epsilon) = \{\bar{B} \in \mathbb{R}^{n\times n} \mid \underline{\sigma}(\bar{B}) \geq \epsilon, \bar{B}_{ij} = 0 \in \mathbb{R}^{n_i\times n_j} \text{ for all } 1 \leq i \neq j \leq q\},$$

for some given scalar $\epsilon > 0$ and

$$\mathcal{H} = \{\bar{H} \in \mathbb{R}^{n\times n} \mid \det(\bar{H}) \neq 0, \bar{H}_{ij} = 0 \in \mathbb{R}^{n_i\times n_j} \text{ for all } 1 \leq i \neq j \leq q\}.$$

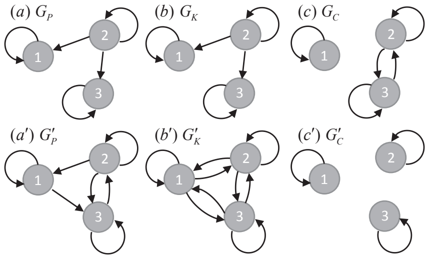

Figure 1: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs, $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs, and $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

Now we can introduce the set $\mathcal{P}$ of plants of interest as the space of all discrete-time linear time-invariant systems

$$x(k+1) = Ax(k) + Bu(k) + Hw(k)\ ;\ x(0) = 0, \tag{1}$$

with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $H \in \mathcal{H}$. With slightly abusing notation, we show a plant $P \in \mathcal{P}$ with triple (A, B, H) since the set $\mathcal{P}$ is clearly isomorph to $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{H}$. We will denote the ordered set of state indices related to subsystem i with $\mathcal{I}_i$, that is, $\mathcal{I}_i := (1 + \sum_{j=1}^{i-1} n_j, \ldots, n_i + \sum_{j=1}^{i-1} n_j)$. For subsystem i, state $\underline{x}_i \in \mathbb{R}^{n_i}$, control input $\underline{u}_i \in \mathbb{R}^{n_i}$, and exogenous input $\underline{w}_i \in \mathbb{R}^{n_i}$ are defined as

$$\underline{x}_i = \begin{bmatrix} x_{\ell_1} \\ \vdots \\ x_{\ell_{n_i}} \end{bmatrix},\ \underline{u}_i = \begin{bmatrix} u_{\ell_1} \\ \vdots \\ u_{\ell_{n_i}} \end{bmatrix},\ \underline{w}_i = \begin{bmatrix} w_{\ell_1} \\ \vdots \\ w_{\ell_{n_i}} \end{bmatrix}$$

where the ordered set of indices $(\ell_1, \ldots, \ell_{n_i}) \equiv \mathcal{I}_i$, and its dynamic is specified by

$$\underline{x}_i(k+1) = \sum_{j=1}^{q} A_{ij}\underline{x}_j(k) + B_{ii}\underline{u}_i(k) + H_{ii}\underline{w}_i(k).$$

An example of a plant graph $G_{\mathcal{P}}$ is given in Figure 1(a). For instance, the plant graph $G_{\mathcal{P}}$ shows that the second subsystem can affect the first and the third subsystems, that is, A_{12} and A_{32} can be nonzero. The first system is also a sink in the plant graph $G_{\mathcal{P}}$. An example of a plant graph $G'_{\mathcal{P}}$ without sink is given in Figure 1(a').

2.2 Controller

Let a control graph $G_{\mathcal{K}}$ with adjacency matrix $S_{\mathcal{K}}$ be given. In this paper, we are interested in dynamic discrete-time linear time-invariant state feedback control laws of the form

$$x_K(k+1) = A_K x_K(k) + B_K x(k)\ ;\ x_K(0) = 0,$$

$$u(k) = C_K x_K(k) + D_K x(k),$$

which can also be represented as the transfer function

$$K \triangleq \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = C_K(zI - A_K)^{-1}B_K + D_K,$$

where z is the symbol for one time-step forward shift operator. The controller K must belong to

$$\begin{aligned}\mathcal{K}(S_{\mathcal{K}}) = \{\bar{K} \in \ (\mathcal{RL}_\infty)^{n\times n} | \bar{K}_{ij} = 0 \in (\mathcal{RL}_\infty)^{n_i \times n_j} \\ \text{for all } 1 \le i, j \le q \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}.\end{aligned}$$

We refer to the set of controllers as $\mathcal{K}$ when adjacency matrix $S_{\mathcal{K}}$ can be deduced from the context or it is not relevant.

Figure 1(b) shows an example of an incomplete control graph $G_{\mathcal{K}}$ that characterizes a set of structured controllers. For instance, using control graph $G_{\mathcal{K}}$, we know that the third subsystem only has access to state measurements of the second subsystem beside its own state measurements, that is, $K_{31} = 0$ while K_{32} and K_{33} can be nonzero.

2.3 Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}$. Let a control design method Γ be partitioned according to subsystems dimensions like

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1q} \\ \vdots & \ddots & \vdots \\ \Gamma_{q1} & \cdots & \Gamma_{qq} \end{bmatrix} \tag{2}$$

and a design graph $G_{\mathcal{C}}$ with adjacency matrix $S_{\mathcal{C}}$ be given. Each element Γ_{ij} is a mapping $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{H} \to (\mathcal{RL}_\infty)^{n_i \times n_j}$. We say that Γ has structure $G_{\mathcal{C}}$ if, for all $1 \le i \le q$, the subsystem i subcontroller is constructed with the knowledge of those subsystems $1 \le j \le q$ plant model such that $(j, i) \in E_{\mathcal{C}}$, that is, the mapping $[\, \Gamma_{i1} \ \cdots \ \Gamma_{iq} \,]$ is only a function of $\{[\, A_{j1} \ \cdots \ A_{jq} \,], B_{jj}, H_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}$. The set of all these limited model information control design methods with structure $G_{\mathcal{C}}$ is denoted by $\mathcal{C}$.

Figure 1(c) shows an example of a design graph $G_{\mathcal{C}}$. For instance, using this design graph $G_{\mathcal{C}}$, we realize that the third subsystem model is available to the designer of the second subsystem controller but not the first subsystem model. Figure 1(c') illustrates an example of a fully disconnected design graph $G'_{\mathcal{C}}$ with self-loops only which shows that the controller of all subsystems are constructed using only their own model information.

2.4 Performance Metric

The considered performance metrics is a modified version of the performance metrics originally defined in [9, 10]. Let us start with introducing the closed-loop performance measure.

To each plant $P = (A, B, H) \in \mathcal{P}$ and controller $K \in \mathcal{K}$, we associate a performance measure which is the H_2 norm of the transfer function between the exogenous input $w(k)$ and the output

$$y(k) = \begin{bmatrix} C^T & 0 \end{bmatrix}^T x(k) + \begin{bmatrix} 0 & D^T \end{bmatrix}^T u(k),$$

where the matrices $C \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{n \times n}$ are block diagonal *full-rank* matrices with each diagonal block entry belonging to $\mathbb{R}^{n_i \times n_i}$. Figure 2 illustrates the feedback system with the given controller K and the overall-plant

$$\hat{P} = \left[\begin{array}{c|cc} A & H & B \\ \hline \hat{C} & 0 & \hat{D} \\ I & 0 & 0 \end{array}\right]$$

where $\hat{C} = \begin{bmatrix} C^T & 0 \end{bmatrix}^T$ and $\hat{D} = \begin{bmatrix} 0 & D^T \end{bmatrix}^T$. Using the notation $\mathcal{F}(\hat{P}, K)$ for the closed-loop transfer function from $w(k)$ to $y(k)$, the performance measure can be written as

$$J_P(K) = \|\mathcal{F}(\hat{P}, K)\|_2. \tag{3}$$

We make the following standing assumption:

Assumption 3 $C = D = I$.

This is without loss of generality because the change of variables $(\bar{x}, \bar{u}) = (Cx, Du)$ transforms the output of the system and its state space representation into

$$y(k) = \begin{bmatrix} I & 0 \end{bmatrix}^T \bar{x}(k) + \begin{bmatrix} 0 & I \end{bmatrix}^T \bar{u}(k),$$

and

$$\bar{x}(k+1) = CAC^{-1}\bar{x}(k) + CBD^{-1}\bar{u}(k).$$

This is done without changing the plant, control, or design graphs because of the block diagonal structure of matrices C and D.

Definition 2.1 (Competitive Ratio) *Let a plant graph $G_{\mathcal{P}}$, a control graph $G_{\mathcal{K}}$, and a constant $\epsilon > 0$ be given. Let us assume that, for each plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that*

$$J_P(K^*(P)) \leq J_P(K), \ \forall K \in \mathcal{K}.$$

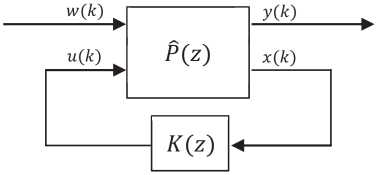

Figure 2: The feedback system with the given controller K and the overall-plant $\hat{P}$.

The competitive ratio of a control design method Γ *is defined as*

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,H)\in\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$*" equals one.*

Definition 2.2 (Domination) *A control design method* Γ' *is said to dominate another control design method* Γ *if*

$$J_P(\Gamma'(P)) \leq J_P(\Gamma(P)),\ \forall\ P = (A, B, H) \in \mathcal{P}, \tag{4}$$

with strict inequality holding for at least one plant in $\mathcal{P}$*. When* $\Gamma \in \mathcal{C}$ *and no control design method* $\Gamma' \in \mathcal{C}$ *exists that satisfies (4), we say that* Γ *is undominated in* $\mathcal{C}$*.*

2.5 Mathematical Problem Formulation

Now we can formulate the primary question concerning the connection between closed-loop performance and limited model information control design strategies. For a given plant graph $G_{\mathcal{P}}$, control graph $G_{\mathcal{K}}$, and design graph $G_{\mathcal{C}}$, we want to solve

$$\arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma). \tag{5}$$

Since the solution to this problem might not be unique, we are interested in finding a minimizer that is also undominated. These solutions are the best worst-case designs with limited model information.

3 Preliminary Results

In order to give the main results of the paper, we need to define a control design strategy and find its competitive ratio.

Definition 2.3 *Let a plant graph* $G_{\mathcal{P}}$ *and a constant* $\epsilon > 0$ *be given. The control design method* Γ^{Θ} *is defined as*

$$\Gamma^{\Theta}(P) = -\operatorname{diag}(W_1(P), \ldots, W_q(P))A, \tag{6}$$

for all plants $P = (A, B, H) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{H}$, *where*

$$W_i(P) = \begin{cases} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii}, & \text{if } i \text{ is a sink,} \\ B_{ii}^{-1}, & \text{otherwise,} \end{cases}$$

and for each sink i *the matrix* X_{ii} *is the unique positive definite solution of the discrete algebraic Riccati equation*

$$A_{ii}^T X_{ii} A_{ii} - A_{ii}^T X_{ii} B_{ii} (I + B_{ii}^T X_{ii} B_{ii})^{-1} B_{ii}^T X_{ii} A_{ii} - X_{ii} + I = 0.$$

The control design method Γ^{Θ} applies the so-called deadbeat strategy [10] to every subsystem that is not a sink (thus those closed-loop subsystems reach origin in just one time-step [11]) and, for every sink, applies the same optimal control law as if the node were decoupled from the rest of the graph.

Lemma 16 *The competitive ratio of the control design method* Γ^{Θ} *defined in (6) is* $r_{\mathcal{P}}(\Gamma^{\Theta}) = \sqrt{1 + 1/\epsilon^2}$ *if one of the following conditions is satisfied:*

(*a*) *the plant graph* $G_{\mathcal{P}}$ *contains no isolated node and the control graph* $G_{\mathcal{K}}$ *is a complete graph;*

(*b*) *the acyclic plant graph* $G_{\mathcal{P}}$ *contains no isolated node and* $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$.

Proof: Let $K_C^*(P)$ denotes the optimal static full-state feedback (centralized) controller for each plant $P \in \mathcal{P}$. According to the proof of the "only if" part of Theorem 3.6 in [10], we have

$$Z \leq A^T B^{-T} B^{-1} A + I, \tag{7}$$

for all plants $P = (A, B, H) \in \mathcal{P}$, where Z is the unique positive definite solution of discrete algebraic Lyapunov equation

$$(A + B\Gamma^{\Theta}(P))^T Z (A + B\Gamma^{\Theta}(P)) - Z + I + \Gamma^{\Theta}(P)^T \Gamma^{\Theta}(P) = 0. \tag{8}$$

Thus, the cost of the control design strategy Γ^{Θ} for each plant $P = (A, B, H)$ is upper-bounded as

$$\begin{aligned} J_P(\Gamma^{\Theta}(P))^2 &= \operatorname{tr}\left(H^T Z H\right) \\ &\leq \operatorname{tr}\left(H^T \left(A^T B^{-T} B^{-1} A + I\right) H\right). \end{aligned} \tag{9}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix. According to Theorem 3.2 in [10], it is evident that

$$A^T B^{-T} B^{-1} A \leq \left(1 + 1/\epsilon^2\right)(X - I),$$

and equivalently

$$\operatorname{tr}(H^T A^T B^{-T} B^{-1} AH) \leq \left(1 + 1/\epsilon^2\right) \operatorname{tr}(H^T(X - I)H), \tag{10}$$

where X is the unique positive definite solution of discrete algebraic Riccati equation

$$A^T XA - A^T XB(I + B^T XB)^{-1} B^T XA = X - I. \tag{11}$$

Putting (10) in (9), we get

$$\begin{aligned} J_P(\Gamma^\Theta(P))^2 &\leq \left(1 + 1/\epsilon^2\right) \operatorname{tr}(H^T XH) \\ &= \left(1 + 1/\epsilon^2\right) J_P(K_C^*(P))^2. \end{aligned}$$

Clearly, because $J_P(K_C^*(P)) \leq J_P(K^*(P))$, irrespective of the control graph $G_\mathcal{K}$, we have

$$J_P(\Gamma^\Theta(P))^2 \leq \left(1 + 1/\epsilon^2\right) J_P(K^*(P))^2,$$

and as a result

$$r_\mathcal{P}(\Gamma^\Theta) = \sup_{P=(A,B,H)\in\mathcal{P}} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*(P))} \leq \sqrt{1 + 1/\epsilon^2}.$$

To show that this upper-bound is tight, we should exhibit plants for which it is attained.

Part a: Condition (a) is satisfied. Since there is no isolated node in the plant graph, we can pick indices $1 \leq i \neq j \leq q$ such that $(s_\mathcal{P})_{ij} \neq 0$. The rest of the proof is given in two different cases.

Case a.1: Node i is not a sink. Pick indices $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(s) = se_{i_1} e_{j_1}^T$, $B = \epsilon I$, and $H = I$. We get

$$r_\mathcal{P}(\Gamma^\Theta) \geq \lim_{s\to\infty} \sqrt{\frac{s^2/\epsilon^2 + n}{s^2/(1 + \epsilon^2) + n}} = \sqrt{1 + 1/\epsilon^2},$$

since the unique positive definite solution of discrete algebraic Riccati equation in (11) is $X = I + [s^2/(1+\epsilon^2)]e_{j_1} e_{j_1}^T$, and as a result $J_P(K^*(P)) = \sqrt{s^2/(1 + \epsilon^2) + n}$.

Case a.2: Node i is a sink. We know $(s_\mathcal{P})_{ii} \neq 0$ since all the self-loops are present. Pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r, s) = re_{i_1} e_{i_1}^T + se_{i_1} e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to Theorem 3.8 in [10], we get

$$J_P(\Gamma^\Theta(P)) = \sqrt{\beta_\Theta(s^2 + r^2) + n},$$

where

$$\beta_\Theta = \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2ar^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 - \epsilon^2 - 1}{2\epsilon^2 r^2}.$$

Again, using Theorem 3.8 in [10], the optimal closed-loop performance is

$$J_P(K^*(P)) = \sqrt{\beta_{K^*}(s^2+r^2)+n},$$

where β_{K^*} is

$$\beta_{K^*} = \frac{\epsilon^2 s^2 + r^2(1+\epsilon^2) - (\epsilon^2+1)^2 + \sqrt{c_+ c_-}}{2\epsilon^2(\epsilon^2+1)(s^2+r^2)},$$
$$c_\pm = \epsilon^2 s^2 + (r^2 \pm 2r)(\epsilon^2+1) + (\epsilon^2+1)^2.$$

Then, we get

$$r_{\mathcal{P}}(\Gamma^\Theta) \geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*(P))} = \sqrt{1+1/\epsilon^2}.$$

Part b: Condition (b) is satisfied. Any acyclic directed graph has at least one sink. Let i denote a sink in plant graph $G_{\mathcal{P}}$. Since there is no isolated node in the plant graph, there exists an index $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r,s) = re_{i_1}e_{i_1}^T + se_{i_1}e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to Lemma 4.1 in [12], we get

$$J_P(K^*_{\mathcal{P}}(P)) = \sqrt{\beta_{K^*}s^2 + \beta_\Theta r^2 + n},$$

where $K^*_{\mathcal{P}}(P)$ is the optimal controller when $G_{\mathcal{K}}$ is equal to $G_{\mathcal{P}}$. This results in

$$\begin{aligned} r_{\mathcal{P}}(\Gamma^\Theta) &\geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*(P))} \\ &\geq \lim_{r\to\infty, \frac{s}{r}\to\infty} \frac{J_P(\Gamma^\Theta(P))}{J_P(K^*_{\mathcal{P}}(P))} = \sqrt{1+1/\epsilon^2} \end{aligned}$$

since clearly $J_P(K^*(P)) \leq J_P(K^*_{\mathcal{P}}(P))$. ∎

Lemma 16 shows that, if we apply the control design strategy Γ^Θ to a particular plant, the performance of the closed-loop system, at most, can be $\sqrt{1+1/\epsilon^2}$ times the cost of the optimal control design strategy K^*.

