# Analysis of Decentralized Potential Field Based Multi-Agent Navigation via Primal-Dual Lyapunov Theory

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***Abstract*— We use a combination of primal and dual Lyapunov theory for almost global asymptotic stabilization and collision avoidance in multi-agent systems. Previous work provided local analysis around the critical points with the use of the dual Lyapunov technique. This paper provides analysis of the whole workspace with the use of the recently introduced combined use of primal and dual Lyapunov functions.**

## I. Introduction

The development of advanced decentralized collision avoidance algorithms for large scale systems is an issue of critical importance in future Air Traffic Control (ATC) architectures, where the increasing number of aircraft will render centralized approaches inefficient. In recent years the application of robotics collision avoidance potential field based methods to ATC has been explored [6],[15] as a promising alternative for such algorithms. A common problem with potential field based path planning algorithms in multi-agent systems is the existence of local minima [7],[9]. The seminal work of Koditschek and Rimon [8] involved navigation of a single robot in an environment of spherical obstacles with guaranteed convergence. In previous work, the closed loop single robot navigation methodology of[8] was extended to multi-agent systems. In [10],[13] this method was extended to take into account the volume of each robot in a centralized multi-agent scheme, while a decentralized version has been presented in [4], [6]. Formation control for point agents using decentralized navigation functions was dealt with in [19], [3]. Decentralized navigation functions were also used for multiple UAV guidance in [2].

In [4],[6] the navigation functions were designed in such a way to allow agents that had already reached their desired destination to cooperate with the rest of the team in the case of a possible collision. In this paper, a construction similar to the initial navigation function construction in [8] is used. Hence each agent no longer participates in the collision avoidance procedure if its initial condition coincides with its desired goal. In essence, the agents might converge to critical points which are no longer guaranteed not to coincide with local minima. In this paper we examine the convergence of the system using a combination of primal and dual [16] Lyapunov techniques. In particular, primal Lyapunov analysis is used to show that the system converges to an arbitrarily small neighborhood of the critical points and the desired goal configuration. Density functions are then used to yield a sufficient condition for the attractors of the undesirable critical points to be sets of measure zero. This combination has been used in [12],[1],[11]. In particular in [11],[12] a density function is provided for a single robot driven by a navigation function in a static obstacle workspace. Primal analysis is used to show convergence to a neighborhood of the critical points and density functions to prove the instability of the undesirable critical points using the properties of the navigation functions. The difference in our case is that we consider a system of multiple moving agents driven by decentralized potential functions and the potential functions are not considered *a priori* navigation functions. On the contrary, the designed potentials are tuned properly to satisfy appropriate conditions to guarantee asymptotic stability from almost all initial conditions. The combination of primal and dual Lyapunov methods as described above was first used in [1] and more recently in [20] for the stabilization of a nonlinear attitude observer.

This paper is a continuation of [5], where we provided a connection between density functions and decentralized navigation functions, by applying the dual Lyapunov theorem of [16] to the critical points of these particular navigation functions, and not the whole configuration space, as in the current paper. Moreover in this paper a less conservative sufficient condition with a different definition of decentralized potential function is derived.

The rest of the paper is organized as follows: Section II presents the system and decentralized multi-agent navigation problem treated in this paper. The necessary mathematical preliminaries are provided in Section III, while Section IV provides the decentralized control design. Section V includes the convergence analysis and a simulated example is found in Section VI. Section VII summarizes the results of the paper and indicates further research directions.

## II. Definitions and Problem Statement

Consider a group of $N$ agents operating in the same planar workspace $W \subset \mathbb{R}^2$. Let $q_i \in \mathbb{R}^2$ denote the position of agent $i$, and let $q = [q_1^T, \ldots, q_N^T]^T$ be the stack vector of all agents' positions. We also denote $u = [u_1^T, \ldots, u_N^T]^T$. Agent motion is described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = \{1, \ldots, N\} \tag{1}$$

where $u_i$ denotes the velocity (control input) for each agent. We consider cyclic agents of specific radius $r_a \geq 0$, which is common for each agent. The results can trivially be extended to the case of agents with not necessarily common radii. Collision avoidance between the agents is meant in the sense

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that no intersections occur between the agents' discs. Thus we want to assure that

$$\|q_i(t) - q_j(t)\| > 2r_a, \forall i, j \in \mathcal{N}, i \neq j \tag{2}$$

for each time instant $t$. For the collision avoidance objective each agent has knowledge of the position of agents located in a cyclic neighborhood of specific radius $d$ at each time instant, where $d > 2r_a$. This set $T_i = \{q : \|q - q_i\| \leq d\}$ is called the *sensing zone* of agent $i$.

