Novel Event-Triggered Strategies for Model Predictive Controllers

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Abstract— **This paper proposes novel event-triggered strategies for the control of uncertain nonlinear systems with additive disturbances under robust Nonlinear Model Predictive Controllers (NMPC). The main idea behind the event-driven framework is to trigger the solution of the optimal control problem of the NMPC, only when it is needed. The updates of the control law depend on the error of the actual and the predicted trajectory of the system. Sufficient conditions for triggering are provided for both, continuous and discrete-time nonlinear systems. The closed-loop system evolves to a compact set where it is ultimately bounded, under the proposed framework. The results are illustrated through a simulated example.**

I. Introduction

The periodic implementation of control tasks is the most common approach for feedback control systems. However, this might be a conservative choice, since the constant sampling period has to guarantee stability in the worst-case scenario. It is apparent that a reduction on the number of the control updates is desirable because it can lead to the alleviation of the energy consumption, or in the case of networks, it can result to amelioration of the network traffic. In recent years the framework of event-driven feedback and sampling has been developed. This results to a more flexible aperiodic sampling, while preserving necessary properties of the system, such as stability and convergence. Related works can be found in [1], [8], [16], [18].

Motivated by the fact that NMPC are widely used control strategies, with conspicuous advantages such as the capability to deal with nonlinearities and constraints, in this paper an event-based framework for this kind of controllers, is investigated. In addition, most NMPC control schemes are computationally demanding, so it would be of great interest if the control law would not be updated at each sampling instant but rather, the already computed control trajectory, would be implemented to the plant until an event occurs. This approach, could be useful in cases, where the computation of the optimal control law is demanding, as in large-scale systems, opposed to the computation of the predicted trajectory. This is for example the case in [17], where an event-based NMPC approach for nonlinear continuous-time systems with nominal dynamics, is presented. The approach is used in order to overcome the bounded delays and information losses that often appear in networked control systems. Although the formulation is event-driven, a criterion for triggering was not provided.

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The contribution of this paper relies in finding sufficient conditions for triggering, in the case of uncertain nonlinear systems with additive disturbances, under robust NMPC strategies. The main assumption for the general event-triggered policies, is the ISS stability of the plant as it can be seen in [2], for discrete-time systems and in [16], for continuous. There has been a lot of research on ISS properties of MPC for discrete-time systems. For linear systems the reader is referred to [6], [10]. More recent results for the ISS properties of nonlinear MPC can be found in [4], [11], and [13]. In [12], the authors presented a robust NMPC controller for constrained discrete-time systems. They also proved that the closed-loop system was ISS, with respect to the uncertainties.The framework proposed in [12], is our starting point here. Although most researchers have focused on the discrete-time frame, the ISS stability of a robust NMPC in continuous-time sampled-data systems was recently presented in [14].

In this work, the triggering condition of a continuous-time system under a robust NMPC control law is given, while a convergence analysis of an uncertain nonlinear system is also provided. We note that the discrete-counterpart will be presented in [3], and is outlined here for the sake of coherence. Although the event-based setup for MPC controllers is quite new, some results have already been presented in [9], [7] and [15].

The remainder of the paper is organized as follows. In Section II, the problem statement for the continuous-time case is presented. Sufficient conditions for triggering of an uncertain continuous-time system under NMPC are provided in Section III. The discrete counterpart of the above framework is reviewed in Section IV, and in Section V some simulation results are presented. Section VI summarizes the results of this paper and indicates further research endeavors.

II. Problem Statement For Continuous-Time Systems

In the following a triggering condition for continuous-time nonlinear systems under NMPC control laws is going to be presented. following the idea behind the analysis proposed in [12] for discrete-time systems, appropriately modified in this case, for continuous-time systems.

Consider a nonlinear continuous time system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \tag{1}$$

$$x(t) \in \mathcal{X} \subset \mathbb{R}^n, \qquad u(t) \in \mathcal{U} \subset \mathbb{R}^m \tag{2}$$

We also assume that $f(x,u)$ is locally Lipschitz in x, with Lipschitz constant L_f and that $f(0,0) = 0$. The whole state $x(t)$, is assumed to be available. Sets $\mathcal{X}$, $\mathcal{U}$ are assumed to be compact and connected, respectively, and $(0,0) \in \mathcal{X} \times \mathcal{U}$.

