Control of Multi-Agent Systems with Applications to Distributed Frequency Control of Power Systems

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Abstract

Multi-agent systems are interconnected control systems with many application domains. The first part of this thesis considers nonlinear multi-agent systems, where the control input can be decoupled into a product of a nonlinear gain function depending only on the agent's own state, and a nonlinear interaction function depending on the relative states of the agent's neighbors. We prove stability of the overall system, and explicitly characterize the equilibrium state for agents with both single- and double-integrator dynamics.

Disturbances may seriously degrade the performance of multi-agent systems. Even constant disturbances will in general cause the agents to diverge, rather than to converge, for many control protocols. In the second part of this thesis we introduce distributed proportional-integral controllers to attenuate constant disturbances in multi-agent systems with first- and second-order dynamics. We derive explicit stability criteria based on the integral gain of the controllers.

Lastly, this thesis presents both centralized and distributed frequency controllers for electrical power transmission systems. Based on the theory developed for multi-agent systems, a decentralized controller regulating the system frequencies under load changes is proposed. An optimal distributed frequency controller is also proposed, which in addition to regulating the frequencies to the nominal frequency, minimizes the cost of power generation.

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Contents

Chapter 1

Introduction

Multi-agent systems, consisting of interconnected sub-systems, arise in several applications and have received overwhelming interest from researchers over the past decade. Multi-robot systems, electrical power systems, see Figure 1.1, and vehicle platoons, see Figure 1.2, are examples of multi-agent systems, to mention a few. In many of the applications of multi-agent systems, it is necessary to control the system in order to achieve the desired properties. Due to the size and complexity of many of these systems, controllers for these systems are often distributed and rely on only the states from the neighboring agents rather than the states of all agents. However, the control objectives are, with few exceptions, global. These control objectives might be for the mobile robots to meet at a common point, or for the frequency of power system generators to converge to a reference frequency. Managing global control specifications with only local measurements is one of the main challenges in multi-agent systems. In this chapter we will introduce the main problems considered in this thesis by some motivating applications, before giving a mathematical problem formulation.

1.1 Motivating applications

A few illustrative examples will be presented here to demonstrate the ubiquitousness of multi-agent systems in engineering applications, and to motivate the problems considered in this thesis. The examples will highlight some of the shortcomings of the state of the art controllers for multi-agent systems, which will be addressed in this thesis.

Example 1.1 (Thermal energy storage in buildings) Thermal energy storage has emerged as a possible method for energy-efficient regulation of temperatures in buildings, as discussed by Zalba et al. (2003). By using a substance which undergoes

Figure 1.1 AC transmission lines of the Nordic power transmission system.

Figure 1.2 Platoon of multiple trucks.

a phase transition near the desired maximum temperature in the building, the temperature may be kept below the maximal desired temperature. While the heat capacity of the air in a building is approximately constant, the total heat capacity of the room is highly nonlinear due to the thermal energy storage. The endothermic and exothermic processes of the phase transitions may be modeled by nonlinear heat capacities, which take the form of a Dirac delta function at the temperature of the phase transition. The model fits well with a consensus protocol for agents with single-integrator dynamics with nonlinear gain and interaction functions. Due to Fourier's law, see e.g., Fourier (1888), the room temperatures are thus well-described by the following nonlinear differential equation

$$\dot{T}_i = -\gamma_i(T_i) \sum_{j \in \mathcal{N}_i} \alpha_{ij}(T_i - T_j), \tag{1.1}$$

where T_i is the temperature of room i, $\alpha_{ij}(T_i - T_j)$ is the heat conductivity between room i and j, where $\alpha_{ij}(\cdot)$ is a nonlinear function $\forall (i,j) \in \mathcal{E}$. $1/\gamma_i(T_i)$ is the temperature-dependent heat capacity of room i, capturing the dynamics of the energy storage. It is of interest to determine the asymptotic temperature in the rooms given their initial temperatures. Furthermore, it is of interest to characterize the convergence rate of the room temperatures towards their final temperature.

Example 1.2 (Autonomous space satellites) Groups of autonomous space satellites may solve tasks in space that require coordination. For a solar power plant in space, this could involve formation control of mirrors, reflecting the sunlight to a solar panel. If the agents are far away from any reference points, it may be assumed that the satellites only have access to their distance and velocity relatively to their neighboring satellites. It is however often important to analyze the dynamical behavior of the satellites from a common reference frame, e.g., the earth. Even if the control laws are linear in the relative velocities in the satellites reference frame, they are generally nonlinear in other reference frames. More specifically, the dynamics of a group of N satellites are assumed to be governed by Newton's second law of motion, resulting in second-order dynamical systems. The raw control signal is the power applied by each agent's engine, P_i. However, the acceleration in an observers reference frame is $a_i = P_i/|v_i|$ due to $P_i = \langle F_i, v_i \rangle$ and F_i being parallel to v_i, where v_i is agent i:s velocity. We assume that the agents only have access to relative measurements. This results in the dynamics

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= -\frac{1}{|v_i|} \sum_{j \in \mathcal{N}_i} \left[\alpha_{ij}\left(x_i - x_j\right) + \beta_{ij}\left(v_i - v_j\right) \right], \end{aligned} \tag{1.2}$$

where $\alpha_{ij}(\cdot)$ and $\beta_{ij}(\cdot)$ are possibly nonlinear interaction functions, $i = 1, \ldots n$ and $\mathcal{N}_i$ denotes the neighbor set of satellite i. Here x_i and v_i denote the position and velocity of satellite i. This networked system motivates the analysis of consensus protocols for agents with double-integrator dynamics with nonlinear gain and interaction functions. It is of particular interest to characterize the asymptotic behavior of the satellites, and study the role of the nonlinearities in the formation of the satellites.

Example 1.3 (Unmanned underwater vehicles) Unmanned underwater vehicles can be used to explore underwater environments where manned vehicles are simply not feasible due to high pressure or extreme temperatures, see Yuh (2000). The exploration of large underwater areas motivates the use of groups of underwater vehicles. In situations where the communication range between the underwater vehicles is limited, the vehicles have to be able to rely only on local and relative measurements. Due to the high viscosity of water, damping due to friction will considerably influence the dynamics of the vehicles. Since the viscosity of the water depends on the water pressure and hence on the operating depth, the damping will in general depend on the state of the underwater vehicle. We thus model the underwater vehicles by double-integrator dynamics with a, possibly nonlinearly, state-dependent damping coefficient. We consider the cooperative task of rendezvous, where the objective of the underwater vehicles is to meet at a common point. For simplicity we only consider rendezvous in one dimension, namely in the depth. Thus, the dynamics of the agents are assumed to be given by

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= -\gamma_i(x)v_i + u_i, \end{aligned} \tag{1.3}$$

where x_i denotes the depth of agent i, and $\gamma_i(x_i)$ is the state-dependent damping coefficient. The controller is assumed to be local and based on the relative states of the agents, and is given by

$$u_i = -\sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j), \tag{1.4}$$

where $\alpha_{ij}(\cdot)$ is a well-behaved nonlinear function. The control input u_i is the vertical force driving the underwater vehicle. This motivates the study of nonlinear control protocols for multi-agent systems with double-integrator dynamics, where the agents dynamics are subject to state-dependent damping. Of particular interest is the stability of the controlled system, and its equilibria.

Example 1.4 (Mobile robot coordination under disturbances) As all control systems, mobile robot systems are susceptible to disturbances. In general, even constant disturbances cause the robot formation to drift, while not achieving the overall objective. We will consider the particular control objective of reaching position-consensus, i.e., rendez-vous. To address the issues caused by disturbances to the robots, a distributed PI controller can be employed. We consider robots with second-order dynamics with damping γ, and a constant disturbance d_i acting on robot i. The disturbance d_i can be caused by e.g., biased sensors or actuators, or a physical disturbance such as an unknown incline. The robots are controlled with a distributed

PI controller. Thus, the dynamics of the robots take the form

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i - \gamma v_i + d_i \\ u_i &= -\sum_{j \in \mathcal{N}_i} \left(\beta(x_i - x_j) + \alpha \int_0^t (x_i(\tau) - x_j(\tau))\, \mathrm{d}\tau \right), \end{aligned} \tag{1.5}$$

where x_i is the position, v_i is the velocity, and z_i is the integrated position of robot i. $\alpha, \beta, \gamma > 0$ are constant parameters. We will investigate when distributed PI controllers can attenuate static disturbances in mobile-robot networks. Furthermore, given the system-specific damping coefficient γ, we would like to characterize under which conditions on the controller gains α and β, the system is stable.

Example 1.5 (Frequency control of power systems) Power systems are among the largest and most complex dynamical systems ever created by mankind, see e.g., Machowski et al. (2008). Whilst being entirely build by humans, the dynamics governing the power systems are very complex. Furthermore, the interconnectivity of power systems poses many challenges when designing controllers. We model the power system by interconnected second-order systems, often referred to as the the swing equation. The swing equation has been used, e.g., in studying transient stability of power systems by Doerfler and Bullo (2011) and fault detection in power systems by Shames et al. (2011). The linearized swing equation is given by

$$m_i \ddot{\delta}_i + d_i \dot{\delta}_i = -\sum_{j \in \mathcal{N}_i} k_{ij}(\delta_i - \delta_j) + p_i^m + u_i, \tag{1.6}$$

where δ_i is the phase angle of bus i, m_i and d_i are the inertia and damping coefficient respectively, p_i^m is the electrical power load at bus i and u_i is the mechanical input power. $k_{ij} = |V_i||V_j|b_{ij}$, where $V_i = |V_i|e^{j\delta_i}$ is the voltage of bus i, and b_{ij} is the susceptance of the line (i, j). The frequency of the power system is denoted $\omega_i = \dot{\delta}_i$. Maintaining a steady frequency is one of the major control problems in power systems. If the frequency is not kept close to the nominal operational frequency, generation and utilization equipment may cease to function properly. The frequency is maintained primarily by automatic generation control (AGC), which is carried out at different levels. In the first level, which is carried out locally at each bus, the power generation is controlled by the deviation from a dynamic reference frequency. At the second level, which is carried out by a central controller, the reference frequency is controlled based on the average frequency in the power system. While the second level controller could easily be automated, it is handled by a human operator in most power systems today.

A simple decentralized frequency control with integral action would take the form:

$$u_i = \alpha(\omega^{\text{ref}} - \omega_i(t)) + \beta \int_0^t (\omega^{\text{ref}} - \omega_i(t'))\, \mathrm{d}t', \tag{1.7}$$

where ω^{ref} is the reference frequency. We would like to guarantee the stability of the above controller, while ensuring that the system frequency reaches the nominal operational frequency, i.e.,

$$\lim_{t\to\infty} \omega_i = \omega^{\text{ref}}\ \forall i \in \mathcal{V}.$$

By providing measurements of the states of the neighboring buses to the controllers, control performance can be improved. We will study how these controllers should be designed, and what control objectives can be fulfilled by adding additional measurements.

1.2 Problem formulation

System model

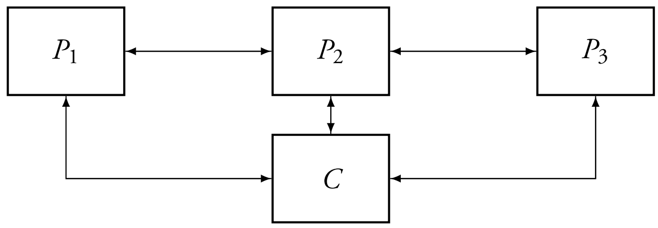
In this thesis we will consider and distinguish between centralized, distributed, and decentralized control of multi-agent systems. We will consider several classes of multi-agent systems, whose common property is that the dynamics of each agent depend on the agent's own state and the states of its neighboring agents. Hence we consider a general multi-agent system model on the form

$$\dot{x}_i = f(x_i, \cup_{j\in\mathcal{N}_i} x_j, u_i) \tag{1.8}$$

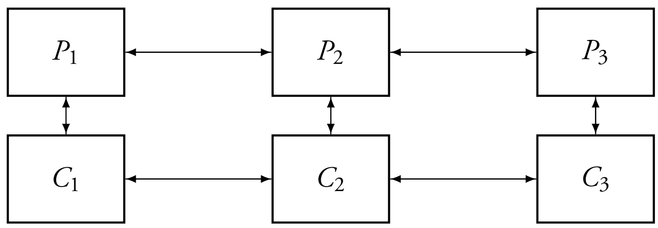
where x_i is the state of agent i and $\mathcal{N}_i$ denotes the neighbor set of agent i. As motivated by the previous examples, we will restrict our analysis to static graphs. Depending on which control architecture is employed, the control input may depend differently on the agents' states. We will distinguish between centralized control, distributed control and decentralized control, illustrated in Figure 1.3. In general we will assume that

$$u_i = \begin{cases} u_i(\cup_{j\in\mathcal{V}} x_j) & \text{(Centralized)} \\ u_i(x_i, \cup_{j\in\mathcal{N}_i} x_j) & \text{(Distributed)} \\ u_i(x_i) & \text{(Decentralized),} \end{cases} \tag{1.9}$$

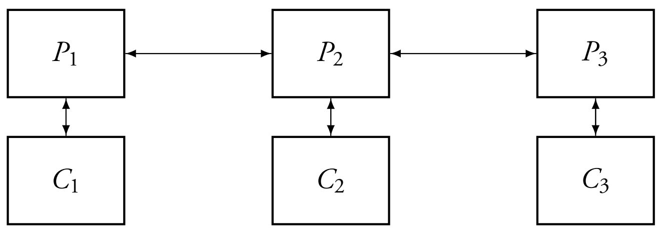
where $\mathcal{V}$ denotes the set of all agents.



(a) Centralized control architecture



(b) Distributed control architecture



(c) Decentralized control architecture

Figure 1.3 Illustration of (a) decentralized, (b) distributed and (c) decentralized control architectures. P_1, P_2 and P_3 represent plants controlled by the controller C or by the controllers C_1, C_2 and C_3 respectively.

Objective

The main objectives of this thesis are threefold, and motivated by the applications discussed earlier. Our first objective is to characterize the stability of nonlinear feedback protocols where the control input can be decoupled into a nonlinear gain depending on the agents own state, and a nonlinear coupling term depending on the relative states of the neighboring agents. Furthermore, we would like to determine under which nonlinear feedback protocols the consensus point of the agents may be determined *a priori*. We will study the problem both for agents with single- and double-integrator dynamics. We will also consider agents with double-integrator dynamics and control protocols with nonlinear coupling and nonlinear, state-dependent damping.

The second objective is the design of distributed feedback protocols which are robust to disturbances. We will focus on constant but unknown disturbances. The overall objective will be for all agents to converge to a common state, i.e., $\lim_{t\to\infty} x_i(t) = x^* \;\; \forall i \in \mathcal{V}$ for single-integrator dynamics, and $\lim_{t\to\infty} v_i(t) = v^* \;\; \forall i \in \mathcal{V}$ for double-integrator dynamics, where x_i denotes the position, and v_i denotes the velocity of agent i.

The third objective is the design of efficient frequency controllers for power systems, which stabilize the power system under unknown load changes. We will model the power system by the swing equation, as mentioned earlier. The control objective will be twofold. First we would like to asymptotically drive the power system frequency towards a nominal reference frequency, i.e., $\lim_{t\to\infty} = \omega^{\text{ref}}$. Second, we would like to asymptotically minimize the cost of power generation in the power system.

1.3 Main Contributions

The main contributions of this thesis are threefold. The first contribution of this thesis is the analysis of distributed nonlinear control protocols for multi-agent systems with single- and double-integrator dynamics. By using integral Lyapunov functions, we prove the stability of a class of distributed control protocols where the control signal is decoupled into a product of a nonlinear gain function which only depends on each agents' own state, and a sum, over the agents' neighbors, of nonlinear interaction functions, each depending on the relative state of the agent and its neighbor. The equilibrium is characterized by invariant integral quantities. The above results have been published in the following proceeding

- M. Andreasson, D. Dimarogonas, and K. H. Johansson. Undamped nonlinear consensus using integral lyapunov functions. In *American Control Conference* (2012a)

The second contribution is the analysis of distributed PI-controllers for multi-agent systems. We introduce distributed PI-controllers for multi-agent systems with single- and double-integrator dynamics. We analyze the stability of the proposed protocols through linear system theory, and give necessary and sufficient stability criteria. The proposed controllers are proven to attenuate constant disturbances in the network. The above results have been published in the following proceeding

- M. Andreasson, H. Sandberg, D. V. Dimarogonas, and K. H. Johansson. Distributed integral action: Stability analysis and frequency control of power systems. In *IEEE Conference on Decision and Control* (2012d)

The two contributions above have been submitted for journal publication as

- M. Andreasson, D. V. Dimarogonas, H. Sandberg, and K. H. Johansson. Distributed control of networked dynamical systems: Static feedback and integral action (2012c). Submitted

The third contribution of this thesis is frequency control of power systems. We propose a decentralized and a distributed frequency controller for power systems, and compare their performance with two centralized controllers. We provide sufficient stability conditions for the proposed control protocols, and provide simulations on the IEEE 30 bus test system. The above results have been submitted for publication partly in Andreasson et al. (2012d) as well as

- M. Andreasson, D. Dimarogonas, K. H. Johansson, and H. Sandberg. Distributed vs. centralized power systems frequency control under unknown load changes (2012b). Submitted

Two other contributions not included in this thesis have been published in

- N. Jayakrishnan, M. Andreasson, L. Andrew, S. Low, and J. Doyle. File fragmentation over an unreliable channel. In *Proceedings IEEE International Conference on Computer Communications, San Diego, March 2010*, 1–9. IEEE (2010)

- M. Andreasson, S. Amin, G. Schwartz, K. H. Johansson, H. Sandberg, and S. Sastry. Correlated failures of power systems : Analysis of the nordic grid. In *Preprints of Workshop on Foundations of Dependable and Secure Cyber-Physical Systems* (2011)

1.4 Outline

The remaining chapters of this thesis are organized as follows. Chapter 2 presents some background in graph theory, nonlinear systems, linear systems, multi-agent systems and power systems, of relevance for this thesis. In Chapter 3, nonlinear controllers for multi-agent systems are presented. In Chapter 4, a distributed PI controllers for multi-agent systems is presented. In Chapter 5, several frequency controllers for power systems are presented. The thesis is concluded in Chapter 6, which also contains a discussion on possible future research directions.

Chapter 2

Background

THE study of multi-agent systems, as presented in this thesis, relies on several results from algebraic graph theory as well as nonlinear and linear system theory. This chapter provides the most important results in the above mentioned areas. Some basic power system theory will also be covered. Recent related work is also presented.

2.1 Notation

We denote by $\mathbb{R}^-/\mathbb{R}^+$ the open left/right real axis, and by $\bar{\mathbb{R}}^-/\bar{\mathbb{R}}^+$ its closure. Let $\mathbb{C}^-/\mathbb{C}^+$ denote the open left/right half complex plane, and $\bar{\mathbb{C}}^-/\bar{\mathbb{C}}^+$ its closure. We will denote the scalar position of agent i as x_i, and its velocity as v_i, and collect them into column vectors $x = (x_1, \dots, x_n)^T$, $v = (v_1, \dots, v_n)^T$. We denote by $c_{n\times m}$ a vector or matrix of dimension $n \times m$ whose elements are all equal to c. I_n denotes the identity matrix of dimension n. A function $f(\cdot)$ with domain $\mathcal{X}$ is said to be globally Lipschitz (continuous) if there exists $K \in \mathbb{R}^+ : \ \forall x, y \in \mathcal{X} : \ \|f(x) - f(y)\| \leq K\|x - y\|$.

2.2 Mathematical preliminaries

Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected, static graph. Let $\mathcal{V} = \{1, \dots, n\}$ denote the node set of $\mathcal{G}$, and $\mathcal{E} = \{1, \dots, m\} \subset (\mathcal{V} \times \mathcal{V})$ denotes the edge set of $\mathcal{G}$. Let $\mathcal{N}_i$ be the set of neighboring nodes to i. The degree of node i is denoted $\deg(i) = |\mathcal{N}_i|$. Two vertices i and j are called adjacent if there is an edge connecting them, i.e., if either $(i, j) \in \mathcal{E}$ or $(j, i) \in \mathcal{E}$. A path is a sequence of edges, such that the starting node of the proceeding edge is the end node of the previous edge. A graph $\mathcal{G}$ is connected if there is a path between any pair of nodes. We denote by $\mathcal{B} = \mathcal{B}(\mathcal{G})$ the node-edge incidence matrix of $\mathcal{G}$. The node-edge incidence matrix of an undirected graph is defined by assigning an arbitrary orientation of each edge. The elements of the node-edge incidence matrix are defined as

$$\mathcal{B}_{vw} = \begin{cases} 1 & \text{if } (v, w) \in \mathcal{E} \\ -1 & \text{if } (w, v) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian matrix of $\mathcal{G}$, is denoted $\mathcal{L}$. Its elements are defined by

$$\mathcal{L}_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise.} \end{cases}$$

For undirected graphs, there is a simple algebraic relation between the Laplacian and the node-edge incidence matrix, as shown by the following lemma.

Lemma 2.1 *For undirected graphs, $\mathcal{L} = \mathcal{B}\mathcal{B}^T$.*

The following result is of great importance for the analysis of multi-agent systems.

Lemma 2.2 *(Diestel (2005)) The eigenvalues of $\mathcal{L}$ are nonnegative. $\mathcal{L}$ has one eigenvalue equal to zero, with the corresponding eigenvector $e = 1_{n\times 1}$. The remaining eigenvalues are nonzero if and only if $\mathcal{G}$ is connected.*

Nonlinear systems

Consider a nonlinear system, described by a nonlinear differential equation:

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{2.1}$$

Assume without loss of generality that $x = 0$ is an equilibrium point of (2.1).

Definition 2.1 (Khalil (2002)) The equilibrium point $x_0 = 0$ of (2.1) is

- stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \ \forall t \geq 0$$

- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$$

Theorem 2.1 *(Khalil (2002)) Let $D \subset \mathbb{R}^n$ be a domain containing* 0*. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \quad \textit{and} \quad V(x) > 0 \textit{ in } D \setminus \{0\}$$
$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V(x)}{\partial x} f(x) \leq 0 \textit{ in } D,$$

then $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \textit{ in } D \setminus \{0\},$$

then $x = 0$ is asymptotically stable.

The function $V(x)$ is often referred to as a Lyapunov function. For some systems, it may be possible to find a Lyapunov function $V(x)$ with only non-positive derivative. Under some conditions, it is still possible to guarantee asymptotic stability with such a $V(x)$. First, we need to define the notion of positive invariance. A set S is said to be *invariant* if $x(0) \in S \Rightarrow x(t) \in S \; \forall t$, and *positively invariant* if $x(0) \in S \Rightarrow x(t) \in S \; \forall t \geq 0$.

Theorem 2.2 *(Khalil (2002)) Let $\Omega \in D$ be a compact set which is positively invariant with respect to* (2.1)*. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

In particular Theorem 2.2, which is commonly referred to as LaSalles invariance principle, implies that if the origin is the largest invariant set in E, then it is asymptotically stable.

Linear time-invariant systems

A linear time-invariant system is defined as a set of linear time-invariant ordinary differential equations. A linear system can be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$. If $u(t)$ is given by linear state feedback $u(t) = -Kx(t)$, the system equation (2.2) becomes

$$\begin{aligned} \dot{x}(t) &= (A - BK)x(t) \\ y(t) &= Cx(t). \end{aligned} \tag{2.3}$$

The linear system (2.3) is indeed a special case of the nonlinear system (2.1).