There is no loss of generality in assuming that the plant graph $G_{\mathcal{P}}$ contains no isolated node since it is always possible to design an optimal controller for an isolated subsystem without any model information about the other subsystems and without affecting them. In particular, this implies that there are $q \geq 2$ vertices in the plant graph.

4 Plant Graph Influence on Achievable Performance

In this section, we study the achievable closed-loop performance, in terms of the competitive ratio and the domination, for different plant interconnection pattern. The next theorem shows that the control design strategy Γ^Θ is an undominated minimizer of the competitive ratio for all given plant graphs $G_{\mathcal{P}}$ when the control graph $G_{\mathcal{K}}$ is a complete graph and the design graph $G_{\mathcal{C}}$ is fully disconnected.

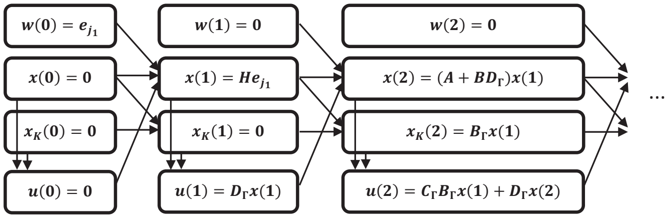

Figure 3: State transition of the closed-loop system and its controller as a function of time for the exogenous input $w(k) = \delta(k)e_{j_1}$.

Theorem 17 *Let the plant graph $G_{\mathcal{P}}$ contain no isolated node, the control graph $G_{\mathcal{K}}$ be a complete graph, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$. Furthermore, the control design strategy Γ^{Θ} is undominated by set of limited model information control design strategies with design graph $G_{\mathcal{C}}$.*

Proof: We use the following notation

$$\Gamma(P) = \left[\begin{array}{c|c} A_\Gamma(P) & B_\Gamma(P) \\ \hline C_\Gamma(P) & D_\Gamma(P) \end{array}\right],$$

to work with different parts of the state-space representation of a control design strategy Γ. The entries $A_\Gamma(P)$, $B_\Gamma(P)$, $C_\Gamma(P)$, and $D_\Gamma(P)$ are matrices with appropriate dimension for each plant $P = (A, B, H) \in \mathcal{P}$. The matrices $A_\Gamma(P)$ and $C_\Gamma(P)$ are block diagonal matrices since different subcontrollers should not share state variables (each controller should be implemented in a decentralized fashion). This realization is not necessarily a minimal realization.

Consider indices $1 \leq i \neq j \leq q$ such that $(s_{\mathcal{P}})_{ij} \neq 0$ (this is always possible since there is no isolated node in the plant graph). The rest of the proof is given in two different cases.

Case 1: Node i is not a sink. Therefore, there exists an index $\ell \neq i$ such that $(s_{\mathcal{P}})_{\ell i} \neq 0$. Pick indices $\ell_1 \in \mathcal{I}_\ell$, $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A(r, s) = se_{i_1}e_{j_1}^T + re_{\ell_1}e_{i_1}^T$ and $B = \epsilon I$. Let $H_{jj} = rI$ and $H_{tt} = I$ for all $t \neq j$. Using the exogenous impulse input $w(k) = \delta(k)e_{j_1}$ and the time-steps given in Figure 3, we get

$$\begin{aligned} J_P(\Gamma(P))^2 &\geq u_{\ell_1}(2)^2 + x_{\ell_1}(3)^2 \\ &= u_{\ell_1}(2)^2 + \left(r^2(s + \epsilon(d_\Gamma)_{i_1 j_1}(s)) + \epsilon u_{\ell_1}(2)\right)^2 \\ &\geq r^4(s + \epsilon(d_\Gamma)_{i_1 j_1}(s))^2/(\epsilon^2 + 1), \end{aligned}$$

because, irrespective of the choice of $u_{\ell_1}(2)$, the function $u_{\ell_1}(2)^2 + (r^2(s+\epsilon(d_\Gamma)_{i_1 j_1}(s)) + \epsilon u_{\ell_1}(2))^2$ is lower-bounded by $r^4(s+\epsilon(d_\Gamma)_{i_1 j_1}(s))^2/(\epsilon^2+1)$. It is worth mentioning that $(d_\Gamma)_{i_1 j_1}(s)$ is only a function of the scalar s and it is independent of the scalar r, since r is in model parameters of subsystems $\ell, j \neq i$ and the design graph is fully disconnected. On the other hand

$$\begin{aligned} J_P(\Gamma^\Delta(P)) &= \sqrt{\operatorname{tr}\left(H^T\left((1/\epsilon^2)A^T A + I\right)H\right)} \\ &= \sqrt{(s^2r^2 + r^2)/\epsilon^2 + n - n_j + n_j r^2}, \end{aligned}$$

where Γ^Δ is the deadbeat control design strategy and it is defined as $\Gamma^\Delta(P) = -B^{-1}A$ [10]. Therefore

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &= \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(K^*(P))} \\ &= \sup_{P\in\mathcal{P}} \left[\frac{J_P(\Gamma(P))}{J_P(\Gamma^\Delta(P))}\frac{J_P(\Gamma^\Delta(P))}{J_P(K^*(P))}\right] \\ &\geq \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma(P))}{J_P(\Gamma^\Delta(P))} \\ &\geq \lim_{r\to\infty} \sqrt{\frac{r^4(s+\epsilon(d_\Gamma)_{i_1 j_1}(s))^2/(\epsilon^2+1)}{(s^2r^2+r^2)/\epsilon^2 + n - n_j + n_j r^2}}. \end{aligned} \tag{12}$$

since $J_P(\Gamma^\Delta(P)) \geq J_P(K^*(P))$ for all plants $P \in \mathcal{P}$. The competitive ratio $r_{\mathcal{P}}(\Gamma)$ is bounded only if $s + \epsilon(d_\Gamma)_{i_1 j_1}(s) = 0$. Therefore, there is no loss of generality in assuming that $(d_\Gamma)_{i_1 j_1}(s) = -s/\epsilon$ because otherwise the $r_{\mathcal{P}}(\Gamma)$ is infinity and the inequality $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^\Theta)$ is trivially satisfied. Now, let us redefine $A(s) = s e_{i_1} e_{j_1}^T$, $H = I$ and $B = \epsilon I$. Since the parameters of the subsystem i is not changed, we have $(d_\Gamma)_{i_1 j_1}(s) = -s/\epsilon$. Therefore, for the same impulse exogenous input $w(k) = \delta(k) e_{j_1}$, we have

$$J_P(\Gamma(P))^2 \geq u_{i_1}(1)^2 = (d_\Gamma)_{i_1 j_1}(s)^2 = s^2/\epsilon^2,$$

and

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{s\to\infty} \sqrt{\frac{s^2/\epsilon^2}{s^2/(1+\epsilon^2)+n}} = \sqrt{1+1/\epsilon^2}, \tag{13}$$

since similar to Case a.1 in the proof of Lemma 16, we have $J_P(K^*(P)) = \sqrt{s^2/(1+\epsilon^2)+n}$.

Case 2: Node i is a sink. We have $(s_{\mathcal{P}})_{ii} \neq 0$ since all the self-loops are present. Let us pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r,s) = r e_{i_1} e_{i_1}^T + s e_{i_1} e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to the proof of the "only if" part of Theorem 3.6 in [10], for this particular family of plants, $\Gamma^\Theta(P)$ is the globally optimal H_2 state-feedback controller. Now using Case a.2 in the proof of Lemma 16, it is easy to see that $r_{\mathcal{P}}(\Gamma) \geq \sqrt{1+1/\epsilon^2}$.

To prove that the control design strategy Γ^{Θ} is undominated by set of limited model information control design strategies $\Gamma \in \mathcal{C}$, we construct plants $P = (A, B, H) \in \mathcal{P}$ that satisfy $J_P(\Gamma(P)) > J_P(\Gamma^{\Theta}(P))$ for any control design method $\Gamma \in \mathcal{C} \setminus \{\Gamma^{\Theta}\}$. The detailed proof of this part is given in [12]. ∎

As an example, consider the limited model information design problem given by the plant graph $G'_{\mathcal{P}}$ in Figure $1(a')$, the control graph $G'_{\mathcal{K}}$ in Figure $1(b')$, and the design graph $G'_{\mathcal{C}}$ in Figure $1(c')$. Theorem 17 shows that the control design strategy Γ^{Θ} is the best control design strategy that one can propose based on the local model of subsystems since it is an undominated minimizer of the competitive ratio.

5 Control Graph Influence on Achievable Performance

In this section, we study the structured controllers and their influence on the achievable closed-loop performance of the limited model information control design strategies. Note that finding the optimal control design strategy $K^*(P)$ is numerically intractable for general plant and control graphs. We use the results in [6, 7] which give an explicit solution to the problem of designing optimal decentralized controller for some special classes of subsystems interconnection and controller structures. Therefore, we assume that the plant graph $G_{\mathcal{P}}$ is an acyclic directed graph and the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. Note that the control design strategy Γ^{Θ} is still applicable in this scenario.

Theorem 18 *Let the acyclic plant graph $G_{\mathcal{P}}$ contain no isolated node, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$. Furthermore, the control design strategy Γ^{Θ} is undominated by set of limited model information control design strategies with design graph $G_{\mathcal{C}}$.*

Proof: Any acyclic directed graph has at least one sink. Let i denote a sink in plant graph $G_{\mathcal{P}}$. Since there is no isolated node in the plant graph, there exists an index $j \neq i$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Pick $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$. Let $A(r, s) = re_{i_1}e_{i_1}^T + se_{i_1}e_{j_1}^T$, $B = \epsilon I$, and $H = I$. According to the proof of the "only if" part of Theorem 3.6 in [10], for this particular family of plants, $\Gamma^{\Theta}(P)$ is the globally optimal H_2 state-feedback controller. Now using Part b of the proof of Lemma 16, it is easy to see that $r_{\mathcal{P}}(\Gamma) \geq \sqrt{1 + 1/\epsilon^2}$.

The detailed proof of the part that control design strategy Γ^{Θ} is undominated is given in [12]. ∎

For instance, consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure $1(a)$, the control graph $G_{\mathcal{K}}$ in Figure $1(b)$, and the design graph $G'_{\mathcal{C}}$ in Figure $1(c')$. Theorem 18 illustrates that the control design strategy Γ^{Θ} is again the best control design strategy that one can propose based on the local model of subsystems, because it is an undominated minimizer of the competitive ratio.

6 Design Graph Influence on Achievable Performance

In this section, we try to determine the amount of the model information that we need in each subsystem to be able to setup a control design strategy Γ with a smaller competitive ratio than the control design strategy Γ^{Θ}.

Theorem 19 *Let the plant graph $G_{\mathcal{P}}$ and the design graph $G_{\mathcal{C}}$ be given and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. If the plant graph $G_{\mathcal{P}}$ contains the path $j \rightarrow i \rightarrow \ell$ with distinct vertices i, j, and ℓ while $(\ell, i) \notin E_{\mathcal{C}}$, then $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$ for all $\Gamma \in \mathcal{C}$.*

Proof: Because of the path $j \rightarrow i \rightarrow \ell$ with distinct vertices i, j, and k, we have $(s_{\mathcal{P}})_{ij} \neq 0$ and $(s_{\mathcal{P}})_{\ell i} \neq 0$. Pick indices $\ell_1 \in \mathcal{I}_\ell$, $i_1 \in \mathcal{I}_i$ and $j_1 \in \mathcal{I}_j$ and define $A(r,s) = s e_{i_1} e_{j_1}^T + r e_{\ell_1} e_{i_1}^T$, $B = \epsilon I$, and $H = I$. Similar to the proof of Theorem 17, using the exogenous impulse input $w(k) = \delta(k) e_{j_1}$ and the time-steps given in Figure 3, we get

$$J_P(\Gamma(P))^2 \geq r^2 (s + \epsilon (d_\Gamma)_{i_1 j_1}(s))^2 / (\epsilon^2 + 1),$$

Again, it should be noted that $(d_\Gamma)_{i_1 j_1}(s)$ is only a function of the scalar s, and it is independent of the scalar r because r has appeared in model matrices of the subsystem $\ell \neq i$, and $(\ell, i) \notin E_{\mathcal{C}}$. We claim that for the competitive ratio to be bounded there should exist a positive constant $\theta \in \mathbb{R}$ independent of scalars s such that $|s + \epsilon (d_\Gamma)_{i_1 j_1}(s)| \leq \theta$. Assume this claim is not true, thus, there exist a sequence of scalars $\{s_z\}_{z=1}^{\infty} \subset \mathbb{R}$ such that

$$\lim_{z \to \infty} |s_z + \epsilon (d_\Gamma)_{i_1 j_1}(s_z)| = +\infty.$$

Clearly, using (12) we get

$$\begin{aligned} r_{\mathcal{P}}(\Gamma) &\geq \lim_{z \to \infty, \frac{r}{s_z} \to \infty} \sqrt{\frac{r^2 |s_z + \epsilon (d_\Gamma)_{i_1 j_1}(s_z)|^2 / (\epsilon^2 + 1)}{(s_z^2 + r^2)/\epsilon^2 + n}} \\ &= +\infty. \end{aligned}$$

since $J_P(\Gamma^{\Delta}(P)) = \sqrt{(s_z^2 + r^2)/\epsilon^2 + n}$. Now, lets redefine $A(s) = s e_{i_1} e_{j_1}^T$. Since the model parameters of the subsystem i is not changed, and its controller is not a function of the model parameters of subsystem ℓ, the design entry $(d_\Gamma)_{i_1 j_1}(s)$ stays the same. Therefore, $|s + \epsilon (d_\Gamma)_{i_1 j_1}(s)| \leq \theta$ for all $s \in \mathbb{R}$, and as a result, for large enough $|s|$, we get $|(d_\Gamma)_{i_1 j_1}(s)| \geq (|s| - \theta)/\epsilon$. Therefore, using the exogenous impulse input $w(k) = \delta(k) e_{j_1}$, we get

$$J_P(\Gamma(P))^2 \geq u_{i_1}(1)^2 = (d_\Gamma)_{i_1 j_1}(s)^2 \geq (|s| - \theta)^2 / \epsilon^2,$$

and

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{s\to\infty} \sqrt{\frac{(|s|-\theta)^2/\epsilon^2}{s^2/(1+\epsilon^2)+n}} = \sqrt{1+1/\epsilon^2}.$$

For this special plant, we know $K_C^*(P) = -\epsilon/(1+\epsilon^2)A$ belongs to the set $\mathcal{K}(S_{\mathcal{K}})$ since the control graph $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$, and consequently $J_P(K^*(P)) \leq J_P(K_C^*(P))$ because $K^*(P)$ has a lower cost than any other controller is $\mathcal{K}(S_{\mathcal{K}})$. On the other hand, clearly, for any plant $J_P(K_C^*(P)) \leq J_P(K^*(P))$. Therefore, for this special plant

$$J_P(K^*(P)) = J_P(K_C^*(P)) = \sqrt{s^2/(1+\epsilon^2)+n}.$$

This concludes the proof. ■

Consider the limited model information design problem given by the plant graph $G'_{\mathcal{P}}$ in Figure 1(a'), the control graph $G'_{\mathcal{K}}$ in Figure 1(b'), and the design graph $G_{\mathcal{C}}$ in Figure 1(c). Note that there is a path $3 \to 2 \to 1$ in the plant graph $G_{\mathcal{P}}$ but the edge $1 \to 2$ is not present in the design graph $G_{\mathcal{C}}$. Therefore, using Theorem 19, it is easy see that $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta})$ for any $\Gamma \in \mathcal{C}$.

7 Extensions

In this section, we relax the assumption that all the subsystems are required to be fully-actuated, that is, $B \in \mathcal{B}(\epsilon)$ is square invertible. To do so, we assume that plant graph $G_{\mathcal{P}}$ is an acyclic directed graph with $c \geq 1$ sinks since any acyclic graph has at least one sink. Accordingly, its adjacency matrix $S_{\mathcal{P}}$ is of the form

$$S_{\mathcal{P}} = \left[\begin{array}{c|c} (S_{\mathcal{P}})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_{\mathcal{P}})_{21} & (S_{\mathcal{P}})_{22} \end{array}\right], \tag{14}$$

where

$$(S_{\mathcal{P}})_{11} = \begin{bmatrix} (s_{\mathcal{P}})_{11} & \cdots & (s_{\mathcal{P}})_{1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q-c,1} & \cdots & (s_{\mathcal{P}})_{q-c,q-c} \end{bmatrix},$$

$$(S_{\mathcal{P}})_{21} = \begin{bmatrix} (s_{\mathcal{P}})_{q-c+1,1} & \cdots & (s_{\mathcal{P}})_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{q,1} & \cdots & (s_{\mathcal{P}})_{q,q-c} \end{bmatrix},$$

and $(S_{\mathcal{P}})_{22} = \mathrm{diag}((s_{\mathcal{P}})_{q-c+1,q-c+1}, \dots, (s_{\mathcal{P}})_{qq})$, where we assume, without loss of generality, that the vertices are numbered such that the sinks are labeled $q-c+1, \dots, q$. We define the set $\mathcal{P}'$ of plants of interest as the set of all triples $(A, B, H) \in \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}'(\epsilon) \times \mathcal{H}$ where

$$\mathcal{B}'(\epsilon) = \{\bar{B} \in \mathbb{R}^{n\times m} \mid \underline{\sigma}(\bar{B}) \geq \epsilon,\, \bar{B}_{ij} = 0 \in \mathbb{R}^{n_i \times m_j} \text{ for all } 1 \leq i \neq j \leq q\}.$$

Each $m_i \in \mathbb{N}$ is the number of control inputs in subsystem i, and consequently $\sum_{i=1}^{q} m_i = m$. Let relax $m_i \leq n_i$ for all $q-c+1 \leq i \leq q$ but force $m_i = n_i$ otherwise. In addition, all matrices A and B must satisfy

(a) (A_{ii}, B_{ii}) is controllable,

(b) $\text{span}(A_{ij}) \subseteq \text{span}(B_{ii})$ for all $j \neq i$ or equivalently there should exist a matrix $W_i \in \mathbb{R}^{m_i \times (n-n_i)}$ such that $[A_{i1} \ \cdots \ A_{i,i-1} \ A_{i,i+1} \ \cdots \ A_{iq}] = B_{ii} W_i$,

for all $q-c+1 \leq i \leq q$. For this new set of plants, the control design strategy Γ^{Θ} is still applicable since it does not require B_{ii} to be invertible for $q-c+1 \leq i \leq q$.