The proposed framework may cover various multi-agent objectives. These include navigation of the agent to desired destination and formation control. The function $\gamma_{di}$ is agent's $i$ goal function which is minimized once the desired objective with respect to this particular agent is fulfilled. In the first case, let $q_{di} \in W$ denote the desired destination point of agent $i$. We then define $\gamma_{di} = \|q_i - q_{di}\|^2$ as the squared distance of the agent's $i$ configuration from its desired destination $q_{di}$. In the formation control case, the objective of agent $i$ is to be stabilized in a desired relative position $c_{ij}$ with respect to each member $j$ of $\mathcal{N}_i$, where $\mathcal{N}_i \subset \mathcal{N}$ is a subset of the rest of the team. The definition of the sets $\mathcal{N}_i$ specifies the desired formation. We assume that $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$ in this paper. In this case, we have $\gamma_{di} = \sum\limits_{j \in \mathcal{N}_i} \|q_i - q_j - c_{ij}\|^2$. While we consider solely the first case here, the overall framework can be extended to the formation control problem as well as other definitions of $\gamma_{di}$.

In order to encode inter-agent collision scenarios, we define a function $\gamma_{ij}$, for $j = 1, \ldots, N, j \neq i$, given by

$$\gamma_{ij}(\beta_{ij}) = \begin{cases} \frac{1}{2}\beta_{ij}, \ 0 \leq \beta_{ij} \leq c^2 \\ \phi(\beta_{ij}), \ c^2 \leq \beta_{ij} \leq d^2 - 4r_a^2 \\ 1, \ d^2 - 4r_a^2 \leq \beta_{ij} \end{cases} \tag{3}$$

where $\beta_{ij} = \|q_i - q_j\|^2 - 4r_a^2$. We also define the function $\gamma_{i0}$ which refers to the workspace boundary (indexed by 0) and is used to maintain the agents within the workspace. We have $\beta_{i0} = (R_W - r_a)^2 - \|q_i\|^2$. The function $\gamma_{i0}$ is defined in the same way as $\gamma_{ij}, j > 0$. The positive scalar parameter $c$ and the function $\phi$ are chosen in such a way so that $\gamma_{ij}$ is everywhere twice continuously differentiable. For example, we can chose $\phi$ to be a fifth degree polynomial function of the form $\phi(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. The coefficients $a_i, i = 0, \ldots, 5$ are calculated so that $\gamma_{ij}$ is everywhere twice continuously differentiable, and in particular, at the points $\beta_{ij} = c^2, \beta_{ij} = d^2 - 4r_a^2$. This provides a system of six linear algebraic equations, the solution of which provides the coefficients in terms of $c^2$ and $d^2 - 4r_a^2$. In the sequel, we will also use the notation $\nabla_i(\cdot) \triangleq \frac{\partial}{\partial q_i}(\cdot)$ for brevity.

Motivated by applications in ATC, and in particular from the need to provide congestion metrics in large scale Air Traffic Control systems [18], we note that in the analysis that follows we consider point agents. This is not necessary for the analysis of the first part (primal Lyapunov analysis), however, it facilitates the calculations for the dual Lyapunov analysis of the second part. Point agents are actually also considered in [19],[3]. A discussion on the extension of this assumption is provided after the convergence analysis.

## III. Mathematical Preliminaries

### A. Dual Lyapunov Theory

For functions $V : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the notation

$$\nabla V = \left[ \begin{array}{ccc} \frac{\partial V}{\partial x_1} & \ldots & \frac{\partial V}{\partial x_n} \end{array} \right]^T$$

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}$$

is used. The dual Lyapunov result of [16] is stated as follows:

*Theorem 1:* Given the equation $\dot{x}(t) = f(x(t))$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a nonnegative density function $\rho \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ such that $\rho(x) f(x) / \|x\|$ is integrable on $\{x \in \mathbb{R}^n : \|x\| \geq 1\}$ and

$$[\nabla \cdot (f\rho)](x) > 0 \text{ for almost all } x \tag{4}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if $\rho$ takes negative values.

In this paper we use a combination of primal and dual Lyapunov techniques for almost global stability. In particular, we use primal Lyapunov analysis to show that the system converges to a set that includes a neighborhood of the critical points and the desired goal configuration. Density functions are then used to yield a sufficient condition for the attractors of the undesirable critical points to be sets of measure zero. This is guaranteed by the satisfaction of condition (4) in a neighborhood of the critical points.

Note that while Theorem 1 applies to the whole $\mathbb{R}^n$, we apply it here for the workspace $W$. The application of density functions to navigation function based systems was also used in [11]. A local version of Theorem 1 was used in [17]. Relaxed conditions for convergence to an equilibrium point in subsets of $\mathbb{R}^n$ were provided in [14].

## IV. Decentralized Control

In [4],[6] the control law allowed agents that had already reached their desired destination to cooperate with the rest of the team in the case of a possible collision. In this paper, we use a construction similar to the initial navigation function construction in [8]. Hence each agent no longer participates in the collision avoidance procedure if its initial condition coincides with its desired destination. As a result, the derived decentralized potential functions are not guaranteed to have the Morse property. A local analysis of the proposed decentralized potential around the critical points was held in [5] using dual Lyapunov theory [16]. In this paper the global stability properties of the closed-loop system are examined.

Specifically, we equip each agent with a decentralized potential function $\varphi_i : \mathbb{R}^{2N} \to [0, 1]$ defined as

$$\varphi_i = \frac{\gamma_{di}}{\left(\gamma_{di}^k + G_i\right)^{1/k}} \tag{5}$$

The exponent $k$ is a scalar positive parameter. The function $G_i$ is constructed in such a way in order to render the motion produced by the negated gradient of $\varphi_i$ with respect to $q_i$ repulsive with respect to the other agents. The control law is of the form

$$u_i = -K\nabla_i\varphi_i \tag{6}$$

where $K > 0$ is a positive scalar gain.