In a realistic formulation though, modeling errors, uncertainties and disturbances may exist. Thus, a perturbed version of (1) is going to be considered as well. The perturbed system can be described as

$$\dot{x}(t) = f(x(t), u(t)) + w(t), \qquad x(0) = x_0 \tag{3}$$

where the additive term $w(t) \in \mathcal{W} \subset \mathbb{R}^n$ is the disturbance at time $t \in \mathbb{R}_{\geq 0}$ and $\mathcal{W}$ is a compact set containing the origin as an interior point. Furthermore, note that $w(t)$ is bounded because it is defined in a compact set $w(t) \in \mathcal{W}$. Thus, there exists $\gamma^{\text{sup}} \in \mathbb{R}_{\geq 0}$ such that $\sup_{t \geq 0} ||w(t)|| \leq \gamma^{\text{sup}}$.

Given the system (1), the predicted state is denoted as $\hat{x}(t_i+\tau, u(\cdot), x(t_i))$. This notation will be equipped hereafter and it accounts for the predicted state at time $t_i + \tau$ with $\tau \geq 0$, based on the measurement of the real state at time t_i while using a control trajectory $u(\cdot; x(t_i))$ for time period t_i until $t_i + \tau$. It holds that $\hat{x}(t_i, u(\cdot), x(t_i)) \equiv x(t_i)$, i.e. the measured state at time t_i.

A. NMPC for Continuous-Time Systems

The main idea behind NMPC is to solve on-line a finite-horizon, open-loop optimal control problem, based on the measurement provided by the plant. At the recalculation time t_i, the actual state of the plant $x(t_i)$, is measured and the following Optimal Control Problem (OCP), is solved:

$$\min_{\tilde{u}(\cdot)} J(\tilde{u}(\cdot), x(t_i)) =$$

$$\min_{\tilde{u}(\cdot)} \int_{t_i}^{t_i+T_p} F(\tilde{x}(\tau), \tilde{u}(\tau)) \, \mathrm{d}\,\tau + E(\tilde{x}(t_i + T_p)), \tag{4a}$$

s.t.

$$\dot{\tilde{x}} = f(\tilde{x}(t), \tilde{u}(t)), \qquad \tilde{x}(t_i) = x(t_i), \tag{4b}$$

$$\tilde{u}(t) \in \mathcal{U}, \tag{4c}$$

$$\tilde{x}(t) \in \mathcal{X}_{t-t_i} \qquad t \in [t_i, t_i + T_p], \tag{4d}$$

$$\tilde{x}(t_i + T_p) \in \mathscr{E}_f, \tag{4e}$$

where $\tilde{\cdot}$ denotes the controller internal variables, corresponding to the nominal dynamics of the system. F and E are the running and terminal costs functions, respectively, with $E \in C^1$, $E(0) = 0$. The terminal constraint set $\mathscr{E}_f \subset \mathbb{R}^n$ is assumed to be closed and connected.

Assume, also, that the cost function F is quadratic of the form $F(x,u) = x^T Q x + u^T R u$, with Q and R being positive definite matrices. Moreover we have $F(0,0) = 0$ and $F(x,u) \geq \lambda_{\min}(Q)||x||^2$, with $\lambda_{\min}(Q)$ being the smallest eigenvalue of Q. Since $\mathcal{X}$ and $\mathcal{U}$ are bounded, the stage cost is Lipschitz continuous in $\mathcal{X} \times \mathcal{U}$, with a Lipschitz constant L_F.

The state constraint set $\mathcal{X}$ of the standard MPC formulation, is being replaced by a restricted constraint set $\mathcal{X}_{t-t_i}$ in (4d). This state constraints' tightening for the nominal system with additive disturbance is a key ingredient of the robust NMPC controller and guarantees that the evolution of the real system will be admissible for all time.