Theorem 2.3 *(Kailath (1980)) The solution of* (2.3)*, starting at* $x(0) = x_0$ *is given by*

$$x(t) = e^{(A-BK)t}x_0,$$

where

$$e^{(A-BK)t} = T^{-1}e^J T.$$

Here, the columns of T^{-1} *consist of the generalized eigenvectors of* $(A - BK)$*, i.e.,* $T^{-1} = [e_1^1, \dots, e_1^{\mu_1}, \dots, e_k^1, \dots, e_k^{\mu_k}]$. $e^J \in \mathbb{R}^{n\times n}$ *is given by*

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 & \dots & 0 \\ 0 & e^{J_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{J_k t} \end{bmatrix} \quad \textit{and} \quad e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \dots & \frac{t^{\mu_i - 1}e^{\lambda_i t}}{(\mu_i-1)!} \\ 0 & e^{\lambda_i t} & \dots & \frac{t^{\mu_i - 2}e^{\lambda_i t}}{(\mu_i-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_i t} \end{bmatrix}$$

where $\lambda_1, \dots, \lambda_k$ *are the eigenvalues of* $(A - BK)$ *of multiplicities* $\mu_1, \dots, \mu_k$*.*

By the previous theorem, the stability of a linear system can be easily determined.

Corollary 2.1 *The system* (2.3) *is asymptotically stable if and only if all eigenvalues of* $(A - BK)$ *lie in the open left half complex plane.*

2.3 Multi-agent systems

Multi-agent systems consist of several coupled sub-systems, so called *agents*. Both the dynamics of the agents, as well as the coupling between the agents can take many different forms. We will here give a general mathematical model of a multi-agent system. We model the coupling of the agents with a graph $\mathcal{G}$. The agents are represented by nodes, and the coupling by edges. Two agents are coupled if and only if they are connected by an edge. The dynamics of an agent i is given by

$$\dot{x}_i = f(x_i, \cup_{j\in\mathcal{N}_i} x_j, u_i) \tag{2.4}$$

where u_i may depend either only on x_i (decentralized control), or on x_i and x_j for all $j \in \mathcal{N}_i$ (distributed control). The control objective depends on the application, and are numerous. One of the most well-studied control problems is the consensus problem, where the control objective of the agents is to reach a common state, i.e., $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \ \forall \ i, j \in \mathcal{G}$. The consensus problem has been studies by, e.g., by Olfati-Saber and Murray (2004), Ren and Beard (2005) and Ren et al. (2007).

The consensus problem may be solved by a linear control protocol. Assuming that the agent dynamics are linear single integrators

$$\dot{x}_i = u_i, \tag{2.5}$$

the controller

$$u_i = \gamma_i \sum_{j \in \mathcal{N}_i} \alpha_{ij}(x_j - x_i), \tag{2.6}$$

where γ_i and α_{ij} are positive constants, satisfies $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \; \forall \; i, j \in \mathcal{G}$ if $\mathcal{G}$ is connected, see e.g. Olfati-Saber and Murray (2004).

Another control problem in the framework of multi-agent systems is formation control, which has been studied by, e.g., Tanner et al. (2003), Olfati-Saber (2006) and Dimarogonas and Johansson (2010). The control objective is here to attain certain, generally nonzero, distances between the agents rather than a zero distance as in the consensus problem. The control objective may be formulated mathematically as $\lim_{t\to\infty} |x_i(t) - x_j(t)| = d_{ij} \; \forall \; i, j \in \mathcal{G}$. It has been shown that the formation control problem may be solved by introducing a potential function which attains its minimum at the desired distances d_{ij}, and letting the control input be given by the negative gradient of the potential function.

When studying more advanced control problems in multi-agent systems, the required control protocols tend to be more involved. We will here briefly discuss two more advanced control problems of interest to this thesis. The first is static nonlinear feedback control protocols, and the second is distributed PI control.

Distributed control with static nonlinear feedback

Distributed control by nonlinear controllers is a natural extension of linear consensus protocols, and a well-studied problem, see e.g. Olfati-Saber et al. (2003); Chen et al. (2009); Hui and Haddad (2008); Moreau (2005), with applications to consensus with preserving connectedness and collision avoidance, see e.g. Tanner et al. (2007); Ji and Egerstedt (2007); Dimarogonas and Kyriakopoulos (2008). Sufficient conditions for the convergence of nonlinear protocols for first-order integrator dynamics are given in Ajorlou et al. (2011) and extended to multidimensional state-spaces in Lin et al. (2007). Consensus on a general function value was introduced in Olfati-Saber and Murray (2004) as χ-consensus, and a solution to the so called χ-consensus problem was presented in Cortés (2006), by using nonlinear gain functions. χ-consensus has applications for instance in weighted power mean consensus, see Cortés (2006); Bauso et al. (2006); Cortés (2008).

The literature on nonlinear controllers has been focused on agents with single-integrator dynamics. However, as we show later, the results can be generalized to double-integrator dynamics. Consensus protocols where the input of an agent can be separated into a product of a positive function of the agents own state were studied in Bauso et al. (2006) for single integrator dynamics. Münz et al. (2011) studied position consensus for agents with double-integrator dynamics under a class of nonlinear interaction functions and nonlinear velocity-damping. In contrast to the references, this thesis will focus on undamped consensus protocols for single- and double-integrator dynamics using integral Lyapunov functions. Xie and Wang (2007) consider double-integrator consensus problems with linear non-homogeneous damping coefficients. We later generalize the results for the corresponding linear damping to hold also for a class of nonlinear damping coefficients.

Distributed control under disturbances

Multi-agent systems, as all control processes, are in general sensitive to disturbances. When only relative measurements are available, disturbances are often spread through the network. It has for example been shown by Bamieh et al. (2012) that vehicular string formations with only relative measurements cannot maintain coherency under disturbances as the size of the formation increases. Young et al. (2010) study the robustness of consensus-protocols under disturbances, but limit their study to the relative states of the agents.

Distributed control of multi-agent systems with integral action for disturbance attenuation has been studied in Freeman et al. (2006). It was shown that the proposed consensus protocol can attenuate constant and some time-varying disturbances to a certain degree. In Yucelen and Egerstedt (2012) the authors take a similar approach to attenuate unknown disturbances. In both papers the analysis is limited to agents with single-integrator dynamics. Our proposed PI controller is related to the consensus protocols studied in Cheng et al. (2008); Hong et al. (2007). However, the models presented in these references do not consider disturbances.

2.4 Power systems

Electrical power systems are multi-agent systems, which often cover a large geographical area. Due to their vital importance to virtually every part of society, power systems are among the most critical infrastructures in a modern society.

While the dynamics of power transmission networks are very complex, they may be well approximated by the swing equation, see e.g. Machowski et al. (2008)

$$m_i\ddot{\delta}_i + d_i\dot{\delta}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}\sin(\delta_i - \delta_j) + p_i^m + u_i, \tag{2.7}$$

where δ_i is the phase angle of bus i, m_i and d_i are the inertia and damping coefficient respectively, p_i^m is the electrical power load at bus i and u_i is the mechanical input power. $k_{ij} = |V_i||V_j|b_{ij}$, where $V_i = |V_i|e^{j\delta_i}$ is the voltage of bus i, and b_{ij} is the susceptance of the line (i, j). By linearizing (2.7) around he equilibrium where $\delta_i = \delta_j \ \forall i, j$, we obtain the linearized swing equation

$$m_i\ddot{\delta}_i + d_i\dot{\delta}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}(\delta_i - \delta_j) + p_i^m + u_i, \tag{2.8}$$

Control of power systems

An AC transmission system must operate at a synchronous frequency $\omega = \dot{\delta}$, which is typically 50 Hz or 60 Hz. Any deviations from the nominal frequency may damage the generation equipment or even cause instability. Hence it is of of major importance to operate the power system close to its nominal frequency. Automatic generation control (AGC), see e.g. Jaleeli et al. (1992); Ibraheem et al. (2005) and frequency controllers, see Liu et al. (2003); Machowski et al. (2008) are two commonly employed control strategies to maintain a constant operation frequency. The commonly employed frequency controllers are mainly centralized, as in Bevrani (2009); Liu et al. (2003), however some efforts towards decentralized control of power system frequencies have been made by Venkat et al. (2008), by employing a distributed MPC. Due to load and generation changes as well as model imperfections, a proportional frequency controller cannot reach the desired reference frequency in general. To attenuate static errors, integrators are used, see Machowski et al. (2008) and the references therein.

Due to the inherent difficulties with distributed PI control, detailed in Morari and Zafiriou (1989), automatic frequency control of power systems is typically carried out at two levels: an inner and an outer level. In the inner control loop, the frequency is controlled with a proportional controller against a dynamic reference frequency. In the outer loop, the reference frequency is controlled with a centralized PI controller to eliminate static errors. While this control architecture works satisfactorily in most of today's situations, future power system developments might render it unsuitable. For instance, large-scale penetration of renewable power generation increases generation fluctuations, creating a need for fast as well as local disturbance attenuation. Decentralized control of power systems might also provide efficient anti-islanding control and self-healing features, even when communication between subsystems is limited or even unavailable, see e.g. Senroy et al. (2006); Yang et al. (2006).

Chapter 3

Distributed control with static nonlinear feedback

In this chapter we will study distributed control protocols using static nonlinear state feedback. The control objective is to drive the states of the agents towards a common state, and to explicitly characterize the limit set of the system. The control input might either be a part of the system's natural dynamics, or it might be an external control input, depending on the application. In the end of this chapter we revisit the motivating applications and demonstrate that the results of this chapter have several applications.

3.1 Distributed control for single-integrator dynamics

We consider agents with the first-order dynamics, and controllers of the form

$$\begin{aligned} \dot{x}_i &= u_i \\ u_i &= -\gamma_i(x_i) \sum_{j \in \mathcal{N}_i} \alpha_{ij}(x_i - x_j). \end{aligned} \tag{3.1}$$

The study of agents with dynamics given by (3.1) is motivated by, e.g., the study of thermal energy storage in smart buildings, as discussed in Chapter 1.1. We make the following technical assumptions of the gain and interaction functions.

Assumption 3.1 γ_i is continuous and $\gamma_i(x) \geq \underline{\gamma} > 0 \quad \forall i \in \mathcal{V}, \ \forall x \in \mathbb{R}$

Assumption 3.2 $\alpha_{ij}(\cdot)$ is Lipschitz continuous $\forall i \in \mathcal{V}, \forall \ (i,j) \in \mathcal{E}$, and furthermore:

1. $\alpha_{ij}(-y) = -\alpha_{ji}(y) \quad \forall (i,j) \in \mathcal{E}, \ \forall y \in \mathbb{R}$
2. $\alpha_{ij}(y) > 0 \quad \forall (i,j) \in \mathcal{E}, \ \forall y > 0,$

3. $\alpha_{ij}(0) = 0$,

Remark 3.1 Assumption 3.2 guarantees that the agents move in the direction of their neighbors, as well as symmetry in the flow. The assumption that $\alpha_{ij}(0) = 0$, ensures that the consensus point where $x_i = x_j \ \forall i, j \in \mathcal{V}$ is an equilibrium.

We are now ready to state the main result of this section.

Theorem 3.1 *Given n agents with dynamics* (3.1)*, where* γ_i *and* α_{ij} *satisfy Assumptions 3.1 and 3.2 respectively, then the agents converge asymptotically to an agreement point* $\lim_{t\to\infty} x_i(t) = x^* \ \forall \ i \in \mathcal{V}$ *depending on the initial condition, where* x^* *is uniquely determined by the integral equation*

$$\sum_{i\in\mathcal{V}} \int_0^{x_i^0} \frac{1}{\gamma_i(y)}\, \mathrm{d}y = \int_0^{x^*} \sum_{i\in\mathcal{V}} \frac{1}{\gamma_i(y)}\, \mathrm{d}y, \tag{3.2}$$

for any condition $x_i(0) = x_i^0, i = 1, \dots, n.$

Proof. Consider the quantity

$$E(x) = \sum_{i\in\mathcal{V}} \int_0^{x_i} \frac{1}{\gamma_i(y)}\, \mathrm{d}y.$$

Differentiating $E(x)$ with respect to time yields

$$\begin{aligned}\frac{\mathrm{d}E(x(t))}{\mathrm{d}t} &= \frac{\partial E(x(t))}{\partial x}\frac{\partial x}{\partial t} = -\left[\frac{1}{\gamma_1(x_1)}, \dots, \frac{1}{\gamma_n(x_n)}\right]\Gamma(x)B\alpha(\mathcal{B}^T x)\\ &= -1_{1\times n}B\alpha(\mathcal{B}^T x) = 0,\end{aligned}$$

where $\Gamma(x) = \mathrm{diag}([\gamma_1(x_1), \dots, \gamma_n(x_n)])$, and $\alpha(\cdot)$ is taken component-wise. Hence $E(x)$ is invariant and the agreement point x^* is given by (3.2). By Assumption 3.1, $E(x^* 1_{n\times 1})$ is strictly increasing in x^*, and hence (3.2) admits a unique solution. Now consider the following candidate Lyapunov function:

$$V(x) = \sum_{i\in\mathcal{V}} \int_{x^*}^{x_i} \frac{y - x^*}{\gamma_i(y)}\, \mathrm{d}y, \tag{3.3}$$

where x^* is the agreement point given by (3.2). It can easily be verified that $V(x^* 1_{n\times 1}) = 0$. To show that $V(x) > 0$ for $x \neq 0$, it suffices to show that $\int_{x^*}^{x_i} y - x^*/\gamma_i(y)\, \mathrm{d}y > 0 \ \forall i \in \mathcal{V}$. Consider first the case when $x_i > x^*$

$$\int_{x^*}^{x_i} \frac{y - x^*}{\gamma_i(y)}\, \mathrm{d}y = \int_0^{x_i - x^*} \frac{z}{\gamma_i(z + x^*)}\, \mathrm{d}z > 0,$$

by the change of variable $z = y - x^*$. The case when $x_i < x^*$ is treated analogously

$$\int_{x^*}^{x_i} \frac{y - x^*}{\gamma_i(y)}\, \mathrm{d}y = \int_0^{x^* - x_i} \frac{z}{\gamma_i(x^* - z)}\, \mathrm{d}z > 0,$$

with the change of variable $z = x^* - y$. This also implies that $V(x) = 0 \Rightarrow x = x^* 1_{n\times 1}$. Now consider $\dot{V}(x)$ along trajectories of the closed loop system:

$$\begin{aligned}\dot{V}(x) &= \sum_{i\in V} \frac{\partial V(x(t))}{\partial x_i}\frac{\partial x_i}{\partial t} = -\sum_{i\in\mathcal{V}} \frac{x_i - x^*}{\gamma_i(x_i)} \cdot \gamma_i(x_i) \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j) \\ &= -\sum_{i\in\mathcal{V}} x_i \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j) + \sum_{i\in\mathcal{V}} x^* \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j).\end{aligned} \tag{3.4}$$

Due to the symmetry property in Assumption 3.2, the first term of (3.4) may be rewritten as

$$\sum_{i\in\mathcal{V}} x_i \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j) = \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} x_i \alpha_{ij}(x_i - x_j) = -\frac{1}{2} \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} (x_i - x_j)\alpha_{ij}(x_i - x_j)$$

Clearly the second term of (3.4) satisfies $\sum_{i\in\mathcal{V}} x^* \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j) = 0$ due to Assumption 3.2. Hence, $\dot{V}(x)$ may be rewritten as

$$\dot{V}(x) = -\frac{1}{2} \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{N}_i} (x_i - x_j)\alpha_{ij}(x_i - x_j) < 0,$$

unless $x_i = x_j \;\forall i, j \in \mathcal{V}$. Hence the agents converge to $x_i = x^* \;\forall i \in \mathcal{V}$. □

Remark 3.2 The agreement protocol (3.1) has an intuitive physical interpretation. If we consider the smart building problem in Example 1.1, and let x_i be the temperature of the rooms, $1/\gamma_i(\cdot)$ is the temperature-dependent heat capacity of the rooms. Analogously, $\alpha_{ij}(\cdot)$ is the thermal conductivity of the walls, being dependent on the temperature gradient between the rooms. The invariant quantity

$$E(x) = \sum_{i\in\mathcal{V}} \int_0^{x_i} \frac{1}{\gamma_i(y)}\, \mathrm{d}y$$

is the total thermal energy of the system, e.g. the floor, which is assumed to be constant.

Remark 3.3 The convergence of the dynamics (3.1) was proven by Shi and Hong (2009). However, as opposed to this reference, we here explicitly characterize the equilibrium set. Furthermore our proof relies on a different Lyapunov function.

3.2 Distributed control for double-integrator dynamics

In this section we consider agents with double-integrator dynamics, and control input given by

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \\ u_i &= -\gamma_i(v_i) \sum_{j \in \mathcal{N}_i} \Big[\alpha_{ij}\left(x_i - x_j\right) + \beta_{ij}\left(v_i - v_j\right) \Big] . \end{aligned} \tag{3.5}$$

The study of consensus protocols for double integrator dynamics of the form (3.5) is motivated by, e.g., distributed coordination of satellites without absolute position or measurements, as discussed in Chapter 1.2. We show that under mild conditions, the consensus protocol (3.5) achieves asymptotic consensus on the velocities v_i. The following theorem generalizes both the literature on linear second-order consensus as in Ren and Beard (2008), as well as the literature on first-order nonlinear consensus as in Bauso et al. (2006). By using an integral Lyapunov function, we are able to prove that the agents reach consensus for the nonlinear consensus protocol also under double-integrator dynamics.

Theorem 3.2 *Consider agents with dynamics* (3.5)*, where* $\alpha_{ij}(\cdot)$ *and* $\gamma_i(\cdot)$ *satisfy Assumptions 3.1 and 3.2, respectively, and* $\beta_{ij}(\cdot)$ *satisfies 3.2, mutatis mutandis. The system achieves consensus with respect to x and v, i.e.,* $|x_i - x_j| \to 0$, $|v_i - v_j| \to 0\ \forall i, j \in \mathcal{G}$ *as* $t \to \infty$ *for any initial condition* $(x(0), v(0))$*. Furthermore, the velocities converge to a common value* $\lim_{t\to\infty} v_i(t) = v^*\ \forall\ i \in \mathcal{V}$ *uniquely determined by*

$$\sum_{i\in\mathcal{V}} \int_0^{v_i^0} \frac{1}{\gamma_i(y)}\,\mathrm{d}y = \int_0^{v^*} \sum_{i\in\mathcal{V}} \frac{1}{\gamma_i(y)}\,\mathrm{d}y. \tag{3.6}$$

Proof. We write (3.5) in vector form as

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\Gamma(v)\Big[\mathcal{B}\alpha(\bar{x}) + \mathcal{B}\beta(\mathcal{B}^T v)\Big], \end{aligned}$$

where $\bar{x} = \mathcal{B}^T x$, and $\alpha(\cdot)$ and $\beta(\cdot)$ are taken component-wise, and $\Gamma(x) = \mathrm{diag}([\gamma_1(x_1), \ldots, \gamma_n(x_n)])$. Consider now the following candidate Lyapunov function, also used by Münz et al. (2011),

$$V(\bar{x}, v) = \sum_{i\in\mathcal{V}} \left(\int_{v^*}^{v_i} \frac{y - v^*}{\gamma_i(y)}\,\mathrm{d}y \right) + \sum_{(i,j)\in\mathcal{E}} \int_0^{\bar{x}_{ij}} \alpha_{ij}(y)\,\mathrm{d}y,$$

where v^* is the common velocity of the agents in steady state, given by (3.6). It is straightforward to verify that $V([0_{1\times m}, v^* 1_{1\times n}]^T) = 0$. By following the proof of the positive semi-definiteness of $V(x)$ in the proof of Theorem 3.1, mutatis mutandis, the positive semi-definiteness of $\sum_{i\in\mathcal{V}}(\int_{v^*}^{v_i} y - v^*/\gamma_i(y)\,\mathrm{d}y)$ follows. For showing the positive semi-definiteness of the second term, it suffices to show that $\int_0^{\bar{x}_{ij}} \alpha_{ij}(y)\,\mathrm{d}y > 0\ \forall (i,j) \in \mathcal{E}$. For $\bar{x}_{ij} > 0$, this inequality clearly holds. When $\bar{x}_{ij} < 0$ we have

$$\int_0^{\bar{x}_{ij}} \alpha_{ij}(y)\,\mathrm{d}y = -\int_{\bar{x}_{ij}}^0 \alpha_{ij}(y)\,\mathrm{d}y = \int_{\bar{x}_{ij}}^0 \alpha_{ji}(-y)\,\mathrm{d}y > 0.$$

We may write $V(\bar{x}, v)$, using the incidence matrix $\mathcal{B}$, as

$$V(\bar{x}, v) = \int_0^{\bar{x}} 1_{1\times n}\mathcal{B}^T\alpha(y)\,\mathrm{d}y + \int_{v^* 1_{n\times 1}}^{v} \tilde{y}^T\Gamma^{-1}(y)1_{n\times 1}\,\mathrm{d}y,$$

where $\tilde{y} = [y_1 - v^*, \ldots, y_n - v^*]^T$. Differentiating $V(x, v)$ with respect to time yields:

$$\begin{aligned}
\frac{dV(\bar{x}, v)}{dt} &= \frac{\partial V(x, v)}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V(x, v)}{\partial v}\frac{\partial v}{\partial t} \\
&= \alpha(\bar{x})^T\mathcal{B}^T v - (v - v^* 1)^T\Gamma^{-1}(v)\Gamma(v)\left[\mathcal{B}\alpha(\bar{x}) + \mathcal{B}\beta(\mathcal{B}^T v)\right] \\
&= -v^T\mathcal{B}\beta(\mathcal{B}^T v) + v^* 1^T\mathcal{B}\beta(\mathcal{B}^T v) = -v^T\mathcal{B}\beta(\mathcal{B}^T v) \le 0
\end{aligned}$$

due to Assumption 3.2, with equality if and only if $\mathcal{B}^T v = 0$. We now invoke LaSalles invariance principle to show that the agreement point satisfies $\dot{v} = 0$. The subspace where $\dot{V}(\bar{x}, v) = 0$ is given by $S_1 = \left\{(\bar{x}, v) | v = c1_{n\times 1}\right\}$. We note that on S_1,

$$\dot{v} = -\Gamma(v)\left[\mathcal{B}\alpha(\bar{x}) + \mathcal{B}\beta(\mathcal{B}^T v)\right] = -\Gamma(v)\mathcal{B}\alpha(\bar{x}) \neq k(t)1_{n\times 1}.$$

To see this, suppose that

$$\dot{v}(t) = -\Gamma(v)\mathcal{B}\alpha(\bar{x}) = k(t)1_{n\times 1} \Leftrightarrow \mathcal{B}\alpha(\bar{x}) = \Gamma^{-1}(v)k(t)1_{n\times 1},$$

where $k(t) \neq 0$. Premultiplying the above equation with $1_{1\times n}$ yields

$$0 = 1_{1\times n}\mathcal{B}\alpha(\bar{x}) = k(t)1^T\Gamma^{-1}(v)1 \neq 0,$$

which is a contradiction since $k(t) \neq 0$ by assumption. Hence the only trajectories contained in S_1 are those where $v = v^* 1_{n\times 1}, \dot{v} = 0$. It can also be shown that no trajectories where $\bar{x} \neq 0$ are contained in S_1. Assume for the sake of contradiction that

$\bar{x} \neq 0$ in S_1. Let $i^- = \min_{j \in \mathcal{V}} x_j$ s.t. $\exists k \in \mathcal{N}_{i^-} : x_k > x_{i^-}$. It is clear that such an i^- exists, since otherwise $\bar{x} = 0$. Consider

$$\begin{aligned}\dot{v}_{i^-} &= -\gamma_{i^-}(v_{i^-}) \sum_{j \in \mathcal{N}_{i^-}} \left[\alpha_{i^-j}\left(x_{i^-} - x_j\right) + \beta_{ij}\left(v_{i^-} - v_j\right)\right] \\ &= -\gamma_{i^-}(v_{i^-}) \sum_{j \in \mathcal{N}_{i^-}} \left[\alpha_{i^-j}\left(x_{i^-} - x_j\right)\right] > -\gamma_{i^-}(v_{i^-})\alpha_{i^-k}\left(x_{i^-} - x_k\right) > 0\end{aligned}$$

by the assumption that $x_k > x_{i^-}$. Thus, any trajectory in S_1 where $\bar{x} \neq 0$ cannot remain in S_1, implying that $|x_i - x_j| \to 0, |v_i - v_j| \to 0 \ \forall i,j \in \mathcal{G}$ as $t \to \infty$ and furthermore $\dot{v}(t) = 0$. Next we show that

$$P(v) = \sum_{i \in \mathcal{V}} \int_0^{v_i} \frac{1}{\gamma_i(y)} \,\mathrm{d}y = \int_0^{v} 1^T \Gamma^{-1}(v) 1 \,\mathrm{d}y$$

is invariant under (3.5). Consider:

$$\begin{aligned}\frac{\mathrm{d}P(v(t))}{\mathrm{d}t} &= \frac{\partial P}{\partial v}\frac{\partial v}{\partial t} = -1^T\Gamma^{-1}(v)\Gamma(v)\left[\mathcal{B}\alpha(\bar{x}) + \mathcal{B}\beta(\mathcal{B}^T v)\right] \\ &= -1^T\mathcal{B}\alpha(\bar{x}) - 1^T\mathcal{B}\beta(\mathcal{B}^T v) = 0.\end{aligned}$$

Thus we conclude that $\lim_{t \mapsto \infty} x(t) = x^*(t)1$ and $\lim_{t \mapsto \infty} v(t) = v^* 1$ with v^* given by the integral equation

$$\sum_{i \in \mathcal{V}} \int_0^{v_i^0} \frac{1}{\gamma_i(y)} \,\mathrm{d}y = \int_0^{v^*} \sum_{i \in \mathcal{V}} \frac{1}{\gamma_i(y)} \,\mathrm{d}y.$$

The existence and uniqueness of the solution to the above integral equation follows from Assumption 3.1, and by the proof of Theorem 3.1, mutatis mutandis. □

Remark 3.4 Theorem 3.2 has a physical interpretation. If we regard $\frac{1}{\gamma_i(v_i)}$ as the velocity dependent mass of agent i, e.g. due to special relativity, then the invariant quantity

$$P(v) = \sum_{i \in \mathcal{V}} \int_0^{v_i} \frac{1}{\gamma_i(y)} \,\mathrm{d}y$$

is the total momentum of the mechanical system.