### A. Construction of the $G_i$ function

In the control law, each agent has a different $G_i$ which represents its relative position with all the other agents. In the sequel we review the construction of $G_i$ for each agent $i$, which was introduced in [4], [6]. In this paper, the function $G_i$ is constructed to take into account the local sensing capabilities of each agent. To encode all possible inter-agent proximity situations, the multi-agent team is associated with an (undirected) graph whose vertices are indexed by the team members. We use the following notions:

*Definition 1:* A *binary relation* with respect to an agent $i$ is an edge between agent $i$ and another agent.

*Definition 2:* A *relation* with respect to agent $i$ is defined as a set of binary relations with respect to agent $i$.

*Definition 3:* The *relation level* is the number of binary relations in a relation with respect to agent $i$.

The complementary set $(R_j^{i,C})_l$ of relation $j$ with respect to agent $i$ is the set that contains all the relations of the same level apart from the specific relation $j$.

The function $\gamma_{ij}$ defined above is called the "Proximity Function" between agents $i$ and $j$.

A "Relation Proximity Function" (RPF) provides a measure of the distance between agent $i$ and the other agents involved in the relation. Each relation has its own RPF. Let $R_k^i$ denote the $k^{th}$ relation of level $l$ with respect to $i$. The RPF of this relation is given by $(b_{R_k^i})_l = \sum_{j\in(R_k^i)_l}\gamma_{ij}$ where $j \in (R_k^i)_l$ denotes the agents that participate in the relation. When it is not necessary to specify the level and the specific relation, we also use the simplified notation $b_r^i = \sum_{j\in P_r}\gamma_{ij}$ for the RPF for simplicity, where $r$ denotes a relation and $P_r$ denotes the set of agents participating in the specific relation with respect to agent $i$.

A "Relation Verification Function" (RVF) is defined by:

$$(g_{R_k^i})_l = (b_{R_k^i})_l + \frac{\lambda(b_{R_k^i})_l}{(b_{R_k^i})_l + (B_{R_k^{i,C}})_l^{1/h}} \tag{7}$$

where $\lambda, h > 0$ and $(B_{R_k^{i,C}})_l = \prod_{m\in(R_k^C)_l}(b_m)_l$ where as previously defined, $(R_k^{i,C})_l$ is the complementary set of relations of level-$l$, i.e. all the other relations with respect to agent $i$ that have the same number of binary relations with the relation $R_k^i$. Again for simplicity we also use the notation $(B_{R_k^{i,C}})_l \equiv \tilde{b}_r^i = \prod_{\substack{s\in S_r\\ s\neq r}} b_s^i$ for the term $(B_{R_k^{i,C}})_l$ where $S_r$ denotes the set of relations in the same level with relation $r$. The RVF can be now also be written as $g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + (\tilde{b}_r^i)^{1/h}}$. It is obvious that for the highest level $l = n-1$ only one relation is possible so that $(R_k^{i,C})_{n-1} = \emptyset$ and $(g_{R_k^i})_l = (b_{R_k^i})_l$ for $l = n-1$. The basic property that we demand from RVF is that it assumes the value of zero if a relation holds, while no other relations of the same or other higher levels hold. In other words it should indicate which of all possible relations holds. We have he following limits of RVF (using the simplified notation): (a) $\lim\limits_{b_r^i\to 0}\lim\limits_{\tilde{b}_r^i\to 0} g_r^i\left(b_r^i, \tilde{b}_r^i\right) = \lambda$ (b) $\lim\limits_{\substack{b_r^i\to 0\\ \tilde{b}_r^i\neq 0}} g_r^i\left(b_r^i, \tilde{b}_r^i\right) = 0$. These limits guarantee that RVF will behave as an indicator of a specific relation.

The function $G_i$ is now defined as $G_i = \prod_{l=1}^{n_L^i}\prod_{j=1}^{n_{R_l}^i}(g_{R_j^i})_l$ where $n_L^i$ the number of levels and $n_{R_l}^i$ the number of relations in level-$l$ with respect to agent $i$. Hence $G_i$ is the product of the RVF's of all relations wrt $i$. Using the simplified notation, we have $G_i = \prod\limits_{r=1}^{N_i} g_r^i$ where $N_i$ is the number of all relation with respect to $i$.

We then have $\nabla_i\varphi_i = \frac{\left(\gamma_{di}^k+G_i\right)^{1/k}\nabla_i\gamma_{di} - \frac{\gamma_{di}}{k}\left(\gamma_{di}^k+G_i\right)^{1/k-1}\left(k\gamma_{di}^{k-1}\nabla_i\gamma_{di}+\nabla_i G_i\right)}{\left(\gamma_{di}^k+G_i\right)^{2/k}}$, so that

$$\nabla_i\varphi_i = \left(\gamma_{di}^k + G_i\right)^{-1/k-1}\left(G_i\nabla_i\gamma_{di} - \frac{\gamma_{di}}{k}\nabla_i G_i\right) \tag{8}$$

We can also compute

$$\nabla_i\varphi_j = \left(\gamma_{dj}^k + G_j\right)^{-1/k-1}\left(-\frac{\gamma_{dj}}{k}\nabla_i G_j\right) \tag{9}$$

A critical point of $\varphi_i$ is defined by $\nabla_i\varphi_i = 0$. The following Proposition will be useful in the following analysis:

*Proposition 1:* For every $\epsilon > 0$ there exists a positive scalar $P(\epsilon) > 0$ such that if $k \geq P(\epsilon)$ then there are no critical points of $\varphi_i$ in the set $F_i = \left\{q \in W | g_r^i \geq \epsilon, \forall r = 1,\ldots,N_i\right\} \setminus \{\gamma_{di}\}$.

