Opinion Consensus of Modified Hegselmann-Krause Models

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Abstract

We consider the opinion consensus problem using a multi-agent setting based on the Hegselmann-Krause (H-K) Model. Firstly, we give a sufficient condition on the initial opinion distribution so that the system will converge to only one cluster. Then, modified models are proposed to guarantee convergence for more general initial conditions. The overall connectivity is maintained with these models, while the loss of certain edges can occur. Furthermore, a smooth control protocol is provided to avoid the difficulties that may arise due to the discontinuous right-hand side in the H-K model.

Key words: opinion dynamics, multi-agent systems

1 Introduction

The opinion consensus problem is about opinion compromise of a certain event by different agents. Assume that opinion is continuous, and that all agents have bounded confidence in the way that they only consider the opinion that is close to their own opinion. Agent-based models of opinion dynamics under these assumptions have been established in the beginning of this century by Hegselmann and Krause [Hegselmann and Krause, 2002] and Weisbuch et al [Weisbuch et al., 2002]. Both models lead to clustering of opinions in a similar way. In this paper we will consider the model of Hegselmann and Krause (H-K).

The H-K model has attracted significant attention in the past few years, e.g., [Forunato, 2004], [Blondel et al., 2009], [Mirtabatabaei and Bullo, 2012], [Canuto et al., 2012], because of its simple model structure and complex evolving behavior. The previous study about the H-K model shows that not all initial opinion distributions corresponding to a connected graph will lead to consensus [Lorenz, 2006], [Lorenz, 2005], [Blondel et al., 2009]. This is due to the fact that during the process the graph can keep disconnected since the neighborhood is based on opinion differences between pairs of agents. An important result of multi-agent rendezvous problem is that the consensus is reached if and only if the switching networks are "ultimately connected" proved by Moreau [Moreau, 2005]. The possible permanent loss of connectivity can yield several clusters of agent opinions in different positions. This phenomenon is also observed in the paper by Hegselmann and Krause [Hegselmann and Krause, 2002]. However, even for the one-dimensional case, few theoretical results have been obtained so far regarding the relationship between this loss of connectivity and the initial opinion distribution. We provide a sufficient condition on the initial state to guarantee consensus in this paper.

On the other hand, instead of imposing constraints on the initial distribution, one can modify the model to guarantee that consensus is achieved for any initial opinions. This is related to the connectivity maintenance problem in the multi-agent systems theory. A way to achieve this is by using potential functions. The main idea is that the force between two agent opinions becomes infinitely large when the difference between the opinions becomes big enough, i.e., near the boundary of confidence. This approach has been used by several researchers in the past few years, e.g., [De Gennaro and Jadbabaie, 2006], [Ji and Egerstedt, 2005]. Bounded controllers for connectivity control are considered in [Kan et al., 2010], [Dimarogonas and Johansson, 2008], [Wang et al., 2013]. The common idea in these papers is that no edge is allowed to break during the process, thus imposing constraints in the relative states of pairs of agents that constitute an edge. However, this is only a sufficient condition for connectiv-

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ity maintenance because the loss of some "non-crucial" edges may not influence the connectivity. In this paper we use topological arguments to guarantee connectivity instead of applying infinite potentials when an edge is bound to break. In particular, inspired by the idea used in [Gustavi et al., 2010], we show that common neighbors play an important role in the problem. If two nodes share some common neighbors, the edge between them can be allowed to break because they are still connected through the common neighbors. On the contrary, if they do not have any common neighbors, then the edge becomes crucial and should not be broken.

The modified model that we provide in this paper guarantees opinion consensus for almost all connected initial opinion distribution, even if the ratio between the opinion diversity and the confidence bound is significant. Usually one obtains clustering behavior, i.e., disconnectedness, of the original H-K model when this ratio is big. This issue is overcome by using the modified model in Section III. A requirement of our model is that two or more agents cannot have the same initial opinions. Since this is a possible scenario in practice, we provide another model to deal with this case. Furthermore, for the original H-K model, the right-hand side is not a continuous function of the state x. This results in a measure zero set of initial conditions from which the solution may not be unique. We introduce in the paper a smooth modification of the model in order to avoid this.

The remainder of the paper is summarized as follows: in Section II we formulate the problem under consideration. The modified version of the H-K model is presented and analyzed in Section III as well as a sufficient condition for guarantee consensus. A smoothed version of this is provided in Section IV. Some discussion about high dimensional scenario is included in Section V. In Section VI we provide simulations that support the derived theoretical results. Finally, a summary of the results of this paper as well as possible directions of future work are included in Section VII.