3.3 Distributed control for double-integrator dynamics with state-dependent damping

In this section we consider agents with double-integrator dynamics, and control input given by:

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= u_i \\ u_i &= -\gamma_i(x_i)v_i - \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j).\end{aligned} \tag{3.7}$$

The study of consensus protocols for double-integrator dynamics with state-dependent damping, as in equation (3.7), is motivated by, e.g., coordination of underwater vehicles, as discussed in Chapter 1.3. The following theorem generalizes the results of Xie and Wang (2007) to include nonlinear state-dependent damping, as well as nonlinear interaction functions. With this framework, we are able to generalize the average consensus to a much broader class of controllers.

Theorem 3.3 *Consider agents with dynamics* (3.7)*, where* $\gamma_i(\cdot)$ *satisfies Asssumption 3.1, and* $\alpha_{ij}(\cdot)$ *satisfies Assumption 3.2. Then the agents converge to a common point for all initial positions* $x_i(0)$*. Furthermore, the consensus point is uniquely determined by*

$$\sum_{i\in\mathcal{V}} \Big(\int_0^{x_i^0} \gamma_i(y)\,\mathrm{d}y + v_i(0) \Big) = \int_0^{x^*} \sum_{i\in\mathcal{V}} \gamma_i(y)\,\mathrm{d}y. \tag{3.8}$$

Proof. We first note that by Assumption 3.1 and 3.2, a unique continuous solution of (3.7) exists for all $t \geq 0$. Consider the candidate Lyapunov function

$$V(x,v) = \sum_{i\in\mathcal{V}} \left[\frac{v_i^2}{2} + \sum_{j\in\mathcal{N}_i} \int_0^{x_i - x_j} \alpha_{ij}(y)\,\mathrm{d}y \right].$$

Differentiating $V(x, v)$ along trajectories of (3.7) yields

$$\begin{aligned}\dot{V}(x,v) &= \sum_{i\in\mathcal{V}} \left[\frac{\partial V(x,v)}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial V(x,v)}{\partial v_i}\frac{\partial v_i}{\partial t} \right] \\ &= \sum_{i\in\mathcal{V}} v_i\big(-\gamma_i(x_i)v_i - \sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j)\big) \\ &\quad + \sum_{i\in\mathcal{V}} \big(\sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j)\big) v_i = -\sum_{i\in\mathcal{V}} \gamma_i(x_i) v_i^2 \leq 0.\end{aligned}$$

It is thus clear that $\exists\ \Omega$ compact, such that $[\bar{x}(t), v(t)] \in \Omega\ \forall t \geq 0$, namely $\{(x, v) : V(\bar{x}, v) \leq V(\bar{x}_0, v_0)\}$, where $\bar{x} = \mathcal{B}x$ and $\bar{x}_0 = \bar{x}(0), v_0 = v(0)$. It remains to ensure that also $[x(t), v(t)]$ evolve in a compact set. Since $\bar{x}$ is bounded, then clearly x is bounded iff $x' = \frac{1}{n}\sum_{i\in\mathcal{V}} x_i$ is bounded. Consider now

$$E(x, v) = \sum_{i\in\mathcal{V}} \left(\int_0^{x_i} \gamma_i(y)\, \mathrm{d}y + v_i \right).$$

Differentiating $E(x, v)$ along trajectories of (3.7) yields

$$\dot{E}(x, v) = \sum_{i\in\mathcal{V}} \left(\frac{\partial E(x, v)}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial E(x, v)}{\partial v_i}\frac{\partial v_i}{\partial t} \right) = -\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} \alpha_{ij}(x_i - x_j) = 0$$

by Assumption 3.2. Denoting the initial condition by $[x_0, v_0]$, we obtain

$$E_0 = E(x_0, v_0) = \sum_{i\in\mathcal{V}} \left(\int_0^{x_i} \gamma_i(y)\, \mathrm{d}y + v_i \right).$$

Since $[\bar{x}(t), v(t)]$ evolve in a compact set, $v_i(t)$ is bounded. Hence $\forall i \in \mathcal{V}\ \exists M \in \mathbb{R}^+ :$ $|v_i(t)| \leq M\ \forall t \geq 0,\ \forall i \in \mathcal{V}$. By assumption 3.1 $\gamma_i(x) \geq \underline{\gamma} > 0 \quad \forall i \in \mathcal{V},\ \forall x \in \mathbb{R}$. Using these inequalities we obtain

$$\left| \int_0^{x_i} \gamma_i(y)\, \mathrm{d}y \right| \leq nM + |E_0|, \tag{3.9}$$

Assume for the sake of contradiction that $x'(t)$ is unbounded. Let us consider the case when $x'(t) \to +\infty$. Since $\bar{x}$ is bounded and $\mathcal{G}$ is connected, $|x_i(t) - x_j(t)|$ is bounded $\forall i, j \in \mathcal{V}$ by let us say M'. Thus $x_i(t) > 0\ \forall i \in \mathcal{V}$ whenever $x'(t) > M'$. Provided that $x'(t) > M'$, we obtain the following inequality:

$$\sum_{i\in\mathcal{V}} \int_0^{x_i} \gamma_i(y)\, \mathrm{d}y \geq \sum_{i\in\mathcal{V}} \underline{\gamma} x_i$$

By assumption, $x'(t)$ is unbounded, implying that also $\sum_{i\in\mathcal{V}} x_i(t)$ is unbounded. Thus $\exists t_1 : \sum_{i\in\mathcal{V}} x_i(t_1) > \max\{\frac{1}{\underline{\gamma}}\left(nM + |E_0|\right), M'\}$. But this contradicts (3.9). Hence $x'(t)$ must be bounded. The cases when $x'(t) \to -\infty$ as well as the case when no limit of $x'(t)$ exists are treated analogously. We conclude that x must be bounded. Denoting the closure of the set in which $[x, v]$ evolves Ω', we note that Ω' is compact by the Heine-Borel Theorem.

Let $E = \{(x, y) | v = 0\}$. Consider any trajectory of (3.7) with $x \neq x^*(t)\mathbf{1}$. By (3.7) and the assumption that $\mathcal{G}$ is connected, $\dot{v}_i \neq 0$ for at least one index i. Thus the largest invariant manifold of E is $\{(x, v)|x = x^*, v = 0\}$. Since Ω' is compact and positively invariant, by LaSalle's invariance principle, see Theorem 2.2, the agents converge to a common point $x_i = x^* \ \forall i \in \mathcal{G}$, with $v_i = 0 \ \forall i \in \mathcal{G}$.

It remains to show that the common point to which the agents converge to is the point given by (3.8), and that the solution is unique. Indeed, consider again the function $E(x, v)$. Since $\dot{E}(x, v) = 0$, and the agents converge to a point x^* with $v_i = 0 \ \forall i \in \mathcal{V}$. It follows that x^* is given by (3.8). Since $\gamma_i(y) > 0$ by assumption, (3.8) admits a unique solution. □

The following corollary follows directly from Theorem 3.3.

Corollary 3.1 *Given n agents starting from rest, i.e., $v_i(0) = 0 \ \forall i \in \mathcal{V}$, and applying the control law* (3.7)*, the agents converge to a common point for all initial positions $x_i(0)$ if and only if the underlying communication graph $\mathcal{G}$ is connected. Furthermore, the consensus point is uniquely determined by*

$$\sum_{i \in \mathcal{V}} \int_0^{x_i^0} \gamma_i(y)\, \mathrm{d}y = \int_0^{x^*} \sum_{i \in \mathcal{V}} \gamma_i(y)\, \mathrm{d}y. \tag{3.10}$$

Remark 3.5 In Theorem 3.1, the consensus point is given by

$$\sum_{i \in \mathcal{V}} \int_0^{x_i^0} \frac{1}{\gamma_i(y)}\, \mathrm{d}y = \int_0^{x^*} \sum_{i \in \mathcal{V}} \frac{1}{\gamma_i(y)}\, \mathrm{d}y,$$

as opposed to (3.10) in Corollary 3.1. The intuition behind this peculiarity is that in (3.1), $\gamma_i(x_i)$ acts as a gain of agent i, where an increased $\gamma_i(x_i)$ will increase the speed of agent i. In (3.7) however, $\gamma_i(x_i)$ acts as a damping on agent i, where an increased $\gamma_i(x_i)$ will decrease the speed of agent i.

3.4 Motivating applications revisited

In this section we revisit some of the motivating applications introduced in Chapter 1.1. We will demonstrate that the results in this Chapter have numerous potential engineering applications.

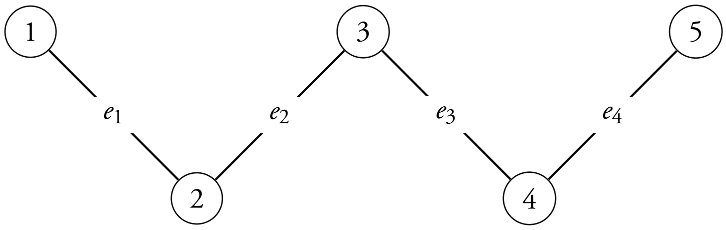


Figure 3.1 Floor topology.

Example 1.1 (Thermal energy storage in buildings, continued) We here return to the example of thermal energy storage in smart buildings. Recall that the temperatures dynamics in the rooms can be described by:

$$\dot{T}_i = -\gamma_i(T_i) \sum_{j \in \mathcal{N}_i} \alpha_{ij}(T_i - T_j), \tag{3.11}$$

In accordance with Fourier's law, the heat conductivity α is assumed to be constant and uniform, implying $\alpha_{ij}(x) = \alpha x \ \forall (i,j) \in \mathcal{E}$, where it is assumed that $\alpha = 0.5$W/K. Consider the floor topology in Figure 3.1. We assume that the desired maximum temperature is given by $t_b = 23°$C. The heat capacity is assumed to be given by Figure 3.2 for $i \in \{\text{Room 2}, \text{Room 5}\}$ due to thermal energy storage installations, and $\frac{1}{\gamma_i(T)} = 50$kJ/K for $i \in \{\text{Room 1}, \text{Room 3}, \text{Room 4}, \text{Room 6}, \text{Corridor}\}$ where no thermal energy storage is installed. The temperatures as a function of time are shown in Figure 3.3 for a given set of initial temperatures. We note that the temperatures in room 2 and 5 never exceed the desired maximum temperature $t_b = 23°$C, due to the thermal energy storage, and that the temperatures converge to a temperature below t_b in all rooms. In fact, this follows as a direct consequence of Theorem 3.1.

Corollary 3.2 *If there exists $\hat{T}$ such that*

$$\sum_{i \in \mathcal{V}} \int_0^{T_i(0)} \frac{1}{\gamma_i(y)} \, \mathrm{d}y \leq \int_0^{\hat{T}} \sum_{i \in \mathcal{V}} \frac{1}{\gamma_i(y)} \, \mathrm{d}y,$$

then the temperatures converge to $T^* \leq \hat{T}$.

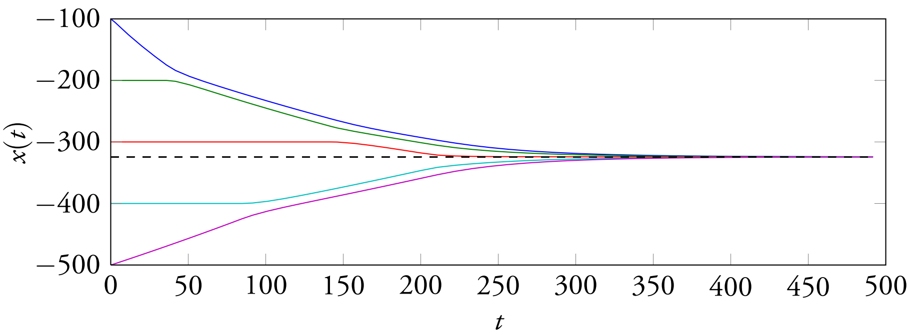


Figure 3.2 The figure shows the heat capacities of Room 2 and Room 5.

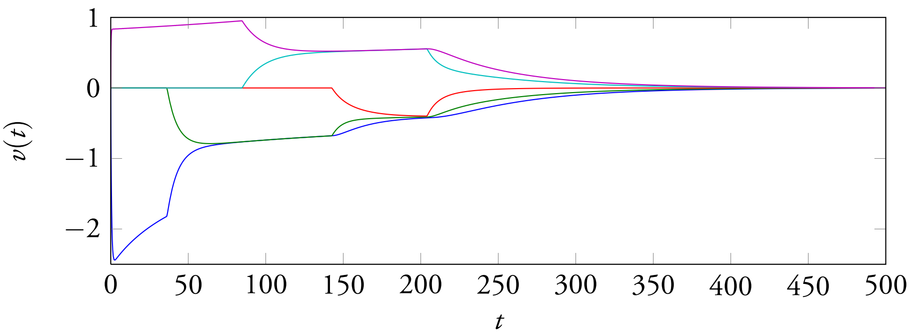


Figure 3.3 The figure shows the temperatures in the building floor. The initial temperatures was 29°C for room 6, 24°C for room 1, 22°C for the corridor and 20°C for the other rooms.

Example 1.2 (Autonomous space satellites, continued) Consider a group of autonomous space satellites with unitary masses. The agents are denoted $1, \ldots, 5$, and their communication topology is given by Figure 3.7. The control objective is to reach position and velocity consensus in one dimension by applying a distributed consensus control law by using only relative position and velocity measurements. The raw control signal is the power applied by each agent's engine, P_i. However, the acceleration in an observers reference frame is $a_i = P_i/|v_i|$ due to $P_i = \langle F_i, v_i \rangle$ and F_i being parallel to v_i, where v_i is agent i:s velocity. We assume that the agents only have access to relative measurements. This scenario can be modeled by the proposed nonlinear consensus protocol (3.5), where the gain function $\gamma_i(y) = 1/(|y| + c) \quad \forall i \in \mathcal{V}$ captures the

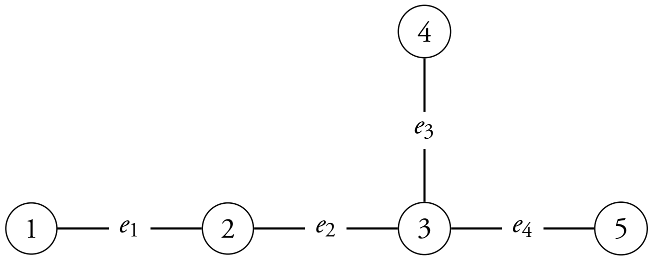


Figure 3.4 Communication topology of the space satellites.

dependence of the agents acceleration on it's absolute speed:

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= \frac{1}{|y| + c}\left(\sum_{j \in \mathcal{N}_i} \alpha_{ij}\left(x_i - x_j\right) + \beta_{ij}\left(v_i - v_j\right)\right),\end{aligned} \tag{3.12}$$

where $c \in \mathbb{R}^+$ is arbitrarily small, and ensures the boundedness of $\gamma_i(y)$ as $|y| \to 0$. Thus the dynamics of the satellites can be described by (3.5). The interaction functions in this example are assumed to be $\alpha_{ij}(y) = 2\beta_{ij}(y) = 20\,(e^y - 1)\,\mathrm{sgn}\,(y) \quad \forall (i, j) \in \mathcal{E}$, which clearly satisfy Assumption 3.2. It is clear that the above dynamics cannot be modeled by any previously proposed linear consensus protocols. The proposed interaction functions $\alpha_{ij}(\cdot)$ and $\beta_{ij}(\cdot)$ grow faster than linear, resulting in faster convergence when the satellites are far away. When the satellites are close, $\alpha_{ij}(\cdot)$ and $\beta_{ij}(\cdot)$ are approximately linear, resulting in smooth exponential convergence. Figure 3.6 shows the state trajectories for different initial conditions. As predicted by Theorem 3.2, consensus is reached, and the final consensus velocity, as seen from an observer, is calculated by (3.8), and is indicated by the dashed line.

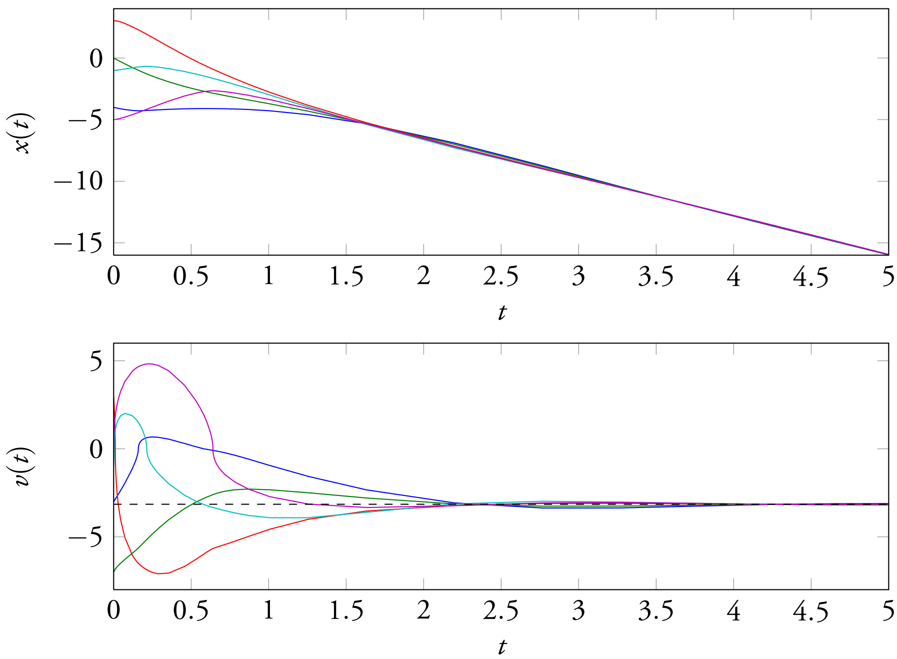


Figure 3.5 The figures show the state trajectories of the space satellites described by (3.12) for the initial conditions $x(0) = [-4, 0, 3, -1, -5]$, $v(0) = [-3, -7, 3, -1, 0]^T$.

Example 1.3 (Unmanned underwater vehicles, continued) Consider again a group of unmanned underwater vehicles. With only relative measurements available, the control objective is to rendezvous at a common depth. Due to the viscosity of water being pressure-dependent, the damping coefficients of the agents will depend on their depth. Let x_i denote the depth of agent i. The dynamics of agent i are given by

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= -(d_0 + k_d x_i) v_i + u_i \\ u_i &= -k \sum_{j \in \mathcal{N}_i} \min(|x_i - x_j|, a) \operatorname{sgn}(x_i - x_j). \end{aligned} \tag{3.13}$$

Clearly the dynamics are on the form (3.7), satisfying assumptions 3.1 and 3.2. The saturation function guarantees an upper bound on the input of each agent, considering that the damping is due to the water resistance. By knowing the degree Δ_i of agent

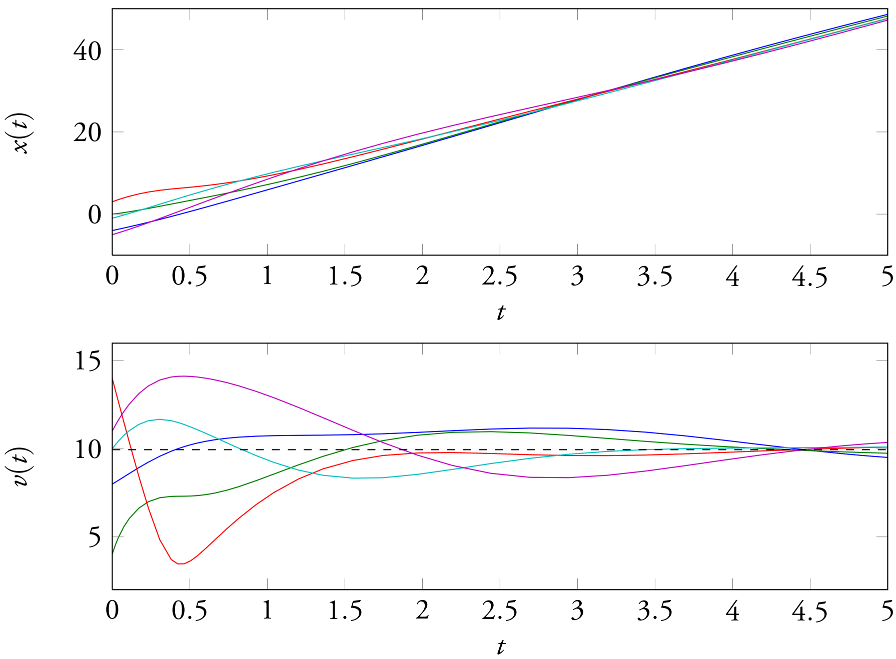


Figure 3.6 The figures show the state trajectories of the space satellites described by (3.12) for the initial conditions $x(0) = [-4, 0, 3, -1, -5]$, $v(0) = [8, 4, 14, 10, 11]^T$.

i, the input u_i^m is bounded by: $|u_i^m| \leq \Delta_i a$. The constants were set to $d_0 = 1$, $k_d = 0.01$, $k = 1$ and $a = 25$. The communication topology of the agents is illustrated in Figure 3.7.Figure 3.8 shows the state trajectories of the agents, starting at rest from $x(0) = [-100, -200, -300, -400, -500]$.

The effect of the saturation of u_i^m is clearly visible in the state trajectories of the vehicles, where the velocities of the agents is almost constant in the beginning, to decrease in magnitude as the agents approach each other.