Notice that the difference between the actual measurement at time $t_i + \tau$ and the predicted state at the same time under some control law $u(t_i + \tau, x(t_i))$, with $0 \leq \tau \leq T_p$, starting at the same initial state $x(t_i)$, can be shown [5] to be upper bounded by

$$||x(t_i + \tau) - \hat{x}(t_i + \tau, u(\cdot), x(t_i))|| \leq \frac{\gamma^{\text{sup}}}{L_f}(e^{L_f \cdot \tau} - 1) \tag{5}$$

Set $\gamma(t) \triangleq \frac{\gamma^{\text{sup}}}{L_f}(e^{L_f \cdot t} - 1) \qquad \forall t \in \mathbb{R}_{\geq 0}$.

The restricted constrained set is then defined as $\mathcal{X}_{t-t_i} = \mathcal{X} \sim \mathscr{B}_{t-t_i}$ where $\mathscr{B}_{t-t_i} = \{x \in \mathbb{R}^n : ||x|| \leq \gamma(t - t_i)\}$, with $t \in [t_i, t_i + T_p]$. The set operator "$\sim$" denotes the Pontryagin difference.

The solution of the OCP at time t_i provides an optimal control trajectory $u^*(t; x(t_i))$, for $t \in [t_i, t_i + T_p]$, where T_p represents the finite prediction horizon. A portion of the optimal control that corresponds to the time interval $[t_i, t_i + \delta_i)$, is then applied to the plant, i.e.,

$$u(t) = u^*(t; x(t_i)), \qquad t \in [t_i, t_i + \delta_i) \tag{6}$$

where δ_i represents the recalculation period that may not be equidistant for every t_i, $\delta_i = \delta(t_i) = t_{i+1} - t_i$. A time instant $t_i \in \mathbb{R}_{\geq 0}$ must be a proper recalculation time, in the sense defined in [17], i.e. a time instant $t_i \in \mathbb{R}_{\geq 0}$ is a proper recalculation time if there exists $\beta \in \mathbb{R}_{\geq 0}$, such that, $0 < \beta \leq t_{i+1} - t_i = \delta_i < T_p, \qquad \forall t_i, t_{i+1} \in \mathbb{R}_{\geq 0}$.

In order to assert that the NMPC strategy results in a robustly stabilizing controller, some stability conditions are stated for the nominal system. Thus, system (1) is supposed to fulfill the following assumption.

Assumption 1.

i) Let the terminal region $\mathscr{E}_f$ from (4e) be a subset of an admissible positively invariant set $\mathscr{E}$ of the nominal system, where $\mathscr{E} \subset \mathcal{X}$ is closed, connected and containing the origin.

ii) Assume that there is a local stabilizing controller $h(x(t))$ for the terminal set $\mathscr{E}_f$. The associated Lyapunov function $E(\cdot)$ has the following properties

$$\frac{\partial E}{\partial x} f(x(\tau), h(x(\tau))) + F(x(\tau), h(x(\tau))) \leq 0 \qquad \forall x \in \mathscr{E}$$

and is Lipschitz in $\mathscr{E}$, with Lipschitz constant L_E.

iii) The set $\mathscr{E}$ is given by $\mathscr{E} = \{x \in \mathbb{R}^n : E(x) \leq \alpha_{\mathscr{E}}\}$ such that $\mathscr{E} \subseteq \mathcal{X} = \{x \in \mathcal{X}_{T_p} : h(x) \in \mathcal{U}\}$. The set $\mathscr{E}_f = \{x \in \mathbb{R}^n : E(x) \leq \alpha_{\mathscr{E}_f}\}$ is such that for all $x \in \mathscr{E}$, $f(x, h(x)) \in \mathscr{E}_f$. Assume also that $\alpha_{\mathscr{E}}, \alpha_{\mathscr{E}_f} \in \mathbb{R}_{\geq 0}$ and is such that $\alpha_{\mathscr{E}} \geq \alpha_{\mathscr{E}_f}$.

iv) $\exists\, T_p$, such that $0 < \beta \leq \delta(t) < T_p$, for some $\beta \in \mathbb{R}_{\geq 0}$.

Note that i)-iii) are standard assumptions for a NMPC system, see for example [14]. Assumption iv), can be verified either experimentally or theoretically for specific systems and it states that every recalculation time is a proper recalculation time.