2 Mathematical preliminaries

2.1 *Basic concepts from graph theory*

In this section, we review some concepts from graph theory that will be used in this paper. These definitions can be otherwise found in a standard textbook on graph theory.

Consider a set of n nodes denoted by $V = \{1,2,\dots,n\}$ and a subset $E \subset V \times V$. We call $G = (V,E)$ a graph with the set of vertices (or nodes) V and the set of edges E. In $G = (V,E)$, the neighbor set of the vertex i is defined by $\mathcal{N}_i = \{j \in V | (j,i) \in E\}$. A graph $G = (V,E)$ is called *undirected* if $(i,j) \in E$ implies $(j,i) \in E$. An undirected graph is called *simple* if it has no loops (edges connected at both ends to the same vertex) and no more than one edge between any two different vertices. All the graphs mentioned in this paper will be simple graphs. In a graph, if there is an edge connecting two vertices, i.e., $(i,j) \in E$, then these two vertices i, j are called *adjacent*. A graph is called *complete* if any two nodes are adjacent. A *path* from a vertex i to another vertex j is a sequence of distinct vertices starting with i and ending with j, in which each vertex is adjacent to its next vertex. Two vertices i and j are called *connected* if there exists a path from i to j. An undirected graph is call a *connected graph* if any pair of vertices in it is connected.

2.2 *Introduction of Hegselmann-Krause model*

Consider a system of n autonomous agents labeled as $1,2,\dots,n$, whose opinions are located in the one-dimensional Euclidean space $\mathbb{R}$. We denote the set of all agents as $V = \{1,2,\dots,n\}$. For an agent $i \in V$, the position of its opinion is denoted by $x_i(t) \in \mathbb{R}$, and has the following dynamics:

$$\dot{x}_i(t) = u_i(t),$$

where $u_i(t)$ is considered as the controller of agent i. The *consensus* problem is to find the controllers $u_i(t)$ so that the stack state $x(t) = (x_1(t)\ x_2(t)\ \cdots\ x_n(t))^T$ will converge to the subspace generated by the vector $\vec{1} = (1\ 1\ \cdots\ 1)^T$ as $t \to \infty$. If the edge set $E \subset V \times V$ is given, one can then define a graph $G = (V,E)$ and generate a basic control protocol for the consensus problem:

$$\dot{x}_i(t) = u_i(t) = \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t)) \tag{1}$$

It is well-known that the system (1) will converge to the equilibrium $x_i(t) = \alpha,\ i = 1,2,\dots,n$ if the graph G is connected, where $\alpha = \frac{1}{n}\sum_{i=1}^n x_i(0)$.

Now assume that the graph $G = (V,E)$ is defined by $V = \{1,2,\dots,n\}$ and $E = \{(i,j) | i \neq j, |x_i - x_j| \leq d\}$ for some $d > 0$. Applying the same control in (1) with this definition of the graph, we obtain the Hegselmann-Krause (H-K) model:

$$\dot{x}_i(t) = \sum_{j : |x_j - x_i| \leq d} (x_j(t) - x_i(t)). \tag{2}$$

Note that the right-hand side of the equation is a discontinuous function with respect to x. In [Blondel et al., 2010], the almost sure existence of solution to this differential equation have been proved as well as the convergence. Here the convergence means that the state will converge to an equilibrium of the system. However, the equilibrium is not necessarily in the form $\alpha \cdot \vec{1}$, as discussed in the introduction. Instead it can consist of several clusters.

From now on, we will call $G(t) = (V,E(t))$ the *corresponding graph* of $x(t)$, where $V = \{1,2,\dots,n\}$ and $E(t) = \{(i,j) | i \neq j, |x_i(t) - x_j(t)| \leq d\}$.

3 Non-smooth model

3.1 Sufficient condition for consensus

In this section, we give a sufficient condition on the initial states (opinions) such that the system will converge to exactly one cluster. The concept of common neighbor will be used in the theorem.

Definition 3.1 *For a simple graph $G = (V,E)$, the set of common neighbors between two nodes i and j is defined as:*

$$\mathcal{N}_{ij} = \{k \in V | (i,k) \in E, (j,k) \in E\} = \mathcal{N}_i \cap \mathcal{N}_j. \tag{3}$$

Note that the set of common neighbors can be defined locally since $N_{ij} \subseteq N_i$, agent i only needs to check the relative distance between any neighbor $k \in N_i$ and the agent j.