Now we are ready to solve the problem (5) for this set of underactuated plants $\mathcal{P}'$.

Theorem 20 *Let the acyclic plant graph $G_{\mathcal{P}}$ contain no isolated node, the control graph $G_{\mathcal{K}}$ be equal to the plant graph $G_{\mathcal{P}}$, and the design graph $G_{\mathcal{C}}$ be a totally disconnected graph. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta}) = \sqrt{1+1/\epsilon^2}$ if $(S_{\mathcal{P}})_{11}$ is not diagonal. Furthermore, the control design strategy Γ^{Θ} is undominated by set of limited model information control design strategies with design graph $G_{\mathcal{C}}$.*

Proof: Similar to (14), we can write any $A \in \mathcal{A}(S_{\mathcal{P}})$ as

$$A = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

where

$$\tilde{A}_{11} = \begin{bmatrix} A_{11} & \cdots & A_{1,q-c} \\ \vdots & \ddots & \vdots \\ A_{q-c,1} & \cdots & A_{q-c,q-c} \end{bmatrix},$$

$$\tilde{A}_{21} = \begin{bmatrix} A_{q-c+1,1} & \cdots & A_{q-c+1,q-c} \\ \vdots & \ddots & \vdots \\ A_{q1} & \cdots & A_{q,q-c} \end{bmatrix},$$

and $\tilde{A}_{22} = \text{diag}(A_{q-c+1,q-c+1}, \ldots, A_{qq})$. Clearly, if we apply deadbeat to all subsystems that are not sinks, the other subsystems (i.e., sinks) become decoupled (see Theorem 3.6 in [10]), and as a result

$$J_P(\Gamma^{\Theta}(P))^2 = J^{(1)}(\tilde{A}_{11}, \tilde{B}_{11}, \tilde{H}_{11}) + J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22})$$

where $H = \text{diag}(\tilde{H}_{11}, \tilde{H}_{22})$, $B = \text{diag}(\tilde{B}_{11}, \tilde{B}_{22})$, $J^{(1)}(\tilde{A}_{11}, \tilde{B}_{11}, \tilde{H}_{11})$ is the cost of applying deadbeat control design to the nodes that are not sinks, and $J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22})$ is the cost of applying the same optimal control law as if the sinks were decoupled from the rest of the graph. Thus, we get

$$J^{(1)}(\tilde{A}_{11}, \tilde{B}_{11}, \tilde{H}_{11}) = \text{tr}(\tilde{H}_{11}^T \tilde{A}_{11}^T \tilde{B}_{11}^{-T} \tilde{B}_{11}^{-1} \tilde{A}_{11} \tilde{H}_{11})$$

and

$$J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22}) \leq \mathrm{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}) + \mathrm{tr}(\tilde{H}_{11}^T \tilde{A}_{21}^T \tilde{B}_{22}^{\dagger T} \tilde{B}_{22}^{\dagger} \tilde{A}_{21} \tilde{H}_{11}) \qquad (15)$$

where $\tilde{B}_{22}^{\dagger} = (B_{22}^T B_{22})^{-1} B_{22}^T$. The inequality in (15) is true since $J^{(2)}(\tilde{A}_{21}, \tilde{A}_{22}, \tilde{B}_{22}, \tilde{H}_{22})$ is the cost of the optimal control law as if the sinks were decoupled from the rest of the graph (see Theorem 3.6 in [10]), and it certainly has a lower cost than any other controller particularly

$$K_2 = -[\tilde{B}_{22}^{\dagger} \tilde{A}_{21} \ \ (I + \tilde{B}_{22}^T Y \tilde{B}_{22})^{-1} \tilde{B}_{22}^T Y \tilde{A}_{22}],$$

where Y is the unique positive definite solution of discrete algebraic Riccati equation

$$\tilde{A}_{22}^T Y \tilde{A}_{22} - \tilde{A}_{22}^T Y \tilde{B}_{22} (I + \tilde{B}_{22}^T Y \tilde{B}_{22})^{-1} \tilde{B}_{22}^T Y \tilde{A}_{22} - Y + I = 0.$$

Note that since $\tilde{A}_{22}$ is block diagonal, the positive definite matrix Y is also block diagonal, and each block is only a function the corresponding subsystem. Thus, we get

$$J_P(\Gamma^{\Theta}(P))^2 \leq \mathrm{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}) + \mathrm{tr}(\tilde{H}_{11}^T (\tilde{A}_{11}^T \tilde{B}_{11}^{-T} \tilde{B}_{11}^{-1} \tilde{A}_{11} + \tilde{A}_{21}^T \tilde{B}_{22}^{\dagger T} \tilde{B}_{22}^{\dagger} \tilde{A}_{21}) \tilde{H}_{11}). \qquad (16)$$

The optimal closed-loop performance is $J_P(K^*(P))^2 = \mathrm{tr}(H^T U H)$ where $U = [I_{n\times n}\ 0] V [I_{n\times n}\ 0]^T$ and V is the unique positive definite solution of discrete algebraic Lyapunov equation

$$\begin{aligned} &\begin{bmatrix} A + BD^*(P) & BC^*(P) \\ B^*(P) & A^*(P) \end{bmatrix}^T V \begin{bmatrix} A + BD^*(P) & BC^*(P) \\ B^*(P) & A^*(P) \end{bmatrix} - V \\ &\qquad + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} D^*(P)^T D^*(P) & D^*(P)^T C^*(P) \\ C^*(P)^T D^*(P) & C^*(P)^T C^*(P) \end{bmatrix} = 0 \end{aligned} \qquad (17)$$

with $A^*(P)$, $B^*(P)$, $C^*(P)$, and $D^*(P)$ as the state-space realization matrices of the optimal control design strategy $K^*(P)$ for a given plant $P \in \mathcal{P}'$. Clearly, we have

$$J_P(K^*(P))^2 = \sum_{t=1}^{n} e_t^T H^T U H e_t = \sum_{t=1}^{n} \sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k),$$

where for each t the vector $y^{(t)}(k)$ is the output of the system to the exogenous impulse input $w^{(t)}(k) = \delta(k) e_t$. This is true because for each t the summation $\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k)$ gives the diagonal element $e_t^T H^T U H e_t$. For any $P = (A, B, H) \in \mathcal{P}'$, we know that $H^T U H \geq H^T X H$ since centralized controller has the least performance cost over all other controllers either dynamic or static. Thus, for each $t \in \mathcal{N} = \bigcup_{z=1}^{q-c} \mathcal{I}_z$, we get $e_t^T H^T U H e_t \geq e_t^T H^T X H e_t$ which shows

$$\sum_{t \in \mathcal{N}} \sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k) \geq \sum_{t \in \mathcal{N}} e_t^T (H^T X H) e_t.$$

According to [13], we have $X \geq A^T(I + BB^T)^{-1}A + I$ for any $P \in \mathcal{P}'$, and consequently

$$\sum_{t\in\mathcal{N}}\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k) \geq \mathrm{tr}(\tilde{H}_{11}^T(\tilde{A}_{11}^T(I + \tilde{B}_{11}\tilde{B}_{11}^T)^{-1}\tilde{A}_{11} + \tilde{A}_{21}^T(I + \tilde{B}_{22}\tilde{B}_{22}^T)^{-1}\tilde{A}_{21})\tilde{H}_{11}).$$

On the other hand, for each $t \in \mathcal{S} = \bigcup_{z=q-c+1}^{q} \mathcal{I}_z$, we know there exists a sink i such that $t \in \mathcal{I}_i$. For each $w^{(t)}(k)$, we get $\underline{x}_j = 0$ for any $j \neq i$ (since i is a sink in $G_{\mathcal{P}}$). The other subsystems cannot use state-measurements of subsystem i because $G_{\mathcal{K}}$ is equal to $G_{\mathcal{P}}$ (and consequently i is a sink in $G_{\mathcal{K}}$). Therefore, at best case scenario, the cost of controlling subsystem i is equal to the cost of optimal controller designed locally (independent of other subsystems). Thus, we get

$$\sum_{t\in\mathcal{S}}\sum_{k=0}^{\infty} y^{(t)}(k)^T y^{(t)}(k) \geq \mathrm{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}).$$

Therefore, we get

$$J_P(K^*(P))^2 \geq \mathrm{tr}(\tilde{H}_{11}^T(\tilde{A}_{11}^T(I + \tilde{B}_{11}\tilde{B}_{11}^T)^{-1}\tilde{A}_{11} + \tilde{A}_{21}^T(I + \tilde{B}_{22}\tilde{B}_{22}^T)^{-1}\tilde{A}_{21})\tilde{H}_{11}) + \mathrm{tr}(\tilde{H}_{22}^T Y \tilde{H}_{22}). \tag{18}$$

Now, lets define the set

$$\mathcal{M} = \{\bar{\beta} \in \mathbb{R} \mid \bar{\beta} J_P(K^*(P)) - J_P(\Gamma^{\Theta}(P)) \geq 0 \ \forall P \in \mathcal{P}'\}.$$

Using the inequalities in (16) and in (18), it is evident if

$$\mathrm{tr}\left(\tilde{H}_{11}^T(\tilde{A}_{11}^T\left[\beta^2(I + \tilde{B}_{11}\tilde{B}_{11}^T)^{-1} - \tilde{B}_{11}^{-T}\tilde{B}_{11}^{-1}\right]\tilde{A}_{11} + \tilde{A}_{21}^T\left[\beta^2(I + \tilde{B}_{22}\tilde{B}_{22}^T)^{-1} - \tilde{B}_{22}^{\dagger T}\tilde{B}_{22}^{\dagger}\right]\tilde{A}_{21})\tilde{H}_{11}\right) \geq 0. \tag{19}$$

for some $\beta \in \mathbb{R}$, then β would belong to $\mathcal{M}$. Thus, $\{\bar{\beta} \in \mathbb{R} \mid \bar{\beta} \geq \sqrt{1 + 1/\epsilon^2}\} \subseteq \mathcal{M}$. This shows that $r_{\mathcal{P}}(\Gamma^{\Theta}) \leq \sqrt{1 + 1/\epsilon^2}$. Now if $(S_{\mathcal{P}})_{11}$ is not diagonal, with the same argument as in the proof of Case 1 in Theorem 18, we get $r_{\mathcal{P}}(\Gamma) \geq r_{\mathcal{P}}(\Gamma^{\Theta}) = \sqrt{1 + 1/\epsilon^2}$ for any $\Gamma \in \mathcal{C}$. This can be done because there are at least two fully-actuated subsystems and we can forget about the underactuated subsystems.

The proof of the part that the control design strategy Γ^{Θ} is undominated is similar to the one given in [12] for fully-actuated subsystems. ■

8 Conclusions

We considered optimal H_2 dynamic control design for interconnected linear systems under limited plant model information. We introduced control design strategies as functions from the set of plants to the set of structured dynamic controller and compared these control design strategies using the competitive ratio as a performance metric. For a large class of system interconnections, controller structure, and design information, we found an explicit undominated minimizer of the competitive ratio.

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PAPER 3

Decentralized Disturbance Accommodation with Limited Plant Model Information

Farhad Farokhi, Cédric Langbort, and Karl H. Johansson

Abstract–The design of optimal disturbance accommodation and servomechanism controllers with limited plant model information is considered in this paper. Their closed-loop performance are compared using a performance metric called competitive ratio which is the worst-case ratio of the cost of a given control design strategy to the cost of the optimal control design with full model information. It was recently shown that when it comes to designing optimal centralized or partially structured decentralized state-feedback controllers with limited model information, the best control design strategy in terms of competitive ratio is a static one. This is true even though the optimal structured decentralized state-feedback controller with full model information is dynamic. In this paper, we show that, in contrast, the best limited model information control design strategy for the disturbance accommodation problem gives a dynamic controller. We find an explicit minimizer of the competitive ratio and we show that it is undominated, that is, there is no other control design strategy that performs better for all possible plants while having the same worst-case ratio. This optimal controller can be separated into a static feedback law and a dynamic disturbance observer. For constant disturbances, it is shown that this structure corresponds to proportional-integral control.

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1 Introduction

Recent advances in networked control systems have created new opportunities and challenges in controlling large-scale systems composed of several interacting subsystems. An example of a networked control system is shown in Figure 1 where P_i denotes the subsystems to be controlled and C_i denotes the controllers. The interactions between the subsystems and the controllers as well as the external disturbances and references are indicated by arrows. For such networked systems, many researchers have considered the problem of decentralized or distributed stabilization or optimal control as well as the effect of communication channel limitations on closed-loop performance [2–11]. However, at the heart of all these methods lies the (sometimes implicit) assumption that the designer has access to the global plant model information when designing a local controller. This assumption might not be warranted, however, in some applications of interest [12, 13], in which the designer is constrained to compute local controllers for a large-scale systems in a distributed manner with access to only a limited or partial model of the plant. This might be due to several reasons, for example, (*i*) the designer wants the parameters of each local controller to only depend on local model information, so that the controllers do not need to be modified if the model parameters of a particular subsystem, which is not directly connected to them, change, (*ii*) the design of each local controller is done by a designer with no access to the global model of plant since at the time of design the complete plant model information is not available or might change later in the design process, or (*iii*) different subsystems belong to different individuals who refuse to share their model information since they consider it private. These situations are very common in practice. For instance, a chemical plant in process industry can have thousands of proportional-integral-derivative controllers. These processes well illustrate Case (*i*), as the tuning of each local controller does not typically require model information from other control loops in order to simplify the maintenance and limit the controller complexity. Case (*ii*) is typical for cooperative driving such as vehicle platooning, where each vehicle has its own local (cruise) controller which cannot be designed based on model information of all possible vehicles that it may cooperate with in future traffic scenarios. Case (*iii*) can be also illustrated by the control of the power grid, where economic incentives might limit the exchange of network model information across regional borders. Therefore, we have started investigating the concept of limited model information control design for large-scale systems [14–17].

Control design strategies, mappings from the set of plants of interest to the set of applicable controllers, with various degrees of model information are compared using the competitive ratio as a performance metric, that is, the worst-case ratio of the cost of a given control design strategy to the cost of the optimal control design with full model information. In control design with limited plant model information, we search for the "best" control design strategy which attains the minimum competitive ratio among all limited model information design strategies. As this minimizer might not be unique, we further want to find an undominated minimizer of the competitive ratio, that is, there is no other control design strategy in the set of all limited model information design strategies with a better closed-loop performance for all possible plants while maintaining the same worst-case ratio. Recent attention has been on limited model information design methods that produce centralized or decentralized static state-feedback controllers with specific structure. This was justified, at first, by being the simplest case to explore [14–16], and then, maybe more surprisingly, by the recently proven fact that the "best" (in the sense of competitive ratio and domination) state-feedback structured H_2- controller for a plant with lower triangular information pattern that can be designed with limited model information is also static [17], even though the best such controller constructed with access to full model information is dynamic [9, 10]. In this paper, we study the problem of limited model information control design for optimal disturbance accommodation and servomechanism, and show that, contrary to the situations mentioned above, the "best" limited model information design method gives dynamic controllers. Optimal disturbance accommodation is a meaningful model for problems such as constant disturbance rejection or step reference tracking, and has been well-studied in the literature [18–22], but with no attention being paid to the model information limitations in the design procedure.

In this paper, specifically, we consider limited model information control design for interconnection of scalar discrete-time linear time-invariant subsystems being affected by scalar decoupled disturbances with a quadratic separable performance criterion. The choice of such a separable cost function is motivated first by the servomechanism and disturbance accommodation literature [18–22], and second by our interest in dynamically-coupled but cost-decoupled plants and their applications in supply chains and shared infrastructure [12, 13] which has been shown to be well-modeled in this fashion. The assumptions on scalar subsystems and scalar disturbances are technical assumptions to make the algebra in the proofs shorter. Since we want each subsystem to be directly controllable (so that designing subcontrollers based on only local model information is possible), we assume that the overall system is fully-actuated.