The event-triggered strategy presented later in this paper, is used in order to enlarge, as much as possible, the inter-calculation period δ_i for the actual system (3). The enlargement of the inter-calculation period results in the overall reduction of the control updates which is desirable in numerous occasions, as for example energy consumption reasons. In an event-based framework the inter-calculation period is not equidistant but is "decided" *ex tempore*, based on the error between the actual state measurement of (3), and the state trajectory of the nominal system, (1). The triggering condition, i.e. how the next calculation time t_{i+1}, is chosen, is presented next.

III. Triggering Condition for the NMPC of Continuous-Time Systems

In this section, the feasibility and the convergence of the closed loop system (3), (6) are provided first. Then, the event-triggering rule for sampling is reached.

A. Feasibility and Convergence

As usual in model predictive control, the proof of stability consists in two separate parts; the feasibility property is guaranteed first and then, based on the previous result, the convergence property is shown. Due to the fact that the system in consideration is perturbed, we only require "ultimate boundedness" results.

The first part will establish that initial feasibility implies feasibility afterwards. Consider two successive triggering events t_i and t_{i+1} and a feasible control trajectory $\bar{u}(\cdot, x(t_{i+1}))$, based on the solution of the OCP in t_i, $u^*(\cdot, x(t_i))$

$$\bar{u}(\tau, x(t_{i+1})) = \\ = \begin{cases} u^*(\tau, x(t_i)) & \forall \tau \in [t_{i+1}, t_i + T_p] \\ h(\hat{x}(t_i + T_p, u^*(\cdot), x(t_i))) & \forall \tau \in [t_i + T_p, t_{i+1} + T_p] \end{cases} \tag{7}$$

From feasibility of $u^*(\cdot, x(t_i))$ it follows that there is $\bar{u}(\tau, x(t_{i+1})) \in \mathcal{U}$, and similar to the procedure in [12] $\hat{x}(t_{i+1} + T_p, \bar{u}(\tau, x(t_{i+1})), x(t_{i+1})) \in \mathscr{E}_f$ provided that the uncertainties are bounded by $\gamma^{\sup} \leq \frac{(\alpha_{\mathscr{E}} - \alpha_{\mathscr{E}_f}) \cdot L_f}{L_E \cdot (e^{L_f \cdot T_p} - 1)}$. Finally, the state constraints must be fulfilled. According to [12] and [14] and considering that $||x(t) - \hat{x}(t, u(\cdot), x(t_i))|| \leq \gamma(t)$, for all $t \geq t_i$, it is verified that since the $\hat{x}(t, u^*(\cdot), x(t_i)) \in \mathcal{X}_{t-t_i}$, then $\hat{x}(t, \bar{u}(\cdot), x(t_{i+1})) \in \mathcal{X}_{t-t_{i+1}}$.

The second part involves proving convergence of the state and is being introduced now. In order to prove stability of the closed-loop system, it must be shown that a proper value function is decreasing starting from a sampling instant t_i. Consider the optimal cost $J^*(u^*(\cdot; x(t_i)), x(t_i)) \triangleq J^*(t_i)$ from (4a) as a Lyapunov function candidate. Then, consider the cost of the feasible trajectory, indicated by $\bar{J}(\bar{u}(\cdot; x(t_{i+1})), x(t_{i+1})) \triangleq \bar{J}(t_{i+1})$, where t_i, t_{i+1} are two successive triggering instants. Also, $\bar{x}(\tau, \bar{u}(\tau; x(t_{i+1})), x(t_{i+1}))$ is introduced, and it accounts for the predicted state at time τ, with $\tau \geq t_{i+1}$, based on the measurement of the real state at time t_{i+1}, while using the control trajectory $\bar{u}(\tau; x(t_{i+1}))$ from (7).

Set $x_1(\tau) = \bar{x}(\tau, \bar{u}(\tau; x(t_{i+1})), x(t_{i+1}))$, $u_1(\tau) = \bar{u}(\tau; x(t_{i+1}))$, $x_2(\tau) = \hat{x}(\tau, u^*(\tau; x(t_i)), x(t_i))$ and $u_2(\tau) = u^*(\tau; x(t_i))$.