Theorem 3.2 *For an initial condition $x(0) \in \mathbb{R}^n$ and the corresponding graph $G(0) = (V,E(0))$, if $G(0)$ is connected and for any pair $(i, j) \in E(0)$, it holds that $|\mathcal{N}_{ij}| \geq \frac{n}{2} - 2$, then the solution to (2) will converge to $\alpha \cdot \vec{1}$, where $\alpha = \frac{1}{n}\vec{1}^T x(0)$.*

Proof: Because the initial graph G is connected by assumption, if no edge in E is lost during the process, the graph $G(t)$ will be always connected. Then it is well-known that the states will converge to the average value of the initial states. What we need to show now is that for any pair of vertices $(i, j) \in E$, the distance $|x_j(t) - x_i(t)|$ will not exceed the threshold d while the number of common neighbors $|\mathcal{N}_{ij}|$ is not smaller than $\frac{n}{2} - 2$ at time t. Due to the continuity of $x(t)$, we consider only the situation that $|x_j(t) - x_i(t)| = d$, and assume that $x_j(t) > x_i(t)$ without loss of generality. Denote $\mathcal{N}_i' = \mathcal{N}_i \setminus (\mathcal{N}_{ij} \cup \{j\})$ and $\mathcal{N}_j' = \mathcal{N}_j \setminus (\mathcal{N}_{ij} \cup \{i\})$. Then we can compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(x_j(t) - x_i(t)) = \dot{x}_j(t) - \dot{x}_i(t) \\
&= \sum_{k \in \mathcal{N}_j} (x_k(t) - x_j(t)) - \sum_{k \in \mathcal{N}_i} (x_k(t) - x_i(t)) \\
&= \sum_{k \in \mathcal{N}_j'} (x_k(t) - x_j(t)) - \sum_{k \in \mathcal{N}_i'} (x_k(t) - x_i(t)) \\
&\quad - (|\mathcal{N}_{ij}| + 2)(x_j(t) - x_i(t)) \\
&\leq (|\mathcal{N}_j'| + |\mathcal{N}_i'|)d - (|\mathcal{N}_{ij}| + 2)d. \\
&\leq (n - (|\mathcal{N}_{ij}| + 2))d - (|\mathcal{N}_{ij}| + 2)d \\
&\leq (n - 2(|\mathcal{N}_{ij}| + 2))d \leq 0.
\end{aligned}$$

As we can see, the distance $x_j(t) - x_i(t)$ will not increase in this case, which proves that the edge (i, j) will not break if $|\mathcal{N}_{ij}| \geq \frac{n}{2} - 2$ at time t.

Now suppose the first edge break happens right after time t for the edge $(i, j) \in E(0)$. This means $|\mathcal{N}_{ij}| < \frac{n}{2} - 2$ at time t. With the initial condition that $|\mathcal{N}_{ij}| \geq \frac{n}{2} - 2$, the number of common neighbors of $|\mathcal{N}_{ij}|$ must have decreased at some time before t. But this can never happen without an edge break, which contradicts the assumption that (i, j) is the first edge to break. Therefore, no original edge will break under the stated assumption. With a positive lower bound of the dwell time between switches among connected graphs, we guarantee the consensus. ■

Remark: The condition in Theorem 3.2 is not a necessary condition for reaching consensus. This can be shown by counterexamples. With the constraint that edges are defined by distance, most vectors $x(0) \in \mathbb{R}^n$ do not satisfy this condition. If we consider infinitely many agents uniformly distributed on an interval of length L, then $L \leq 2d$ is required in Theorem 3.2, which coincides with the $2R$ conjecture in [Blondel et al., 2007].