3.5 Summary

In this section we have studied a class of nonlinear controllers for multi-agent systems, endowed with single- and double-integrator dynamics. In particular, we have studied distributed controllers where the control input i separated into a product of

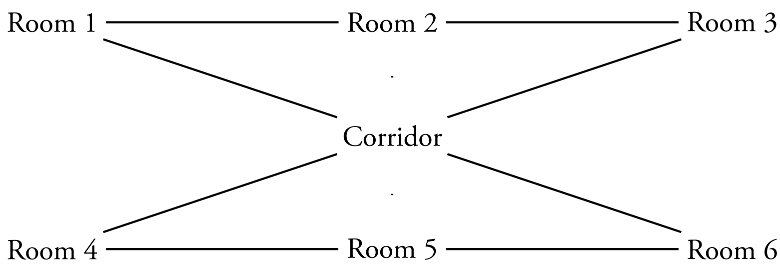


Figure 3.7 Communication topology of the underwater vehicles.

a nonlinear gain function depending only on the agents own state, and a sum of nonlinear interaction functions depending on the relative states of its neighbors. We proved stability for the proposed protocols by Lyapunov analysis, and characterized the convergence point by invariant functionals, for which we provided physical interpretations in terms of the constant quantities energy and momentum. We have also considered nonlinear control protocols for agents with double-integrator dynamics and state-dependent damping. We proved stability for the control protocol, and characterized the convergence point by an invariant functional. We have demonstrated how the obtained results can be applied in control of autonomous space satellites, control of underwater vehicles and in building temperature control.

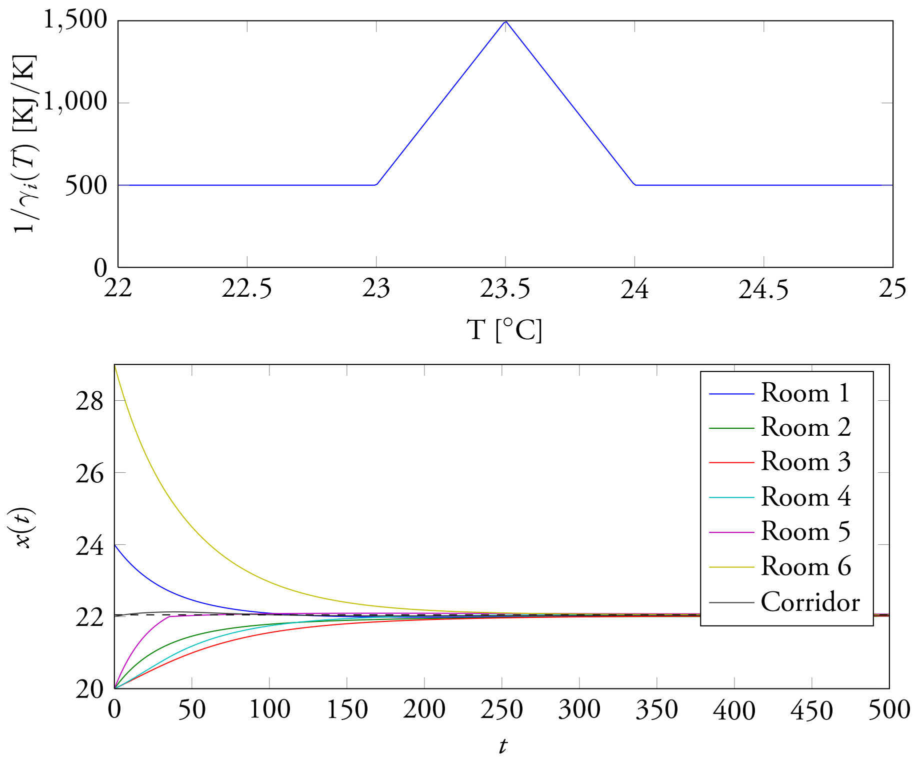


Figure 3.8 State trajectories of the underwater vehicles governed by (3.13), with $x(0) = [-100, -200, -300, -400, -500]$, $v(0) = [0, 0, 0, 0, 0]^T$

Chapter 4

Distributed control with integral action

Multi-agent systems are, like most control processes, sensitive to disturbances. Generally, static distributed control protocols cannot stabilize multi-agent systems in the presence of even constant disturbances. In this section we propose a control protocol for single- and double-integrator dynamics that drives the agents to a common state under static disturbances. By using distributed integral action, we are able to cancel the disturbances in a distributed setting. Moreover, with the proposed control algorithm, the agents reach the average of their initial positions for arbitrary initial velocities in the absence of disturbances. We study the properties of the control protocols and derive necessary and sufficient conditions under which the multi-agent system is stable in terms of the controller gains.

4.1 Distributed integral action for single-integrator dynamics

Consider agents with single-integrator dynamics, and control input given by:

$$\begin{aligned} \dot{x}_i &= u_i + d_i \\ u_i &= -\sum_{j\in\mathcal{N}_i}\left(\beta(x_i - x_j) + \alpha\int_0^t (x_i(\tau) - x_j(\tau))\,\mathrm{d}\tau\right) - \delta(x_i - x_i(0)) \end{aligned} \tag{4.1}$$

where $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+$, $\delta \in \bar{\mathbb{R}}^+$ are fixed parameters, and $d_i \in \mathbb{R}$ is an unknown disturbance. Note that if $\delta = 0$, absolute position measurements are unavailable, and (4.1) is completely distributed. The study of control protocols of the form (4.1) is motivated by mobile robot coordination under constant disturbances, as discussed in

Chapter 1.4. Having made the above definitions, we are now ready to state the first theorem of this Chapter.

Theorem 4.1 *Under the dynamics* (4.1)*, the agents converge to a common value x^* for any constant disturbance d_i and any initial condition. If $d_i = 0$ $\forall i \in \mathcal{V}$, the agents converge to*

$$x^* = \frac{1}{n}\sum_{i\in\mathcal{V}} x_i(0) \quad \forall\, v_i(0), z_i(0) = 0 \quad \forall i \in \mathcal{V},$$

for any initial condition. If absolute position measurements are not present, i.e., $\delta = 0$, it still holds that $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ $\forall i, j \in \mathcal{V}$ *for any set of disturbances d_i and any $\alpha, \beta \in \mathbb{R}^+$. However the absolute states are unbounded, i.e.,* $\lim_{t\to\infty} |x_i(t)| = \infty$ $\forall i \in \mathcal{V}$*, unless* $1_{1\times n} d = 0$.

Proof. First consider the case where $\delta = 0$ and $d_i = 0$ $\forall i \in \mathcal{V}$. By introducing the integral states $z = [z_1, \dots, z_n]^T$ we may rewrite the dynamics (4.1) in vector form as

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n\times n} & I_n \\ -\alpha\mathcal{L} & -\beta\mathcal{L} \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ x \end{bmatrix}, \tag{4.2}$$

together with the initial condition $z(0) = 0_{n\times 1}$. By elementary column operations we note that the characteristic equation of A is given by $0 = \det\big((\beta s + \alpha)\mathcal{L} + s^2 I_n\big)$. By comparing the characteristic polynomial with the characteristic equation of $\mathcal{L}$, being $0 = \det(\mathcal{L} - \kappa I_n)$, with solutions $\kappa = \lambda_i \geq 0$, we obtain the equation $0 = s^2 + \lambda_i\beta s + \lambda_i\alpha$. This equation has two solutions $s = 0$ if $\lambda_i = 0$, and solutions $s \in \mathbb{C}^-$ if $\lambda_i > 0$. Since the above equation has exactly two solutions for every λ_i, it follows that the algebraic multiplicity of the eigenvalue 0 must be equal to two. It is well-known that for connected graphs $\mathcal{G}$, λ_1 is the only zero-eigenvalue of the Laplacian $\mathcal{L}$. By straightforward calculations we obtain that $e_1^1 = [1_{1\times n},\ 0_{1\times n}]^T$ is an eigenvector and $e_1^2 = [0_{1\times n},\ 1_{1\times n}]^T$ is a generalized eigenvector of A corresponding to the eigenvalue 0. It can also be verified that $v_1 = \frac{1}{n}[1_{1\times n},\ 0_{1\times n}]$ and $v_2 = \frac{1}{n}[0_{1\times n},\ 1_{1\times n}]$ are a generalized left eigenvector and an eigenvector of A, respectively, corresponding to the eigenvalue 0, and that $v_1 e_1^1 = 1,\ v_2 e_1^2 = 1$ and $v_2 e_1^1 = 0,\ v_1 e_1^2 = 0$. If we let P be an orthonormal matrix consisting of the normalized eigenvectors of A, we can chose the first columns of P to be e_1^1 and e_1^2, and the first rows of P^{-1} to be v_1 and v_2, respectively.

Since all remaining eigenvalues of A have strictly negative real part we obtain

$$
\begin{aligned}
\lim_{t\to\infty} e^{At} &= \lim_{t\to\infty} Pe^{Jt}P^{-1} = P \lim_{t\to\infty} \begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & e^{J' t} \end{bmatrix} P^{-1} \\
&= \lim_{t\to\infty} P \begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & 0_{(2n-2)\times(2n-2)} \end{bmatrix} P^{-1} \\
&= \lim_{t\to\infty} \frac{1}{n} \begin{bmatrix} 1_{n\times n} & t1_{n\times n} \\ 0_{n\times n} & 1_{n\times n} \end{bmatrix}
\end{aligned}
$$

Thus, given an initial position $x(0) = x_0$, we obtain

$$
\lim_{t\to\infty} x_i(t) = \frac{1}{n} \sum_{i\in\mathcal{V}} x_{0,i} \quad \forall\, i \in \mathcal{V}
$$

i.e., the agents converge to the average of their initial positions.

We now consider the case where $\delta = 0$ and $d_i \neq 0$ $\forall i \in \mathcal{V}$. Define the output of the system

$$
\begin{bmatrix} y_z \\ y_x \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{B}^T & 0_{m\times n} \\ 0_{m\times n} & \mathcal{B}^T \end{bmatrix}}_{\triangleq C} \begin{bmatrix} z \\ x \end{bmatrix}
$$

and consider the linear coordinate change:

$$
\begin{aligned}
x &= \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} u \qquad & u &= \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1\times n} \\ S^T \end{bmatrix} x \\
z &= \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} w \qquad & w &= \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1\times n} \\ S^T \end{bmatrix} z
\end{aligned}
\tag{4.3}
$$

where S is a matrix such that $[\frac{1}{\sqrt{n}} 1_{n\times 1}, S]$ is an orthonormal matrix. In the new coordinates, the system dynamics (4.2) become:

$$\begin{aligned}\dot{w} &= u \\ \dot{u} &= \begin{bmatrix} 0 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & -\alpha S^T \mathcal{L} S \end{bmatrix} w \\ &\quad + \begin{bmatrix} 0 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & -\beta S^T \mathcal{L} S \end{bmatrix} u + \begin{bmatrix} \frac{1}{n} 1_{1\times n} \\ S^T \end{bmatrix} d.\end{aligned} \tag{4.4}$$

We also note that the states u_1 and w_1 are both unobservable and uncontrollable. We thus omit these states to obtain a minimal realization by defining the new coordinates $u' = [u_2, \ldots, u_n]^T$ and $w' = [w_2, \ldots, w_n]^T$, thus obtaining the system dynamics

$$\begin{bmatrix} \dot{w}' \\ \dot{u}' \end{bmatrix} = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{(n-1)} \\ -\alpha S^T \mathcal{L} S & -\beta S^T \mathcal{L} S \end{bmatrix} \begin{bmatrix} w' \\ u' \end{bmatrix} + \begin{bmatrix} 0_{(n-1)\times 1} \\ S^T d \end{bmatrix}.$$

Clearly $x^T S^T \mathcal{L} S x \geq 0$, with equality only if $Sx = k 1_{n\times 1}$. However, since $[\frac{1}{\sqrt{n}} 1_{n\times 1}, S]$ is orthonormal, $1_{1\times n} S x = 0_{1\times n} x = 0 = k 1_{1\times n} 1_{n\times 1} = kn$, which implies $k = 0$. Hence $S^T \mathcal{L} S$ is positive definite and thus invertible, and we may define

$$\begin{bmatrix} w'' \\ u'' \end{bmatrix} = \begin{bmatrix} w' \\ u' \end{bmatrix} - \begin{bmatrix} 0_{(n-1)\times 1} \\ \frac{1}{\alpha} (S^T \mathcal{L} S)^{-1} S^T d \end{bmatrix}.$$

It is easily verified that the origin is the only equilibrium of the system dynamics, which in the new coordinates are given by

$$\begin{bmatrix} \dot{w}'' \\ \dot{u}'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{(n-1)} \\ -\alpha S^T \mathcal{L} S & -\beta S^T \mathcal{L} S \end{bmatrix}}_{\triangleq A''} \begin{bmatrix} w'' \\ u'' \end{bmatrix}.$$

By elementary column operations, the characteristic polynomial in κ of A'' is given by $\det(\kappa^2 I_{(n-1)} + (\beta\kappa + \alpha) S^T \mathcal{L} S)$. By comparing this polynomial with the characteristic polynomial $\det(sI + S^T \mathcal{L} S)$, which since $S^T \mathcal{L} S$ is positive definite has solutions $-s_i < 0$, we know that the eigenvalues of A'' must satisfy $\kappa^2 + s_i \beta \kappa + s_i \alpha = 0$, with solutions $\kappa \in \mathbb{C}^-$. Thus A'' is Hurwitz. From the dynamics (4.4), it is clear that $\dot{u}_1 = \frac{1}{n} 1_{1\times n} d$. Hence $\lim_{t\to\infty} u_1(t) = \pm\infty$ unless $1_{1\times n} d = 0$. Since u' is bounded, by the coordinate change (4.3), x is bounded if and only if $1_{1\times n} d = 0$.

Now consider the case where $\delta > 0$ and $d_i = 0 \ \forall i \in \mathcal{V}$. The dynamics can be written as:

$$\begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n\times n} & I_n \\ -\alpha\mathcal{L} & -\beta\mathcal{L} - \delta I \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ x \end{bmatrix}.$$

By elementary column operations, the characteristic polynomial of A may be written as $0 = \det\big((\beta s + \alpha)\mathcal{L} + (s^2 + \delta s)I_n\big)$. By similar arguments used in the previous parts of the proof, A has a simple eigenvalue 0, with the corresponding eigenvector $e_1 = [1_{1\times n},\ 0_{1\times n}]^T$ and the left eigenvector $v_1 = \frac{1}{n}[1_{1\times n},\ 0_{1\times n}]$, whereas all other eigenvalues have negative real part. We see that $v_1 e_1 = 1$, and hence it follows that

$$\begin{aligned} \lim_{t\to\infty} e^{At} &= \lim_{t\to\infty} Pe^{Jt}P^{-1} = P \lim_{t\to\infty} \begin{bmatrix} 1 & 0_{1\times(2n-1)} \\ 0_{(2n-1)\times 1} & e^{J't} \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} 1 & 0_{1\times(2n-1)} \\ 0_{(2n-1)\times 1} & 0_{(2n-1)\times(2n-1)} \end{bmatrix} P^{-1} = \frac{1}{n} \begin{bmatrix} 1_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}. \end{aligned}$$

Given any initial position $x(0) = x_0$, it immediately follows that $\lim_{t\to\infty} x(t) = 0$.

Now let $\delta > 0$ and $d_i \neq 0$ for at least one $i \in \mathcal{V}$. Since the proof is analogous to the case when $\delta = 0$ and $d_i \neq 0$, it is omitted. Finally, if $d_i = 0 \ \forall i \in \mathcal{V}$, the stationarity of $x(t)$ implies $\lim_{t\to\infty} 1_{1\times n} \left(-\alpha\mathcal{L}z(t) - \beta\mathcal{L}x(t) - \delta x(t) + \delta x(0)\right) = 0$, so $nx^* = \sum_{i\in\mathcal{V}} x_i(0)$, which concludes the proof. □

Remark 4.1 Theorem 4.1 guarantees that the agents converge to a common state, even in the presence of constant disturbances. If absolute position measurements are available, i.e., when $\delta > 0$, the agents converge to a constant, bounded state. Since the agents converge to the average of their initial states in the absence of disturbances, and the system remains stable for any integral gain, there are no immediate performance degradations by the introduction of integral action, in the sense that it does not affect the stability nor the equilibria of the system, compared with a system with zero integral gain.

4.2 Distributed integral action for double-integrator dynamics

In this section we will generalize the results of Section 4.1 to agents with double-integrator dynamics. Consider agents with velocity-damped double-integrator dynamics, and control input given by a distributed PI-controller:

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= u_i - \gamma v_i + d_i \\ u_i &= -\sum_{j\in\mathcal{N}_i}\left(\beta(x_i - x_j) + \alpha\int_0^t (x_i(\tau) - x_j(\tau))\,\mathrm{d}\tau\right) - \delta(x_i - x_i^0)\end{aligned} \tag{4.5}$$

where $x_i^0 = x_i(0)$, $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+$, $\gamma \in \mathbb{R}^+$, $\delta \in \bar{\mathbb{R}}^+$ and $d_i \in \mathbb{R}$ is an unknown scalar disturbance. Note that if $\delta = 0$, (4.5) is completely distributed. The study of control protocols of the form (4.5) is motivated by mobile robot coordination under constant disturbances, as discussed in Chapter 1.4.

Theorem 4.2 *Under the dynamics* (4.5)*, the agents converge to an agreement point for any constant disturbance d_i and any initial condition, provided that $\alpha < \beta\gamma$. If $d_i = 0\ \forall i \in \mathcal{V}$, the agents converge to*

$$x^* = \frac{1}{n}\sum_{i\in\mathcal{V}} x_i(0) \quad \forall\ v_i(0).$$

If absolute position measurements are not present, i.e., $\delta = 0$, we still have $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0\ \forall i,j \in \mathcal{V}$ for any set of disturbances d_i. However the absolute states are in general unbounded, i.e., $\lim_{t\to\infty} |x_i(t)| = \infty\ \forall i \in \mathcal{V}$. Also, in this case the agents converge to a common value if and only if $\alpha < \beta\gamma$.

Proof. The proof follows the same principle ideas as the proof of Theorem 4.1. However, as we consider second-order dynamics, the problem is inherently different to first-order dynamics. First consider the case where $\delta = 0$. Let also $d_i = 0\ \forall i \in \mathcal{V}$. By introducing the state vector $z = [z_1, \dots, z_n]^T$ we may rewrite the dynamics:

$$\begin{bmatrix}\dot{z}\\ \dot{x}\\ \dot{v}\end{bmatrix} = \underbrace{\begin{bmatrix}0_{n\times n} & I_n & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & I_n\\ -\alpha\mathcal{L} & -\beta\mathcal{L} & -\gamma I_n\end{bmatrix}}_{\triangleq A}\begin{bmatrix}z\\ x\\ v\end{bmatrix},$$

together with the initial condition $z(0) = 0_{n\times 1}$. By elementary column operations it is easily shown that the characteristic polynomial of A can be written as $0 = \det((\alpha + \beta s)L + s^2(s+\gamma)I)$, where I is the identity matrix of appropriate dimensions. Comparing the above equation with the characteristic polynomial of $\mathcal{L}$, we get that $0 = s^3 + \gamma s^2 + \lambda_i\beta s + \lambda_i\alpha$, where λ_i is an eigenvalue of $\mathcal{L}$. If $\lambda_i > 0$, the above equation has all its solutions $s \in \mathbb{C}^-$ if and only if $\alpha < \beta\gamma$, and $\alpha, \beta, \gamma > 0$ by the Routh-Hurwitz stability criterion. Since $\mathcal{G}$ by assumption is connected, $\lambda_1 = 0$ and $\lambda_i > 0 \ \forall i = 2, \dots, n$. For $\lambda_1 = 0$, the above equation has the solutions $s = 0, s = -\gamma$. By straightforward calculations it can be shown that $e_1^1 = [1_{1\times n},\ 0_{1\times n},\ 0_{1\times n}]^T$ and $e_1^2 = [0_{1\times n},\ 1_{1\times n},\ 0_{1\times n}]^T$ are an eigenvector and a generalized eigenvector of A, respectively, corresponding to the eigenvalue 0. Furthermore $v_1 = \frac{1}{\gamma^2 n}[\gamma^2 1_{1\times n},\ 0_{1\times n},\ -1_{1\times n}]$ and $v_2 = \frac{1}{\gamma n}[0_{1\times n},\ \gamma 1_{1\times n},\ 1_{1\times n}]$ are a generalized left eigenvector and a left eigenvector of A corresponding to the eigenvalue 0. Furthermore $v_1 e_1^1 = 1,\ v_2 e_1^2 = 1$ and $v_2 e_1^1 = 0,\ v_1 e_1^2 = 0$. Hence the first columns of P can be chosen as e_1^1 and e_1^2, and the first rows of P^{-1} can be chosen to be v_1 and v_2. Since all other eigenvalues of A have strictly negative real part we obtain

$$
\begin{aligned}
\lim_{t\to\infty} e^{At} &= \lim_{t\to\infty} P e^{Jt} P^{-1} \\
&= P \lim_{t\to\infty} \begin{bmatrix} 1 & t & 0_{1\times(3n-2)} \\ 0 & 1 & 0_{1\times(3n-2)} \\ 0_{(3n-2)\times 1} & 0_{(3n-2)\times 1} & e^{J't} \end{bmatrix} P^{-1} \\
&= \lim_{t\to\infty} P \begin{bmatrix} 1 & t & 0_{1\times(3n-2)} \\ 0 & 1 & 0_{1\times(3n-2)} \\ 0_{(3n-2)\times 1} & 0_{(3n-2)\times 1} & 0_{(3n-2)\times(3n-2)} \end{bmatrix} P^{-1} \\
&= \lim_{t\to\infty} \frac{1}{n} \begin{bmatrix} 1_{n\times n} & t1_{n\times n} & \frac{t\gamma-1}{\gamma^2} 1_{n\times n} \\ 0_{n\times n} & 1_{n\times n} & \frac{1}{\gamma} 1_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}.
\end{aligned}
$$

Given any initial position $x(0) = x_0,\ \ v(0) = v_0$, we obtain $\lim_{t\to\infty} x_i(t) = \frac{1}{n}\sum_{i\in\mathcal{V}} x_{0,i} + \frac{1}{\gamma n}\sum_{i\in\mathcal{V}} v_{0,i} \quad \forall\ i \in \mathcal{V}$. Now let us turn our attention to the case where where $\delta = 0$ and $d_i \neq 0\ \forall i \in \mathcal{V}$. We define the output of the system as

$$
\begin{bmatrix} y_z \\ y_x \\ y_v \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{B}^T & 0_{m\times n} & 0_{m\times n} \\ 0_{m\times n} & \mathcal{B}^T & 0_{m\times n} \\ 0_{m\times n} & 0_{m\times n} & \mathcal{B}^T \end{bmatrix}}_{\triangleq C} \begin{bmatrix} z \\ x \\ v \end{bmatrix},
$$

and consider the same linear coordinate change of z, x and v as applied to z and x in the proof of Theorem 4.1. In the new coordinates the system dynamics are

$$\begin{aligned}
\dot{z}' &= x' \\
\dot{x}' &= v' \\
\dot{v}' &= \begin{bmatrix} 0 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & -\alpha S^T \mathcal{L} S \end{bmatrix} z' + \begin{bmatrix} 0 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & -\beta S^T \mathcal{L} S \end{bmatrix} x' \\
&\quad - \gamma v' + \begin{bmatrix} \frac{1}{n} 1_{1\times(n)} \\ S^T \end{bmatrix} d.
\end{aligned} \tag{4.6}$$

We note that the states z'_1, x'_1 and v'_1 are unobservable and uncontrollable. We thus omit these states to obtain a minimal realization by defining the new coordinates $z'' = [z'_2, \dots, z'_n]^T$, $x'' = [x'_2, \dots, x'_n]^T$ and $v'' = [v'_2, \dots, v'_n]^T$ we obtain the system dynamics

$$\begin{bmatrix} \dot{z}'' \\ \dot{x}'' \\ \dot{v}'' \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{(n-1)^2} & I_{(n-1)^2} & 0_{(n-1)^2} \\ 0_{(n-1)^2} & 0_{(n-1)^2} & I_{(n-1)^2} \\ -\alpha S^T \mathcal{L} S & -\beta S^T \mathcal{L} S & -\gamma I_{(n-1)^2} \end{bmatrix}}_{\triangleq A''} \begin{bmatrix} z'' \\ x'' \\ v'' \end{bmatrix} + \begin{bmatrix} 0_{(n-1)\times 1} \\ 0_{(n-1)\times 1} \\ S^T d \end{bmatrix}.$$

We now shift the state space by defining

$$\begin{bmatrix} z^{(3)} \\ x^{(3)} \\ v^{(3)} \end{bmatrix} = \begin{bmatrix} z'' \\ x'' \\ v'' \end{bmatrix} - \begin{bmatrix} 0_{(n-1)\times 1} \\ 0_{(n-1)\times 1} \\ \frac{1}{\alpha}(S^T \mathcal{L} S)^{-1} S^T d \end{bmatrix}.$$

It is easily verified that the origin is the only equilibrium of the system dynamics, and that the dynamics in the new coordinates are also characterized by the matrix A''. By a similar argument used when showing that A has eigenvalues with non-positive real part, we may show that A'' has eigenvalues with non-positive real part. But since $S^T \mathcal{L} S$ is full-rank, A'' must also be full-rank, and hence A'' is Hurwitz. This implies that $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0 \ \forall i, j \in \mathcal{V}$ even in the presence of disturbances d_i. It is also clear that whenever $\alpha \geq \beta\gamma$, at least one eigenvalue will have non-negative real part, and that its (generalized) eigenvector will be distinct from e_1^1 and e_1^2. Thus consensus is not reached. From the dynamics (4.6), it is clear that $\dot{x}'_1 = \frac{1}{n} 1_{1\times n} d$. Hence $\lim_{t\to\infty} x'_1(t) = \pm\infty$ unless $1_{1\times n} d = 0$. Since x'' is bounded, by the coordinate change (4.3), x is bounded if and only if $1_{1\times n} d = 0$.