The difference between the optimal cost and the feasible cost is

$$\begin{aligned} &\bar{J}(t_{i+1}) - J^*(t_i) = \\ &\int_{t_{i+1}}^{t_{i+1}+T_p} F(x_1(\tau), u_1(\tau)) \, \mathrm{d}\tau + E(x_1(t_{i+1} + T_p)) \\ &- \int_{t_i}^{t_i+T_p} F(x_2(\tau), u_2(\tau)) \, \mathrm{d}\tau - E(x_2(t_i + T_p)) \\ &= \int_{t_{i+1}}^{t_i+T_p} F(x_1(\tau), u_1(\tau)) \, \mathrm{d}\tau + E(x_1(t_{i+1} + T_p)) \\ &+ \int_{t_i+T_p}^{t_{i+1}+T_p} F(x_1(\tau), u_1(\tau)) \, \mathrm{d}\tau \\ &- \int_{t_i}^{t_{i+1}} F(x_2(\tau), u_2(\tau)) \, \mathrm{d}\tau \\ &- \int_{t_{i+1}}^{t_i+T_p} F(x_2(\tau), u_2(\tau)) \, \mathrm{d}\tau - E(x_2(t_i + T_p)) \end{aligned} \tag{8}$$

From (7), we have that $u_1(t) \equiv u_2(t) \equiv \bar{u}(t)$ for $t \in [t_{i+1}, t_i + T_p]$, so imposing this control law to the system (1), it yields

$$\begin{aligned} &||x_1(t) - x_2(t)|| = ||x(t_{i+1}) + \int_{t_{i+1}}^{t} f(\bar{x}(\tau), \bar{u}(\tau)) \, \mathrm{d}\tau \\ &- x(t_i) - \int_{t_i}^{t_{i+1}} f(\hat{x}(\tau), u^*(\tau)) \, \mathrm{d}\tau \\ &- \int_{t_{i+1}}^{t} f(\hat{x}(\tau), \bar{u}(\tau)) \, \mathrm{d}\tau|| \end{aligned} \tag{9}$$

Note that for the nominal system (1), it holds that

$$\hat{x}(t_{i+1}, u^*(\cdot), x(t_i)) = x(t_i) + \int_{t_i}^{t_{i+1}} f(\hat{x}(\tau), u^*(\tau)) \, \mathrm{d}\tau$$

Also, we have

$$\begin{aligned} &|| \int_{t_{i+1}}^{t} f(\bar{x}(\tau), \bar{u}(\tau)) \, \mathrm{d}\tau - \int_{t_{i+1}}^{t} f(\hat{x}(\tau), \bar{u}(\tau)) \, \mathrm{d}\tau|| \\ &\leq \gamma(t - t_{i+1}) \qquad \forall t \geq t_{i+1} \end{aligned} \tag{10}$$

Define the error $e(t, x(t_i))$ as the difference between the actual state measurement at time $t \geq t_i$ and the predicted state measurement at the same time, i.e.,

$$e(t, x(t_i)) = ||x(t) - \hat{x}(t, u^*(\cdot), x(t_i))|| \tag{11}$$

Obviously we have $e(t_i, x(t_i)) = 0$.

Then, (9) with the help of (10), (11) and $t \in [t_{i+1}, t_i + T_p]$ is

$$||x_1(t) - x_2(t)|| \leq e(t_{i+1}, x(t_i)) + \gamma(t - t_{i+1}) \tag{12}$$

The difference between the running costs, with the help of (12), is

$$\begin{aligned}
&\int_{t_{i+1}}^{t_i+T_p} F(x_1(\tau),u_1(\tau))\ \mathrm{d}\,\tau - \int_{t_{i+1}}^{t_i+T_p} F(x_2(\tau),u_2(\tau))\ \mathrm{d}\,\tau \\
&\le \int_{t_{i+1}}^{t_i+T_p} ||F(x_1(\tau),\bar{u}(\cdot)) - F(x_2(\tau),\bar{u}(\cdot))||\ \mathrm{d}\,\tau \\
&\le L_F \int_{t_{i+1}}^{t_i+T_p} ||x_1(\tau)-x_2(\tau)||\ \mathrm{d}\,\tau \\
&\le L_F \cdot e(t_{i+1},x(t_i)) \cdot (t_i+T_p-t_{i+1}) + L_F \cdot \mu(t_{i+1})
\end{aligned} \tag{13}$$

Where $\mu(t) \triangleq \frac{\gamma^{\sup}}{L_f}[\frac{1}{L_f}(e^{L_f\cdot(t_i+T_p)} - e^{L_f\cdot(t)}) - (t_i+T_p-t)]$.