**Proof**: At a critical point, we have $\nabla_i\varphi_i = 0$, or $G_i\nabla_i\gamma_{di} = \frac{\gamma_{di}}{k}\nabla_i G_i$, which implies $2kG_i = \sqrt{\gamma_{di}}\left\|\nabla_i G_i\right\|$. since $\left\|\nabla_i\gamma_{di}\right\| = 2\sqrt{\gamma_{di}}$. A sufficient condition for this equality not to hold in $F_i$ is given by $k > \frac{\sqrt{\gamma_{di}}\|\nabla_i G_i\|}{2G_i}$, $\forall q \in F_i$. An upper bound for the right hand side is given by $\frac{\sqrt{\gamma_{di}}\|\nabla_i G_i\|}{2G_i} \leq \frac{\sqrt{\gamma_{di}}}{2}\sum\limits_{r=1}^{N_i}\frac{\left\|\nabla_i g_r^i\right\|}{g_r^i} \leq \frac{1}{2\varepsilon}\max\limits_W\left\{\sqrt{\gamma_{di}}\right\}\sum\limits_{r=1}^{N_i}\max\limits_W\left\{\left\|\nabla_i g_r^i\right\|\right\} \triangleq P$, since $g_r^i \geq \epsilon, \forall r = 1,\ldots,N_i$. Note that the terms $\max\limits_W\left\{\sqrt{\gamma_{di}}\right\}$, $\max\limits_W\left\{\left\|\nabla_i g_r^i\right\|\right\}$ are bounded due to the boundedness of the workspace. $\diamondsuit$

This proposition also implies that all the critical points are restricted to the set $B_r^i(\epsilon) = \{q : 0 \leq g_r^i < \epsilon\}$ for some $r = 1,\ldots,N_i$. Note that since $G_i$ is a product of bounded functions $g_r^i$, it is straightforward that for all $\epsilon > 0$, there exists $\epsilon_1(\epsilon) > 0$ such that $0 \leq G_i < \epsilon_1(\epsilon)$ implies $0 \leq g_r^i < \epsilon$ for at least one relation $r = 1,\ldots,N_i$.

## V. Convergence Analysis

The convergence analysis of the overall system consists of two parts. The first part uses primal (classic) Lyapunov analysis to show that the system converges to an arbitrarily small neighborhood of the critical points. We then use dual Lyapunov analysis to show that the set of initial conditions that drives the system to points other than the goal configurations is of zero measure.

## A. Primal Lyapunov Analysis

Note that the closed loop kinematics of system (1) under the control law (6) are given by

$$\dot{q} = f(q) = \begin{bmatrix} -K\left(\gamma_{d1}^k + G_1\right)^{-1/k-1}\left\{G_1\nabla_1\gamma_{d1} - \frac{\gamma_{d1}}{k}\nabla_1 G_1\right\} \\ \vdots \\ -K\left(\gamma_{dN}^k + G_N\right)^{-1/k-1}\left\{G_N\nabla_N\gamma_{dN} - \frac{\gamma_{dN}}{k}\nabla_N G_N\right\} \end{bmatrix}$$

Define $\varphi = \sum_i \varphi_i$. The derivative of $\varphi$ can be computed by $\dot{\varphi} = (\nabla\varphi)^T\dot{q} = -K\sum_{i=1}^N(\nabla_i\varphi)^T(\nabla_i\varphi_i) = -K\sum_{i=1}^N\sum_{j=1}^N(\nabla_i\varphi_i)^T(\nabla_i\varphi_j)$ where $\varphi_i$ is defined in (5). Consider $\varepsilon > 0$. Then we can further compute

$$\begin{aligned}\dot{\varphi} &= -K\sum_{i=1}^N(\|\nabla_i\varphi_i\|^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j)) \\ &= -K\sum_{i:\|\nabla_i\varphi_i\|>\varepsilon}(\|\nabla_i\varphi_i\|^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j)) \\ &\quad -K\sum_{i:\|\nabla_i\varphi_i\|\leq\varepsilon}(\|\nabla_i\varphi_i\|^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j)) \\ &\leq -K\sum_{i:\|\nabla_i\varphi_i\|>\varepsilon}(\varepsilon^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j)) \\ &\quad -K\sum_{i:\|\nabla_i\varphi_i\|\leq\varepsilon}(\sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j))\end{aligned}$$