We now turn our attention to the case where $\delta > 0$ and $d_i = 0 \ \forall i \in \mathcal{V}$. The dynamics (4.5) can then be written in vector form as

$$\begin{bmatrix} \dot{z} \\ \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n \\ -\alpha\mathcal{L} & -\beta\mathcal{L} - \delta I & -\gamma I_n \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ x \\ v \end{bmatrix}.$$

By performing elementary column operations the characteristic polynomial of A can be written as $\det((\alpha+\beta s)\mathcal{L}+(s^3+\gamma s^2+\delta s)I)$. By comparing the characteristic polynomial of A with the characteristic polynomial of $\mathcal{L}$, it can be seen that the eigenvalues s of A satisfy $0 = s^3 + \gamma s^2 + (\delta + \lambda_i\beta)s + \lambda_i\alpha$, where $\lambda_i \in \text{spec}(\mathcal{L})$ Since $\mathcal{G}$ by assumption is connected, $\mathcal{L}$ has a single simple eigenvalue $\lambda_1 = 0$, which gives the characteristic equation $0 = s(s^2 + \gamma s + \delta)$, with one solution $s = 0$, and two solutions $s \in \mathbb{C}^-$. As all other eigenvalues are strictly positive, the solutions s corresponding to strictly positive λ_i satisfy $s \in \mathbb{C}^-$ iff $\lambda_i\alpha + \delta < \lambda_i\beta\gamma$. This is satisfied whenever $\alpha < \beta\gamma$. It can be verified that $e_1 = [1_{1\times n},\ 0_{1\times n},\ 0_{1\times n}]^T$ and $v_1 = \frac{1}{n}[\delta 1_{1\times n},\ \gamma 1_{1\times n},\ 1_{1\times n}]^T$ are a right and left eigenvector of A, corresponding to the eigenvalue 0. Furthermore $v_1 e_1 = 1$. Since all other eigenvalues have strictly negative real part, we have

$$\begin{aligned} \lim_{t\to\infty} e^{At} &= \lim_{t\to\infty} Pe^{Jt}P^{-1} = P \lim_{t\to\infty} \begin{bmatrix} 1 & 0_{1\times(3n-1)} \\ 0_{(3n-1)\times 1} & e^{J't} \end{bmatrix} P^{-1} \\ &= \frac{1}{n} \begin{bmatrix} \delta 1_{n\times n} & \gamma 1_{n\times n} & 1_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \end{bmatrix}. \end{aligned}$$

Given any initial position $x(0) = x_0,\ v(0) = v_0$, we obtain that $\lim_{t\to\infty} x(t) = 0$.

Now let $\delta > 0$ and $d_i \neq 0$ for at least one $i \in \mathcal{V}$. Since the proof is analogous to the case when $\delta = 0$ and $d_i \neq 0$, it is omitted.

Finally, if $d_i = 0 \ \forall i \in \mathcal{V}$, the stationarity of $v(t)$ implies:
$\lim_{t\to\infty} 1_{1\times n} \left(-\alpha\mathcal{L}z - \beta\mathcal{L}x - \delta x + \delta x(0) - \gamma v\right) = 0$, so $nx^* = \sum_{i\in\mathcal{V}} x_i(0)$, which concludes the proof. □

4.3 Motivating application revisited

In this section we revisit some of the motivating applications introduced in Chapter 1.1. We will demonstrate that the results in this Chapter have potential applications.

Example 1.4 (Mobile robot coordination under disturbances, continued) In this section we revisit the example of mobile robots from Section 1.4. The dynamics of the robots are given by

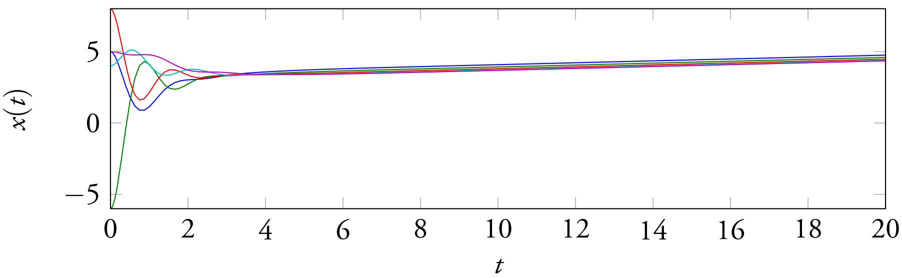
$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i - \gamma v_i + d_i \\ u_i &= -\sum_{j \in \mathcal{N}_i} \left(\beta(x_i - x_j) + \alpha \int_0^t (x_i(\tau) - x_j(\tau))\,\mathrm{d}\tau \right), \end{aligned} \tag{4.7}$$

Let the damping coefficient be given by $\gamma = 3$, and the static gain $\beta = 5$. We consider the system with a constant disturbance $d = [1, 0, 0, 0, 0]$, for the different integral gains $\alpha = 0$, $\alpha = 1$, and $\alpha = 15$. The initial conditions are given by $x(0) = [5, -6, 8, 4, 5]$, $v(0) = [0, 0, 0, 0, 0]^T$. The setup we will consider consists of a string of 5 mobile robots, whose communication topology is a string graph.

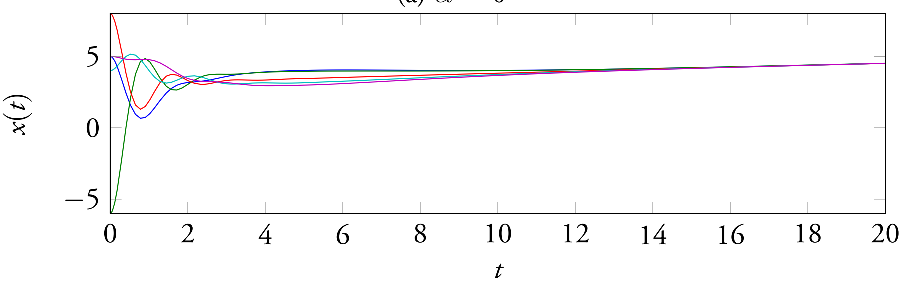
By Theorem 4.2 stability is guaranteed if and only if $\alpha < \beta\gamma$. In Figure 4.1 the state trajectories are shown for different choices of α. We observe that asymptotic consensus amongst the mobile robots is only reached when $\alpha = 1$. When $\alpha = 0$, consensus is not reached due to the static disturbance. When $\alpha = 1$, the disturbance is attenuated, and asymptotic consensus is reached. However, as we increase α to $15 = \beta\gamma$, the system becomes marginally unstable, i.e., stable but not asymptotically stable. By increasing α further, the system becomes unstable, in accordance with Theorem 4.2.

4.4 Summary

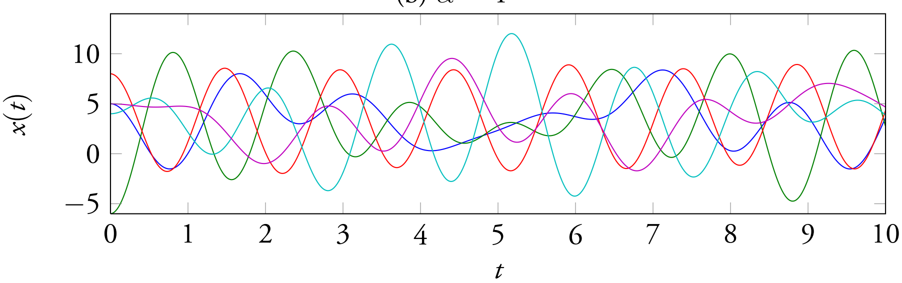
In this section we have studied distributed PI-controllers for multi-agent systems with constant disturbances. We have considered both agents with single- and double-integrator dynamics, and agents with velocity-damped double-integrator dynamics. We have provided necessary and sufficient conditions for the integral gains, under which the system is stable. Furthermore we have shown that the relative states of the agents converge to zero if the integral gain is strictly positive. We have demonstrated our results on mobile robot coordination under disturbances.



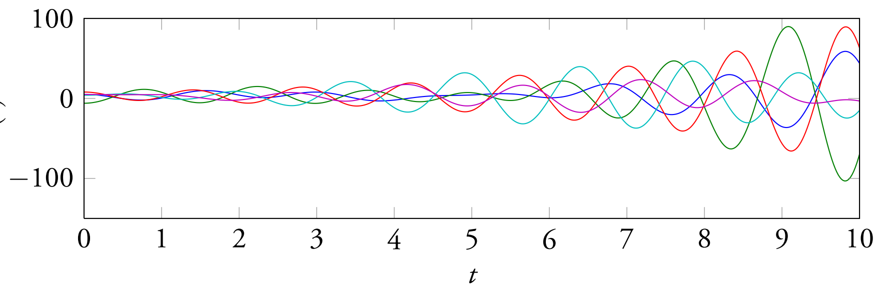
(a) $\alpha = 0$



(b) $\alpha = 1$



(c) $\alpha = 15$



(d) $\alpha = 20$

Figure 4.1 The figures show the state trajectories of (1.5) for $\alpha = 0$, $\alpha = 1$, $\alpha = 15$ and for $\alpha = 20$.

Chapter 5

Frequency control of power systems

MAINTAINING a constant frequency under varying power loads is one of the main control problems in today's power systems. Generators and electric machines will take damage if operated far from the nominal frequency. Keeping the frequency close to the nominal frequency is thus of major importance in any power system. Frequency control is traditionally carried out by automatic generation control (AGC), which employs distributed proportional controllers with a dynamic reference value, set by a centralized PI controller or even a human operator. Hence, even though parts of the controller are distributed, the overall control architecture is centralized. Furthermore, the AGC will in general not consider generation costs, i.e., an optimal load profile might become suboptimal in steady-state after a change in load occurs. Moreover, the centralized controller or its communication to the buses may be sensitive to link failures. A distributed controller structure may provide more redundancy to controller and link failures. A centralized control architecture might be particularly susceptible to failure under islanding, when the power system is split into two or more disconnected components. Maintaining stability under islanding is particularly motivated by emerging micro-grid applications, see e.g. Katiraei et al. (2005). A micro-grid is typically a part of the main electrical power system, but can operate autonomously in case of isolation. This requires controllers that can stabilize the micro-grid both when it is connected with the main grid and when operating autonomously. Distributed control for micro-grids that can be disconnected from the main grid has been studied by, e.g., Lopes et al. (2006). However, the aspect of power sharing or optimal generation is not treated in a distributed fashion within the micro-grid.

In this chapter we will address the above mentioned shortcomings with centralized AGC, by introducing distributed frequency controllers. For comparison purposes we also consider centralized controllers. We first compare the performance of a model of an AGC with a proposed suboptimal decentralized frequency controller. Later we propose a centralized and decentralized frequency controller which minimize the power generation cost in steady state, and compare the their performance.

5.1 Power system model

Consider a power system modeled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each node, here referred to as a bus, is assumed to obey the swing equation, as described by Machowski et al. (2008)

$$m_i\ddot{\delta}_i + d_i\dot{\delta}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}\sin(\delta_i - \delta_j) + p_i^m + u_i, \tag{5.1}$$

where δ_i is the phase angle of bus i, m_i and d_i are the inertia and damping coefficient respectively, p_i^m is the power load at bus i and u_i is the mechanical input. By convention we will define injected power to have a positive sign, and power load to have a negative sign. $k_{ij} = |V_i||V_j|b_{ij}$, where V_i is the voltage of bus i, and b_{ij} is the susceptance of the line (i, j). By linearizing (5.1) around the equilibrium where $\delta_i = \delta_j \ \forall i, j \in \mathcal{V}$, we obtain the linearized swing equation

$$m_i\ddot{\delta}_i + d_i\dot{\delta}_i = -\sum_{j\in\mathcal{N}_i} k_{ij}(\delta_i - \delta_j) + p_i^m + u_i. \tag{5.2}$$

By defining $\delta = [\delta_1 \dots, \delta_n]^T$, we may rewrite (5.1) in state-space form:

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & I_n \\ -M\mathcal{L}_k & -MD \end{bmatrix} \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ Mp^m \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ Mu \end{bmatrix} \tag{5.3}$$

where $M = \text{diag}(\frac{1}{m_1}, \dots, \frac{1}{m_n})$, $D = \text{diag}(d_1, \dots, d_n)$, $\mathcal{L}_k$ is the weighted Laplacian with edge weights k_{ij}, $p^m = [p_1^m, \dots, p_n^m]^T$, $u = [u_i, \dots, u_n]^T$.

5.2 Suboptimal centralized PI control

We will here present a a centralized frequency control protocol for power systems and analyze its stability properties. Traditionally, the AGC of a power systems is carried out at two levels, see e.g. Machowski et al. (2008). In the first level, the frequency is controlled with a proportional controller against a reference frequency. At the second level, the reference frequency is controlled with a proportional controller to eliminate static errors. We model the first level, proportional controller of bus i as:

$$u_i = \alpha(\hat{\omega} - \omega_i) \tag{5.4}$$

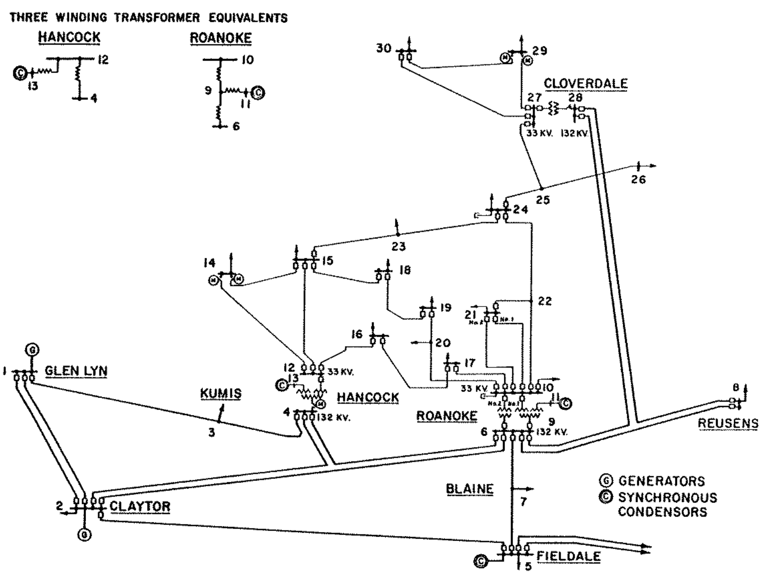


Figure 5.1 The IEEE 30-bus test system, an example of an electrical power system.

The outer proportional controller, regulating $\hat{\omega}$ is assumed to be given by:

$$\dot{\hat{\omega}} = \beta\left(\omega^{\text{ref}} - \frac{1}{n}\sum_{i\in\mathcal{V}}\omega_i\right), \tag{5.5}$$

where we have assumed that the average frequency of the buses is measured by the central controller[1]. The centralized controller architecture is illustrated in Figure 5.2.

Theorem 5.1 *The power system described by* (5.3) *where u_i is given by* (5.4)–(5.5)*, is stable for $\alpha, \beta > 0$. Furthermore $\lim_{t\to\infty}\omega_i(t) = \omega^{ref}$ for any set of initial conditions.*

[1]In reality the frequency is often measured at a specific bus. This will typically lead to longer delays, since disturbances need to propagate through the system before control action can be taken.

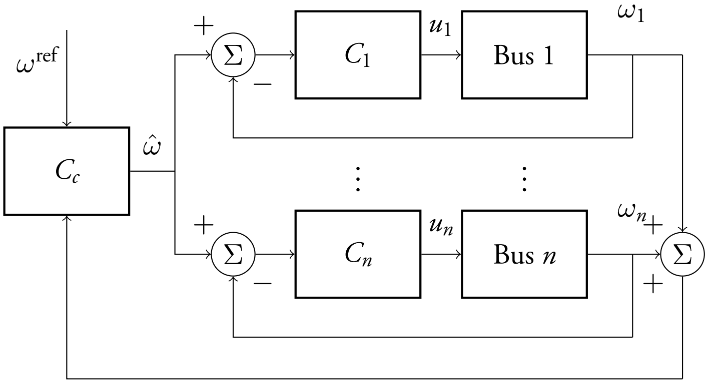


Figure 5.2 Centralized control architecture

Proof. We may write (5.3) with u given by (5.4)–(5.5) as:

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta} \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0_{1\times n} & -\frac{\beta}{n} 1_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} & I_n \\ \alpha 1_{n\times 1} & -M\mathcal{L}_k & -MD - \alpha I_n \end{bmatrix}}_{\triangleq A} \begin{bmatrix} \hat{\omega} \\ \delta \\ \omega \end{bmatrix} + \begin{bmatrix} \beta\omega^{\text{ref}} \\ 0_{n\times 1} \\ p^m \end{bmatrix}.$$

We now consider the matrix A' defined as

$$A' \triangleq \begin{bmatrix} 0 & 0_{1\times n} & -\frac{\beta}{n} 1_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} & I_n \\ \alpha 1_{n\times 1} & -M\mathcal{L}_k & -\alpha I_n \end{bmatrix}.$$

By elementary column operations on A', we may write the characteristic equation of A' as

$$s \det \left(\left(M\mathcal{L}_k + \frac{\alpha n}{\beta} 1_{n\times n} \right) + (s^2 + \alpha s) I_n \right) = 0.$$

The first factor has the solution $s = 0$. Comparing the second factor with the characteristic polynomial of $(M\mathcal{L}_k + \alpha n/\beta 1_{n\times n})$, we see that s satisfies $s^2 + s\alpha + t_i = 0$, where $t_i > 0$ is an eigenvalue of $(M\mathcal{L}_k + \alpha n/\beta 1_{n\times n})$. Also this equation has solutions $s \in \mathbb{C}^-$. Thus A' is Hurwitz, implying that A is Hurwitz. Stationarity implies that the equilibrium solution satisfies $\omega = \omega^{\text{ref}} 1_{n\times 1}$. □

5.3 Suboptimal decentralized PI control

In this section we analyze a decentralized AGC, where each bus controls its own frequency based only on local phase and frequency measurements. Thus, no frequency measurements need to be sent to a central controller, and there is no need to send control signals or reference values to the buses. This architecture might be favorable due to security concerns when sending unencrypted frequency measurements and control signals over large areas. Another benefit is improved performance when the tripping of one or several power lines causes the network to be split up into two or more sub-networks, so called islanding. The controller of node i is assumed to be given by Equation (1.7), here written as

$$\dot{z}_i = \omega^{\text{ref}} - \omega_i \tag{5.6}$$

$$u_i = \alpha(\omega^{\text{ref}} - \omega_i) + \beta z_i. \tag{5.7}$$

The controller architecture is illustrated in Figure 5.3. The decentralized controller

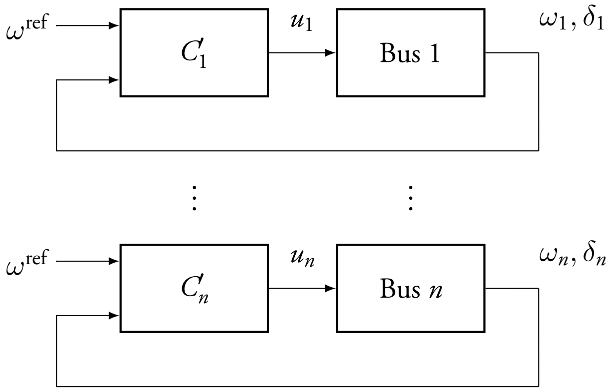


Figure 5.3 Decentralized control architecture.

(5.6)–(5.7) is typically not practically feasible with only frequency measurements available at the generation buses. Even the slightest measurement error will be integrated and cause instability, see, e.g., Machowski et al. (2008). However, with recent advances in phasor measurement unit (PMU) technology however, phase measurements are becoming more likely to be available to all generator buses, see, e.g., Phadke (1993). By employing optimal PMU placement, the number of PMUs needed for complete observability can be drastically reduced, as described by Nuqui and Phadke (2005). By integrating (5.6) we obtain

$$z_i = \omega^{\text{ref}} t - \delta_i.$$

This implies that in order to accurately estimate the integral state z_i, each generator bus needs access only to accurate time and phase measurements, both provided by PMU's.

Theorem 5.2 *The power system described by* (5.3) *where u_i is given by* (5.6)–(5.7), *is stable for any choice of $\alpha, \beta > 0$. Furthermore $\lim_{t\to\infty} \omega_i(t) = \omega^{ref}$.*

Proof. If we consider $[\mathcal{B}^T\delta, \omega]$ to be the output, the dynamics of (5.3) may be modified as long as the dynamics of $[\mathcal{B}^T\delta, \omega]$ are left unchanged. We thus may rewrite (5.3) with u given by (5.6)–(5.7):

$$\begin{bmatrix} \dot{z} \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n\times n} & I_n \\ -M\mathcal{L}_k - \beta M & -MD - \alpha M \end{bmatrix}}_{\triangleq A} \begin{bmatrix} z \\ \omega \end{bmatrix} + \begin{bmatrix} -\omega^{\text{ref}} 1_{n\times 1} \\ M(p^m + \alpha\omega^{\text{ref}} 1_{n\times 1}) \end{bmatrix},$$

since $\dot{\delta} - \dot{\omega} = \omega^{\text{ref}} 1_{n\times 1}$, implying that $\delta - \omega = t\omega^{\text{ref}} 1_{n\times 1}$. Since $\mathcal{L}_k 1_{n\times 1} = 0_{n\times 1}$, the output dynamics of the above equation is equivalent to that of (5.3) with respect to the output $[\mathcal{B}^T\delta, \omega]$. Here, the i'th element of β is β_i. Let $\underline{m} = \min_i m_i$ and $\underline{d} = \min_i d_i$. We can now write $MD = \underline{m}\,\underline{d} I_n + D'$, where D' is a diagonal positive-semidefinite matrix. Define:

$$A' \triangleq \begin{bmatrix} 0_{n\times n} & I_n \\ -M\mathcal{L}_k - \beta M & -\underline{m}\,\underline{d} I_n \end{bmatrix}.$$

By elementary column operations, the eigenvalues of A' are given by the roots of $0 = \det((s^2 + s\underline{m}\,\underline{d})I_n + M\mathcal{L} + \beta I_{n\times n})$. Comparing this with the characteristic polynomial of $M\mathcal{L}_k + \text{diag}(\beta)$, we conclude that s must satisfy $s^2 + s\underline{m}\,\underline{d} + t_i = 0$, where $t_i \geq 0$ is an eigenvalue of $M\mathcal{L}_k + \text{diag}(\beta)$. Since $M\mathcal{L}_k + \text{diag}(\beta)$ is positive definite, the above equation has all its solutions in $\mathbb{C}^-$. It follows that also A is Hurwitz. Now consider the coordinate shift

$$\begin{bmatrix} z' \\ \omega' \end{bmatrix} = \begin{bmatrix} z \\ \omega \end{bmatrix} - \begin{bmatrix} z_o \\ \omega_o \end{bmatrix}$$

where

$$\begin{aligned} z_0 &= (\beta I_n + \mathcal{L}_k)^{-1}(D\omega^{\text{ref}} 1_{n\times 1} - p^m) \\ \omega_0 &= \omega^{\text{ref}} 1_{n\times 1}. \end{aligned}$$

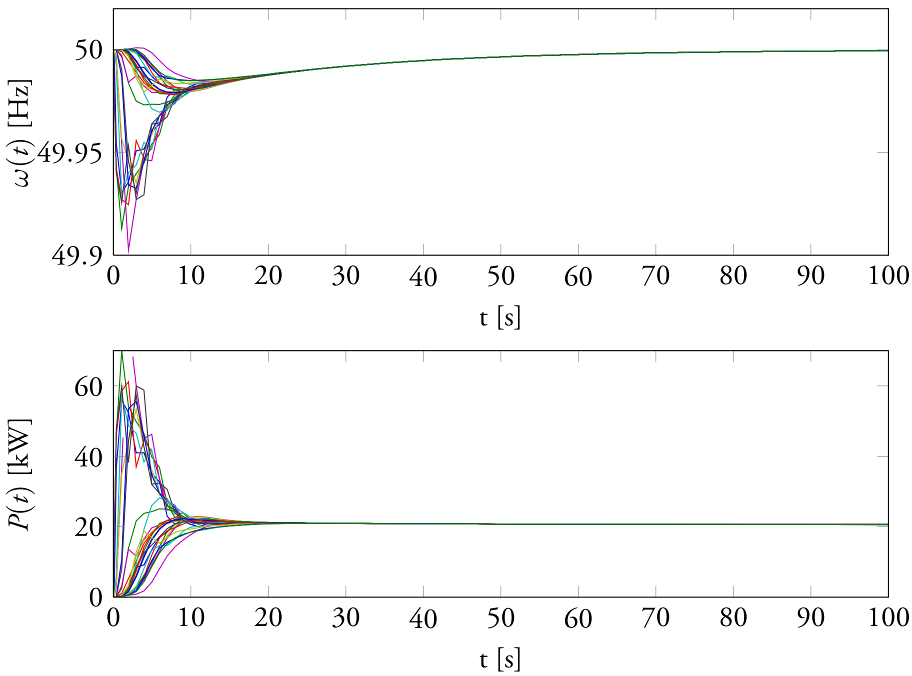


Figure 5.4 The figures show the bus frequencies and control signals respectively under centralized frequency control for $\alpha = 0.8$, $\beta = 0.04$ and $\gamma = 0.04$.