We start with the case that each subcontroller is only designed with the corresponding subsystem model information. We prove that, in the case where the plant graph contains no sink and the control graph is a supergraph of the plant graph, the so-called dynamic deadbeat control design strategy is an undominated minimizer of the competitive ratio. For any fixed plant, the controller given by the deadbeat control design strategy can be separated into a static feedback law and a dynamic disturbance observer. For constant disturbances, it is shown that this structure corresponds to a proportional-integral controller. However, the deadbeat control design strategy is dominated when the plant graph has sinks. We present an undominated limited model information control design method that takes advantage of the knowledge of the sinks' location to achieve a better closed-loop performance. We further show that this control design strategy has the same competitive ratio as the deadbeat control design strategy. Later, we characterize the amount of model information needed to achieve a better competitive ratio than the deadbeat control

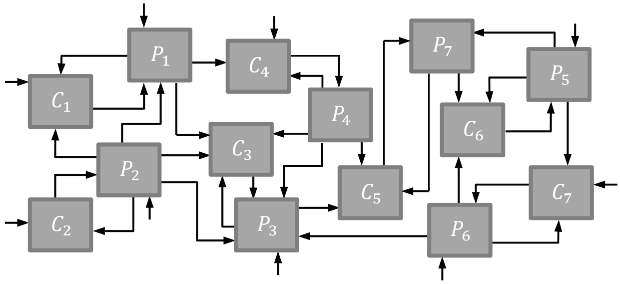

Figure 1: Illustrative example of a networked control system.

design strategy. The amount of information is captured using the design graph, that is, a directed graph which indicates the dependency of each subcontroller on different parts of the global dynamical model. It turns out that, to achieve a better competitive ratio than the deadbeat control design strategy, each subsystem's controller should, at least, has access to the model of all those subsystems that can affect it.

This paper is organized as follows. We formulate the problem and define the performance metric in Section 2. In Section 3, we introduce two specific control design strategies and study their properties. We characterize the best limited model information control design method as a function of the subsystems interconnection pattern in Section 4. In Section 5, we study the influence of the amount of the information available to each subsystem on the quality of the controllers that they can produce. We discuss special cases of constant-disturbance rejection, step-reference tracking, and proportional-integral control in Section 6. Finally, we end with conclusions in Section 7.

1.1 Notation

The set of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. All other sets are denoted by calligraphic letters, such as $\mathcal{P}$ and $\mathcal{A}$. Particularly, the letter $\mathcal{R}$ denotes the set of proper real rational functions.

Matrices are denoted by capital roman letters such as A. A_j will denote the j^{th} row of A. A_{ij} denotes a submatrix of matrix A, the dimension and the position of which will be defined in the text. The entry in the i^{th} row and the j^{th} column of the matrix A is a_{ij}.

Let $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$) be the set of symmetric positive definite (positive semidefinite) matrices in $\mathbb{R}^{n\times n}$. $A > (\geq) 0$ means that the symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite (positive semidefinite) and $A > (\geq) B$ means that $A - B > (\geq) 0$.

$\underline{\sigma}(Y)$ and $\overline{\sigma}(Y)$ denote the smallest and the largest singular values of the matrix Y, respectively. Vector e_i denotes the column-vector with all entries zero except the i^{th} entry, which is equal to one.

All graphs considered in this paper are directed, possibly with self-loops, with vertex set $\{1, ..., q\}$ for some positive integer q. If $G = (\{1, ..., q\}, E)$ is a directed graph, we say that i is a sink if there does not exist $j \neq i$ such that $(i, j) \in E$. The adjacency matrix $S \in \{0, 1\}^{q \times q}$ of graph G is a matrix whose entries are defined as $s_{ij} = 1$ if $(j, i) \in E$ and $s_{ij} = 0$ otherwise. Since the set of vertices is fixed for all considered graphs, a subgraph of a graph G is a graph whose edge set is a subset of the edge set of G and a supergraph of a graph G is a graph of which G is a subgraph. We use the notation $G' \supseteq G$ to indicate that G' is a supergraph of G.

2 Mathematical Formulation

2.1 Plant Model

We are interested in discrete-time linear time-invariant dynamical systems described by

$$x(k+1) = Ax(k) + B(u(k) + w(k))\ ;\ x(0) = x_0, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^n$ is the control input, $w(k) \in \mathbb{R}^n$ is the disturbance vector and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ are appropriate model matrices. Furthermore, we assume that the dynamic disturbance can be modeled as

$$w(k+1) = Dw(k)\ ;\ w(0) = w_0, \tag{2}$$

where $w_0 \in \mathbb{R}^n$ is unknown to the controller (and the control designer). Let a plant graph $G_{\mathcal{P}}$ with adjacency matrix $S_{\mathcal{P}}$ be given. We define the following set of matrices

$$\mathcal{A}(S_{\mathcal{P}}) = \{\bar{A} \in \mathbb{R}^{n \times n} \mid \bar{a}_{ij} = 0 \text{ for all } 1 \leq i, j \leq n \text{ such that } (s_{\mathcal{P}})_{ij} = 0\}.$$

Also, let us define

$$\mathcal{B}(\epsilon) = \{\bar{B} \in \mathbb{R}^{n \times n} \mid \underline{\sigma}(\bar{B}) \geq \epsilon, \bar{b}_{ij} = 0 \text{ for all } 1 \leq i \neq j \leq n\},$$

for some given scalar $\epsilon > 0$ and

$$\mathcal{D} = \{\bar{D} \in \mathbb{R}^{n \times n} \mid \bar{d}_{ij} = 0 \text{ for all } 1 \leq i \neq j \leq n\}.$$

Now, we can introduce the set of plants of interest $\mathcal{P}$ as the set of all discrete-time linear time-invariant systems (1)–(2) with $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $D \in \mathcal{D}$, $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathbb{R}^n$. With a slight abuse of notation, we will henceforth identify a plant $P \in \mathcal{P}$ with its corresponding tuple (A, B, D, x_0, w_0).

The variables $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$, and $w_i \in \mathbb{R}$ are the state, input, and disturbance of scalar subsystem i whose dynamics are given by

$$\begin{aligned} x_i(k+1) &= \sum_{j=1}^{n} a_{ij} x_j(k) + b_{ii}(u_i(k) + w_i(k)), \\ w_i(k+1) &= d_{ii} w_i(k). \end{aligned}$$

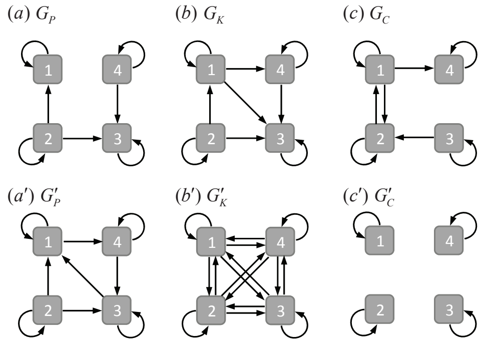

Figure 2: $G_{\mathcal{P}}$ and $G'_{\mathcal{P}}$ are examples of plant graphs, $G_{\mathcal{K}}$ and $G'_{\mathcal{K}}$ are examples of control graphs, and $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ are examples of design graphs.

We call $G_{\mathcal{P}}$ the plant graph since it illustrates the interconnection structure between different subsystems, that is, subsystem j can affect subsystem i only if $(j,i) \in E_{\mathcal{P}}$. Note that we assume that the global system is fully-actuated; i.e., all the matrices $B \in \mathcal{B}(\epsilon)$ are square invertible matrices. This assumption is motivated by the fact that we need all subsystems to be directly controllable. Moreover, we make the standing assumption that the plant graph $G_{\mathcal{P}}$ contain no isolated node. There is no loss of generality in assuming that there is no isolated node in the plant graph $G_{\mathcal{P}}$, since it is always possible to design a controller for an isolated subsystem without any model information about the other subsystems and without influencing the overall system performance. Note that, in particular, this implies that there are $q \geq 2$ vertices in the graph because for $q = 1$ the only subsystem that exists is an isolated node in the plant graph.

Figure 2(a) shows an example of a plant graph $G_{\mathcal{P}}$. Each node represents a subsystem of the system. For instance, the second subsystem in this example affects the first subsystem and the third subsystem, that is, submatrices A_{12} and A_{32} can be nonzero. Note that the first subsystem in Figure 2(a) represents a sink of $G_{\mathcal{P}}$. The plant graph $G'_{\mathcal{P}}$ in Figure 2(a') has no sink.

2.2 Controller Model

The control laws of interest in this paper are discrete-time linear time-invariant dynamic state-feedback control laws of the form

$$x_K(k+1) = A_K x_K(k) + B_K x(k) \; ; \; x_K(0) = 0, \tag{3}$$
$$u(k) = C_K x_K(k) + D_K x(k). \tag{4}$$

Each controller can also be represented by a transfer function

$$K \triangleq \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] = C_K(zI - A_K)^{-1} B_K + D_K,$$

where z is the symbol for the one time-step forward shift operator. Let a control graph $G_{\mathcal{K}}$ with adjacency matrix $S_{\mathcal{K}}$ be given. Each controller K belongs to

$$\mathcal{K}(S_{\mathcal{K}}) = \{K \in \mathcal{R}^{n\times n} \mid k_{ij} = 0 \text{ for all } 1 \leq i, j \leq n \text{ such that } (s_{\mathcal{K}})_{ij} = 0\}.$$

When the adjacency matrix $S_{\mathcal{K}}$ is not relevant or can be deduced from context, we refer to the set of controllers as $\mathcal{K}$. Since it makes sense for each subcontroller to use at least its corresponding subsystem state-measurements, we make the standing assumption that in each design graph $G_{\mathcal{K}}$, all the self-loops are present.

An example of a control graph $G_{\mathcal{K}}$ is given in Figure 2(b). Each node represents a subsystem–controller pair of the overall system. For instance, $G_{\mathcal{K}}$ shows that the first subcontroller can use state measurements of the second subsystem beside its corresponding subsystem state-measurements. Figure 2(b') shows a complete control graph $G'_{\mathcal{K}}$. This control graph indicates that each subcontroller has access to full state measurements of all subsystems, that is, $\mathcal{K}(S_{\mathcal{K}}) = \mathcal{R}^{n\times n}$.

2.3 Control Design Methods

A control design method Γ is a map from the set of plants $\mathcal{P}$ to the set of controllers $\mathcal{K}$. Any control design method Γ has the form

$$\Gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix}, \tag{5}$$

where each entry γ_{ij} represents a map $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{D} \to \mathcal{R}$.

Let a design graph $G_{\mathcal{C}}$ with adjacency matrix $S_{\mathcal{C}}$ be given. We say that Γ has structure $G_{\mathcal{C}}$, if for all i, subcontroller i is computed with knowledge of the plant model of only those subsystems j such that $(j, i) \in E_{\mathcal{C}}$. Equivalently, Γ has structure $G_{\mathcal{C}}$, if for all i, the map $\Gamma_i = [\gamma_{i1} \ \cdots \ \gamma_{in}]$ is only a function of $\{[a_{j1} \ \cdots \ a_{jn}], b_{jj}, d_{jj} \mid (s_{\mathcal{C}})_{ij} \neq 0\}$. When $G_{\mathcal{C}}$ is not a complete graph, we refer to $\Gamma \in \mathcal{C}$ as being a "limited model information control design method". Since it makes sense for the designer of each subcontroller to have access to at least its corresponding subsystem model parameters, we make the standing assumption that in each design graph $G_{\mathcal{C}}$, all the self-loops are present.

The set of all control design strategies with structure $G_{\mathcal{C}}$ will be denoted by $\mathcal{C}$, which is considered as a subset of all maps from $\mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{D}$ to $\mathcal{K}(S_{\mathcal{K}})$ because a design method with structure $G_{\mathcal{C}}$ is not a function of the initial state x_0 or the initial disturbance w_0. We use the notation $\Gamma(A, B, D)$ instead of $\Gamma(P)$ for each plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ to emphasize this fact.

To simplify the notation, we assume that any control design strategy Γ has a state-space realization of the form

$$\Gamma(A, B, D) = \left[\begin{array}{c|c} A_\Gamma(A, B, D) & B_\Gamma(A, B, D) \\ \hline C_\Gamma(A, B, D) & D_\Gamma(A, B, D) \end{array}\right],$$

where $A_\Gamma(A,B,D)$, $B_\Gamma(A,B,D)$, $C_\Gamma(A,B,D)$, and $D_\Gamma(A,B,D)$ are matrices of appropriate dimension for each plant $P = (A,B,D,x_0,w_0) \in \mathcal{P}$. The matrices $A_\Gamma(A,B,D)$ and $C_\Gamma(A,B,D)$ are block diagonal matrices since subcontrollers do not share state variables. This realization is not necessarily minimal.

An example of a design graph $G_\mathcal{C}$ is given in Figure 2(c). Each node represents a subsystem–controller pair of the overall system. For instance, $G_\mathcal{C}$ shows that the second subsystem's model is available to the designer of the first subsystem's controller but not the third and the forth subsystems' model. Figure 2(c') shows a fully disconnected design graph $G'_\mathcal{C}$. A local designer in this case can only rely on the model of its corresponding subsystem.

2.4 Performance Metric

The goal of this paper is to investigate the influence of the plant graph on the properties of controllers derived from limited model information control design methods. We use two performance metrics to compare different control design methods, which are adapted from the notions of competitive ratio and domination recently introduced in [14–17]. Let us start with introducing the closed-loop performance criterion.

To each plant $P = (A,B,D,x_0,w_0) \in \mathcal{P}$ and controller $K \in \mathcal{K}$, we associate the performance criterion

$$J_P(K) = \sum_{k=0}^{\infty} \left[x(k)^T Q x(k) + (u(k)+w(k))^T R (u(k)+w(k)) \right], \tag{6}$$

where $Q \in \mathcal{S}^n_{++}$ and $R \in \mathcal{S}^n_{++}$ are diagonal matrices. We make the following standing assumption:

Assumption 3.1 $Q = R = I$.

This is without loss of generality because the change of variables $(\bar{x},\bar{u},\bar{w}) = (Q^{1/2}x, R^{1/2}u, R^{1/2}w)$ transforms the closed-loop performance measure and state-space representation into

$$J_P(K) = \sum_{k=0}^{\infty} \left[\bar{x}(k)^T \bar{x}(k) + (\bar{u}(k)+\bar{w}(k))^T (\bar{u}(k)+\bar{w}(k)) \right], \tag{7}$$

and

$$\begin{aligned} \bar{x}(k+1) &= Q^{1/2} A Q^{-1/2} \bar{x}(k) + Q^{1/2} B R^{-1/2} (\bar{u}(k)+\bar{w}(k)) \\ &= \bar{A}\bar{x}(k) + \bar{B}(\bar{u}(k)+\bar{w}(k)), \end{aligned}$$

without affecting the plant, control, or design graphs, due to Q and R being diagonal matrices.

Definition 3.1 (Competitive Ratio) *Let a plant graph $G_{\mathcal{P}}$, a control graph $G_{\mathcal{K}}$, and a constant $\epsilon > 0$ be given. Assume that, for every plant $P \in \mathcal{P}$, there exists an optimal controller $K^*(P) \in \mathcal{K}$ such that*

$$J_P(K^*(P)) \leq J_P(K),\ \forall K \in \mathcal{K}.$$

The competitive ratio of a control design method Γ is defined as

$$r_{\mathcal{P}}(\Gamma) = \sup_{P=(A,B,D,x_0,w_0)\in\mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))},$$

with the convention that "$\frac{0}{0}$" equals one.

Note that the optimal control design strategy (with full plant model information) K^* does not necessarily belong to the set $\mathcal{C}$.

Definition 3.2 (Domination) *A control design method Γ is said to dominate another control design method Γ' if*

$$J_P(\Gamma(A,B,D)) \leq J_P(\Gamma'(A,B,D)), \quad \forall\ P = (A,B,D,x_0,w_0) \in \mathcal{P}, \tag{8}$$

with strict inequality holding for at least one plant in $\mathcal{P}$. When $\Gamma' \in \mathcal{C}$ and no control design method $\Gamma \in \mathcal{C}$ exists that satisfies (8), we say that Γ' is undominated in $\mathcal{C}$ for plants in $\mathcal{P}$.

In the remainder of this paper, we determine optimal control design strategies

$$\Gamma^* \in \arg\min_{\Gamma\in\mathcal{C}} r_{\mathcal{P}}(\Gamma), \tag{9}$$

for a given plant, control, and design graph. Since several design methods may achieve this minimum, we are interested in determining which ones of these strategies are undominated.

3 Preliminary Results

Before stating the main results of the paper, we introduce two specific control design strategies and study their properties.

3.1 Optimal Centralized Control Design Strategy

The problem of designing optimal constant input-disturbance accommodation control for linear time-invariant continuous-time systems was solved earlier in [20, 22]. To the best of our knowledge, this was not the case for arbitrary dynamic disturbance accommodation when dealing with linear time-invariant discrete-time systems. As we need it later, we start by developing the optimal centralized (i.e,

$G_{\mathcal{K}}$ is a complete graph) disturbance accommodation controller $K^*(P)$ for a given plant $P \in \mathcal{P}$. First, let us define the auxiliary variables $\xi(k) = u(k) + w(k)$ and $\bar{u}(k) = u(k+1) - Du(k)$. It then follows that

$$\begin{aligned} \xi(k+1) &= u(k+1) + w(k+1) \\ &= u(k+1) + Dw(k) \\ &= Du(k) + Dw(k) + \bar{u}(k) \\ &= D\xi(k) + \bar{u}(k). \end{aligned} \tag{10}$$

Augmenting the state-transition in (10) with the state-space representation of the system in (1) results in

$$\begin{bmatrix} x(k+1) \\ \xi(k+1) \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u}(k). \tag{11}$$

Besides, we can write the performance measure in (7) as

$$J_P(K) = \sum_{k=0}^{\infty} \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}. \tag{12}$$

To guarantee the existence and uniqueness of the optimal controller $K^*(P)$, we need the following lemma.