3.2 Weighted model

As mentioned above, Theorem 3.2 holds for a limited number of initial conditions. Instead of finding a condition on the initial states, we propose a slight modification on the model to guarantee opinion consensus for any $x(0)$ with a corresponding connected graph. Consider the following model:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} \frac{1}{|\mathcal{N}_{ij}| + 1}(x_j(t) - x_i(t)). \tag{4}$$

Remark: Note that both $\mathcal{N}_i$ and $\mathcal{N}_{ij}$ are state dependent and the right-hand side is not continuous with respect to the state, thus the existence and uniqueness of the solution can become an issue. We notice that the key to the proof of the existence and uniqueness of the solution of (2) in [Blondel et al., 2010] is to show the existence of the upper bound on the number of transitions taking place during any given time interval. For (4), we can use a similar argument after noticing the following fact:

If an edge (i, j) from a graph G_0 is broken at time T and a new graph G_1 is formed consequently, the distance between x_i and x_j will keep increasing after T. Without loss of generality, we assume $x_j(T) - x_i(T) = d$. Then

$$\begin{aligned}
&\dot{x}_j(T^+) - \dot{x}_i(T^+) \\
&= \sum_{k \in \mathcal{N}_j(T^+)} \frac{x_k(T^+) - x_j(T^+)}{|\mathcal{N}_{jk}(T^+)| + 1} - \sum_{k \in \mathcal{N}_i(T^+)} \frac{x_k(T^+) - x_i(T^+)}{|\mathcal{N}_{ik}(T^+)| + 1} \\
&= \sum_{k \in \mathcal{N}_j'(T^+)} \frac{x_k(T^+) - x_j(T^+)}{|\mathcal{N}_{jk}(T^+)| + 1} - \sum_{k \in \mathcal{N}_i'(T^+)} \frac{x_k(T^+) - x_i(T^+)}{|\mathcal{N}_{ik}(T^+)| + 1} \\
&\quad + \sum_{k \in \mathcal{N}_{ij}(T^+)} \left(\frac{x_k(T^+) - x_j(T^+)}{|\mathcal{N}_{jk}(T^+)| + 1} + \frac{x_i(T^+) - x_k(T^+)}{|\mathcal{N}_{ik}(T^+)| + 1} \right) \\
&\geq \dot{x}_j(T) - \dot{x}_i(T) + \frac{2(x_j(T) - x_i(T))}{|\mathcal{N}_{ij}(T)| + 1} > 0.
\end{aligned} \tag{5}$$

Similarly, one can prove that if an edge between two agents is created at time T, it will be maintained after T. Therefore, by using a similar argument as in [Blondel et al., 2010], we can obtain a positive lower bound on the time for a broken edge to be reconstructed or for a new edge to be broken. Since there is a limited number of edges that can be added or broken, a positive lower bound of the dwell time in each graph topology can also be derived. The existence and uniqueness of the solution of the switching system in hand can then be guaranteed.

In order to prove our main theorem, we also need the following proposition and definition.

Proposition 3.3 *Given an initial condition $x(0) \in \mathbb{R}^n$ satisfying $x_i(0) \neq x_j(0)$ for any $i \neq j$, then $x_i(T) \neq x_j(T)$ for any $i \neq j$ and any $T \in [0,\infty)$, where $x(t)$ is the solution to the differential equation (4).*

The proof of Proposition 3.3 can be found in [Yang et al., 2012] and is omitted due to the space limitation.

Definition 3.4 *For an undirected graph $G = (V,E)$, an edge $(i,j) \in E$ is called* crucial *if $\mathcal{N}_{ij} = \phi$, i.e. there does not exist $k \in V$ such that $(i,k) \in E$ and $(k,j) \in E$ simultaneously.*

Theorem 3.5 (Main Theorem) *For any initial condition $x(0) \in \mathbb{R}^n$ such that:*

1. *the corresponding graph is connected;*
2. *$x_i(0) \neq x_j(0)$ for any $i \neq j$,*

the system (4) will converge to the equilibrium $\alpha \cdot \vec{\mathbf{1}}$, where $\alpha = \frac{1}{n}\vec{1}^T x(0)$, i.e., consensus is reached.