In the translated coordinates, the origin is the only equilibrium of the system. Hence $\lim_{t\to\infty} \omega_i(t) = \omega^{\text{ref}} \; \forall i \in \mathcal{V}$. $\square$

Example 1.5 (Frequency control of power systems — Suboptimal PI control) The centralized and decentralized frequency control algorithms were tested on the IEEE 30 bus test system, illustrated in Figure 5.1. The line admittances were extracted from the IEEE 30 bus test system, and the voltages were assumed to be 132 kV for all buses. The values of M and D were assumed to be given by $m_i = 10^5$ kg m^2 and $d_i = 1\ s^{-1}\ \forall i \in \mathcal{V}$. The power system is initially in an operational equilibrium, until the power load is increased by a step of 200 kW in the buses 2, 3 and 7. This will immediately result in decreased frequencies at the extra load buses. The frequency controllers at the buses will then control the frequencies towards the desired frequency of $\omega^{\text{ref}} = 50$ Hz. For the centralized controller the parameters were set to $\alpha = 0.8$, $\beta = 0.04$, $\gamma = 0.04$, while for the decentralized control architecture the parameters were $\alpha = 0.8$, $\beta = 0.04$. The

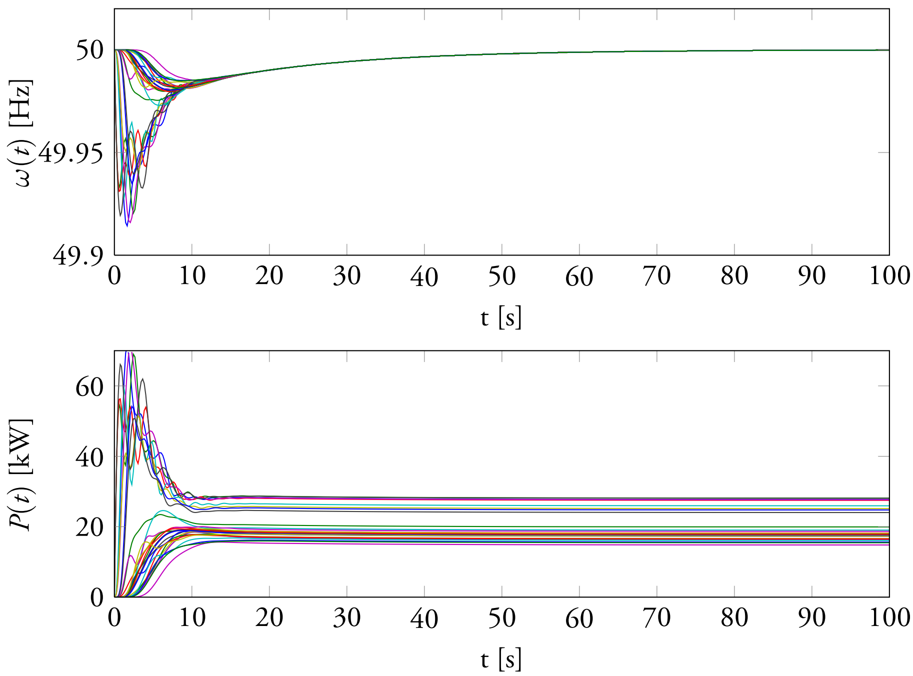


Figure 5.5 The figures show the bus frequencies and control signals respectively under decentralized frequency control for $\alpha = 0.8$ and $\beta = 0.04$.

step responses of the frequencies are plotted in Figure 5.4 for the centralized controller, and in Figure 5.5 for the decentralized controller. We note that if there is a centralized PI controller for the reference frequency, the generation is increased uniformly among the generators. If however the integral action is distributed amongst the generators, some generators will increase their generation more than others. Figure 5.6 and 5.7 show the step response under much larger integral action for the centralized and the decentralized controller respectively. For the centralized controller the parameters were set to $\alpha = 0.8$, $\beta = 0.8$, $\gamma = 0.8$, while for the decentralized control architecture the parameters were $\alpha = 0.8$, $\beta = 0.8$. We notice that the step response of the decentralized controller shows better performance compared to the centralized controller. As the centralized controller only has knowledge about the average frequency in the power system, the reference frequency $\hat{\omega}$ will be too large for some buses, decreasing generation too much at those buses. On the other hand, $\hat{\omega}$ will be too small for other buses, increasing the generation too much at those buses. Neither the proposed centralized nor the decentralized controller take into account

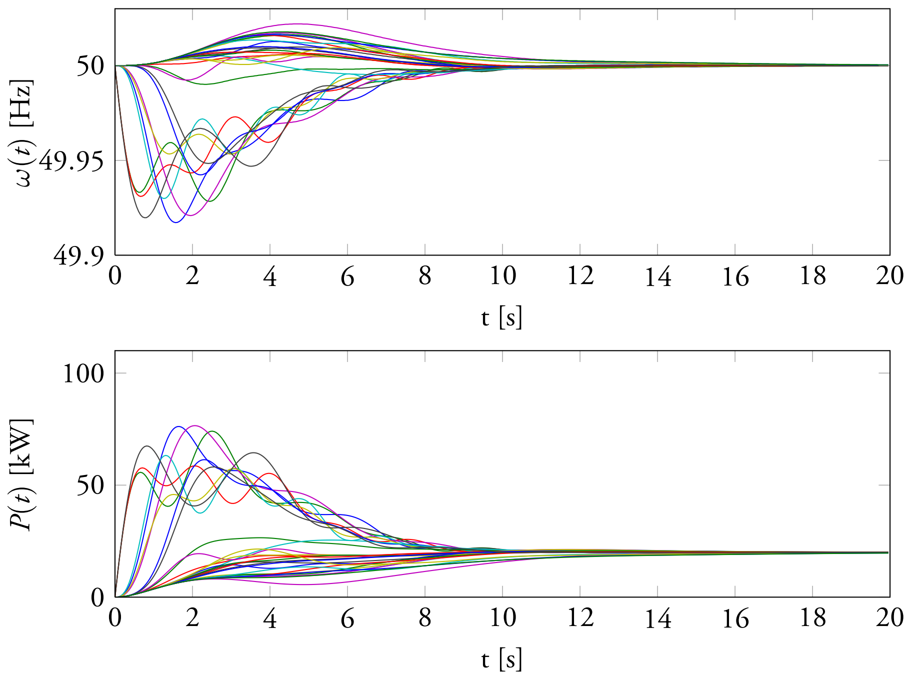


Figure 5.6 The figures show the bus frequencies and control signals respectively under centralized frequency control for $\alpha = 0.8$, $\beta = 0.8$ and $\gamma = 0.8$.

the cost of power generation. As the frequency of the power system at steady state is determined by the total power generation, this allows for an optimal distribution of the power generation to minimize a certain cost.

Load sharing

It is well-known that a centralized PI frequency controller has the property of load sharing, see e.g. Kundur (1994). This means that if the load is increased at some bus in the power system, all generators will increase their generation equally, provided that the proportional controller gain α is uniform. However, as seen in Example 1.5, the proposed decentralized PI controller does not have the load sharing property in general. However, we show that in the limit when the integral gain β approaches zero, the decentralized controller has the load sharing property. On the other hand, as the

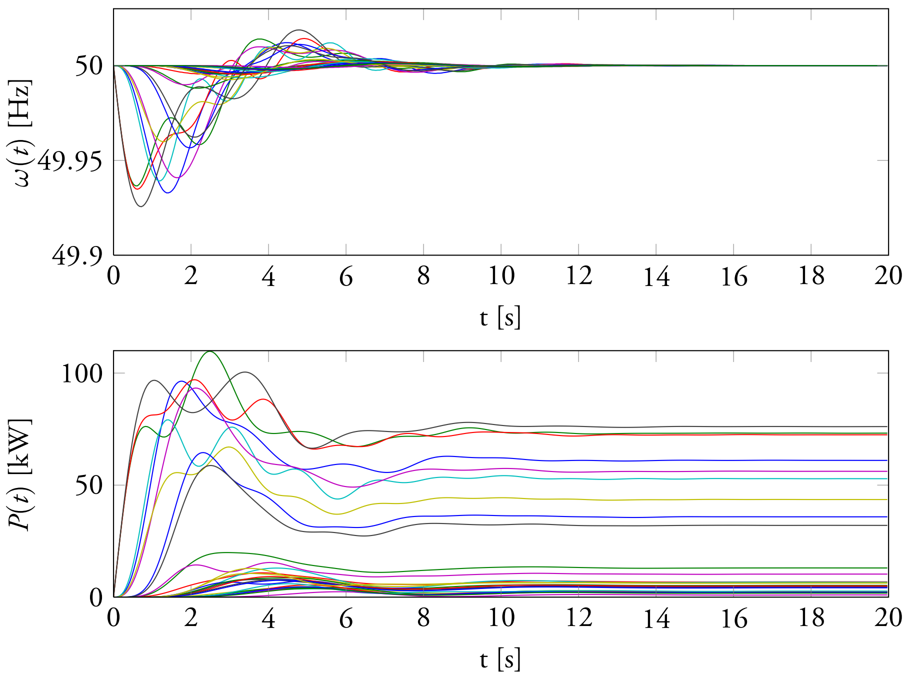


Figure 5.7 The figures show the bus frequencies and control signals respectively under decentralized frequency control for $\alpha = 0.8$ and $\beta = 0.8$.

integral gain β approaches infinity, load changes will result only in local increase of power generation.

Theorem 5.3 *Consider the power system described by* (5.3) *where u_i is given by* (5.6)–(5.7). *Assume that the power system is initially operating at ω^{ref}, after which the load is changed by p^m. The power generation u the satisfies:*

$$\lim_{\frac{\beta}{\alpha}\to\infty} \lim_{t\to\infty} u(t) = (D\omega^{ref} 1_{n\times 1} - p^m)$$

$$\lim_{\frac{\beta}{\alpha}\to 0} \lim_{t\to\infty} u(t) = \left(1_{1\times n}(D\omega^{ref} 1_{n\times 1} - p^m)\right) 1_{n\times 1}$$

Proof. By the proof of Theorem 5.2, the steady state solution of y (5.3) where u is given by (5.6)–(5.7), is given by

$$
\begin{aligned}
z_0 &= (\beta I_n + \mathcal{L}_k)^{-1}(D\omega^{\text{ref}} 1_{n\times 1} - p^m) \\
\omega_0 &= \omega^{\text{ref}} 1_{n\times 1}.
\end{aligned}
$$

The control signal is given by (5.7), i.e.,

$$
u_i = \alpha(\omega^{\text{ref}} - \omega_i) + \beta z_i,
$$

where $\alpha > 0$ is fixed. This implies that in steady state, the control signal is given by

$$
u = \beta(\beta I_n + \mathcal{L}_k)^{-1}(D\omega^{\text{ref}} 1_{n\times 1} - p^m). \tag{5.8}
$$

Consider first the first case, when $\beta \to \infty$. Clearly:

$$
\lim_{\beta\to\infty} \beta\,(\beta I_n + \mathcal{L}_k)^{-1} = \lim_{\beta\to\infty} \left(I_n + \frac{1}{\beta}\mathcal{L}_k \right)^{-1} = I_n^{-1} = I_n.
$$

Hence, by (5.8), the control signal in steady state is given by

$$
u = (D\omega^{\text{ref}} 1_{n\times 1} - p^m).
$$

Consider now the first case, when $\beta \to 0$. The characteristic equation of $(\beta I_n + \mathcal{L}_k)$ is given by

$$
0 = \det\left(\beta I_n + \mathcal{L}_k - sI_n\right) = \det\left(\mathcal{L}_k - (s-\beta)I_n\right), \tag{5.9}
$$

implying that the solutions to (5.9) are given by $\lambda_l = \lambda_l(\mathcal{L}_k) + \beta$, where $\lambda_l(\mathcal{L}_k)$ denotes the l:th eigenvalue of $\mathcal{L}_k$. $(\beta I_n + \mathcal{L}_k)$ can easily be shown to have full rank, implying that we may write

$$
(D\omega^{\text{ref}} 1_{n\times 1} - p^m) = \sum_{l=1}^{n} a_l v_l,
$$

where v_l denotes the l:th eigenvector of $(\beta I_n + \mathcal{L}_k)$. Using the above substitution in (5.8) we obtain

$$
\begin{aligned}
u &= \beta(\beta I_n + \mathcal{L}_k)^{-1}(D\omega^{\text{ref}} 1_{n\times 1} - p^m) = \beta(\beta I_n + \mathcal{L}_k)^{-1} \left(\sum_{l=1}^{n} a_l v_l \right) \\
&= \sum_{l=1}^{n} \beta \frac{a_l}{\lambda_l} v_l = \sum_{l=1}^{n} \beta \frac{a_l}{\lambda_l(\mathcal{L}_k) + \beta} v_l = a_1 v_1 + \sum_{l=2}^{n} \beta \frac{a_l}{\lambda_l(\mathcal{L}_k) + \beta} v_l,
\end{aligned}
$$

since the smallest eigenvalue of $(\beta I_n + \mathcal{L}_k)$ is given by β, as the smallest eigenvalue of $\mathcal{L}_k$ is zero. Clearly, the above expression converges to $a_1 v_1$ as $\beta \rightarrow 0$. a_1 is given by the projection of $(D\omega^{\text{ref}} 1_{n\times 1} - p^m)$ onto v_1. v_1 is easily shown to be equal to $1_{n\times 1}$, since

$$(\beta I_n + \mathcal{L}_k) 1_{n\times 1} = \beta I_n 1_{n\times 1} + \mathcal{L}_k 1_{n\times 1} = \beta 1_{n\times 1},$$

and zero is an eigenvalue of $\mathcal{L}_k$, with the corresponding eigenvector $1_{n\times 1}$. Thus

$$a_1 = v_1^T (D\omega^{\text{ref}} 1_{n\times 1} - p^m) = 1_{1\times n}(D\omega^{\text{ref}} 1_{n\times 1} - p^m),$$

which implies that in steady state it holds that

$$u = 1_{1\times n}(D\omega^{\text{ref}} 1_{n\times 1} - p^m) 1_{n\times 1}.$$

□

Remark 5.1 Theorem 5.3 implies that when $\frac{\beta}{\alpha} \rightarrow \infty$, power is always generated where it is consumed, whereas when $\frac{\beta}{\alpha} \rightarrow 0$, power generation is shared equally amongst the generators.

5.4 Optimal centralized frequency control

Motivated my discussion in the previous section, we will in addition to the objective of maintaining a constant reference frequency also consider the objective of minimizing a cost function of the power generation. This generalizes the concept of load sharing, as setting the costs of all buses equal will achieve load sharing. Assume that there is a quadratic cost $f_i^c(x) = \frac{1}{2} C_i x^2$ of generating power at bus i. The control objective is to design a distributed control protocol that asymptotically attains the reference frequency at all buses, while minimizing the accumulated generation cost. We formalize these requirements by the two conditions below.

Condition 5.1 The controller asymptotically regulates the bus frequencies to the reference frequency ω^{ref}, i.e.,

$$\lim_{t\rightarrow\infty} \omega_i(t) = \omega^{\text{ref}} \; \forall i \in \mathcal{V}. \tag{5.10}$$

Condition 5.2 The power generation minimizes the accumulate generation cost in steady state of (5.3), i.e.,

$$\lim_{t\rightarrow\infty} u(t) = u^*, \tag{5.11}$$

where u^* is the minimizer of

$$\sum_{i\in\mathcal{V}} \frac{1}{2} C_i u_i^2 \quad \text{s.t.} \quad \mathcal{L}_k\delta - u = P^m - \omega^{\text{ref}} D 1_{n\times 1}. \tag{5.12}$$

We propose the following centralized controller to solve the frequency control problem:

$$\begin{aligned} u_i &= \alpha(\hat{\omega}_i - \omega_i) \\ \dot{\hat{\omega}}_i &= \beta\left(u^* - \alpha(\hat{\omega}_i - \omega_i)\right) + \gamma(\omega^{\text{ref}} - \omega_i), \end{aligned} \tag{5.13}$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$, and u^* is given by

$$\begin{aligned} [u^*, \delta^*] = \arg\min_{[u,\delta]} & \sum_{i\in\mathcal{V}} \frac{1}{2} C_i u_i^2 \\ \text{s.t.} \quad \mathcal{L}_k\delta - u &= P^m - \omega^{\text{ref}} D 1_{n\times 1}. \end{aligned}$$

Remark 5.2 The centralized controller (5.13) requires global information about the total power load profile P^m, as well as global information about the power generation costs C, in addition to the exact model of the power system.

In the following sections we will show that the controller (5.13) satisfies conditions 5.1 and 5.2, and derive conditions under which it stabilizes the power system.

Sufficient stability criterion based on eigenvalues

In this section we study the stability of (5.3) with the control given by (5.13). We will give sufficient conditions for the stability of the proposed control protocol based on linear system theory.

Theorem 5.4 *The power system* (5.3) *with control input* (5.13) *satisfies Conditions 5.1 and 5.2 for any initial condition* $(\delta(0), \omega(0))$ *if the matrix*

$$A \triangleq \begin{bmatrix} -\alpha\beta I_n & 0_{n\times n} & (\alpha\beta - \gamma) I_n \\ 0_{n\times n} & 0_{n\times n} & I_n \\ \alpha M & -M\mathcal{L}_k & -M(D+\alpha I_n) \end{bmatrix},$$

has exactly one eigenvalue equal to 0 *and all other eigenvalues in the left half complex plane.*

Proof. Assume that A has exactly one zero eigenvalue, and all other eigenvalues in the left half complex plane. It can be verified that the dynamics of the system (5.3) with the control given by (5.25) can be written as

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta} \\ \dot{\omega} \end{bmatrix} = A \begin{bmatrix} \hat{\omega} \\ \delta \\ \omega \end{bmatrix} + \begin{bmatrix} \beta u^* + \gamma \omega^{\text{ref}} 1_{n\times 1} \\ 0_{n\times 1} \\ MP^m \end{bmatrix}. \tag{5.14}$$

Consider the linear change of coordinates:

$$\delta = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} \delta'$$
$$\delta' = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1\times n} \\ S^T \end{bmatrix} \delta.$$

where S is a matrix such that $[\frac{1}{\sqrt{n}} 1_{n\times 1} \quad S]$ is an orthonormal matrix. In the new coordinates the system dynamics are given by:

$$\dot{\hat{\omega}} = -\alpha\beta\hat{\omega} + (\alpha\beta I_n - \gamma I_n)\omega + \beta u^* + \gamma 1_{n\times 1}\omega^{\text{ref}}$$
$$\dot{\delta}' = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1\times n} \\ S^T \end{bmatrix} \omega$$
$$\dot{\omega} = \alpha M\hat{\omega} - M\mathcal{L}_k \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} \delta' - M(D + \alpha I_n)\omega + MP.$$

By defining the output of the system (5.3) and (5.25) as

$$y = \begin{bmatrix} \mathcal{L}_k\delta \\ \omega \end{bmatrix} = \begin{bmatrix} \mathcal{L}_k \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} \delta' \\ \omega \end{bmatrix} = \begin{bmatrix} [0 \ \mathcal{L}_K S]\, \delta' \\ \omega \end{bmatrix},$$

which are the system states of interest, we note that δ'_1 is unobservable. Hence we may omit this state by defining $\delta'' = [\delta'_2, \ldots, \delta'_n]$. In the new coordinates the system dynamics are given by

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta}'' \\ \dot{\omega} \end{bmatrix} = A' \begin{bmatrix} \hat{\omega} \\ \delta'' \\ \omega \end{bmatrix} + \underbrace{\begin{bmatrix} \beta u^* + \gamma \omega^{\text{ref}} 1_{n\times 1} \\ 0_{n\times 1} \\ MP^m \end{bmatrix}}_{\triangleq b}, \tag{5.15}$$

where

$$A' = \begin{bmatrix} -\alpha\beta I_n & 0_{n\times(n-1)} & (\alpha\beta-\gamma)I_n \\ 0_{(n-1)\times n} & 0_{(n-1)\times(n-1)} & S^T \\ \alpha M & -M\mathcal{L}_k S & -M(D+\alpha I_n) \end{bmatrix}.$$

We now show that A' has full rank. Consider

$$A' \begin{bmatrix} \hat{\omega} \\ \delta' \\ \omega \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ 0_{(n-1)\times 1} \\ 0_{n\times 1} \end{bmatrix}.$$

The second row of the above equation gives $S^T\omega = 0_{(n-1)\times 1}$, implying $\omega = k1_{n\times 1}$. Multiplying the third row with β and adding it to the first row yields:

$$\beta\mathcal{L}_k S\delta + (\beta D + \gamma I_n)\omega = 0_{n\times 1}.$$

Premultiplying the above equation with $1_{1\times n}$ and substituting $\omega = k1_{n\times 1}$ yields:

$$\left(\beta \sum_{i\in\mathcal{V}} d_i + \gamma n\right) k = 0,$$

implying $k = 0$. Thus, substituting $\omega = 0_{n\times 1}$ in the first row immediately gives $\hat{\omega} = 0_{n\times_1}$. Finally the third row with $\hat{\omega} = 0_{n\times_1}$ and $\omega = 0_{n\times_1}$ gives $\delta' = 0_{(n-1)\times 1}$ Since by the change of coordinates, the eigenvalues of A remain the same, we also conclude that A' has the same eigenvalues as A, except the zero eigenvalue. It follows that A' is Hurwitz iff A has exactly one zero eigenvalue, and all other eigenvalues in the left half complex plane. We now shift the state-space by defining

$$\begin{bmatrix} \hat{\omega} \\ \delta''' \\ \omega \end{bmatrix} = \begin{bmatrix} \hat{\omega} \\ \delta'' \\ \omega \end{bmatrix} - A'^{-1}b.$$

It follows that in these new coordinates, the system dynamics are

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta}''' \\ \dot{\omega} \end{bmatrix} = A' \begin{bmatrix} \hat{\omega} \\ \delta''' \\ \omega \end{bmatrix}. \tag{5.16}$$

The equilibrium solution of (5.28) satisfies

$$\beta(u^* + \alpha(\omega - \hat{\omega})) + \gamma(\omega^{\text{ref}} 1_{n\times 1} - \omega) = 0_{n\times 1} \tag{5.17}$$

$$S^T\omega = 0_{n\times 1} \tag{5.18}$$

$$\alpha(\omega - \hat{\omega}) + \mathcal{L}_k S\delta' + D\omega = 0_{n\times 1}. \tag{5.19}$$

As the rows of S^T are orthonormal to $1_{1\times n}$, (5.18) implies that $\omega = c_1 1_{n\times 1}$, where $c_1 \in \mathbb{R}$. Noting that $u = \alpha(\hat{\omega} - \omega)$, (5.17) becomes

$$\beta(u^* - u) + \gamma(\omega^{\text{ref}} 1_{n\times 1} - \omega) = 0_{n\times 1}. \tag{5.20}$$

We also note that by (5.12), u^* satisfies

$$\mathcal{L}_k\delta - u^* = P^m - \omega^{\text{ref}} D 1_{n\times 1}. \tag{5.21}$$

Since $\mathcal{L}_k S\delta' = \mathcal{L}_k\delta$, substituting (5.21) in (5.19) yields

$$(u - u^*) + D(\omega^{\text{ref}} 1_{n\times 1} - \omega) = 0. \tag{5.22}$$

Multiplying (5.22) with β and adding to (5.20) while substituting $\omega = c_1 1_{n\times 1}$ yields

$$(\beta D + \gamma I_n) 1_{n\times 1}(\omega^{\text{ref}} - c_1) = 0_{n\times 1},$$

which implies $c_1 = \omega^{\text{ref}}$. By (5.17) we get that $u = u^*$, which concludes the proof. □

Explicit sufficient stability criterion

While Theorem 5.4 provides a straightforward condition weather a given set of parameters result in a stable system, it does not give any implication on how to stabilize an unstable system. The following theorem gives a sufficient conditions for when A has all eigenvalues except one in the open left half complex plane.