Lemma 3.1 *The pair* $(\tilde{A}, \tilde{B})$*, with*

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{13}$$

is controllable for any given $P = (A, B, D, x_0, w_0) \in \mathcal{P}$*.*

Proof: The pair $(\tilde{A}, \tilde{B})$ is controllable if and only if

$$\left[\begin{array}{c:c} \tilde{A} - \lambda I & \tilde{B} \end{array}\right] = \left[\begin{array}{cc:c} A - \lambda I & B & 0 \\ 0 & D - \lambda I & I \end{array}\right]$$

is full-rank for all $\lambda \in \mathbb{C}$. This condition is always satisfied since all matrices $B \in \mathcal{B}(\epsilon)$ are full-rank matrices. ■

Now the problem of minimizing the cost function in (12) subject to plant dynamics in (11) becomes a state-feedback linear quadratic optimal control with a unique solution of the form

$$\bar{u}(k) = G_1 x(k) + G_2 \xi(k),$$

where $G_1 \in \mathbb{R}^{n \times n}$ and $G_2 \in \mathbb{R}^{n \times n}$ satisfy

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} = -(\tilde{B}^T X \tilde{B})^{-1} \tilde{B}^T X \tilde{A} \tag{14}$$

and X is the unique positive-definite solution of the discrete algebraic Riccati equation

$$\tilde{A}^T X \tilde{B}(\tilde{B}^T X \tilde{B})^{-1} \tilde{B}^T X \tilde{A} - \tilde{A}^T X \tilde{A} + X - I = 0. \tag{15}$$

Therefore, we have

$$\begin{aligned} u(k+1) &= Du(k) + \bar{u}(k) \\ &= Du(k) + G_1 x(k) + G_2 \xi(k). \end{aligned} \tag{16}$$

Using the identity $\xi(k) = B^{-1}(x(k+1) - Ax(k))$ in (16), we get

$$\begin{aligned} u(k+1) &= Du(k) + G_1 x(k) + G_2 \xi(k) \\ &= Du(k) + G_1 x(k) + G_2 B^{-1}(x(k+1) - Ax(k)) \\ &= Du(k) + (G_1 - G_2 B^{-1} A) x(k) + G_2 B^{-1} x(k+1). \end{aligned} \tag{17}$$

Putting a control signal of the form $u(k) = x_K(k) + D_K x(k)$ in (17), we get

$$x_K(k+1) = Dx_K(k) + (DD_K + G_1 - G_2 B^{-1} A) x(k) + (G_2 B^{-1} - D_K) x(k+1).$$

Now, we enforce the condition $G_2 B^{-1} - D_K = 0$, as $x_K(k+1)$ can only be a function of $x(k)$ and $x_K(k)$, see (3). Therefore, the optimal controller $K^*(P)$ becomes

$$\begin{aligned} x_K(k+1) &= Dx_K(k) + [G_1 + DG_2 B^{-1} - G_2 B^{-1} A] x(k), \\ u(k) &= x_K(k) + G_2 B^{-1} x(k), \end{aligned}$$

with $x_K(0) = 0$.

Lemma 3.2 *Let the control graph $G_{\mathcal{K}}$ be a complete graph. Then, the cost of the optimal controller $K^*(P)$ for each plant $P \in \mathcal{P}$ is lower-bounded as*

$$J_P(K^*(P)) \geq \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2 B^{-2} & -D(W + B^{-2}) \\ -(W + B^{-2})D & W + B^{-2} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix},$$

where

$$W = A^T (I + B^2)^{-1} A + I.$$

Proof: Define

$$\bar{J}_P(K, \rho) = \sum_{k=0}^{\infty} \left(\begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} + \rho \bar{u}(k)^T \bar{u}(k) \right),$$

and

$$\bar{K}_\rho^*(P) = \underset{K \in \mathcal{K}}{\arg\min}\, \bar{J}_P(K, \rho).$$

Using Lemma 3.1, we know that $\bar{K}^*_\rho(P)$ exists and is unique. We can find $\bar{J}_P(\bar{K}^*_\rho(P),\rho)$ using $X(\rho)$ as the unique positive definite solution of the discrete algebraic Riccati equation

$$\tilde{A}^T X(\rho)\tilde{B}(\rho I+\tilde{B}^T X(\rho)\tilde{B})^{-1}\tilde{B}^T X(\rho)\tilde{A}-\tilde{A}^T X(\rho)\tilde{A}+X(\rho)-I=0. \qquad (18)$$

According to [23], the positive-definite matrix $X(\rho)$ is lower-bounded by

$$\begin{aligned} X(\rho)-I &\geq \tilde{A}^T\left(\bar{X}(\rho)^{-1}+\rho^{-1}\tilde{B}\tilde{B}^T\right)^{-1}\tilde{A} \\ &= \tilde{A}^T\left(\bar{X}(\rho)-\bar{X}(\rho)\tilde{B}\left(\rho I+\tilde{B}^T\bar{X}(\rho)\tilde{B}\right)^{-1}\tilde{B}^T\bar{X}(\rho)\right)\tilde{A}, \end{aligned}$$

where

$$\bar{X}(\rho) = \tilde{A}^T\left(I+\rho^{-1}\tilde{B}\tilde{B}^T\right)^{-1}\tilde{A}=\begin{bmatrix} A^TA+I & A^TB \\ BA & B^2+D^2\frac{\rho}{\rho+1}+I \end{bmatrix}.$$

Basic algebraic calculations show that

$$\lim_{\rho\to 0}\left[\bar{X}(\rho)-\bar{X}(\rho)\tilde{B}(\rho I+\tilde{B}^T\bar{X}(\rho)\tilde{B})^{-1}\tilde{B}^T\bar{X}(\rho)\right]=\begin{bmatrix} A^T(I+B^2)^{-1}A+I & 0 \\ 0 & 0 \end{bmatrix}.$$

According to [24], we know that

$$\lim_{\rho\to 0^+}\bar{J}_P(\bar{K}^*_\rho(P),\rho)=J_P(K^*(P)),$$

and as a result

$$X=\lim_{\rho\to 0}X(\rho)\geq\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^T\begin{bmatrix} A^T(I+B^2)^{-1}A+I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}+I. \qquad (19)$$

where X is the unique positive-definite solution of the discrete algebraic Riccati equation in (15) and consequently

$$J_P(K^*(P))=\begin{bmatrix} x_0 \\ \xi(0) \end{bmatrix}^T\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}\begin{bmatrix} x_0 \\ \xi(0) \end{bmatrix}$$

with X being partitioned as

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}.$$

We know that

$$\xi(0)=u(0)+w_0=G_2B^{-1}x_0+w_0=-(X_{22}^{-1}X_{12}^T+DB^{-1})x_0+w_0.$$

Thus, the cost of the optimal control design $J_P(K^*(P))$ becomes

$$\begin{aligned}
&\begin{bmatrix} x_0 \\ -(X_{22}^{-1}X_{12}^T + DB^{-1})x_0 + w_0 \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ -(X_{22}^{-1}X_{12}^T + DB^{-1})x_0 + w_0 \end{bmatrix} \\
&\quad = \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}^T \begin{bmatrix} X_{11} - X_{12}X_{22}^{-1}X_{12}^T + B^{-1}DX_{22}DB^{-1} & -B^{-1}DX_{22} \\ -X_{22}DB^{-1} & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \\
&\quad = \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}^T \begin{bmatrix} B^{-1}(X_{22} + DX_{22}D - I)B^{-1} & -B^{-1}DX_{22} \\ -X_{22}DB^{-1} & X_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}
\end{aligned} \tag{20}$$

The second equality is true because of the following equation extracted from the discrete algebraic Riccati equation in (15)

$$X_{22} = I + BX_{11}B - BX_{12}X_{22}^{-1}X_{12}^T B,$$

which is equivalent to

$$X_{11} - X_{12}X_{22}^{-1}X_{12}^T = B^{-1}(X_{22} - I)B^{-1}. \tag{21}$$

Using (19), it is evident that

$$X_{22} \geq B[A^T(I + B^2)^{-1}A + I]B + I = BWB + I,$$

and as a result, the inner-matrix in (20) is lower-bounded by

$$\begin{aligned}
&\begin{bmatrix} B^{-1}(X_{22} + DX_{22}D - I)B^{-1} & -B^{-1}DX_{22} \\ -X_{22}DB^{-1} & X_{22} \end{bmatrix} \\
&\qquad = \begin{bmatrix} B^{-1}(X_{22} - I)B^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B^{-1}DX_{22}DB^{-1} & -B^{-1}DX_{22} \\ -X_{22}DB^{-1} & X_{22} \end{bmatrix} \\
&\qquad = \begin{bmatrix} B^{-1}(X_{22} - I)B^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -B^{-1}D \\ I \end{bmatrix} X_{22} \begin{bmatrix} -B^{-1}D \\ I \end{bmatrix}^T \\
&\qquad \geq \begin{bmatrix} B^{-1}(BWB)B^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -B^{-1}D \\ I \end{bmatrix} (BWB + I) \begin{bmatrix} -B^{-1}D \\ I \end{bmatrix}^T \\
&\qquad = \begin{bmatrix} W + DWD + D^2B^{-2} & -D(WB + B^{-1}) \\ -(BW + B^{-1})D & BWB + I \end{bmatrix}
\end{aligned}$$

Finally, we get

$$\begin{aligned}
J_P(K^*(P)) &\geq \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2B^{-2} & -D(WB + B^{-1}) \\ -(BW + B^{-1})D & BWB + I \end{bmatrix} \begin{bmatrix} x_0 \\ w_0 \end{bmatrix} \\
&= \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2B^{-2} & -D(W + B^{-2}) \\ -(W + B^{-2})D & W + B^{-2} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}.
\end{aligned}$$

This statement concludes the proof. ■

3.2 Deadbeat Control Design Strategy

In this subsection, we introduce the deadbeat control design strategy and calculate its competitive ratio.

Definition 3.3 *The deadbeat control design strategy* $\Gamma^{\Delta} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \times \mathcal{D} \to \mathcal{K}$ *is defined as*

$$\Gamma^{\Delta}(A, B, D) \triangleq \left[\begin{array}{c|c} D & -B^{-1}D^2 \\ \hline I & -B^{-1}(A+D) \end{array}\right].$$

It should be noted that using the deadbeat control design strategy, irrespective of the value of the initial state x_0 and the initial disturbance w_0, the closed-loop system reaches the origin in just two time-steps. The closed-loop system with deadbeat control design strategy is shown in Figure 3(a). This feedback loop can be rearranged as the one in Figure 3(b) which has two separate components. One component is a static deadbeat control design strategy for regulating the state of the plant and the other one is a deadbeat observer for canceling the disturbance. This structure is further discussed in Section 6, where it is shown that it corresponds to proportional-integral control in some cases. First, we need to calculate an expression for the cost of the deadbeat control design strategy.

Lemma 3.3 *The cost of the deadbeat control design strategy* Γ^{Δ} *for each plant* $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ *is*

$$J_P(\Gamma^{\Delta}(A, B, D)) = \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix},$$

where

$$Q_{11} = I + D^2(I + B^{-2}) + A^T B^{-2} A + DA^T B^{-2} AD + A^T B^{-2} D + DB^{-2} A, \quad (22)$$
$$Q_{12} = -D - A^T B^{-2} - DB^{-2} - DA^T B^{-2} A, \quad (23)$$
$$Q_{22} = A^T B^{-2} A + B^{-2} + I. \quad (24)$$

Proof: First, it should be noted that the state of the closed-loop system with $\Gamma^{\Delta}(A, B, D)$ in feedback reaches the origin in two time-steps. Now, using the system state transition, one can calculate the deadbeat control design strategy cost as

$$\begin{aligned} J_P(\Gamma^{\Delta}(A, B, D)) = x_0^T x_0 + (u(0) + w_0)^T (u(0) + w_0) \\ + x(1)^T x(1) + (u(1) + w(1))^T (u(1) + w(1)), \end{aligned}$$

where $x(1) = -Dx_0 + Bw_0$, $u(0) = -B^{-1}(A + D)x_0$, and $u(1) = -B^{-1}(A + D)x(1) - B^{-1}D^2 x_0$. The rest of the proof is a trivial simplification. ■

We need the following lemma in order to calculate the competitive ratio of the deadbeat control design strategy Γ^{Δ} when the control graph $G_{\mathcal{K}}$ is a supergraph

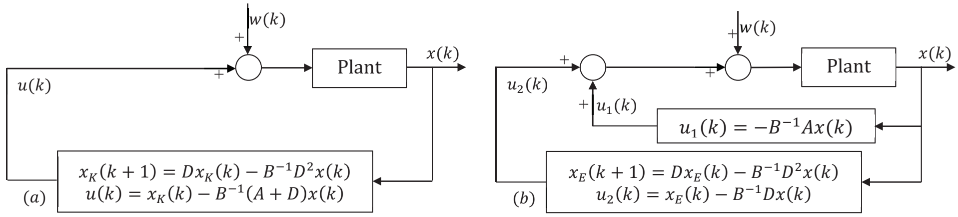

Figure 3: The closed-loop system with (a) the deadbeat control design strategy Γ^{Δ}, and (b) rearranging this control design strategy as a static deadbeat control design and a deadbeat observer design.

of the plant graph $G_{\mathcal{P}}$. As the notation $K^*(P)$ is reserved for the optimal control design strategy for a given control graph $G_{\mathcal{K}}$, from now on, we will use K_C^* to denote the centralized optimal control design strategy (i.e., the optimal control design strategy with access to full-state measurement).

Lemma 3.4 *Let $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$, and $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ be a plant with A being a nilpotent matrix of degree two. Then, $J_P(K^*(P)) = J_P(K_C^*(P))$.*

Proof: When matrix A is nilpotent, the unique positive-definite solution of the discrete algebraic Riccati equation (15) is

$$X = \begin{bmatrix} A^TA + I & A^TB \\ BA & BA^T(I+B^2)^{-1}AB + I + B^2 \end{bmatrix}.$$

Consequently, the optimal centralized controller gains in (14) are

$$G_1 = 0, \qquad G_2 = -(I+B^2)^{-1}BAB - D,$$

and as a result, the optimal centralized controller $K_C^*(P)$ is

$$\begin{aligned} K_C^*(P) &= \left[\begin{array}{c|c} D & D(I+B^2)^{-1}B^{-1}A - B^{-1}D^2 \\ \hline I & -(I+B^2)^{-1}BA - B^{-1}D \end{array}\right] \\ &= (zI - D)^{-1}D(I+B^2)^{-1}B^{-1}A - B^{-1}D^2 - (I+B^2)^{-1}BA - B^{-1}D. \end{aligned}$$

Thus, $K_C^*(P) \in \mathcal{K}(S_{\mathcal{K}})$ because the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. Now, considering that $K^*(P)$ is the global optimal decentralized controller, it has a lower cost than any other decentralized controller $K \in \mathcal{K}(S_{\mathcal{K}})$, specially $K_C^*(P) \in \mathcal{K}(S_{\mathcal{K}})$ for this particular plant. Hence,

$$J_P(K^*(P)) \leq J_P(K_C^*(P)). \tag{25}$$

On the other hand, it is evident that

$$J_P(K_C^*(P)) \leq J_P(K^*(P)). \tag{26}$$

This concludes the proof. ■

Remark 3.1 *Finding the optimal structured controller is intractable in general, even when the global model is known. In this paper, we concentrate on the cases where the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$, because it is relatively easier to solve the optimal control design problem under limited model information in this case. In addition, although, in this paper, we may not be able to find the optimal structured controller $K^*(P)$ for a particular plant in some of the cases, we can still compute the competitive ratio $r_{\mathcal{P}}$. Thus, in a sense, this makes the competitive ratio a quite powerful tool.*

Next, we derive the competitive ratio of the deadbeat control design method.