Proof: If no crucial edge breaks during the process, the graph remains connected. Therefore, we just need to check when a crucial edge is going to break, i.e., $|x_i(t) - x_j(t)| = d$, and $\mathcal{N}_{ij} = \phi$, which implies $1/(|\mathcal{N}_{ij}|+1) = 1$. Assuming that $x_j(t) > x_i(t)$ without loss of generality, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(x_j(t) - x_i(t)) = \dot{x}_j(t) - \dot{x}_i(t) \\
&= \sum_{k\in\mathcal{N}_j} \frac{x_k(t) - x_j(t)}{|\mathcal{N}_{kj}|+1} - \sum_{k\in\mathcal{N}_i} \frac{x_k(t) - x_i(t)}{|\mathcal{N}_{ki}|+1} \\
&= \sum_{k\in\mathcal{N}_j\backslash\{i\}} \frac{1}{|\mathcal{N}_{kj}|+1}(x_k(t) - x_j(t)) \\
&\quad - \sum_{k\in\mathcal{N}_i\backslash\{j\}} \frac{1}{|\mathcal{N}_{ki}|+1}(x_k(t) - x_i(t)) - 2(x_j(t) - x_i(t)) \\
&\leq \left(\sum_{k\in\mathcal{N}_j\backslash\{i\}} \frac{1}{|\mathcal{N}_{kj}|+1} + \sum_{k\in\mathcal{N}_i\backslash\{j\}} \frac{1}{|\mathcal{N}_{ki}|+1} - 2 \right) d.
\end{aligned}$$

In order to prove $\frac{d}{dt}(x_j(t) - x_i(t)) \leq 0$, we only need to show that

$$\sum_{k\in\mathcal{N}_j\backslash\{i\}} \frac{1}{|\mathcal{N}_{kj}|+1} + \sum_{k\in\mathcal{N}_i\backslash\{j\}} \frac{1}{|\mathcal{N}_{ki}|+1} \leq 2. \tag{6}$$

We will now show that for all $k \in \mathcal{N}_j\backslash\{i\}$, we have that $|\mathcal{N}_{kj}| = |\mathcal{N}_j| - 2$, if $|\mathcal{N}_j| \geq 2$.

Since we assumed that there is no pair of agents with the same initial opinion, there will not be any pair of agents reaching the same state in finite time according to Proposition 3.3. Thus, we have $x_i(t) \neq x_j(t)$ for any $i \neq j$ and $t < \infty$. If (i,j) is a crucial edge with $x_j(t) - x_i(t) = d$, agent j has only one neighbor to its left which is agent i. Then all the other neighbors of j must be located to its right. If $|\mathcal{N}_j| \geq 2$, every j's right neighbor j' is a common neighbor of j and another right neighbor j''. This is because $|x_{j'}(t) - x_j(t)| \leq d$ and $|x_{j'}(t) - x_{j''}(t)| \leq \max\{x_{j'}(t) - x_j(t), x_{j''}(t) - x_j(t)\} \leq d$. We have $i \notin \mathcal{N}_{kj}$ according to the definition of a crucial edge. Therefore, we have $\mathcal{N}_{kj} = \mathcal{N}_j\backslash\{i,k\}$, which implies $|\mathcal{N}_{kj}| = |\mathcal{N}_j| - 2$ for $k \in \mathcal{N}_j\backslash\{i\}$. Equivalently one can get $|\mathcal{N}_{ki}| = |\mathcal{N}_i| - 2$ for $k \in \mathcal{N}_i\backslash\{j\}$ if $|\mathcal{N}_i| \geq 2$. By plugging these results into the left-hand side of inequality (6), we get

$$\begin{aligned}
&\sum_{k\in\mathcal{N}_j\backslash\{i\}} \frac{1}{|\mathcal{N}_{kj}|+1} + \sum_{k\in\mathcal{N}_i\backslash\{j\}} \frac{1}{|\mathcal{N}_{ki}|+1} \\
&= \sum_{k\in\mathcal{N}_j\backslash\{i\}} \frac{1}{|\mathcal{N}_j|-1} + \sum_{k\in\mathcal{N}_i\backslash\{j\}} \frac{1}{|\mathcal{N}_i|-1} \\
&= \frac{|\mathcal{N}_j|-1}{|\mathcal{N}_j|-1} + \frac{|\mathcal{N}_i|-1}{|\mathcal{N}_i|-1} = 2.
\end{aligned}$$

Note that if $|\mathcal{N}_j| = 1$, which means $\mathcal{N}_j = \{i\}$ and $\mathcal{N}_j\backslash\{i\} = \phi$, then (6) is also true since the first term on the left-hand side is equal to 0. ■

Remark: In the model (4), the assignment of weights may not be realistic in many cases. However, in some situations, putting more weight to the common friends, i.e., following the majority, may not be a good strategy, as mentioned in the book *The Wisdom of Crowds* by James Surowiecki [Surowiecki, 2005]. Here instead of modeling the reality, our aim is to find a protocol that will guarantee opinion consensus if the graph is initially connected.