Theorem 5.5 *A has exactly one zero eigenvalue, and all other eigenvalues in the left half complex plane if the following condition is satisfied*

$$\beta\overline{m}\lambda_{max}(\mathcal{L}_k) < (\gamma + \beta\underline{D})(\alpha\underline{D} + \alpha\beta\underline{m}).$$

where $\underline{m} = \min_i m_i$, $\underline{m} = \max_i m_i$ *and* $\underline{D} = \min_i D_i$.

Remark 5.3 There always exists $\beta > 0$, such that the controller (5.13) stabilizes the power system.

Proof. The characteristic equation of A is given by:

$$
\begin{aligned}
0 &= \begin{vmatrix} (-\alpha\beta - s)I_n & 0_{n\times n} & (\alpha\beta-\gamma)I_n \\ 0_{n\times n} & -sI_n & I_n \\ \alpha M & -M\mathcal{L}_k & -MD - \alpha M - sI_n \end{vmatrix} \\
&= \begin{vmatrix} (-\alpha\beta - s)I_n & 0_{n\times n} & (-\gamma - s)I_n \\ 0_{n\times n} & -sI_n & I_n \\ \alpha M & -M\mathcal{L}_k & -MD - sI_n \end{vmatrix} \\
&= \frac{1}{s^n}\begin{vmatrix} (-\alpha\beta - s)I_n & 0_{n\times n} & (-\gamma s - s^2)I_n \\ 0_{n\times n} & -sI_n & 0_{n\times n} \\ \alpha M & -M\mathcal{L}_k & -sMD - s^2 I_n - M\mathcal{L}_k \end{vmatrix} \\
&= \begin{vmatrix} (-\alpha\beta - s)I_n & 0_{n\times n} & (-\gamma s - s^2)I_n \\ 0_{n\times n} & -I_n & 0_{n\times n} \\ \alpha M & 0_{n\times n} & -sMD - s^2 I_n - M\mathcal{L}_k \end{vmatrix} \\
&= \begin{vmatrix} (-\alpha\beta - s)I_n & (-\gamma s - s^2)I_n \\ \alpha M & -sMD - s^2 I_n - M\mathcal{L}_k \end{vmatrix} \\
&= \alpha \det M \det\Big((\alpha\beta + s)(M\mathcal{L}_k sMD + s^2 I_n) + (\gamma s + s^2 I_n)\Big).
\end{aligned} \tag{5.23}
$$

Clearly the above characteristic equation has a solution only if

$$
x^T\Big((\alpha\beta + s)(M\mathcal{L}_k sMD + s^2 I_n) + (\gamma s + s^2 I_n)\Big)x = 0. \tag{5.24}
$$

has a solution. Hence if (5.24) has all its solutions in $\mathbb{C}^-$ for all $\|x\| = 1$, then (5.23) has all its solutions in $\mathbb{C}^-$. This condition thus becomes that the equation

$$
\underbrace{x^T\beta\mathcal{L}_k x}_{a_0} + s\underbrace{x^T\left(\gamma I_n + \frac{1}{\alpha}\mathcal{L}_k + \beta D\right)x}_{a_1} + s^2\underbrace{x^T\left(I_n + \frac{1}{\alpha}\right)x}_{a_2} + s^3\underbrace{\frac{1}{\alpha}x^T M^{-1}x}_{a_3} = 0,
$$

has all its solutions in $\mathbb{C}^-$. We distinguish between the two cases: $x^T\mathcal{L}_k x = 0$ and $x^T\mathcal{L}_k x \neq 0$. Starting with the former case, equation (5.32) may be written as

$$
sa_1 + s^2 a_2 + s^3 a_3 = s(a_1 + sa_2 + s^2 a_3) = 0
$$

If $a_i > 0$ for $i = 1, 2, 3$, the above equation has one solution $s = 0$, and two solutions $s \in \mathbb{C}^-$ if and only if $a_i > 0$, $i = 1, 2, 3$ by the Routh-Hurwitz stability criterion.

We now proceed with the case when $x^T\mathcal{L}_k x \neq 0$. Since $x^T\mathcal{L}_k x \geq 0$, we conclude that $x^T\mathcal{L}_k x > 0$. The Routh-Hurwitz stability criterion is $a_i > 0$ for $i = 0, 1, 2, 3$, and $a_0 a_3 < a_1 a_2$. Clearly $a_i > 0$ $i = 0, 1, 2, 3$, and the latter condition becomes

$$x^T \beta \mathcal{L}_k x \frac{1}{\alpha} x^T M^{-1} x < x \left(\gamma I_n + \frac{1}{\alpha} \mathcal{L}_k + \beta D \right) x x^T \left(I_n + \frac{1}{\alpha} \right) x.$$

A sufficient condition for the above equation to hold is obtained by upper bounding the left hand side and lower bounding the right hand side, which yields

$$\beta \overline{m} \lambda_{\max}(\mathcal{L}_k) < (\gamma + \beta \underline{D})(\alpha \underline{D} + \alpha \beta \underline{m}).$$

□

5.5 Optimal distributed frequency control

In this section we propose a distributed control protocol, which satisfies Conditions 5.1 and 5.2. The proposed controller consists of a proportional and an integral part, which compares the local frequency to a global reference frequency, and compares the marginal cost of power generation with its neighboring buses. We propose the following control protocol:

$$\begin{aligned} u_i &= \alpha(\hat{\omega}_i - \omega_i) \\ \dot{\hat{\omega}}_i &= \beta \left(\sum_{j \in \mathcal{N}_i} k_{ij} \alpha (C_j(\hat{\omega}_j - \omega_j) - C_i(\hat{\omega}_i - \omega_i)) \right) + \gamma(\omega^{\text{ref}} - \omega_i) \end{aligned} \tag{5.25}$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$. We will show that the controller (5.25) satisfies conditions 5.1 and 5.2.

Remark 5.4 The control protocol (5.25) is distributed.

Remark 5.5 C_i can be interpreted as the marginal cost of power generation for bus i.

Sufficient stability criterion based on eigenvalues

In this section we study the stability of (5.3) with the control given by (5.25). We first give sufficient conditions for the stability of the proposed control protocol based on linear system theory.

Theorem 5.6 *The power system* (5.3) *with control input* (5.25) *satisfies Conditions 5.1 and 5.2 for any initial condition* $(\delta(0), \omega(0))$ *if the matrix*

$$A \triangleq \begin{bmatrix} -\alpha\beta\mathcal{L}_k C & 0_{n\times n} & \alpha\beta\mathcal{L}_k C - \gamma I_n \\ 0_{n\times n} & 0_{n\times n} & I_n \\ \alpha M & -M\mathcal{L}_k & -M(D+\alpha I_n) \end{bmatrix}$$

where $C = \operatorname{diag}[c_1, \ldots, c_n]$. *has exactly one eigenvalue equal to zero and all other eigenvalues in the open left half complex plane.*

Proof. Assume that A has exactly one zero eigenvalue, and all other eigenvalues in the left half complex plane. It can be verified that the dynamics of the system (5.3) with the control given by (5.25) can be written as

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta} \\ \dot{\omega} \end{bmatrix} = A \begin{bmatrix} \hat{\omega} \\ \delta \\ \omega \end{bmatrix} + \begin{bmatrix} \gamma\omega^{\text{ref}} 1_{n\times 1} \\ 0_{n\times 1} \\ MP^m \end{bmatrix}. \tag{5.26}$$

Consider the linear change of coordinates:

$$\delta = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} \delta'$$
$$\delta' = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1\times n} \\ S^T \end{bmatrix} \delta.$$

where S is a matrix such that $[\frac{1}{\sqrt{n}} 1_{n\times 1} \quad S]$ is an orthonormal matrix. In the new coordinates the system dynamics are given by:

$$\dot{\hat{\omega}} = -\alpha\beta\mathcal{L}_k C\hat{\omega} + (\alpha\beta\mathcal{L}_k C - \gamma I_n)\omega + \gamma 1_{n\times 1}\omega^{\text{ref}}$$
$$\dot{\delta}' = \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{1\times n} \\ S^T \end{bmatrix} \omega$$
$$\dot{\omega} = \alpha M\hat{\omega} - M\mathcal{L}_k \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} & S \end{bmatrix} \delta' - M(D+\alpha I_n)\omega + MP.$$

By defining the output of the system (5.3) and (5.25) as

$$y = \begin{bmatrix} \mathcal{L}_k\delta \\ \omega \end{bmatrix} = \begin{bmatrix} \mathcal{L}_k \begin{bmatrix} \frac{1}{\sqrt{n}} 1_{n\times 1} \; S \end{bmatrix} \delta' \\ \omega \end{bmatrix} = \begin{bmatrix} [0 \; \mathcal{L}_K s]\, \delta' \\ \omega \end{bmatrix}$$

which are the system states of interest, we note that δ'_1 is unobservable. Hence we may omit this state by defining $\delta'' = [\delta'_2, \ldots, \delta'_n]$. In the new coordinates the system dynamics are given by

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta}'' \\ \dot{\omega} \end{bmatrix} = A' \begin{bmatrix} \hat{\omega} \\ \delta'' \\ \omega \end{bmatrix} + \underbrace{\begin{bmatrix} \gamma 1_{n\times 1} \\ 0_{(n-1)\times 1} \\ MP \end{bmatrix}}_{\triangleq b}, \tag{5.27}$$

where

$$A' = \begin{bmatrix} -\alpha\beta\mathcal{L}_k C & 0_{n\times(n-1)} & \alpha\beta\mathcal{L}_k C - \gamma I_n \\ 0_{(n-1)\times n} & 0_{(n-1)\times(n-1)} & S^T \\ \alpha M & -M\mathcal{L}_k S & -M(D+\alpha I_n) \end{bmatrix}.$$

We now show that A' has full rank. Consider

$$A' \begin{bmatrix} \hat{\omega} \\ \delta' \\ \omega \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ 0_{(n-1)\times 1} \\ 0_{n\times 1} \end{bmatrix}.$$

The second row of the above equation gives $S^T\omega = 0_{(n-1)\times 1}$, implying $\omega = k1_{n\times 1}$. The first row gives $\alpha\beta\mathcal{L}_k C\hat{\omega} = (\alpha\beta\mathcal{L}_k C - \gamma I_n)k1_{n\times 1}$, which implies $k = 0$ since $1_{n\times 1}$ does not lie i the range of $\mathcal{L}_k$. Finally, the third row gives $M\mathcal{L}_k S\delta' = 0_{(n-1)\times 1}$, implying $\delta' = 0_{(n-1)\times 1}$. Since by the change of coordinates, the eigenvalues of A remain the same, we also conclude that A' has the same eigenvalues as A, except the zero eigenvalue. It follows that A' is Hurwitz iff A has exactly one zero eigenvalue, and all other eigenvalues in the left half complex plane. We now shift the state-space by defining

$$\begin{bmatrix} \hat{\omega} \\ \delta''' \\ \omega \end{bmatrix} = \begin{bmatrix} \hat{\omega} \\ \delta'' \\ \omega \end{bmatrix} - A'^{-1}b.$$

It follows that in these new coordinates, the system dynamics are

$$\begin{bmatrix} \dot{\hat{\omega}} \\ \dot{\delta}''' \\ \dot{\omega} \end{bmatrix} = A' \begin{bmatrix} \hat{\omega} \\ \delta''' \\ \omega \end{bmatrix}. \tag{5.28}$$

The equilibrium solution of (5.28) satisfies

$$\alpha\beta\mathcal{L}_k C(\omega - \hat{\omega}) - \gamma\omega = \gamma 1_{n\times 1}\omega^{\text{ref}} \tag{5.29}$$

$$S^T\omega = 0_{n\times 1}. \tag{5.30}$$

As the rows of S^T are orthonormal to $1_{1\times n}$, (5.30) implies that $\omega = c_1 1_{n\times 1}$, where $c_1 \in \mathbb{R}$. Substituting this in (5.29) yields

$$\alpha\beta\mathcal{L}_k C(\omega - \hat{\omega}) - \gamma c_1 1_{n\times 1} = \gamma 1_{n\times 1}\omega^{\text{ref}}.$$

Since $1_{n\times 1}$ does not lie in the range of $\mathcal{L}_k$, we conclude that $c_1 = \omega^{\text{ref}}$, implying that (5.10) is satisfied. Furthermore (5.29) implies $C(\omega - \hat{\omega}) = c_2 1_{n\times 1}$. The KKT conditions Karush (1939) of the convex constrained optimization problem (5.12) are

$$Cu = C\alpha(\omega - \hat{\omega}) = \lambda 1_{n\times 1},$$

where λ is the Lagrange multiplier. Since the equilibrium of (5.28) implies the KKT conditions, and the KKT conditions are necessary and sufficient optimality conditions, the equilibrium pf (5.26) must be the optimal solution of (5.12). □

Explicit sufficient stability criterion

While Theorem 5.6 provides a relatively straightforward condition whether a given set of parameters result in a stable system, it does not suggest how to stabilize an unstable system. In the following section we give sufficient conditions for when A has all eigenvalues except one in the left complex plane.

Theorem 5.7 *A has exactly one zero eigenvalue, and all other eigenvalues in the left half complex plane if the following conditions are satisfied*

$$\begin{aligned}
&\beta\lambda_{max}(\mathcal{L}_k C\mathcal{L}_k)\underline{m} < \alpha\left(\beta\lambda_{min}\left(\frac{1}{2}(L_kCD + DCL_k)\right) + \gamma\right)\cdot \\
&\left(\beta\lambda_{min}\left(\frac{1}{2}(L_kCM^{-1} + M^{-1}CL_k)\right) + 1 + \frac{\underline{D}}{\alpha}\right) \\
&\beta\lambda_{min}\left(\frac{1}{2}(L_kCD + DCL_k)\right) + \gamma > 0 \\
&\beta\lambda_{min}\left(\frac{1}{2}(L_kCM^{-1} + M^{-1}CL_k)\right) + 1 + \frac{\underline{D}}{\alpha} > 0
\end{aligned} \tag{5.31}$$

where $\underline{m} = \min_i m_i$ *and* $\underline{D} = \min_i D_i$.

Remark 5.6 Theorem 5.7 reveals some interesting qualitative rules of thumb when designing the control parameters of (5.25). The maximum integral gain β is inversely proportional to the minimum rotor mass $\underline{m}$. Furthermore, there always exists $\beta > 0$, such that the controller (5.25) stabilizes the power system.

Proof. The characteristic equation of A is given by:

$$
\begin{aligned}
0 &= \begin{vmatrix} sI_n + \alpha\beta\mathcal{L}_k C & 0_{n\times n} & -\alpha\beta\mathcal{L}_k C + \gamma I_n \\ 0_{n\times n} & sI_n & -I_n \\ -\alpha M & -M\mathcal{L}_k & sI_n + M(D+\alpha I_n) \end{vmatrix} \\
&= \begin{vmatrix} sI_n + \alpha\beta\mathcal{L}_k C & 0_{n\times n} & (\gamma + s)I_n \\ 0_{n\times n} & sI_n & -I_n \\ -\alpha M & -M\mathcal{L}_k & sI_n + MD) \end{vmatrix} \\
&= \frac{1}{s^n}\begin{vmatrix} sI_n+\alpha\beta\mathcal{L}_k C & 0_{n\times n} & (\gamma s + s^2)I_n \\ 0_{n\times n} & sI_n & 0_{n\times n} \\ -\alpha M & 0_{n\times n} & s^2 I_n+sMD+M\mathcal{L}_k) \end{vmatrix} \\
&= \begin{vmatrix} sI_n+\alpha\beta\mathcal{L}_k C & 0_{n\times n} & (\gamma s + s^2)I_n \\ 0_{n\times n} & I_n & 0_{n\times n} \\ -\alpha M - sI_n - \alpha\beta\mathcal{L}_k C & 0_{n\times n} & sMD - s\gamma I_n + M\mathcal{L}_k) \end{vmatrix} \\
&= \begin{vmatrix} sI_n + \alpha\beta\mathcal{L}_k C & (\gamma s + s^2)I_n \\ -\alpha M & s^2 I_n + sMD + M\mathcal{L}_k \end{vmatrix} \\
&= \det(-\alpha M)\det\left[I_n(\gamma s + s^2) + (sI_n + \alpha\beta\mathcal{L}_k C)\frac{1}{\alpha}M^{-1}(s^2 I_n + sMD + M\mathcal{L}_k)\right] \\
&= \det\Bigg(\beta\mathcal{L}_k C\mathcal{L}_k + s(\gamma I_n + \frac{1}{\alpha}\mathcal{L}_k + \beta\mathcal{L}_k CD) + \\
&\qquad s^2(I_n + \frac{1}{\alpha}D + \beta\mathcal{L}_k CM^{-1}) + s^3\frac{1}{\alpha}M^{-1}\Bigg) \triangleq \det Q,
\end{aligned}
$$

where we have used standard properties of determinants, see e.g., Silvester (2000). A necessary condition for the above equation to have a solution is that $\exists x : x^T Q(s)x = 0$.

We may without loss of generality assume $x^T x = 1$. Hence we consider

$$0 = x^T Q x = \underbrace{x^T(\beta \mathcal{L}_k C \mathcal{L}_k) x}_{\triangleq a_0} + s \underbrace{x^T \left(\gamma I_n + \frac{1}{\alpha}\mathcal{L}_k + \beta \mathcal{L}_k C D \right) x}_{\triangleq a_1}$$
$$+ s^2 \underbrace{x^T \left(I_n + \frac{1}{\alpha} D + \beta \mathcal{L}_k C M^{-1} \right) x}_{\triangleq a_2} + s^3 \underbrace{x^T \left(\frac{1}{\alpha} M^{-1} \right) x}_{\triangleq a_3}. \quad (5.32)$$

We distinguish between two cases. $x^T \mathcal{L}_k C \mathcal{L}_k x = 0$, and $x^T \mathcal{L}_k C \mathcal{L}_k x \neq 0$. First consider the case when $x^T \mathcal{L}_k C M \mathcal{L}_k x = 0$. Equation (5.32) may now be written

$$s a_1 + s^2 a_2 + s^3 a_3 = s(a_1 + s a_2 + s^2 a_3) = 0.$$

The above equation has one solution $s = 0$, and two solutions $s \in \mathbb{C}^-$ if and only if $a_i > 0$, $i = 1, 2, 3$ by the Routh-Hurwitz stability criterion. We now proceed with the case when $x^T \mathcal{L}_k C \mathcal{L}_k x \neq 0$. Since $x^T \mathcal{L}_k C \mathcal{L}_k x \geq 0$, we must have that $x^T \mathcal{L}_k C \mathcal{L}_k x > 0$. The Routh-Hurwitz stability criterion is $a_i > 0$ for $i = 0, 1, 2, 3$, and $a_0 a_3 < a_1 a_2$. Clearly $a_0 > 0$ and $a_3 > 0$. Consider:

$$a_1 = \gamma + x^T \frac{1}{\alpha} \mathcal{L}_k x + x^T \beta \mathcal{L}_k C D x.$$

Clearly $x^T \frac{1}{\alpha} \mathcal{L}_k x \geq 0$, and since $x^T \beta \mathcal{L}_k C D x = \frac{1}{2} \beta x^T (\mathcal{L}_k C D + D C \mathcal{L}_k) x$, we conclude that $a_1 > 0$ if

$$\beta \lambda_{\min} \left(\frac{1}{2} (L_k C D + D C L_k) \right) + \gamma > 0.$$

By similar arguments it can be shown that $a_2 > 0$ if

$$\beta \lambda_{\min} \left(\frac{1}{2} (L_k C M^{-1} + M^{-1} C L_k) \right) + 1 + \frac{D}{\alpha} > 0.$$

Finally the condition $a_0 a_3 < a_1 a_2$ can be guaranteed by bounding the left hand side from above, and the right hand side from below. The following bounds are easily verified:

$$\begin{aligned}
a_0 &\leq \beta\lambda_{\max}(\mathcal{L}_k C\mathcal{L}_k)\\
a_3 &\leq \frac{m}{\alpha}\\
a_1 &\geq \beta\lambda_{\min}\left(\frac{1}{2}(L_k CD + DCL_k)\right) + \gamma\\
a_2 &\geq \beta\lambda_{\min}\left(\frac{1}{2}(L_k CM^{-1} + M^{-1}CL_k)\right) + 1 + \frac{D}{\alpha}.
\end{aligned}$$

By substituting $a_i,\ i = 0, 1, 2, 3$ with the above bounds we obtain (5.31). □

Example 1.5 (Frequency control of power systems — Optimal PI control) The centralized and the distributed frequency control algorithms were tested on the IEEE 30 bus test system, illustrated in Figure 5.1. The line admittances were extracted from the IEEE 30 bus test system and the voltages were assumed to be 132 kV for all buses. M and D were set to reasonable numerical values. All buses were assumed to be synchronous motors. The power system is initially in an operational equilibrium, until the power load is increased by a step of 200 kW in the buses 2, 3 and 7. This will immediately result in decreased frequencies at the load buses. The frequency controllers at the buses will then control the frequencies towards the desired frequency of $\omega^{\text{ref}} = 50$ Hz. When simulating the centralized controller the parameters were set to $\alpha = 5\cdot 10^4$, $\beta = 5\cdot 10^{-11}$, $\gamma = 0.02$, while when simulating the decentralized control architecture the parameters were $\alpha = 5\cdot 10^4$, $\beta = 5\cdot 10^{-6}$, $\gamma = 0.2$. The choice of parameters was verified to stabilize the power system using Theorems 5.7 and 5.5, respectively.