Theorem 3.5 *Let $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of the deadbeat control design method Γ^{Δ} is equal to*

$$r_{\mathcal{P}}(\Gamma^{\Delta}) = \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2+1}}{2\epsilon^2}.$$

Proof: First, let us define the set of all real numbers that are greater than or equal to the competitive ratio of the deadbeat control design strategy

$$\mathcal{M} = \left\{ \beta \in \mathbb{R} \;\middle|\; \frac{J_P(\Gamma^{\Delta}(A,B,D))}{J_P(K^*(P))} \leq \beta \; \forall P \in \mathcal{P} \right\}.$$

It is evident that

$$J_P(K_C^*(P)) \leq J_P(K^*(P))$$

for each plant $P \in \mathcal{P}$ irrespective of the control graph $G_{\mathcal{K}}$, and as a result

$$\frac{J_P(\Gamma^{\Delta}(A,B,D))}{J_P(K^*(P))} \leq \frac{J_P(\Gamma^{\Delta}(A,B,D))}{J_P(K_C^*(P))}. \tag{27}$$

Using (27) and Lemmas 3.3 and 3.2, β belongs to the set $\mathcal{M}$ if

$$\frac{\begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}}{\begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}^T \begin{bmatrix} W + DWD + D^2B^{-2} & -D(W+B^{-2}) \\ -(W+B^{-2})D & W+B^{-2} \end{bmatrix} \begin{bmatrix} x_0 \\ Bw_0 \end{bmatrix}} \leq \beta, \tag{28}$$

for all $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $D \in \mathcal{D}$, $x_0 \in \mathbb{R}^n$, and $w_0 \in \mathbb{R}^n$ where Q_{11}, Q_{12}, and Q_{22} are matrices defined in (22)–(24). The condition (28) is satisfied, if and only if, for all $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$, we have

$$\begin{bmatrix} \beta(W + DWD + D^2B^{-2}) - Q_{11} & -\beta D(W+B^{-2}) - Q_{12} \\ -\beta(W+B^{-2})D - Q_{12}^T & \beta(W+B^{-2}) - Q_{22} \end{bmatrix} \geq 0.$$

Using Schur complement [25], β belongs to the set $\mathcal{M}$ if

$$\begin{aligned} Z &= \beta(W + B^{-2}) - Q_{22} \\ &= \beta(A^T(I+B^2)^{-1}A + I + B^{-2}) - A^TB^{-2}A - B^{-2} - I \\ &= A^T(\beta(I+B^2)^{-1} - B^{-2})A + (\beta-1)(B^{-2}+I) \geq 0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} -\left[-\beta D(W+B^{-2}) - Q_{12}\right]\left[\beta(W+B^{-2}) - Q_{22}\right]^{-1}\left[-\beta(W+B^{-2})D - Q_{12}^T\right] \\ +\beta(W + DWD + D^2B^{-2}) - Q_{11} \geq 0, \end{aligned} \tag{30}$$

for all $A \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$. We can do the simplification

$$\begin{aligned} -\beta D(W+B^{-2}) - Q_{12} &= -\beta D(A^T(I+B^2)^{-1}A + I + B^{-2}) \\ &\quad -(-D - A^TB^{-2} - DB^{-2} - DA^TB^{-2}A) \\ &= -(\beta-1)D(I+B^{-2}) + A^TB^{-2} \\ &\quad -DA^T(\beta(I+B^2)^{-1} - B^{-2})A \\ &= -DZ + A^TB^{-2}, \end{aligned}$$

and as a result, the condition (30) is equivalent to

$$\beta(W + DWD + D^2B^{-2}) - Q_{11} - [-DZ + A^TB^{-2}]Z^{-1}[-ZD + B^{-2}A] \geq 0, \tag{31}$$

where Z is defined in (29). Furthermore, we can simplify $\beta(W+DWD+D^2B^{-2}) - Q_{11}$ as

$$\begin{aligned} A^T(\beta(I+B^2)^{-1} - B^{-2})A + (\beta-1)[I + D^2B^{-2} + D^2] \\ +DA^T(\beta(I+B^2)^{-1} - B^{-2})AD - A^TB^{-2}D - DB^{-2}A, \end{aligned}$$

which helps us to expand condition (31) to

$$\begin{aligned} A^T\left(\beta(I+B^2)^{-1} - B^{-2}\right)A + (\beta-1)\left(I + D^2B^{-2} + D^2\right) \\ + DA^T\left(\beta(I+B^2)^{-1} - B^{-2}\right)AD - A^TB^{-2}D - DB^{-2}A \\ - D\left(A^T\left(\beta(I+B^2)^{-1} - B^{-2}\right)A + (\beta-1)(B^{-2}+I)\right)D \\ + A^TB^{-2}D + DB^{-2}A - A^TB^{-2}Z^{-1}B^{-2}A \geq 0. \end{aligned} \tag{32}$$

Hence, it follows from (32) that (31) can be simplified as

$$A^T\left(\beta(I+B^2)^{-1} - B^{-2}\right)A - A^TB^{-2}Z^{-1}B^{-2}A \geq 0. \tag{33}$$

The condition (29) is satisfied, for all plants $P \in \mathcal{P}$, if $\beta \geq 1 + 1/\epsilon^2$, since in this case $\beta(I+B^2)^{-1} - B^{-2} \geq 0$ (recall that any matrix B is diagonal and its diagonal elements are lower-bounded by ϵ). Furthermore, for all $\beta \geq 1 + 1/\epsilon^2$, it is easy to see that $Z \geq (\beta-1)(B^{-2}+I)$. As a result, it can be shown that the condition (33) is satisfied if

$$A^T\left(\beta(I+B^2)^{-1} - B^{-2} - (\beta-1)^{-1}B^{-2}(B^{-2}+I)^{-1}B^{-2}\right)A + (\beta-1)I \geq 0. \quad (34)$$

Now, the condition (34) is satisfied if

$$\beta(I+B^2)^{-1} - B^{-2} - (\beta-1)^{-1}B^{-2}(B^{-2}+I)^{-1}B^{-2} \geq 0. \quad (35)$$

Noting that the matrix $B = \mathrm{diag}(b_{11}, \ldots, b_{nn})$, one can rewrite (35) as

$$\frac{\beta}{1+b_{ii}^2} - \frac{1}{b_{ii}^2} - \frac{1}{\beta-1}\frac{1}{b_{ii}^2(1+b_{ii}^2)} \geq 0. \quad (36)$$

for all $b_{ii} \geq \epsilon$. Retracing our steps backward, it easy to see that the set

$$\left\{\beta \mid \beta \geq 1 + \frac{1}{\epsilon^2} \text{ and (36) satisfied}\right\} = \left\{\beta \geq \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}\right\} \subseteq \mathcal{M}.$$

Therefore, we get

$$r_{\mathcal{P}}(\Gamma^\Delta) = \sup_{P\in\mathcal{P}} \frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} \leq \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}. \quad (37)$$

Now, we have to show that this upper bound can be achieved by a family of plants. Consider a one-parameter family of matrices $\{A(r)\}$ defined as $A(r) = re_je_i^T$ for each $r \in \mathbb{R}$. It is always possible to find indices i and j such that $i \neq j$ and $(s_{\mathcal{P}})_{ji} \neq 0$, because of the assumption that there be no isolated node in the plant graph. Let $B = \epsilon I$ and $D = I$. For each $r \in \mathbb{R}$, the matrix $A(r)$ is a nilpotent matrix of degree two, that is, $A(r)^2 = 0$. Thus, using Lemma 3.4, we get

$$J_P(K_C^*(P)) = J_P(K^*(P))$$

for this special plant. The solution to the discrete algebraic Riccati equation in (15) is

$$X = \begin{bmatrix} A(r)^TA(r)+I & \epsilon A(r)^T \\ \epsilon A(r) & \epsilon^2/(1+\epsilon^2)A(r)^TA(r)+(\epsilon^2+1)I \end{bmatrix}.$$

Thus, if we assume that

$$x_0 = \frac{(\epsilon^2+1)(\sqrt{4\epsilon^2+1}+1)}{2\epsilon r}e_i, \quad (38)$$

and

$$w_0 = \frac{(\epsilon^2+1)(\sqrt{4\epsilon^2+1}+1)}{2\epsilon^2 r}e_i - e_j, \quad (39)$$

the cost of the optimal control design strategy is

$$J_P(K^*(P)) = \frac{(\epsilon^2+1)\sqrt{4\epsilon^2+1}+5\epsilon^2+4\epsilon^4+1}{2\epsilon^2} + \frac{(2\epsilon^2+\sqrt{4\epsilon^2+1}+1)\sqrt{4\epsilon^2+1}}{2\epsilon^2 r^2}, \tag{40}$$

and the cost of the deadbeat control design strategy is

$$J_P(\Gamma^\Delta(A,B,D)) = \frac{(\epsilon^2+1)(3\epsilon^2\sqrt{4\epsilon^2+1}+5\epsilon^2+4\epsilon^4+\sqrt{4\epsilon^2+1}+1)}{2\epsilon^4} + \frac{(\epsilon^2+1)(\epsilon^2\sqrt{4\epsilon^2+1}+\epsilon^4\sqrt{4\epsilon^2+1}+\epsilon^2+3\epsilon^4+2\epsilon^6)}{2\epsilon^4 r^2}. \tag{41}$$

This results in

$$\lim_{r\to\infty}\frac{J_P(\Gamma^\Delta(A,B,D))}{J_P(K^*(P))} = \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}. \tag{42}$$

Equation (37) together with (42) conclude the proof. ∎

Remark 3.2 *Consider the limited model information design problem given by the plant graph $G_\mathcal{P}$ in Figure 2(a) and the control graph $G_\mathcal{K}$ in Figure 2(b). Theorem 3.5 shows that, if we apply the deadbeat control design strategy to this particular problem, the performance of the deadbeat control design strategy, at most, can be $(2\epsilon^2+1+\sqrt{4\epsilon^2+1})/(2\epsilon^2)$ times the cost of the optimal control design strategy K^*. In fact, Theorem 3.5 states that this relationship between the performance of the deadbeat control design and the optimal control design with full model information holds for a rather general class of systems. For the case that $\mathcal{B}=\{I\}$, the relationship is given by $(3+\sqrt{5})/2 \approx 2.62$, so the deadbeat control design strategy is never worse than two or three times the optimal.*

With this characterization of Γ^Δ in hand, we are now ready to tackle problem (9).

4 Plant Graph Influence on Achievable Performance

In this section, we study the relationship between the plant graph and the achievable closed-loop performance in terms of the competitive ratio as a performance metric and the domination as a partial order on the set of limited model information control design strategies. To this end, we first state and prove two lemmas which will simplify further developments.

Lemma 3.6 *Fix real numbers $a\in\mathbb{R}$ and $b\in\mathbb{R}$. For any $x\in\mathbb{R}$, we have $x^2+(a+bx)^2 \geq a^2/(1+b^2)$.*

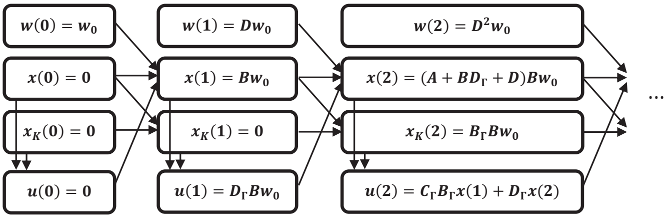

Figure 4: State evolution of the closed-loop system with any control design strategy Γ when $x_0 = 0$.

Proof: Consider the function $x \mapsto x^2 + (a + bx)^2$. Since this function is both continuously differentiable and strictly convex, we can find its unique minimizer as $\bar{x} = -ab/(1 + b^2)$ by setting its derivative to zero. As a result, we get

$$x^2 + (a + bx)^2 \geq \bar{x}^2 + (a + b\bar{x})^2 = a^2/(1 + b^2).$$

This concludes the proof. ■

Lemma 3.7 *Let the design graph $G_\mathcal{C}$ be a totally disconnected graph, and $G_\mathcal{K} \supseteq G_\mathcal{P}$. Furthermore, assume that node i is not a sink in the plant graph $G_\mathcal{P}$. Then, the competitive ratio of a control design strategy $\Gamma \in \mathcal{C}$ is bounded only if $a_{ij} + b_{ii}(d_\Gamma)_{ij}(A, B, D) = 0$ for all $j \neq i$ and all matrices $A \in \mathcal{A}(S_\mathcal{P})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$.*

Proof: The proof is by contrapositive. Let us assume that there exist matrices $\bar{A} \in \mathcal{A}(S_\mathcal{P})$, $B \in \mathcal{B}(\epsilon)$, $D \in \mathcal{D}$, and indices i and j such that $i \neq j$ and $\bar{a}_{ij} + b_{ii}(d_\Gamma)_{ij}(\bar{A}, B, D) \neq 0$. Let $1 \leq \ell \leq n$ be an index such that $\ell \neq i$ and $(s_\mathcal{P})_{\ell i} \neq 0$ (such an index always exists because node i is not a sink in the plant graph $G_\mathcal{P}$). Define matrix A such that $A_i = \bar{A}_i$, $A_\ell = re_i^T$, and $A_t = 0$ for all $t \neq i, \ell$. Because the design graph is a totally disconnected graph, we know that $\Gamma_i(\bar{A}, B, D) = \Gamma_i(A, B, D)$. Using the structure of the cost function in (7) and plant dynamics in (1), the cost of this control design strategy for $w_0 = e_j$ and $x_0 = 0$ is lower-bounded by

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A, B, D)) &\geq (u_\ell(2) + w_\ell(2))^2 + x_\ell(3)^2 \\ &= (u_\ell(2) + w_\ell(2))^2 + (rx_i(2) + b_{\ell\ell}[u_\ell(2) + w_\ell(2)])^2 . \end{aligned}$$

Based on Lemma 3.6 and the fact that $x_i(2) = (a_{ij} + b_{ii}(d_\Gamma)_{ij}(A, B, D))b_{jj}$ (see Figure 4), we get

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A, B, D)) &\geq r^2 x_i(2)^2/(1 + b_{\ell\ell}^2) \\ &= (a_{ij} + b_{ii}(d_\Gamma)_{ij}(A, B, D))^2 b_{jj}^2 r^2/(1 + b_{\ell\ell}^2). \end{aligned}$$

On the other hand, the cost of the deadbeat control design strategy is

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma^{\Delta}(A,B,D)) &= e_j^T B^T (A^T B^{-2} A + B^{-2} + I) B e_j \\ &= b_{jj}^2 + 1 + a_{ij}^2 b_{jj}^2 / b_{ii}^2. \end{aligned}$$

Note that the deadbeat control design strategy is applicable here since the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. This gives

$$\begin{aligned} r_p(\Gamma) &= \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))} \\ &= \sup_{P \in \mathcal{P}} \left[\frac{J_P(\Gamma(A,B,D))}{J_P(\Gamma^{\Delta}(A,B,D))} \frac{J_P(\Gamma^{\Delta}(A,B,D))}{J_P(K^*(P))} \right] \\ &\geq \sup_{P \in \mathcal{P}} \frac{J_P(\Gamma(A,B,D))}{J_P(\Gamma^{\Delta}(A,B,D))} \\ &\geq \frac{(a_{ij} + b_{ii}(d_\Gamma)_{ij}(A,B,D))^2 b_{jj}^2 / (1 + b_{\ell\ell}^2)}{b_{jj}^2 + 1 + a_{ij}^2 b_{jj}^2 / b_{ii}^2} \lim_{r \to \infty} r^2 = \infty. \end{aligned} \tag{43}$$

This inequality proves the statement by contrapositive as the competitive ratio is not bounded in this case. ∎

4.1 Plant Graphs without Sinks

First, we assume that there is no sink in the plant graph and try to characterize the optimal control design strategy in terms of the competitive ratio and domination.

Theorem 3.8 *Let the plant graph $G_{\mathcal{P}}$ contain no sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2}.$$

Proof: Consider a one-parameter family of matrices $\{A(r)\}$ defined as $A(r) = r e_j e_i^T$ for each $r \in \mathbb{R}$. It is always possible to find indices i and j such that $i \neq j$ and $(s_{\mathcal{P}})_{ji} \neq 0$, because of the assumption that there is no isolated node in the plant graph. Let $B = \epsilon I$ and $D = I$. Let $\Gamma \in \mathcal{C}$ be a control design strategy with design graph $G_{\mathcal{C}}$. Without loss of generality, we can assume that $\gamma_{ji}(A,B,D) = -r/\epsilon$ since otherwise, using Lemma 3.7, we get that $r_{\mathcal{P}}(\Gamma)$ is infinity, and as a result the inequality in the theorem statement is trivially satisfied. Thus, for each $r \in \mathbb{R}$, the cost of the control design strategy Γ for x_0 in (38) and w_0 in (39) is lower-bounded by

$$
\begin{aligned}
J_P(\Gamma(A,B,D)) &\geq (u_j(0)+w_j(0))^2 + x_j(1)^2 \\
&= \left(\frac{(\epsilon^2+1)(\sqrt{4\epsilon^2+1}+1)}{2\epsilon^2}+1\right)^2 + \epsilon^2 \\
&= \frac{(\epsilon^2+1)(3\epsilon^2\sqrt{4\epsilon^2+1}+5\epsilon^2+4\epsilon^4+\sqrt{4\epsilon^2+1}+1)}{2\epsilon^4}.
\end{aligned}
$$

On the other hand, for each $r \in \mathbb{R}$, the matrix $A(r)$ is a nilpotent matrix of degree two, that is, $A(r)^2 = 0$. Consequently, using Lemma 3.4, the cost of the optimal control design strategy $K^*(P)$ for x_0 in (38) and w_0 in (39) is given by (40). This results in

$$
r_{\mathcal{P}}(\Gamma) \geq \lim_{r\to\infty} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))} = \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}.
$$

■

Theorem 3.8 shows that the deadbeat control design method Γ^Δ is a minimizer of the competitive ratio $r_{\mathcal{P}}$ as a function over the set of limited model information design methods $\mathcal{C}$. The following theorem shows that it is also undominated by methods of this type, if and only if, the plant graph $G_{\mathcal{P}}$ has no sink.