In Theorem 3.5, it is required that no two agents have the exact same initial opinion. Although this is a set of measure zero in the state space, it is a more common scenario in reality since the opinions are usually not in a continuum space. To accommodate this scenario and at the same time avoid some numerical difficulties encountered when the agents are very close to each other, one can treat all the agents with the same state as one agent. This is equivalent to saying that we still consider agents with the same state separately but with a weight inversely proportional to the number of agents sharing the same opinion. If M_j is defined as the number of agents which have the same opinion as agent j, then we regard j as only $1/M_j$ agent. By applying this, we obtain the second weighted model:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} \frac{1}{(|\tilde{\mathcal{N}}_{ij}|+1)M_j}(x_j(t) - x_i(t)), \tag{7}$$

where $|\tilde{\mathcal{N}}_{ij}|$ is the number of common opinion clusters between i and j. Here two agents belong to the same cluster if and only if they have the same opinion. $|\tilde{\mathcal{N}}_{ij}|$ can be computed by $|\tilde{\mathcal{N}}_{ij}| = \sum_{k\in\mathcal{N}_{ij}\cup\{i,j\}} 1/M_k - 2$. We can then prove the following corollary:

Corollary 3.6 *For any initial condition $x(0) \in \mathbb{R}^n$ whose corresponding graph is connected, the control protocol (7) will guarantee consensus.*

However, on the right-hand side of (7) the weight is not symmetric. So in Corollary 3.6 the equilibrium is actually not the initial average. This is due to the fact that we ignore the weight of those agents who have the same opinion.

4 Smoothed model

Another issue for the original H-K model (2) is the discontinuous right-hand side. In the theory of differential equations, the condition of Lipschitz continuity is essential for the existence and uniqueness of the solution. As stated in [Blondel et al., 2010], the convergence of the solution to (2) is guaranteed for almost all initial conditions, which implies there can exist a set (with measure zero) of singular points. Moreover, some numerical problems may arise from this discontinuity when one wants to implement the model. For example, a small error may change the connectivity of the whole graph when the distance between a pair of nodes is around the threshold d in the H-K model. A common remedy for these problems is to approximate the original function by a continuous (even differentiable in some cases) function. The approximation has the same value as the original function except around the points where the discontinuity occurs. In these areas, a smoothing function is used to replace the original function, e.g., [Saber and Murray, 2003]. We rewrite the original model as:

$$\dot{x}_i(t) = \sum_{j\neq i} \rho_{ij}(x)(x_j(t) - x_i(t)), \tag{8}$$

where ρ_{ij} is a 0-1 function depending on the distance between i and j:

$$\rho_{ij}(x) = \begin{cases} 1, & |x_i(t) - x_j(t)| \leq d, \\ 0, & |x_i(t) - x_j(t)| > d. \end{cases}$$

We can modify $\rho_{ij}(x)$ in the following way: we denote by $\beta_{ij} = (x_i - x_j)^2$ the square distance between agent i and agent j and introduce a potential function between i and j as:

$$r(\beta_{ij}) = \begin{cases} \beta_{ij}, & 0 \leq \beta_{ij} \leq d^2, \\ \varphi(\beta_{ij}), & d^2 < \beta_{ij} \leq (d+\varepsilon)^2, \\ c, & (d+\varepsilon)^2 < \beta_{ij} < \infty. \end{cases} \tag{9}$$

where c is a positive constant and $\varphi(\beta)$ is a chosen monotonically increasing function on the interval $[(d^2, (d+\varepsilon)^2]$ to make $r(\beta_{ij})$ differentiable for any $\beta_{ij} \in (0, \infty)$ (e.g., high order polynomials). Define

$$\begin{aligned} \rho_{ij}(x) &= \frac{\partial r(\beta_{ij})}{\partial \beta_{ij}} \\ &= \begin{cases} 1, & 0 \leq \beta_{ij} \leq d^2, \\ \varphi'(\beta_{ij}), & d^2 < \beta_{ij} \leq (d+\varepsilon)^2, \\ 0, & (d+\varepsilon)^2 < \beta_{ij} < \infty. \end{cases} \end{aligned} \tag{10}$$

If we consider the model (8) with the choice of $\rho_{ij}(x)$ in (10), then the right-hand side is a Lipschitz continuous function, which will ensure the existence and the uniqueness of the solution to the differential equation. In [Ceragioli and Frasca, 2012], this smoothed H-K model is also called continuous H-K model. The convergence of the solution is also analyzed in [Ceragioli and Frasca, 2012] by using the LaSalle's invariance principle. This result dose not necessarily imply consensus, since the largest invariant set that can be described as $\{x | \rho_{ij}(x)(x_i - x_j) = 0, \forall i, j\}$ contains more than one point.