The terms in the first sum, where $\|\nabla_i\varphi_i\| > \varepsilon$, are lower bounded as follows:

$$\varepsilon^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j) \geq \varepsilon^2 - \varepsilon\sum_{j\neq i}\|\nabla_i\varphi_j\|$$

Using (9) we have

$$\|\nabla_i\varphi_j\| = \left(\gamma_{dj}^k + G_j\right)^{-1/k-1}\left(\frac{\gamma_{dj}}{k}\|\nabla_i G_j\|\right)$$

For $\gamma_{dj} > \gamma_{\min}$, $k > 1$, the term $(\gamma_{dj}^k + G_j)^{1/k+1}$ in the above equation is minimized by $\gamma_{\min}^2$ so that

$$\varepsilon^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j) \geq \varepsilon^2 - \varepsilon\frac{1}{k\gamma_{\min}^2}\sum_{j\neq i}\gamma_{dj}\|\nabla_i G_j\|$$

We want to achieve a bound of the form $\varepsilon^2 + \sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j) \geq \rho_1 > 0$, where $0 < \rho_1 < \varepsilon^2$. A sufficient condition for this to hold is $\frac{N-1}{k\gamma_{\min}^2}\max_{j\neq i}\{\gamma_{dj}\|\nabla_i G_j\|\} \leq \frac{\varepsilon^2-\rho_1}{\varepsilon}$, or equivalently

$$k \geq \frac{\varepsilon}{\varepsilon^2 - \rho_1}\frac{N-1}{\gamma_{\min}^2}\max_{j\neq i}\{\gamma_{dj}\|\nabla_i G_j\|\} \tag{10}$$

We next compute a lower bound on the terms in the second sum, where $\|\nabla_i\varphi_i\| \leq \varepsilon$. Note first that

$$\begin{aligned}&(\nabla_i\varphi_i)^T(\nabla_i\varphi_j) = \\ &= \frac{\left(G_i\nabla_i\gamma_{di} - \frac{\gamma_{di}}{k}\nabla_i G_i\right)^T\left(-\frac{\gamma_{dj}}{k}\nabla_i G_j\right)}{\left(\gamma_{di}^k + G_i\right)^{1/k+1}\left(\gamma_{dj}^k + G_j\right)^{1/k+1}} \\ &= \frac{-\frac{\gamma_{dj}G_i}{k}\nabla_i\gamma_{di}^T\nabla_i G_j + \frac{\gamma_{dj}\gamma_{di}}{k^2}\nabla_i G_i^T\nabla_i G_j}{\left(\gamma_{di}^k + G_i\right)^{1/k+1}\left(\gamma_{dj}^k + G_j\right)^{1/k+1}}\end{aligned}$$

so that

$$(\nabla_i\varphi_i)^T(\nabla_i\varphi_j) \geq \frac{1}{\gamma_{\min}^4}(-\frac{\gamma_{dj}G_i}{k}\|\nabla_i\gamma_{di}\|\|\nabla_i G_j\| - \frac{\gamma_{dj}\gamma_{di}}{k^2}\|\nabla_i G_i\|\|\nabla_i G_j\|)$$

We want to achieve a bound of the form $\sum_{j\neq i}(\nabla_i\varphi_i)^T(\nabla_i\varphi_j) \geq -2\rho_2$, where $\rho_2 > 0$. A sufficient condition for this to hold is that $\frac{1}{\gamma_{\min}^4}\frac{\gamma_{dj}G_i}{k}\|\nabla_i\gamma_{di}\|\|\nabla_i G_j\| \leq \rho_2$ and $\frac{1}{\gamma_{\min}^4}\frac{\gamma_{dj}\gamma_{di}}{k^2}\|\nabla_i G_i\|\|\nabla_i G_j\| \leq \rho_2$ or equivalently, that both

$$k \geq \frac{\max_{j\neq i}\{\gamma_{dj}G_i\|\nabla_i\gamma_{di}\|\|\nabla_i G_j\|\}}{\rho_2\gamma_{\min}^4} \tag{11}$$

and

$$k \geq \sqrt{\frac{\max_{j\neq i}\{\gamma_{di}\gamma_{dj}\|\nabla_i G_i\|\|\nabla_i G_j\|\}}{\rho_2\gamma_{\min}^4}} \tag{12}$$

hold. Provided that $k$ satisfies (10),(11),(12) we have

$$\dot{\varphi} \leq -K\rho_1 + K(N-1)^2\rho_2$$

assuming there exists at least one agent such that $\|\nabla_i\varphi_i\| > \varepsilon$. The latter is strictly negative for $0 < (N-1)^2\rho_2 < \rho_1 < \varepsilon$.

In essence, $\dot{\varphi}$ can be rendered strictly negative as long as there exists at least one agent with $\|\nabla_i\varphi_i\| > \varepsilon$. Thus the system converges to an arbitrarily small region of the critical points, provided that $0 < (N-1)^2\rho_2 < \rho_1 < \varepsilon$ and the conditions on $k$ hold. We have:

*Proposition 2:* Consider the system (1) with the control law (6). Assume that $\gamma_{di} \geq \gamma_{\min} > 0$. Pick $\varepsilon > 0, \rho_1, \rho_2 > 0$ satisfying $0 < (N-1)^2\rho_2 < \rho_1 < \varepsilon$ and assume that (10),(11),(12) hold. Then the system converges to the set $\|\nabla_i\varphi_i\| \leq \varepsilon$ for all $i$ in finite time.