Integrating the inequality from Assumption 1ii) for $t \in [t_i+T_p, t_{i+1}+T_p]$, the following result can be obtained

$$\begin{aligned}
&\int_{t_i+T_p}^{t_{i+1}+T_p} F(x_1(\tau),u_1(\tau))\ \mathrm{d}\,\tau + E(x_1(t_{i+1}+T_p)) \\
&- E(x_2(t_i+T_p)) - E(x_1(t_i+T_p)) + E(x_1(t_i+T_p)) \\
&\le E(x_1(t_i+T_p)) - E(x_2(t_i+T_p)) \\
&\le L_E||x_1(t_i+T_p) - x_2(t_i+T_p)|| \\
&\le L_E \cdot e(t_{i+1},x(t_i)) + L_E \cdot \gamma(t_i+T_p-t_{i+1})
\end{aligned} \tag{14}$$

Relying on the fact that function F is positive definite, it can be concluded that

$$\int_{t_i}^{t_{i+1}} F(x_2(\tau),u_2(\tau))\ \mathrm{d}\,\tau \ge \lambda_{\min}(Q)\cdot L_Q(t_{i+1}) \ge 0 \tag{15}$$

with $L_Q(t) \triangleq \lambda_{\min}(Q)\cdot\int_{t_i}^{t} ||\hat{x}(\tau,u^*(\tau;x(t_i)),x(t_i))||^2\ \mathrm{d}\,\tau$ for $t \ge t_i$.

Substituting (13), (14), (15) to (8), the following is derived

$$\begin{aligned}
&\bar{J}(t_{i+1}) - J^*(t_i) \\
&\le (L_F(t_i+T_p-t_{i+1}) + L_E)\cdot e(t_{i+1},x(t_i)) \\
&+ L_F\cdot\mu(t_{i+1}) + L_E\cdot(t_i+T_p-t_{i+1}) - L_Q(t_{i+1})
\end{aligned} \tag{16}$$

The optimality of the solution results to

$$J^*(t_{i+1}) - J^*(t_i) \le \bar{J}(t_{i+1}) - J^*(t_i) \tag{17}$$

Thus, it holds that the optimal cost $J^*(\cdot)$ is a Lyapunov function that has been proven to be decreasing, thus the closed-loop system converges to a compact set $\mathscr{E}_f$, where it is ultimately bounded.

B. Triggering Condition

In the following, the triggering condition will be provided. Consider that at time t_i an event is triggered. In order to achieve the desired convergence property, the Lyapunov function $J^*(\cdot)$ must be decreasing. For some triggering instant t_i and some time t, with $t \in [t_i, t_i+T_p]$, we have

$$\begin{aligned}
&J^*(t) - J^*(t_i) \\
&\le (L_F(t_i+T_p-t) + L_E)\cdot e(t,x(t_i)) \\
&+ L_F\cdot\mu(t) + L_E\cdot(t_i+T_p-t) - L_Q(t)
\end{aligned} \tag{18}$$

where $e(t,x(t_i))$ as in (11), and $x(t)$ is the state of the actual system, continuously measured.

Suppose that the error is restricted to satisfy

$$\begin{aligned}
&(L_F(t_i+T_p-t) + L_E)\cdot e(t,x(t_i)) \\
&+ L_F\cdot\mu(t) + L_E\cdot(t_i+T_p-t) \le \sigma L_Q(t)
\end{aligned} \tag{19}$$

with $0 < \sigma < 1$. Plugging in (19) to (18) we get

$$J^*(t) - J^*(t_i) \le (\sigma-1)\cdot L_Q(t) \tag{20}$$

This suggests that provided $\sigma < 1$, the convergence property is still guaranteed.

This triggering rule states that when (19) is violated, the next event is triggered at time t_{i+1}, i.e., the OCP is solved again using the current measure of the state $x(t_{i+1})$ as the initial state. During the inter-event interval, the control trajectory $u(t) = u^*(t, x(t_i))$ with $t \in [t_i, t_{i+1}]$, is applied to the plant.