Theorem 3.9 *Let the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the control design strategy Γ^Δ is undominated if and only if there is no sink in the plant graph $G_{\mathcal{P}}$.*

Proof: First, we have to prove the sufficiency part of the theorem. Assume that there is no sink in the plant graph. For proving this claim, we are going to prove that for any control design method $\Gamma \in \mathcal{C} \setminus \{\Gamma^\Delta\}$, there exists a plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ such that $J_P(\Gamma(A,B,D)) > J_P(\Gamma^\Delta(A,B,D))$. First, assume that there exist matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$ and an index j such that $\bar{A}_j + b_{jj}(D_\Gamma)_j(\bar{A},B,D) + d_{jj}e_j^T \neq 0$. Without loss of generality, we can assume that $\bar{a}_{jj} + b_{jj}(d_\Gamma)_{jj}(\bar{A},B,D) + d_{jj} \neq 0$, because otherwise, using Equation (43) in the proof of Lemma 3.7, we know that, if there exists $\ell \neq j$ such that $\bar{a}_{j\ell} + b_{jj}(d_\Gamma)_{j\ell}(\bar{A},B,D) \neq 0$, the ratio of the cost of the control design strategy Γ to the cost of the deadbeat design strategy Γ^Δ is unbounded. Therefore, the control design strategy Γ cannot dominate the deadbeat control design strategy Γ^Δ. Pick an index $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. It is always possible to pick such index i because there is no sink in the plant graph. Define matrix A such that $A_j = \bar{A}_j$, $A_i = re_j^T$, and $A_\ell = 0$ for all $\ell \neq i, j$. It should be noted that $\Gamma_j(A,B,D) = \Gamma_j(\bar{A},B,D)$ because the design graph is a totally disconnected graph. We know that $r + b_{ii}(d_\Gamma)_{ij}(A,B,D) = 0$ because otherwise the control design strategy Γ cannot dominate the deadbeat control design strategy. The cost of this control design strategy for $w = e_j$ and $x_0 = 0$ satisfies

$$\begin{aligned} J_P(\Gamma(A,B,D)) &\geq (u_i(1)+w_i(1))^2 + (u_i(2)+w_i(2))^2 + x_i(3)^2 \\ &= r^2 b_{jj}^2/b_{ii}^2 + (u_i(2)+w_i(2))^2 + (x_j(2)r + b_{ii}[u_i(2)+w_i(2)])^2, \end{aligned}$$

because of the structure of the cost function (7) and the plant dynamics (1). Now, using Lemma 3.6, we have

$$J_P(\Gamma(A,B,D)) \geq r^2 b_{jj}^2/b_{ii}^2 + x_j(2)^2 r^2/(1+b_{ii}^2).$$

As a result

$$\begin{aligned} & J_P(\Gamma(A,B,D)) - J_P(\Gamma^\Delta(A,B,D)) \qquad (44) \\ & \quad \geq (\bar{A}_{jj} + b_{jj}(d_\Gamma)_{jj}(\bar{A},B,D) + d_{jj})^2 b_{jj}^2 r^2/(1+b_{ii}^2) - (b_{jj}^2 + 1 + a_{jj}^2), \end{aligned}$$

since $x_j(2) = (\bar{A}_{jj} + b_{jj}(d_\Gamma)_{jj}(\bar{A},B,D) + d_{jj})b_{jj}$ (see Figure 4) and

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma^\Delta(A,B,D)) &= e_j^T B^T (A^T B^{-2} A + B^{-2} + I) B e_j \\ &= b_{jj}^2 + 1 + r^2 b_{jj}^2/b_{ii}^2 + a_{jj}^2. \end{aligned}$$

Thus, if we pick r large enough, the difference in (44) becomes positive, which shows that the control design strategy Γ cannot dominate the deadbeat control design strategy Γ^Δ. Now, assume that there exist matrices $\bar{A} \in \mathcal{A}(S_\mathcal{P})$, $B \in \mathcal{B}(\epsilon)$, and $\bar{D} \in \mathcal{D}$ and an index j such that $\bar{A}_j + b_{jj}(D_\Gamma)_j(\bar{A},B,\bar{D}) + \bar{d}_{jj}e_j^T = 0$ but $\Gamma_j(\bar{A},B,\bar{D}) \neq \Gamma_j^\Delta(\bar{A},B,\bar{D})$. Define matrix A such that $A_j = \bar{A}_j$ and $A_\ell = 0$ for all $\ell \neq j$ and matrix D as $d_{jj} = \bar{d}_{jj}$ and $d_{\ell\ell} = 0$ for all $\ell \neq j$. Let $x_0 = 0$. If there exists an index $i \neq j$ such that $\gamma_{ij}(\bar{A},B,D) \neq \gamma_{ij}^\Delta(\bar{A},B,D)$ pick $w_0 = e_i$, otherwise, pick $w_0 = e_j$. For this special case, the state of the closed-loop system with the controller $\Gamma(A,B,D)$ is equal to the state of the closed-loop system with the controller $\Gamma^\Delta(A,B,D)$ for the first and the second time-steps (see Figure 4 and Figure 5). As a result, the state of the subsystem j reaches zero in two time-steps. Now, since $\Gamma_j(\bar{A},B,\bar{D}) \neq \Gamma_j^\Delta(\bar{A},B,\bar{D})$, in the next time-step the state of the subsystem j becomes non-zero again. This results in a performance cost greater than the performance cost of the control design strategy Γ^Δ. Thus, the control design Γ^Δ is undominated by the control design method Γ.

Now, we have to prove the necessary part of the theorem. Proving this part is equivalent to proving that if there exists (a sink) j such that for every $i \neq j$, $(s_\mathcal{P})_{ij} = 0$, then there exists a control design strategy Γ which can dominate the deadbeat control design strategy. Without loss of generality, let $j = n$; i.e., assume that $(s_\mathcal{P})_{in} = 0$ for all $i \neq n$. In this situation, we can rewrite the matrix A as

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & 0 \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix},$$

Define $\bar{x}_0 = [x_1(0) \ \cdots \ x_{n-1}(0)]^T$ and $\bar{w}_0 = [w_1(0) \ \cdots \ w_{n-1}(0)]^T$. Let $\Gamma(A, B, D)$ be defined as $A_\Gamma(A, B, D) = D$, $C_\Gamma(A, B, D) = I$,

$$B_\Gamma(A,B,D) = \begin{bmatrix} -\frac{d_{11}^2}{b_{11}} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\frac{d_{n-1,n-1}^2}{b_{n-1,n-1}} & 0 \\ (b_\Gamma)_{n1} & \cdots & (b_\Gamma)_{n,n-1} & (b_\Gamma)_{nn} \end{bmatrix},$$

$$D_\Gamma(A,B,D) = \begin{bmatrix} -\frac{a_{11}+d_{11}}{b_{11}} & \cdots & -\frac{a_{1,n-1}}{b_{11}} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{a_{n-1,1}}{b_{n-1,n-1}} & \cdots & -\frac{a_{n-1,n-1}+d_{n-1,n-1}}{b_{n-1,n-1}} & 0 \\ (d_\Gamma)_{n1} & \cdots & (d_\Gamma)_{n,n-1} & (d_\Gamma)_{nn} \end{bmatrix},$$

where $\bar{B}_\Gamma = [(b_\Gamma)_{n1} \ \cdots \ (b_\Gamma)_{nn}]$ and $\bar{D}_\Gamma = [(d_\Gamma)_{n1} \ \cdots \ (d_\Gamma)_{nn}]$ are tunable gains for the last subsystem. We denote the cost of applying the deadbeat controller to subsystems $1, \ldots, n-1$ by $J^{(1)}_{(A,B,D,\bar{x}_0,\bar{w}_0)}$. This cost is independent of the control design parameters $\bar{B}_\Gamma$ and $\bar{D}_\Gamma$, because the last subsystem is a sink and it cannot affect the other subsystems. The overall cost of the controller is

$$J_{(A,B,x_0,w_0)}(\Gamma(A,B,D)) = J^{(1)}_{(A,B,D,\bar{x}_0,\bar{w}_0)} + J^{(2)}_{(A,B,D,x_0,w_0)}(\bar{B}_\Gamma, \bar{D}_\Gamma),$$

where $J^{(2)}_{(A,B,D,x_0,w_0)}(\bar{B}_\Gamma, \bar{D}_\Gamma)$ is the cost of the controller designed for the last subsystem. This cost $J^{(2)}_{(A,B,D,x_0,w_0)}(\bar{B}_\Gamma, \bar{D}_\Gamma)$ is independent of the rest of the system's model, because the deadbeat (for subsystems $1, \ldots, n-1$) cancel out all dependencies in matrix A, thus, one can design the optimal controller for the lower part of the system without the model information of the upper part. Now, we can use the method mentioned in Subsection 3.1 to design the optimal controller for the lower part and find the optimal gains

$$\bar{B}_\Gamma = \frac{d_{nn}}{b_{nn}} \left((\alpha+1)A_n - D_n\right), \qquad \bar{D}_\Gamma = \frac{1}{b_{nn}} \left(\alpha A_n - D_n\right),$$

where

$$\alpha = \frac{2}{b_{nn}^2 + a_{nn}^2 + 1 + \sqrt{a_{nn}^4 + 2a_{nn}^2 b_{nn}^2 - 2a_{nn}^2 + b_{nn}^4 + 2b_{nn}^2 + 1}} - 1.$$

Note that this new control design strategy is always applicable since the control graph $G_\mathcal{K}$ is supergraph of the plant graph $G_\mathcal{P}$. Therefore, there exists a control design strategy which satisfies

$$J_{(A,B,D,x_0,w_0)}(\Gamma(A,B,D)) \leq J_{(A,B,D,x_0,w_0)}(\Gamma^\Delta(A,B,D)),$$

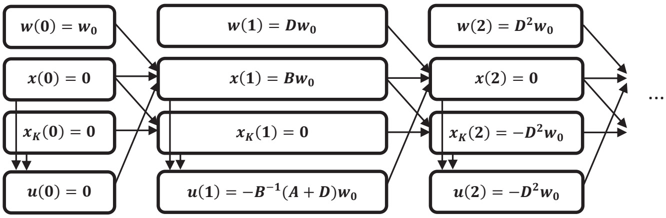

Figure 5: State evolution of the closed-loop system with deadbeat control design strategy Γ^Δ when $x_0 = 0$.

for all matrices $A \in \mathcal{A}(S_\mathcal{P})$, $B \in \mathcal{B}(\epsilon)$, and $D \in \mathcal{D}$ and all vectors $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathbb{R}^n$. Consider the matrix $A \in \mathcal{A}(S_\mathcal{P})$ such that $A_n = re_n^T$ and $A_\ell = 0$ for all $\ell \neq n$. Let $B = \epsilon I$ and $D = I$. For this special system, for all $r > 0$, we have

$$\begin{aligned} J_{(A,B,D,0,e_n)}(\Gamma(A,B,D)) &= \frac{\sqrt{r^4 + 2r^2\epsilon^2 - 2r^2 + \epsilon^4 + 2\epsilon^2 + 1} + r^2 + \epsilon^2 + 1}{2} \\ &< r^2 + \epsilon^2 + 1 \\ &= J_{(A,B,D,0,e_n)}(\Gamma^\Delta(A,B,D)). \end{aligned}$$

Thus, the control design strategy Γ dominates the deadbeat control design strategy Γ^Δ. ■

Remark 3.3 *Consider the limited model information design problem given by the plant graph $G'_\mathcal{P}$ in Figure 2(a′), the control graph $G'_\mathcal{K}$ in Figure 2(b′), and the design graph $G'_\mathcal{C}$ in Figure 2(c′). Theorems 3.8 and 3.9 show that the deadbeat control design strategy Γ^Δ is the best control design strategy that one can propose based on local model of the subsystems and the plant graph, because the deadbeat control design strategy is the minimizer of the competitive ratio and it is undominated.*

We use the construction in proof of the "only if" part of Theorem 3.9 to build a control design strategy for the plant graphs with sinks in the next subsection.

4.2 Plant Graphs with Sinks

In this section, we study the case where there are $c \geq 1$ sinks in the plant graph. By renumbering the sinks as subsystems number $n - c + 1, \cdots, n$ the matrix $S_\mathcal{P}$ can be written as

$$S_\mathcal{P} = \left[\begin{array}{c|c} (S_\mathcal{P})_{11} & 0_{(q-c)\times(c)} \\ \hline (S_\mathcal{P})_{21} & (S_\mathcal{P})_{22} \end{array}\right], \tag{45}$$

where

$$(S_\mathcal{P})_{11} = \begin{bmatrix} (s_\mathcal{P})_{11} & \cdots & (s_\mathcal{P})_{1,n-c} \\ \vdots & \ddots & \vdots \\ (s_\mathcal{P})_{n-c,1} & \cdots & (s_\mathcal{P})_{n-c,n-c} \end{bmatrix},$$

$$(S_{\mathcal{P}})_{21} = \begin{bmatrix} (s_{\mathcal{P}})_{n-c+1,1} & \cdots & (s_{\mathcal{P}})_{n-c+1,n-c} \\ \vdots & \ddots & \vdots \\ (s_{\mathcal{P}})_{n,1} & \cdots & (s_{\mathcal{P}})_{n,n-c} \end{bmatrix},$$

and

$$(S_{\mathcal{P}})_{22} = \begin{bmatrix} (s_{\mathcal{P}})_{n-c+1,n-c+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (s_{\mathcal{P}})_{nn} \end{bmatrix}.$$

From now on, without loss of generality, we assume that the structure matrix is the one defined in (45). The control design method Γ^{Θ} for this type of systems is defined as

$$\Gamma^{\Theta}(A,B,D) = \left[\begin{array}{c|c} D & B^{-1}D(F(A,B)+I)A - B^{-1}D^2 \\ \hline I & B^{-1}(F(A,B)A - D) \end{array}\right], \; \forall P \in \mathcal{P}, \tag{46}$$

where

$$F(A,B) = \mathrm{diag}(0,\ldots,0,f_{n-c+1}(A,B),\ldots,f_n(A,B))$$

and

$$f_i(A,B) = \frac{2}{b_{ii}^2 + a_{ii}^2 + 1 + \sqrt{a_{ii}^4 + 2a_{ii}^2 b_{ii}^2 - 2a_{ii}^2 + b_{ii}^4 + 2b_{ii}^2 + 1}} - 1 \tag{47}$$

for all $i = n-c+1, \cdots, n$.

The control design strategy Γ^{Θ} applies the deadbeat to every subsystem that is not a sink and, for every sink, applies the same optimal control law as if the node was isolated. We will show that when the plant graph contains sinks, the control design method Γ^{Θ} has, in the worst case, the same competitive ratio as the deadbeat strategy. However, unlike the deadbeat strategy, it has the additional property of being undominated by limited model information methods for plants in $\mathcal{P}$ when the plant graph $G_{\mathcal{P}}$ has sinks.

Theorem 3.10 *Let the plant graph $G_{\mathcal{P}}$ contain at least one sink, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of the design method Γ^{Θ} introduced in (46) is*

$$r_{\mathcal{P}}(\Gamma^{\Theta}) = \begin{cases} \frac{2\epsilon^2+1+\sqrt{4\epsilon^2+1}}{2\epsilon^2}, & \text{if } (S_{\mathcal{P}})_{11} \neq 0 \text{ is not diagonal,} \\ 1, & \text{if both } (S_{\mathcal{P}})_{11} = 0 \text{ and } (S_{\mathcal{P}})_{22} = 0. \end{cases}$$

Proof: Based on Theorem 3.5, we know that

$$J_{(A,B,D,x_0,w_0)}(K^*(P)) \geq \frac{2\epsilon^2}{2\epsilon^2+1+\sqrt{4\epsilon^2+1}} J_{(A,B,D,x_0,w_0)}(\Gamma^{\Delta}(A,B,D)), \tag{48}$$

and by the proof of the "only if" part of Theorem 3.9, we know that

$$J_{(A,B,D,x_0,w_0)}(\Gamma^{\Delta}(A,B,D)) \geq J_{(A,B,D,x_0,w_0)}(\Gamma^{\Theta}(A,B,D)), \tag{49}$$

for all $x_0 \in \mathbb{R}^n$ and $w_0 \in \mathbb{R}^n$. Putting (49) into (48) results in

$$J_{(A,B,D,x_0,w_0)}(K^*(P)) \geq \frac{2\epsilon^2}{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}} J_{(A,B,D,x_0,w_0)}(\Gamma^\Theta(A, B, D)),$$

and, therefore, in

$$\frac{J_{(A,B,D,x_0,w_0)}(\Gamma^\Theta(A, B, D))}{J_{(A,B,D,x_0,w_0)}(K^*(P))} \leq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2}, \ \forall P = (A, B, x_0, w) \in \mathcal{P}.$$

As a result

$$r_\mathcal{P}(\Gamma^\Theta) = \sup_{P \in \mathcal{P}} \frac{J_{(A,I,x_0,w)}(\Gamma^\Theta(A, B, D))}{J_{(A,I,x_0,w)}(K_*(P))} \leq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2}.$$

If $(S_\mathcal{P})_{11}$ has an off-diagonal entry, then there exist $1 \leq i, j \leq n - c$ and $i \neq j$ such that $(s_\mathcal{P})_{ij} \neq 0$. Define $A(r)$ such that $A(r) = re_j e_i^T$. In this case, using the proof of Theorem 3.8, we know

$$r_\mathcal{P}(\Gamma^\Theta) = \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2},$$

because the control design Γ^Θ acts as the deadbeat controller on that part of the system. Using both these inequalities proves the statement.

If $(S_\mathcal{P})_{11} = 0$ and $(S_\mathcal{P})_{22} = 0$, every matrix A with structure matrix $(S_\mathcal{P})$ is a nilpotent matrix of degree two. Thus, using Lemma 3.4, we get

$$J_P(K^*(P)) = J_P(K_C^*(P)).$$

Now, based on the proof of Lemma 3.4, we also know that the optimal controller gain for this plant model is

$$K_C^*(P) = \left[\begin{array}{c|c} D & D(I + B^2)^{-1}B^{-1}A - B^{-1}D^2 \\ \hline I & -(I + B^2)^{-1}BA - B^{-1}D \end{array}\right].$$

For control design strategy Γ^Θ, we will have

$$\begin{aligned} \Gamma^\Theta(A, B, D) &= \left[\begin{array}{c|c} D & B^{-1}D(B(I + B^2)^{-1}B - I)A - B^{-1}D^2 \\ \hline I & B^{-1}(B(I + B^2)^{-1}BA - D) \end{array}\right] \\ &= \left[\begin{array}{c|c} D & D(I + B^2)^{-1}B^{-1}A - B^{-1}D^2 \\ \hline I & -(I + B^2)^{-1}BA - B^{-1}D \end{array}\right] \end{aligned}$$

based on (46). Thus, $r_\mathcal{P}(\Gamma^\Theta) = 1$. ■

Theorem 3.11 *Let the plant graph $G_{\mathcal{P}}$ contain at least one sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ satisfies*

$$r_{\mathcal{P}}(\Gamma) \geq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2},$$

if $(S_{\mathcal{P}})_{11}$ is not diagonal.