As seen in Figure 5.8, the centralized controller quickly regulates the bus frequencies towards a common frequency, which is subsequently regulated towards the reference frequency, whereas the distributed controller regulates the frequencies to a frequency lower than the nominal frequency. As seen in Figure 5.9, the frequencies are subsequently regulated towards the nominal frequency by the distributed controller. Also both controllers asymptotically minimize the power generation costs, as seen in Figure 5.10, while the distributed controller is notably slower than the centralized.

The centralized controller is able to regulate the system much faster because the optimal generation profile is assumed to be known a priori, whereas it is unknown a priori for the distributed controller. Calculation of the optimal generation profile by (5.12) would require global knowledge of the current generation cost C and the current power demand P^m, as well as the static system parameters, including $\mathcal{L}_k$ and D. By defining an appropriate cost of a suboptimal power generation profile, this cost can we weighted against the investment costs of the new communication infrastructure required to implement the centralized controller. Both the investment costs as well as the control costs are in general system dependent, and may serve as a decision making tool to determine if the investment in new communication infrastructure is economically feasible.

Example 1.5 (Frequency control of power systems — Control under islanding) Islanding refers to the event when one or several power transmission lines or other components fail, rendering parts of the power system isolated from the remaining grid. In this example we will study islanding of the IEEE 30 bus power system. In particular, we will assume that all loads are initially zero. At time $t = 0$ s, the loads of the buses 2,3,7 and 30, as seen in in Figure 5.1, are set to 200 kW and all other loads remain zero. At time $t = 50$ s, the power lines connecting bus 25 and 27, 6 and 28 as well as 8 and 28, as seen in Figure 5.1, are disconnected. It is assumed that the centralized controller cannot communicate with the isolated buses 27, 28, 29 and 30.

The two upper figures in Figure 5.11 show the frequencies and the power generation at the buses when using the optimal centralized controller. Before the islanding, all frequencies converge to the nominal frequency 50 HZ. However, after the islanding, the centralized secondary frequency controller has no longer access to frequency measurements from the isolated buses. As a result of power deficit in the isolated area, since the local controllers are only proportional controllers, the frequencies in the isolated area immediately drop and stabilize at a value lower than the nominal frequency.

The lower two figures in Figure 5.11 show the frequencies and the power generation at the buses when using the optimal distributed controller. We note that the frequencies converge to the nominal frequency before the islanding. Immediately after the islanding, the frequencies drop in the underpowered area and rise in the overpowered area. The frequencies in both areas are however regulated towards the nominal frequency by the distributed PI controllers.

In Figure 5.12, the power generation costs are shown when using the centralized and distributed controller respectively. We see that the cost of power generation is minimized in each isolated area by the distributed frequency controller, while the centralized controller is suboptimal.

5.6 Summary

In this section we have studied frequency control of electrical power systems. We have proposed a centralized and a decentralized frequency controller which need only local phase and frequency measurements. The proposed decentralized controller showed very fast response in frequency regulation. However, having access only to local phase and frequency measurements, the controllers were not optimal in terms of power generation cost. To overcome this issue, we have introduced a centralized and a decentralized frequency controller which asymptotically minimize a cost function of the power generation. We have studied the stability of the proposed protocols, and derived sufficient stability conditions based on eigenvalue problems and on scalar inequalities. The controllers were simulated on the IEEE 30 bus test system. Simulations suggest that the centralized controller can achieve faster convergence than the decentralized controller. However, decentralized control possesses the advantage of not requiring global knowledge of the state, nor of the system parameters.

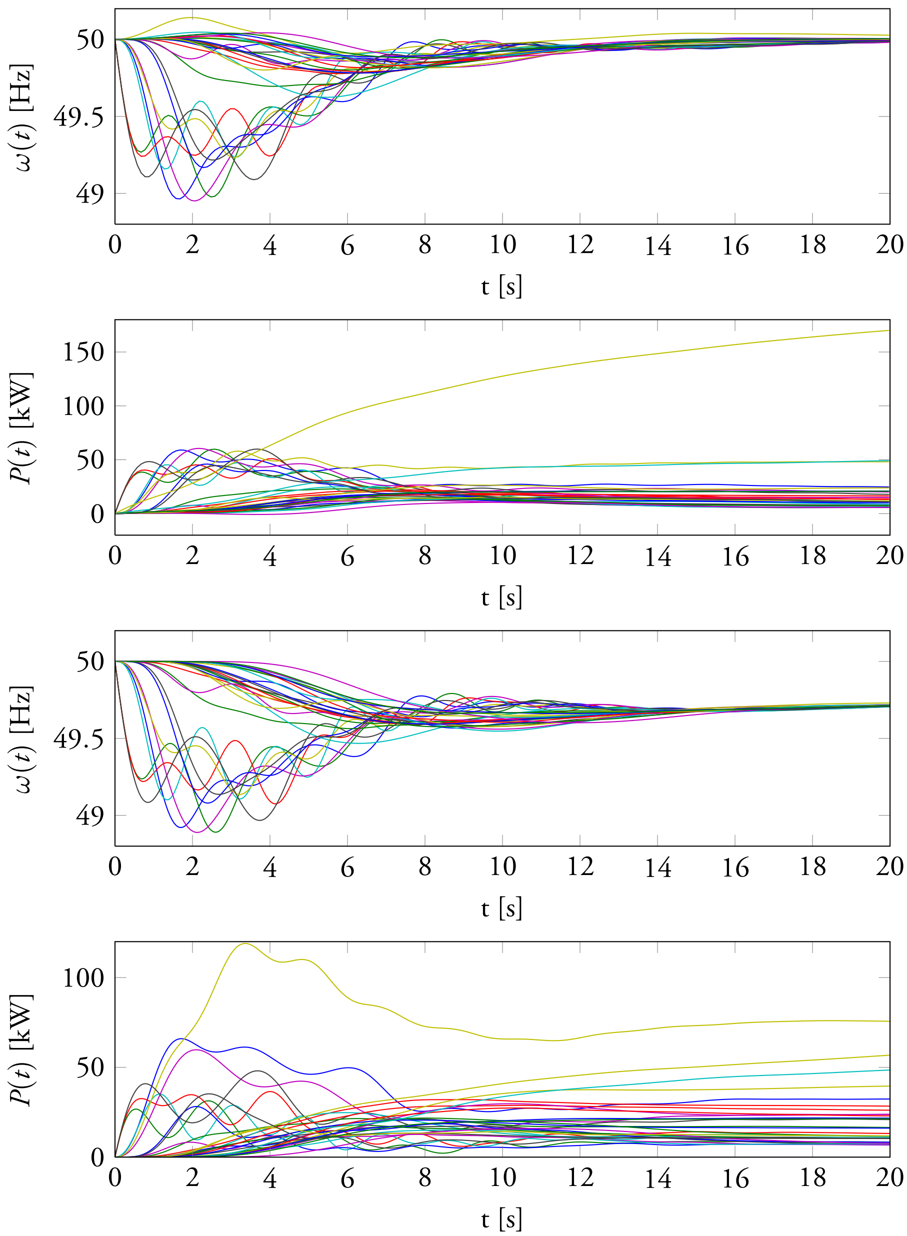


Figure 5.8 The figures show the transient performance of the centralized controller (5.13) versus the performance of the distributed controller (5.25). The two upper figures show the transient bus frequencies and control inputs of the centralized controller, while the two lower figures show the transient bus frequencies and control inputs of the distributed controller.

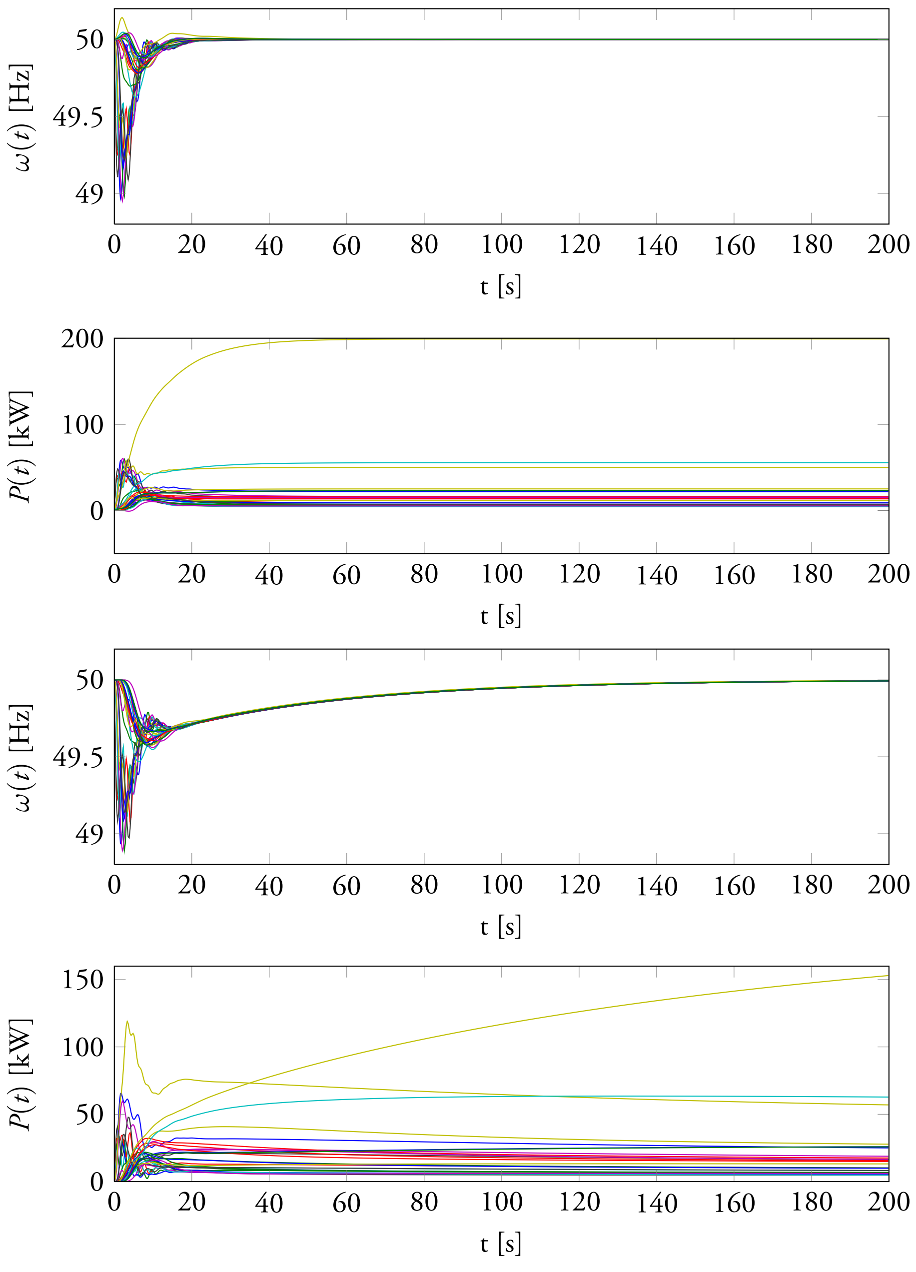


Figure 5.9 The figures show the long term performance of the centralized controller (5.13) versus the performance of the distributed controller (5.25). The two upper figures show the transient bus frequencies and control inputs of the centralized controller, while the two lower figures show the transient bus frequencies and control inputs of the distributed controller.

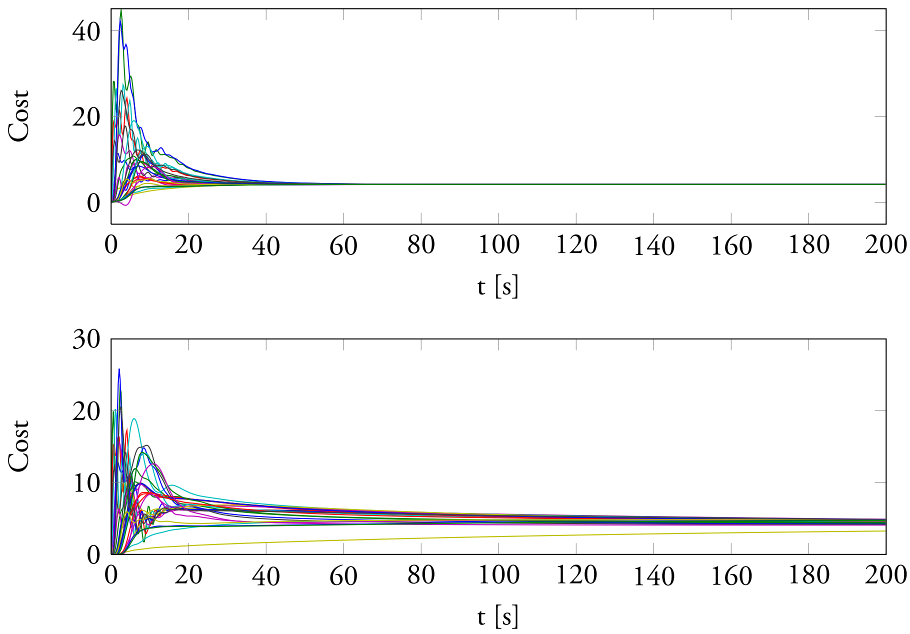


Figure 5.10 The upper figure shows the costs of the power generation of the buses when using the centralized controller, while the lower figure shows the generation costs for the decentralized controller.

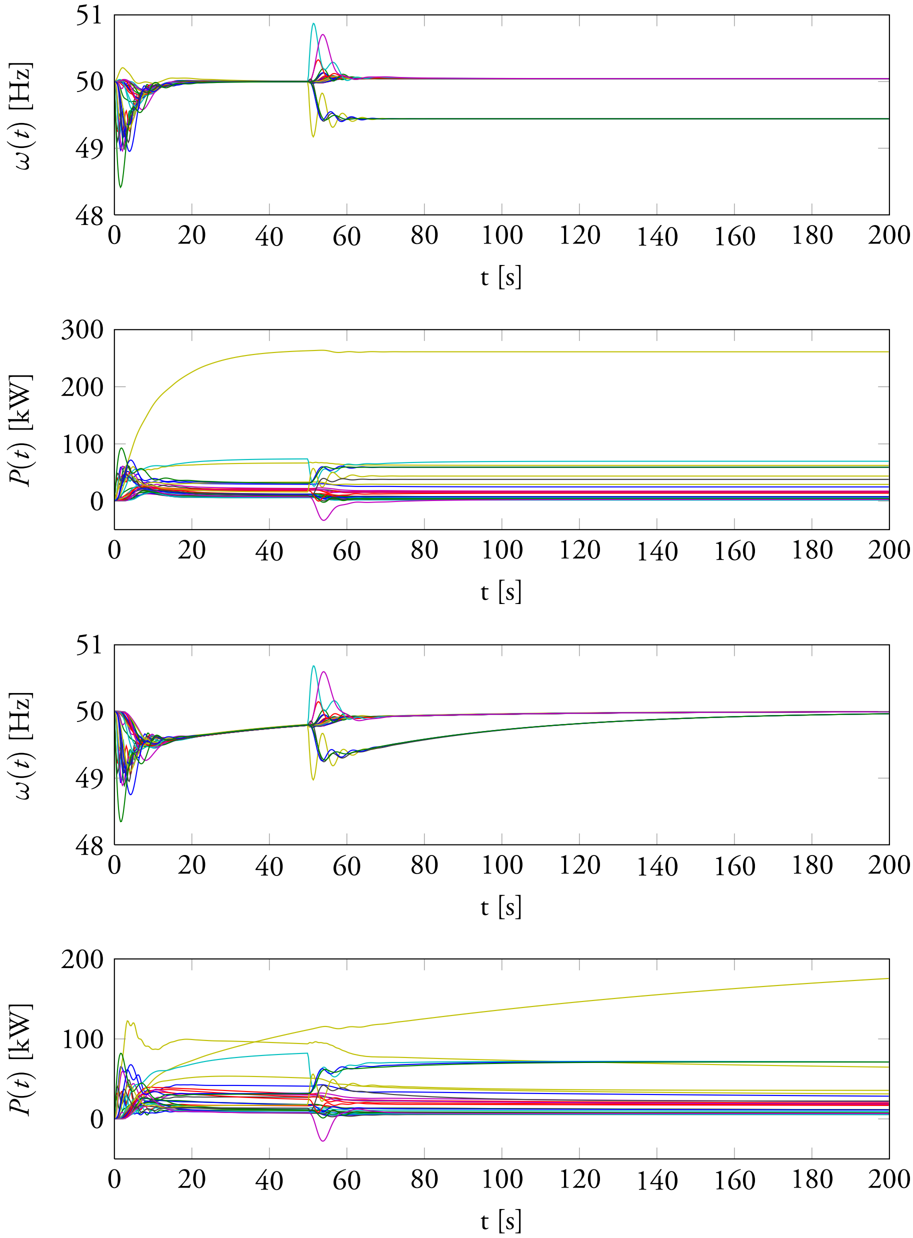


Figure 5.11 The upper two figures show the bus frequencies and control signals respectively under the centralized optimal frequency controller, while the lower two figures show the bus frequencies and control signals respectively under the distributed optimal frequency controller.. At time $t = 50$ s, three transmission lines fail, dividing the power system into two isolated parts.

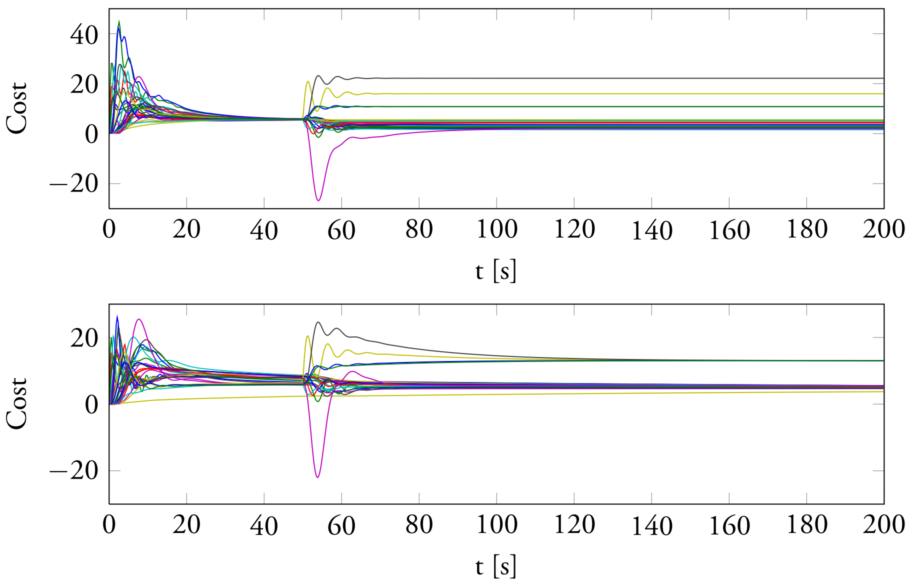


Figure 5.12 The upper figure shows the costs of the power generation of the buses when using the optimal centralized controller, while the lower figure shows the generation costs for the optimal distributed controller.

Chapter 6

Conclusions

THIS chapter concludes the thesis by summarizing the main results presented in chapters 3–5, as well as discussing some interesting possible future research topics.

6.1 Summary

Distributed control with static nonlinear feedback

Distributed nonlinear feedback controllers have been presented for multi-agent systems where the control objective of the agents is to reach a common state. In particular, controllers that can be separated into a product of a nonlinear but positive gain function, and a sum of nonlinear odd interaction functions of the relative states of the relative states of the neighboring agents. We have shown by Lyapunov analysis that the proposed controllers stabilize multi-agent systems with single- and double-integrator dynamics. In the case when the agents' dynamics are single-integrators the agents converge to a common position, while in the case when the agents' dynamics are double integrators the agents converge to a point moving with a constant velocity. In both cases, the agents final position and velocity, respectively, are determined by an integral equation of only the agents' gain functions.

Agents with double-integrator dynamics and state-dependent damping are also considered. A distributed nonlinear controller, similar to the previously mentioned controllers, is proposed. It is shown that the proposed controller stabilizes the multi-agent system, and that the agents converge to a common point which is determined by an integral equation of the damping coefficients.

The results have been applied to control of unmanned underwater vehicles, control of autonomous space satellites and thermal energy storage in smart buildings, where the validity and practical implications of the results are demonstrated by simulations.

Distributed control with integral action

Distributed PI controllers have been presented for multi-agent systems where the control objective of the agents is to reach a common state, in spite of constant disturbances acting on the system. In particular, we have studied agents with single-integrator dynamics and with damped double-integrator dynamics. We have derived necessary and sufficient stability criteria for the proposed controllers by linear system theory. In particular, PI controllers for agents with single-integrator dynamics were shown to be stabilizing for any proportional and integral gains. PI controllers for agents with double-integrator dynamics however, are only stabilizing if the integral gain is lower that a threshold value, determined by the proportional gain and the damping coefficient. Whenever the stability conditions are fulfilled, the agents converge to a common point, despite the presence of constant disturbances.

The results have been applied to mobile robot coordination under disturbances. The robots were modeled by double-integrator dynamics with damping. By the derived results, the robots would fulfill the control objective even under constant disturbances if and only if the integral gain of the PI controller is nonzero.

Frequency control of power systems

A centralized and a decentralized frequency controller have been proposed. The centralized controller employs distributed P control, where the reference value is determined by a central PI controller. The decentralized controller however employs distributed PI control by frequency and phase measurements. Both the centralized and the decentralized controller, when stabilizing, were shown to regulate the frequency of the power system to the reference frequency under unknown load changes. Sufficient stability criteria were derived in terms of the control parameters. Simulations show equal performance of the two controllers with respect to frequency regulation.

A distributed and a novel centralized controller were also presented, by allowing either communication or measurements between buses directly connected by power lines. By the additional measurements, the distributed controller is able to regulate the frequency of the power system to the reference frequency under unknown load changes, while minimizing a cost function of the power generation. Sufficient stability criteria for the proposed controllers were derived in terms of both the eigenvalues of the linear system matrix, and the control parameters. Simulations show that the centralized controller is able to perform faster than the distributed controller, while requiring a good model and global state information.

6.2 Future work

Many interesting open questions remain in the area of multi-agent systems. Many of the results of Chapter 3 and 4 can most likely be extended to switching topologies with standard connectivity assumptions, such as e.g., uniform joint connectivity. As the Lyapunov function used in the proof of Theorem 3.1 does not depend on the edge set of the graph, it will be continuous under switching topologies. This allows the results to be easily extended also to switching topologies. The Lyapunov functions used in the proofs of Theorem 3.2 and 3.3, where the dynamics are of second-order, however depend on the graph topology. This may cause the Lyapunov function to increase discontinuously at switching instances. To prove stability for the second-order nonlinear dynamics, one would have to consider other Lyapunov functions, which are independent of the graph topology. As for distributed PI-controllers, the controllers must be defined properly to function under switching topologies. Proving stability would require a common Lyapunov function for all switching topologies. The extension of the results of Chapter 3 and 4 to switching topologies would be of practical importance, since the communication topology in many of the motivating applications cannot always be justified as static, even if this is true for instance for most power transmission systems.

Integral action is often employed when there are unknown disturbances, constant or time-varying. It would be of interest to study PI control of multi-agent systems under time-varying disturbances. Also, distributed internal model control would be an interesting option when the disturbance model is known but not necessary being constant.

While none of the proposed controllers in this thesis are model-based, the stability of the closed loop system depends on the model. Hence it would be interesting to study the robustness of multi-agent systems under PI control with respect to model errors. In this setting, it would also be of interest to study general interconnected linear systems. It is well-known that integrators may suffer from windup due to saturation of the control signal, seriously degrading the performance of the PI-controller. It is not clear how windup may affect multi-agent systems with distributed PI-controllers. Hence, it would be of interest to study the affects of integral windup in these systems, and possible solutions such as e.g., distributed anti-windup schemes.

In the area of power systems, many interesting research questions remain open. An interesting problem is whether the proposed distributed PI controllers for power systems can be analyzed in the context of the nonlinear swing equation or possibly more advanced power system models, or not. Also, developing faster control algorithms for optimal frequency control would also be a research topic of high priority. An interesting open research topic is whether it is possible to implement decentralized frequency controllers which achieve both frequency regulation and power sharing without communication between the buses.

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