## B. Dual Lyapunov Analysis

Having established convergence to an arbitrarily small neighborhood of the critical points, density functions are now used to pose sufficient conditions that the attractors of undesirable critical points are sets of measure zero.

For $\varphi = \sum_i \varphi_i$, define $\rho = \varphi^{-\alpha}$,$\alpha > 0$ which is defined for all points in $W$ other than the desired equilibrium $\gamma_{di} = 0$, for all $i \in \mathcal{N}$. Note also that each $\varphi_i$ is $C^2$ and takes values in $[0, 1]$ and thus both the function $\varphi$ and its gradients are bounded functions in $W$. Hence, $\rho$ fulfils the integrability condition of Theorem 1 and is a suitable density function for the equilibrium point $\gamma_{di} = 0, \forall i \in \mathcal{N}$. Note that the use of a navigation function as a candidate density function in sphere worlds was also used in [11], involving a single agent navigating in a static obstacle environment.

We have $\nabla\rho = -\alpha\varphi^{-\alpha-1}\nabla\varphi$ and $\nabla\cdot(f\rho) = \nabla\rho\cdot f + \rho\nabla\cdot f == -\alpha\varphi^{-\alpha-1}\nabla\varphi\cdot f + \varphi^{-\alpha}\nabla\cdot f$. Whenever $\nabla_i\varphi_i = 0$ for all $i \in \mathcal{N}$, we have $f = 0$ and

$$\nabla\cdot(f\rho) = \varphi^{-\alpha}\nabla\cdot f = -\varphi^{-\alpha}\sum_i K\left(\frac{\partial^2\varphi_i}{\partial x_i^2} + \frac{\partial^2\varphi_i}{\partial y_i^2}\right)$$

A sufficient condition for the right hand side of the last equation to be strictly positive is

$$\frac{\partial^2 \varphi_i}{\partial x_i^2} + \frac{\partial^2 \varphi_i}{\partial y_i^2} < 0 \tag{13}$$

for all $i \in \mathcal{N}$. We have

$$\begin{aligned}
&\tfrac{\partial^2 \varphi_i}{\partial x_i^2} + \tfrac{\partial^2 \varphi_i}{\partial y_i^2} < 0 \Leftrightarrow \\
&G_i \left( \tfrac{\partial^2 \gamma_{di}}{\partial x_i^2} + \tfrac{\partial^2 \gamma_{di}}{\partial y_i^2} \right) - \tfrac{\gamma_{di}}{k} \left( \tfrac{\partial^2 G_i}{\partial x_i^2} + \tfrac{\partial^2 G_i}{\partial y_i^2} \right) < 0 \Leftrightarrow \\
&4G_i - \tfrac{\gamma_{di}}{k} \left( \tfrac{\partial^2 G_i}{\partial x_i^2} + \tfrac{\partial^2 G_i}{\partial y_i^2} \right) < 0
\end{aligned}$$

since $\frac{\partial^2 \gamma_{di}}{\partial x_i^2} = \frac{\partial^2 \gamma_{di}}{\partial y_i^2} = 2$, and

$$\begin{aligned}
&\nabla_i \varphi_i = 0 \Rightarrow \\
&\tfrac{\partial^2 \varphi_i}{\partial x_i^2} = \left( \gamma_{di}^k + G_i \right)^{-2(1/k+1)} \left( G_i \tfrac{\partial^2 \gamma_{di}}{\partial x_i^2} - \tfrac{\gamma_{di}}{k} \tfrac{\partial^2 G_i}{\partial x_i^2} \right), \\
&\tfrac{\partial^2 \varphi_i}{\partial y_i^2} = \left( \gamma_{di}^k + G_i \right)^{-2(1/k+1)} \left( G_i \tfrac{\partial^2 \gamma_{di}}{\partial y_i^2} - \tfrac{\gamma_{di}}{k} \tfrac{\partial^2 G_i}{\partial y_i^2} \right).
\end{aligned}$$

Therefore, in order to have $[\nabla \cdot (f\rho)] > 0$, it suffices that

$$4G_i - \frac{\gamma_{di}}{k} \left( \frac{\partial^2 G_i}{\partial x_i^2} + \frac{\partial^2 G_i}{\partial y_i^2} \right) < 0 \tag{14}$$

In previous work we provided a sufficient condition for (14) to hold for a simpler construction of the $G_i$ function. It turns out that a less conservative condition can be derived with the definition of $G_i$ used in the current paper.