We are now ready to introduce the main stability result for the event-based NMPC controller.

***Theorem** 1:* Consider the system (3), subject to (2) under an NMPC strategy and assume that Assumption 1 holds. Then the NMPC control law provided by (4a)-(4e) is applied to the plant in an open-loop manner, until the rule (19) is violated and a new event is triggered. The overall event-based NMPC control scheme drives the closed loop system towards a compact set $\mathscr{E}_f$ where it is ultimately bounded.

IV. Review of the Event-Triggered Formulation for Discrete-Time Systems

The discrete counterpart of the above analysis is presented in the following. A brief recap of the event-based NMPC for discrete-time systems is provided for the sake of coherence, while the results will be presented in [3]. Wherever the mathematical proofs are omitted, they can be found in [3]. Note that in [3], a decentralized implementation of the discrete time NMPC is also reported.

A general uncertain system is considered here as well. The ISS stability with respect to the uncertainties of such systems was proven in [12] while, a modification of that analysis is followed, in order to find a triggering condition.

Consider that the plant to be controlled is described by the nonlinear model

$$x_{k+1} = f(x_k, u_k) + w_k \tag{21}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $w_k \in \mathcal{W} \subseteq \mathbb{R}^n$ denotes the system's state, the control variables and its additive disturbance, respectively. Uncertainties are assumed to be bounded by $\gamma_d \in \mathbb{R}_{\ge 0}$. Assumptions on the constraints are similar to the continuous-time case. The nominal model of the system without the additive disturbance is of the form $x_{k+1} = f(x_k, u_k)$. It is also assumed that $f(0,0) = 0$ and that $f(x,u)$ is locally Lipschitz in x in the domain $\mathcal{X} \times \mathcal{U}$, with Lipschitz constant L_{f_d}.

The predicted state of the nominal system is denoted as $\hat{x}(k+j+1|k)$, where the prediction of the state at time $k+j+1$ is based on the measurement of the state of the system at time k, given a control sequence u_{k+j}, i.e., $\hat{x}(k+j+1|k) = f(\hat{x}(k+j|k), u_{k+j})$. The norm of the difference between the predicted and the real evolution of the state is the error denoted as e and will be equipped in the following analysis. In order to address for the specific time step the double subscript notation is going to be used here, as well. Thus, the error is defined as

$$e(k+j|k) = ||x_{k+j} - \hat{x}(k+j|k)|| \tag{22}$$

The OCP in the discrete-time case, consists in minimizing, with respect to a control sequence $u_F(k) \triangleq [u(k|k), u(k+1|k), \dots, u(k+N-1|k)]$, a cost function $J_N(x_k, u_F(k))$,

$$\min_{u_F(k)} J_N(\cdot) = \min_{u_F(k)} \sum_{i=0}^{i=N-1} L(\tilde{x}(k+i|k), u(k+i|k)) + V(\tilde{x}(k+N|k)) \tag{23a}$$

subject to

$$\tilde{x}(k+j|k) \in \mathcal{X}_j \qquad \forall j = 1, \dots, N-1 \tag{23b}$$

$$u(k+j|k) \in \mathcal{U} \qquad \forall j = 0, \dots, N-1 \tag{23c}$$

$$\tilde{x}(k+N|k) \in \mathcal{X}_f \tag{23d}$$

where $N \in \mathbb{Z}_{\geq 0}$ denotes the prediction horizon and $\mathcal{X}_f$ is the terminal constraint set.

Similar assumptions as in the continuous time frame must be made for the robust NMPC controller for discrete-time systems. Following [12], it is assumed that

Assumption 2.

i) The stage cost $L(x,u)$ is Lipschitz continuous in $\mathcal{X} \times \mathcal{U}$, with a Lipschitz constant L_c and it is $L(0,0) = 0$. Also assume that there are positive integers $\alpha > 0$ and $\omega \geq 1$, such that $L(x,u) \geq \alpha ||(x,u)||^{\omega}$.