Similarly to section 3.1, there is a sufficient condition for the initial states to guarantee consensus by using the control protocol (8). However, it is not always true that the pairwise force $\rho_{ij}(x)(x_j - x_i)$ reaches its maximum absolute value when $|x_j - x_i| = d$. So here we need to add some constraints for the function φ to get the next theorem. We want $\beta_{ij} = d^2$ to be the maximum point of the interval $[d^2, (d+\varepsilon)^2]$ for $|\rho_{ij}(x)(x_j - x_i)|$, which implies

$$0 \leq \varphi'(\beta) \leq \frac{d}{\sqrt{\beta}}, \text{ for all } d^2 \leq \beta \leq (d+\varepsilon)^2. \tag{11}$$

Although in general people can define an edge between two nodes when the weight between them is nonzero, we here keep the previous definition of the edge set, which is $E = \{(i,j) | |x_i - x_j| \leq d\}$. The smoothed version of Theorem 3.2 is given as follows.

Theorem 4.1 *For an initial condition $x(0) \in \mathbb{R}^n$, if the corresponding graph $G(0) = (V, E(0))$ is connected and for any pair $(i,j) \in E(0)$, $|\mathcal{N}_{ij}| \geq \frac{n}{2} - 2$, then with the chosen $\varphi(\beta)$ satisfying (11), the solution to (8) will converge to $\alpha \cdot \vec{1}$, where $\alpha = \frac{1}{n}\vec{1}^T x(0)$.*

Proof: Similar to the proof of Theorem 3.2, we prove the theorem by showing that no edge will be broken during the time evolution. Due to the page limitation, the proof is omitted. ∎

5 High dimensional cases

In the general m-dimensional Euclidean space $\mathbb{R}^m$, one can define the distance by the 2-norm and then extend the H-K model to the following form

$$\dot{x}_i(t) = \sum_{j:\|x_j - x_i\| \leq d} (x_j(t) - x_i(t)), \qquad (12)$$

where $x_i(t) \in \mathbb{R}^m$, for $i = 1, \cdots, n$. We define again the edges of the graph by the distance, i.e., $(i, j) \in E$ if $\|x_i - x_j\| \leq d$.

5.1 Non-smoothed model

Since there are not many papers in the literature discussing high dimensional H-K models, the properties of the solution to the differential equation (12) is not well-studied. In [Como and Fagnani, 2011,Canuto et al., 2012], the authors study the continuum model with both discrete-time and continuous-time setting. The convergence of the discrete-time model is also studied in [Nedic and Touri, 2012]. For the continuous-time case, the existence and uniqueness of the solution cannot be guaranteed due to the discontinuity on the right-hand side. Nevertheless, a sufficient condition for consensus can be obtained for any finite dimensional spaces, if the solution to the differential equation exists. The following is an extension of Theorem 3.2 to higher dimensions.

Corollary 5.1 *For an initial condition $x_i(0) \in \mathbb{R}^m$ and the corresponding graph $G(0) = (V, E(0))$, if $G(0)$ is connected and for any pair $(i, j) \in E$, it holds that $|\mathcal{N}_{ij}| \geq \frac{n}{2} - 2$, then the solution to (12), $x(t)$, will converge to $\alpha \otimes \vec{1}$, where $\alpha = \frac{1}{n}\sum_{i=1}^{n} x_i(0)$.*

5.2 Smoothed models

The model (8) with the choice of ρ_{ij} in (10) is still valid if we redefine the parameter β_{ij} by $\beta_{ij} = \|x_i - x_j\|^2$.