We can compute $\frac{\partial G_i}{\partial x_i} = \sum\limits_{r=1}^{N_i} \bar{g}_r^i \frac{\partial g_r^i}{\partial x_i}$ and $\frac{\partial^2 G_i}{\partial x_i^2} = \sum\limits_{r=1}^{N_i} \left\{ \frac{\partial \bar{g}_r^i}{\partial x_i} \frac{\partial g_r^i}{\partial x_i} + \bar{g}_r^i \frac{\partial^2 g_r^i}{\partial x_i^2} \right\}$ Similar relations hold for the $y_i$-partial derivatives. For $G_i = 0$ we have $g_{r^*}^i = 0$ for one and only one relation of agent $i$, i.e., $g_r^i \neq 0$ for all $r \neq r^*$. In this case, it is easy to check $g_r^i \geq \lambda$ for all $r \neq r^*$, so that

$$\frac{\partial^2 G_i}{\partial x_i^2} \geq \lambda^{N_i - 1} \frac{\partial^2 g_{r^*}^i}{\partial x_i^2} + \sum_{r=1}^{N_i} \left\{ \frac{\partial \bar{g}_r^i}{\partial x_i} \frac{\partial g_r^i}{\partial x_i} \right\}$$

Remembering that $|P_r|$ denotes the number of binary relations in a relation, we get

$$\frac{\partial^2 g_{r^*}^i}{\partial x_i^2} = |P_{r^*}| + \lambda \frac{\partial^2}{\partial x_i^2} \frac{b_{r^*}^i}{b_{r^*}^i + (\tilde{b}_{r^*}^i)^{1/h}}$$

Note that for $g_{r^*}^i = 0$ we have $b_{r^*}^i = 0$ and $\tilde{b}_{r^*}^i > 0$. Assuming $r_a = 0$ we then also have $\frac{\partial b_{r^*}^i}{\partial x_i} = 0$, and $\frac{\partial g_{r^*}^i}{\partial x_i} = 0$. In this case we can show that the right term is strictly positive. In particular, considering $b_{r^*}^i = 0$, after some calculations we have $\frac{\partial^2}{\partial x_i^2} \frac{b_{r^*}^i}{b_{r^*}^i + (\tilde{b}_{r^*}^i)^{1/h}} = (\tilde{b}_{r^*}^i)^{-2/h} ((\tilde{b}_{r^*}^i)^{1/h} |P_{r^*}| - 2 || \frac{\partial b_{r^*}^i}{\partial x_i} ||^2 - (2/h) (\tilde{b}_{r^*}^i)^{1/h - 1} \frac{\partial b_{r^*}^i}{\partial x_i} \frac{\tilde{b}_{r^*}^i}{\partial x_i})$. The first term of the right hand side is strictly positive while the second and third ones are cancelled due to the existence of $\frac{\partial b_{r^*}^i}{\partial x_i}$. Therefore we have $\frac{\partial^2 g_{r^*}^i}{\partial x_i^2} > |P_{r^*}|$ and $\frac{\partial^2 G_i}{\partial x_i^2} > \lambda^{N_i - 1} |P_{r^*}| + \sum\limits_{r=1}^{N_i} \left\{ \frac{\partial \bar{g}_r^i}{\partial x_i} \frac{\partial g_r^i}{\partial x_i} \right\}$. We can also show that in the case where $r_a = 0$, the second term is equal to zero. Just note that each summation term is a product containing either $\frac{\partial g_{r^*}^i}{\partial x_i} = 0$ or $g_{r^*}^i = 0$. Therefore $\frac{\partial^2 G_i}{\partial x_i^2} > \lambda^{N_i - 1} |P_{r^*}|$, so that $\frac{\partial^2 G_i}{\partial x_i^2} > 0$ is strictly positive for $G_i = 0$. Note also that for all relations, $|P_r| \geq 1$. By continuity, we then know that for small enough $\epsilon_1(\epsilon)$, we have $\frac{\partial^2 G_i}{\partial x_i^2} \geq \lambda^{N_i - 1} > 0$ for $0 \leq G_i < \epsilon_1(\epsilon)$ which implies that $\frac{\partial^2 G_i}{\partial x_i^2} \geq \lambda^{N_i - 1} > 0$ for $q \in B_r^i(\epsilon)$ for some $r = 1, \ldots, N_i$. Similarly, we can show that $\frac{\partial^2 G_i}{\partial y_i^2} \geq \lambda^{N_i - 1} > 0$. Since we can restrict the critical points of $\varphi$ to occur at the region $q \in B_r^i(\epsilon)$ (Proposition 1) for some $r = 1, \ldots, N_i$, we have that $\frac{\partial^2 G_i}{\partial x_i^2} + \frac{\partial^2 G_i}{\partial y_i^2} \geq 2\lambda^{N_i - 1}$ for $\nabla_i \varphi_i = 0$.

For the analysis that follows, we use the notations

$$\begin{gathered}
\max_W \{\sqrt{\gamma_{di}}\} \sum_{r=1}^{N_i} \max_W \left\{ \left\| \nabla_i g_r^i \right\| \right\} = M_1, \\
\max_{j \neq i} \{\gamma_{dj} \|\nabla_i G_j\|\} = M_2, \\
\max_{j \neq i} \{\gamma_{dj} G_i \|\nabla_i \gamma_{di}\| \|\nabla_i G_j\|\} = M_3, \\
\sqrt{\max_{j \neq i} \{\gamma_{di} \gamma_{dj} \|\nabla_i G_i\| \|\nabla_i G_j\|\}} = M_4.
\end{gathered}$$

We can now choose $k$ according to

$$k = k_{\max} = \max \left\{ \frac{M_1}{2\epsilon}, \frac{\varepsilon(N-1)M_2}{(\varepsilon^2 - \rho_1)\gamma_{\min}^2}, \frac{M_3}{\rho_2 \gamma_{\min}^4}, \frac{M_4}{\sqrt{\rho_2} \gamma_{\min}^2} \right\}$$

which satisfies Proposition 1 and the conditions (10),(11),(12).