Proof: First, suppose that $(S_{\mathcal{P}})_{11} \neq 0$ and $(S_{\mathcal{P}})_{11}$ is not a diagonal matrix, then there exist $1 \leq i, j \leq n - c$ and $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$. Consider the family of matrices $A(r)$ defined by $A(r) = r e_i e_j^T$. Based on Lemma 3.7, if we want to have a bounded competitive ratio, the control design strategy should satisfy $r + b_{ii}(d_\Gamma)_{ij}(A(r), B, D) = 0$ (because node $1 \leq i \leq n - c$ is not a sink). The rest of the proof is similar to the proof of Theorem 3.8. ■

Remark 3.4 *Combining Theorem 3.10 and Theorem 3.11 implies that if $(S_{\mathcal{P}})_{11} \neq 0$ is not diagonal (i.e., the nodes that are not sink can affect each other), control design method Γ^Θ is a minimizer of the competitive ratio over the set of limited model information control methods and consequently a solution to the problem (9). Furthermore, if $(S_{\mathcal{P}})_{11}$ and $(S_{\mathcal{P}})_{22}$ are both zero, then the Γ^Θ becomes equal to K^*, which shows that, Γ^Θ is a solution to the problem (9), in this case too. The rest of the cases are still open here.*

The next theorem shows that Γ^Θ is a more desirable control design method than the deadbeat when plant graph $G_{\mathcal{P}}$ has sinks, since it is then undominated by limited model information design methods for plants in $\mathcal{P}$.

Theorem 3.12 *Let the plant graph $G_{\mathcal{P}}$ contain at least one sink, the design graph $G_{\mathcal{C}}$ be a totally disconnected graph, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Then, the control design method Γ^Θ is undominated by all limited model information control design methods.*

Proof: Assume that there are $c \geq 1$ sink in the plant graph. For proving this claim, we are going to prove that for any control design method $\Gamma \in \mathcal{C} \backslash \{\Gamma^\Theta\}$, there exits a plant $P = (A, B, D, x_0, w_0) \in \mathcal{P}$ such that $J_P(\Gamma(A, B, D)) > J_P(\Gamma^\Theta(A, B, D))$. We will proceed in several steps, which require us to partition the set of limited model information control design strategies $\mathcal{C}$ as follows

$$\mathcal{C} = \mathcal{W}_2 \cup \mathcal{W}_1 \cup \mathcal{W}_0 \cup \{\Gamma^\Delta\},$$

where

$$\mathcal{W}_2 := \{\Gamma \in \mathcal{C} \mid \exists j, n - c + 1 \leq j \leq n, \text{ such that } \Gamma_j(A, B, D) \neq \Gamma_j^\Theta(A, B, D)\},$$

$$\mathcal{W}_1 := \{\Gamma \in \mathcal{C} \setminus \mathcal{W}_2 \mid \exists j, 1 \leq j \leq n - c, \\ \text{and } \exists P \in \mathcal{P}, (D_\Gamma)_j(A, B, D) \neq (D_\Gamma^\Theta)_j(A, B, D)\},$$

and

$$\mathcal{W}_0 := \{\Gamma \in \mathcal{C} \setminus \mathcal{W}_2 \cup \mathcal{W}_1 \mid \exists j, 1 \leq j \leq n-c, \exists P \in \mathcal{P}, \\ \text{such that } \Gamma_j(A,B,D) \neq \Gamma_j^{\Theta}(A,B,D)\}.$$

First, we prove that the Γ^{Θ} is undominated by control design strategies in $\mathcal{W}_2$. We assume that there exist index $n-c+1 \leq j \leq n$ and matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, $\bar{D} \in \mathcal{D}$ such that $\Gamma_j(\bar{A},B,\bar{D}) \neq \Gamma_j^{\Theta}(\bar{A},B,\bar{D})$. Consider matrices A and D defined as $A_j = \bar{A}_j$ and $A_i = 0$ for all $i \neq j$ and $d_{jj} = \bar{d}_{jj}$ and $d_{ii} = 0$. For this particular matrix A, any x_0, and any w_0, we know from the proof of the "only if" part of Theorem 3.9 that $\Gamma^{\Theta}(A,B,D,x_0,w_0)$ is the globally optimal controller with limited model information. Hence, every other control design method in $\mathcal{C}$ leads to a controller with greater performance cost than Γ^{Θ} for this particular type of plants. Therefore, the control design Γ^{Θ} is undominated by control design methods in $\mathcal{W}_2$.

Second, we prove that the control design strategy Γ^{Θ} is undominated by the control design strategies in $\mathcal{W}_1$. Let Γ be a control design strategy in $\mathcal{W}_1$ and let index $1 \leq j \leq n-c$ be such that $\bar{A}_j + b_{jj}(D_\Gamma)_j(\bar{A},B,\bar{D}) + \bar{d}_{jj}e_j^T \neq 0$ for some matrices $\bar{A} \in \mathcal{A}(S_{\mathcal{P}})$, $B \in \mathcal{B}(\epsilon)$, and $\bar{D} \in \mathcal{D}$. It is always possible to pick an index $i \neq j$ such that $(s_{\mathcal{P}})_{ij} \neq 0$ because node j is not a sink in the plant graph. If $1 \leq i \leq n-c$, the proof is the same as the proof of the "if" part of Theorem 3.9, therefore, without any loss of generality, we assume that $n-c+1 \leq i \leq n$. Again, with the same argument as in the proof of the "if" part of Theorem 3.9, without loss of generality, we can assume that $a_{jj} + b_{jj}(d_\Gamma)_{jj}(A,B,D) + d_{jj} \neq 0$ (because otherwise the ratio of the cost the control design strategy Γ to the cost of the control design strategy Γ^{Θ} becomes infinity). Define matrix A such that $A_j = \bar{A}_j$, $A_i = re_j^T$, and $A_\ell = 0$ for all $\ell \neq i,j$. Let $D \in \mathcal{D}$ be such that $d_{jj} = \bar{d}_{jj}$ and $d_\ell = 0$ for all $\ell \neq j$. It should be noted that $\Gamma_j(A,B,D) = \Gamma_j(\bar{A},B,\bar{D})$ because the design graph is a totally disconnected graph. The cost of this control design strategy for $w_0 = e_j$ and $x_0 = 0$ would satisfy

$$\begin{aligned} J_P(\Gamma(A,B,D)) &\geq (u_i(1)+w_i(1))^2 + x_i(2)^2 + (u_i(2)+w_i(2))^2 + x_i(3)^2 \\ &= r^2 b_{jj}^2/(b_{ii}^2+1) + (u_i(2)+w_i(2))^2 + (x_j(2)r + b_{ii}[u_i(2)+w_i(2)])^2 \\ &\geq (r^2 b_{jj}^2 + x_j(2)^2 r^2)/(1+b_{ii}^2), \end{aligned}$$

This results in

$$\begin{aligned} &J_{(A,I,B,D,0,e_j)}(\Gamma(A,B,D)) - J_{(A,I,B,D,0,e_j)}(\Gamma^{\Theta}(A,B,D)) \\ &\qquad \geq (a_{jj} + b_{jj}(d_\Gamma)_{jj}(A,B,D) + d_{jj})^2 b_{jj}^2 r^2/(1+b_{ii}^2) - \kappa(A_j, b_{jj}). \end{aligned}$$

where $\kappa(A_j, b_{jj})$ is only a function A_j and b_{jj} and represents the part of the cost of the control design strategy Γ^{Θ} that is related to subsystem j only. If we pick r large enough, the difference would become positive, which shows that the control design strategy Γ cannot dominate the control design strategy Γ^{Θ}.

Finally, we prove that the control design strategy Γ^{Θ} is undominated by the control design strategies in $\mathcal{W}_0$. The same argument as in the proof of the "if" part of Theorem 3.9 holds here too. ■

Remark 3.5 *Consider the limited model information design problem given by the plant graph $G_{\mathcal{P}}$ in Figure 2(a), the control graph $G'_{\mathcal{K}}$ in Figure 2(b′), and the design graph $G'_{\mathcal{C}}$ in Figure 2(c′). Theorems 3.10, 3.11, and 3.12 together show that, the control design strategy Γ^{Θ} is the best control design strategy that one can propose based on local subsystems' model and the plant graph, because the control design strategy Γ^{Θ} is a minimizer of the competitive ratio and it is undominated.*

5 Design Graph Influence on Achievable Performance

In the previous section, we approached the optimal control design under limited model information when $G_{\mathcal{C}}$ is a totally disconnected graph. The next step is to determine the necessary amount of the model information needed in each subcontroller to be able to setup a control design strategy with a smaller competitive ratio than the deadbeat control design strategy. We tackle this question here.

Theorem 3.13 *Let the plant graph $G_{\mathcal{P}}$ and the design graph $G_{\mathcal{C}}$ be given, and $G_{\mathcal{K}} \supseteq G_{\mathcal{P}}$. Assume that the plant graph $G_{\mathcal{P}}$ contains the path $i \to j \to \ell$ with distinct nodes i, j, and ℓ while $(\ell, j) \notin E_{\mathcal{C}}$. Then, we have*

$$r_{\mathcal{P}}(\Gamma) \geq \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2 + 1}}{2\epsilon^2}.$$

Proof: Let i, j, and k be three distinct nodes such that $(s_{\mathcal{P}})_{ji} \neq 0$ and $(s_{\mathcal{P}})_{\ell i} \neq 0$ (i.e., the path $i \to j \to \ell$ is contained in the plant graph $G_{\mathcal{P}}$). Define the 2-parameter family of matrices $A(r,s) = r e_j e_i^T + s e_\ell e_j^T$. Let $B = \epsilon I$, $D = I$, and $\Gamma \in \mathcal{C}$ be a limited model information with design graph $G_{\mathcal{C}}$. The cost of this control design strategy for $w_0 = e_i$ and $x_0 = 0$ satisfies

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A,B,D)) &\geq (u_\ell(2) + w_\ell(2))^2 + x_\ell(3)^2 \\ &= (u_\ell(2) + w_\ell(2))^2 + (s x_j(2) + \epsilon[u_\ell(2) + w_\ell(2)])^2, \end{aligned}$$

because of the structure of the cost function in (7) and the system dynamic in (1). Now, using Lemma 3.6 and the fact that $x_j(2) = (r + \epsilon (d_\Gamma)_{ji}(r))\epsilon$ (see Figure 4), we get

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma(A,B,D)) &\geq s^2 x_j(2)^2/(1+\epsilon^2) \\ &= (r + \epsilon (d_\Gamma)_{ji}(r))^2 \epsilon^2 s^2/(1+\epsilon^2). \end{aligned}$$

Note that $(d_\Gamma)_{ji}(r)$ is only a function of r and not s since $(\ell, j) \notin E_{\mathcal{C}}$. On the other hand, the cost of the deadbeat control design strategy is

$$\begin{aligned} J_{(A,B,D,0,e_j)}(\Gamma^{\Delta}(A,B,D)) &= e_i^T B^T (A^T B^{-2} A + B^{-2} + I) B e_i \\ &= \epsilon^2 + 1 + r^2. \end{aligned}$$

Note that the deadbeat control design strategy is applicable here since the control graph $G_{\mathcal{K}}$ is a supergraph of the plant graph $G_{\mathcal{P}}$. Using (43), we get

$$r_p(\Gamma) \geq \frac{(r + \epsilon (d_\Gamma)_{ji}(r))^2 \epsilon^2/(1+\epsilon^2)}{\epsilon^2 + 1 + r^2} \lim_{s \to \infty} s^2.$$

Using (50) it is easy to see that the competitive ratio $r_{\mathcal{P}}(\Gamma)$ is bounded only if $r + \epsilon (d_\Gamma)_{ji}(r) = 0$, for all $r \in \mathbb{R}$. Therefore, there is no loss of generality in assuming that $(d_\Gamma)_{ji}(r) = -r/\epsilon$ because otherwise the $r_{\mathcal{P}}(\Gamma)$ is infinity and the inequality in the statement of the theorem is trivially satisfied. Now, let us fix $s = 0$ and use the notation $A(r) = r e_j e_i^T$. Since the parameters of the subsystem j is not changed and $(\ell, j) \notin E_{\mathcal{C}}$, we have $(d_\Gamma)_{ji}(r) = -r/\epsilon$. Therefore, for each $r \in \mathbb{R}$, similar to the proof of Theorem 3.8, the cost of the control design strategy Γ for x_0 in (38) and w_0 in (39) is lower-bounded by

$$J_P(\Gamma(A,B,D)) \geq \frac{(\epsilon^2+1)(3\epsilon^2\sqrt{4\epsilon^2+1} + 5\epsilon^2 + 4\epsilon^4 + \sqrt{4\epsilon^2+1} + 1)}{2\epsilon^4},$$

On the other hand, for each $r \in \mathbb{R}$, the matrix $A(r)$ is a nilpotent matrix of degree two, that is, $A(r)^2 = 0$. Similar to the proof of Theorem 3.8, for x_0 in (38) and w_0 in (39), we get

$$J_P(K^*(P)) = \frac{(\epsilon^2+1)\sqrt{4\epsilon^2+1} + 5\epsilon^2 + 4\epsilon^4 + 1}{2\epsilon^2} + \frac{(2\epsilon^2 + \sqrt{4\epsilon^2+1} + 1)\sqrt{4\epsilon^2+1}}{2\epsilon^2 r^2},$$

since $J_P(K^*(P)) = J_P(K_C^*(P))$ according to Lemma 3.4. This results in

$$r_{\mathcal{P}}(\Gamma) \geq \lim_{r \to \infty} \frac{J_P(\Gamma(A,B,D))}{J_P(K^*(P))} = \frac{2\epsilon^2 + 1 + \sqrt{4\epsilon^2+1}}{2\epsilon^2}.$$

This finishes the proof. ■

Remark 3.6 *Consider the limited model information design problem given by the plant graph $G'_{\mathcal{P}}$ in Figure 2(a′), the control graph $G_{\mathcal{K}}$ in Figure 2(b), and the design graph $G_{\mathcal{C}}$ in Figure 2(c). Theorem 3.13 shows that, because the plant graph $G_{\mathcal{P}}$ contains the path $2 \to 1 \to 4$ but the design graph $G_{\mathcal{C}}$ does not contain $4 \to 1$, the competitive ratio of any control design strategy $\Gamma \in \mathcal{C}$ would be greater than or equal to $r_{\mathcal{P}}(\Gamma^{\Delta})$.*

Remark 3.7 *Theorem 3.13 shows that, when $G_{\mathcal{P}}$ and $G_{\mathcal{K}}$ is a complete graph, achieving a better competitive ratio than the deadbeat design strategy requires each subsystem to have full knowledge of the plant model when constructing each subcontroller.*

6 Proportional-Integral Deadbeat Control Design Strategy

In this section, we use some of the results of the paper on familiar control design problems like constant-disturbance rejection and step reference-tracking.

6.1 Constant-Disturbance Rejection

For the case of constant-disturbance rejection, we can model the disturbance as in (2) with matrix $D = I$. For each plant $P = (A, B, I, x_0, w_0) \in \mathcal{P}$, the deadbeat controller design strategy is

$$\Gamma^{\Delta}(A, B, I) \triangleq \left[\begin{array}{c|c} I & -B^{-1} \\ \hline I & -B^{-1}(A+I) \end{array}\right],$$

This controller can be realized as

$$u(k) = -B^{-1}Ax(k) - B^{-1}\sum_{i=0}^{k} x(i).$$

which is a proportional-integral controller. Thus, we call the restricted mapping $\Gamma^{\Delta}_{\text{const}} : \mathcal{A}(S_{\mathcal{P}}) \times \mathcal{B}(\epsilon) \to \mathcal{K}(S_{\mathcal{K}})$, defined as $\Gamma^{\Delta}_{\text{const}}(A, B) = \Gamma^{\Delta}(A, B, I)$, the proportional-integral deadbeat control design strategy. The proportional term regulates the states of the system and the integral term compensates for the disturbance. For instance, in this case, Theorem 3.8 shows that when the plant graph $G_{\mathcal{P}}$ contains no sink and the design graph $G_{\mathcal{C}}$ is a totally disconnected graph, the deadbeat proportional-integral control design strategy is an undominated minimizer of the competitive ratio. Note that the integral part of this control design strategy is fully decentralized and the proportional part only needs the neighboring subsystems state-measurements.

6.2 Step Reference-Tracking

Consider the case that we are interested in tracking a constant reference signal $r \in \mathbb{R}^n$. We need to define the difference $\bar{x}(k) = x(k) - r$ which gives

$$\bar{x}(k+1) = x(k+1) - r = Ax(k) + Bu(k) - r = A\bar{x}(k) + Bu(k) + Ar - r.$$

Now if the subsystems do not want to share the reference points with each other, we can think of the additional term $Ar - r$ as the constant-disturbance vector $w(k) = B^{-1}(Ar - r)$. Thus, we have

$$\bar{x}(k+1) = A\bar{x}(k) + B(u(k) + w(k)).$$

The subsystems only need to transmit the relative error between the state-measurements and reference points. In this case, we can use the cost function

$$J_P(K) = \sum_{k=0}^{\infty}[\bar{x}(k)^T\bar{x}(k) + (u(k) + w(k))^T(u(k) + w(k))], \tag{50}$$

to make sure that the error $\bar{x}(k)$ goes to zero as time tends to infinity. Note that if we want to have a complete state regulation $\lim_{k\to\infty} \bar{x}(k) = 0$, the control signal should have a limit as

$$\lim_{k\to\infty} u(k) = -B^{-1}(Ar - r).$$

Thus, the second term of the cost function (50) only penalizes the difference of the control signal and its steady-state value.

7 Conclusions

We studied the design of optimal disturbance-rejection and servomechanism dynamic controllers under limited plant model information. We investigated the relationship between closed-loop performance and the control design strategies with limited model information using the performance metric called competitive ratio. We found an explicit minimizer of the competitive ratio and showed that this minimizer is also undominated. This optimal control design is a dynamic control design strategy composed of a static part for regulating the state of the system and a dynamic part for canceling the effect of the disturbance. Possible future work will focus on extending the present framework to situations where the subsystems and disturbances are not scalar.

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