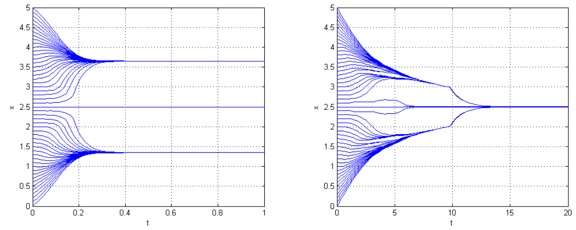
Corollary 5.2 *For any initial condition $x_i(0) \in \mathbb{R}^m$, there exists a vector $x^* \in \mathbb{R}^{mn}$ such that the solution $x(t)$ to the differential equation (8) with the choice of ρ_{ij} in (10) will converge to x^* as $t \to \infty$, while $\beta_{ij} = \|x_i - x_j\|^2$.*

Corollary 5.2 is a direct extension to higher dimensions of the convergence result of continuous H-K model. The proof is similar to that in [Ceragioli and Frasca, 2012]. By choosing the Lyapunov function $V = \sum_{i=1}^{n} \|x_i\|^2$ and using LaSalle's invariance principle, we can show the convergence.

6 Simulations

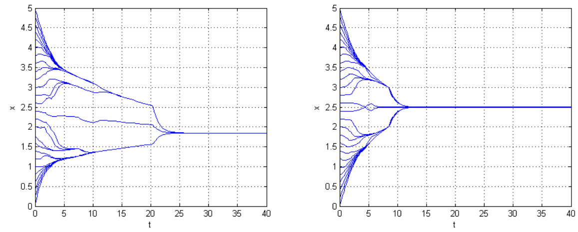
We will present some simulation results of the weighted one dimensional non-smooth models and show one example in the two dimensional space in this section.

In the first example, 51 agent opinions are initially uniformly spaced on an interval of length 5. The interaction radius d is chosen to be 0.98 to avoid singularities from the discontinuous right-hand side in the non-smooth models. We use both



(a) (b)

Fig. 1. Time evolution of 51 agent opinions according to (case (a)) model (2) and (case (b)) model (4). The initial opinions are uniformly spaced on an interval of length 5. The interaction radius d is chosen to be 0.98.



(a) (b)

Fig. 2. Time evolution of 46 agent opinions according to (case (a)) model (4) and (case (b)) model (7). The initial opinions of 26 agents are uniformly spaced on an interval of length 5, while the remaining 20 agents are all initially positioned at 1. The interaction radius d is chosen to be 0.98.

the original H-K model (2) and the weighted model (4). Fig. 1 shows the simulation result. The original H-K model diverges to three clusters (in (a)) and the modified model (4) reaches consensus (in (b)).

In the second example, we want to show how the coincidence of initial opinions affects the simulation result by using the two modified models: (4) and (7). 26 agent opinions are uniformly spread on the interval of length 5, while the other 20 opinions are all located at position 1 initially. d is chosen to be 0.98 again. The initial opinion average α is approximately 1.85. Although the initial distribution does not fulfill the condition in Theorem 3.5, the system does converge to the initial opinion average by using the control protocol (4) (in Fig. 2(a)). If only the opinion cluster is considered, these 20 agent opinions are ignored since there is also one agent opinion positioned at 1 among the first 26 agent opinions. So using model (7), we get a symmetric result in Fig. 2(b), and the compromised opinion is 2 in the end.

7 Conclusions and future work

In this paper, we first gave a sufficient condition for opinions consensus for the original Hegselmann-Krause model, which also holds in any finite dimensional cases. The condition is also valid for smoothed Hegselmann-Krause models if the smooth function satisfies a certain constraint. Furthermore, we provided two modified versions of the Hegselmann-

Krause model such that consensus is guaranteed for any initial configuration corresponding to a connected graph. The essence of the protocol is that one weighs most on the opinion of his "friend" that is not shared by any other friend, while the standard H-K model treats all friends equally. The fact that our protocol guarantees consensus to the average of the initial values, provided the initial graph is connected, gives foundation to many potential distributed applications in which the goal is to reach just such a consensus, for example, distributed estimation using sensor networks.

Future work will focus on the case of higher dimensional spaces, and in particular the two dimensional space. Some of the results in one dimension can be easily extended to any finite dimensional space as we showed in the paper, e.g., Theorem 3.2. But due to the lack of knowledge about properties of solutions to non-smoothed Hegselmann-Krause model in higher dimension, we can hardly draw any further conclusions. In particular, Theorem 3.5 is not extendable to the higher dimensional case in a straightforward manner because of the line structure is used in the proof. So the extension to higher dimensions requires more effort in this case. Moreover, even in one dimensional case, how to extend Theorem 3.5 to a continuous setting is also an open problem. One approach can be providing a continuous definition of the number of neighbors and common neighbors while the consensus can be guaranteed at the same time.

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