A sufficient condition for (14) to hold at the critical points is then given by

$$\lambda > \left( \frac{2\epsilon_1(\epsilon) k_{\max}}{\gamma_{\min}} \right)^{\frac{1}{N_i - 1}} \tag{15}$$

We now use the argument of [11] mentioning that since (14) is satisfied exactly at the critical points, it is satisfied also in an arbitrary small neighborhood around them. From the primal Lyapunov analysis, we know that indeed the system converges to an arbitrarily small neighborhood of the critical points. The dual Lyapunov analysis guarantees that the attractors of the undesirable critical points are sets of measure zero. The following then holds:

*Proposition 3:* Consider the system (1) with the control law (6). Assume that the assumptions of Propositions 1,2 hold and that (15) also holds. Then the closed-loop system is asymptotically stable for almost all initial conditions.

*Remark 1:* The previous analysis holds for the case of point agents, i.e., $r_a = 0$. The extension to the case of arbitrary $r_a$ holds for the primal analysis case, however it is *not* straightforward in the dual analysis, since in this case the terms $\frac{\partial^2}{\partial x_i^2} \frac{b_{r^*}^i}{b_{r^*}^i + (\tilde{b}_{r^*}^i)^{1/h}}$ and $\sum\limits_{r=1}^{N_i} \left\{ \frac{\partial \bar{g}_r^i}{\partial x_i} \frac{\partial g_r^i}{\partial x_i} \right\}$ are not guaranteed to be positive and zero respectively. For the case of non-point agents, a conservative condition was derived in [5]. The extension of the less conservative result of this paper to non-point agents is currently under investigation. Note also that the agents are only guaranteed to converge to a neighborhood of the destination points, since $\gamma_{\min} > 0$ should be bounded away from zero to prevent $k$ and $\lambda$ from exploding to infinity.

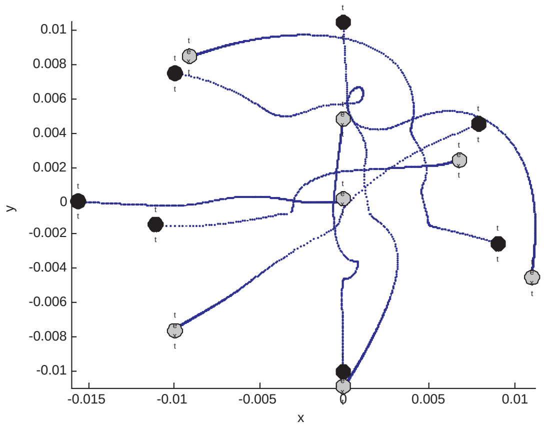


Fig. 1. Seven agents navigate under the control law (6). The black circles represent the initial positions of the agents while their final locations are denoted by a grey circle.

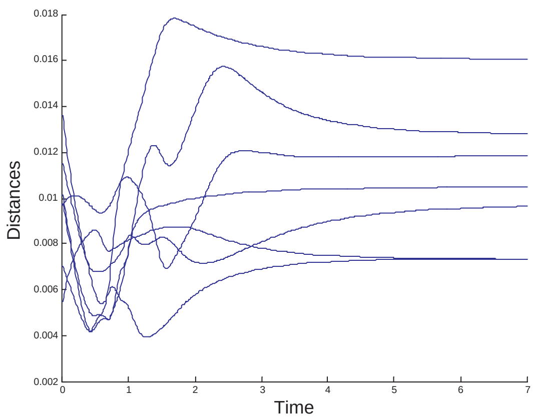


Fig. 2. Distances of a particular agent from each of the remaining ones as the closed-loop system evolves.

This further justifies the need of an additional element to the $\gamma_i$ functions so that agent $i$ cooperates with the rest of the team in the case of a possible collision even once it is already arbitrarily close to its desired destination. The reader is referred to the relevant design in [4],[6].

## VI. Simulations

In order to illustrate the effectiveness of the designed controller we have set up a simulation involving seven agents, with controller parameters given by $k = 4, \lambda = 1, h = 1, R_w = 1$. The evolution of the system is depicted in Figure 1, where the black circles represent the initial positions of the agents and their final destinations are denoted by a grey circle. Figure 2 depicts the distances of a particular agent from each of the remaining ones. One can see that interagent distances satisfy the collision avoidance property. After some time, inter-agent distances become constant since agents converge to their corresponding desired destinations.

## VII. Conclusions

We used a combination of primal and dual Lyapunov Theory to derive sufficient conditions for asymptotic stabilization from almost all initial conditions in multi-agent systems driven by decentralized navigation-like functions. The primal Lyapunov analysis guaranteed convergence of the system to an arbitrary small neighborhood of the critical points. The dual analysis then provided sufficient conditions for the undesirable critical points to have attractors of measure zero.

Future research involves extending the dual analysis result to the case of non-point agents, as well as applying the decentralized navigation functions' framework to the design of [15].

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