ii) Let the terminal region $\mathcal{X}_f$ from (23d) be a subset of an admissible positively invariant set Φ of the nominal system. Assume that there is a local stabilizing controller $h^d(x_k)$ for the terminal state $\mathcal{X}_f$. The associated Lyapunov function $V(\cdot)$ has the following properties $V(f(x_k, h^d(x_k)) - V(x_k) \leq -L(x_k, h^d(x_k)), \forall x_k \in \Phi$, and is Lipschitz in Φ, with Lipschitz constant L_V. The set Φ is given by $\Phi = \{x \in \mathbb{R}^n : V(x) \leq \alpha\}$ such that $\Phi = \{x \in \mathcal{X}_{N-1} : h^d(x) \in \mathcal{U}\}$. The set $\mathcal{X}_f = \{x \in \mathbb{R}^n : V(x) \leq \alpha_\nu\}$ is such that for all $x \in \Phi$, $f(x, h^d(x)) \in \mathcal{X}_f$.

The restricted constraint set $\mathcal{X}_j$ from (23b) is such that $\mathcal{X}_j = \mathcal{X} \sim \mathscr{B}_j$ where $\mathscr{B}_j = \{x \in \mathbb{R}^n : ||x|| \leq \frac{L_f^{j-1} - 1}{L_f - 1} \cdot \gamma_d\}$ and it guarantees that if the nominal state evolution belongs to $\mathcal{X}_j$, then the perturbed trajectory of the system fulfills the constraint $x \in \mathcal{X}$.

Using the framework of [12] it can be proven that system (21) subject to constraints, which satisfies the Assumption 2, is ISS stable with respect to measurement errors, under an NMPC strategy. This can be concluded since it has been proven in [3], that $J_N^*(k) - J_N^*(k-1) \leq L_{Z_0} \cdot e(k|k-1) - \alpha||x_{k-1}||^{\omega}$, with the optimal cost $J_N^*(\cdot)$ to be considered as an ISS Lyapunov function for time steps $k-1$ and k. The constant L_{Z_0} is given by $L_{Z_j} \triangleq L_V L_{f_d}^{(N-1)-j} + L_C \frac{L_{f_d}^{(N-1)-j} - 1}{L_{f_d} - 1}$ for $j \in [0, N-1]$. As this is valid only for the first step, it must be ensured that the value function is still decreasing for the next consecutive steps, in order to maintain stability. Thus, the triggering rule can be stated as

$$L_{Z_j} \cdot e(k+j|k-1) \leq \sigma \cdot \alpha \cdot \sum_{i=0}^{j} ||x_{k-i+j}||^{\omega} \tag{24a}$$

and

$$L_{Z_j} \cdot e(k+j|k-1) - \sigma \cdot \alpha \cdot \sum_{i=0}^{j} ||x_{k-i+j}||^{\omega} \leq L_{Z_{j-1}} \cdot e(k+j-1|k-1) - \sigma \cdot \alpha \cdot \sum_{i=0}^{j-1} ||x_{k-i+j}||^{\omega} \tag{24b}$$

The next OCP is triggered whenever condition (24a) or (24b) is violated. Note, that the state vector x_k is assumed to be available through measurements and that it provides the current plant information.

Hence we can state the following result. Consider the system (21), subject to the constraints, under an NMPC strategy and assume that the previously presented Assumption 2 holds. Then the NMPC control law given by (23a)-(23d) along with the triggering rule (24a)-(24b), drives the closed loop system towards a compact set where it is ultimately bounded.

V. EXAMPLE

In this section, a simulated example of the proposed design on a robotic manipulator is presented. The objective is to provide an efficient NMPC controller, triggered whenever (24a) or (24b) is violated, in order to stabilize the robotic manipulator, in a desired equilibrium configuration. Consider a general manipulator of r degrees of freedom (d.o.f.), which does not interact with the environment. The joint-space dynamic model of these types of manipulators is described as:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = \tau \tag{25}$$

where B is the inertia matrix, C is the Coriolis term, g is the gravity term, F is a positive definite diagonal matrix of viscous friction coefficients at the joints, $q = [q_1, \dots, q_r]$, $\dot{q} = [\dot{q}_1, \dots, \dot{q}_r]$ and $\ddot{q} = [\ddot{q}_1, \dots, \ddot{q}_r]$ are the vectors of the arm joint position, velocity and acceleration, respectively. Finally, $\tau \in \mathbb{R}^r$ are the joint torque inputs. We consider a two-link, planar robotic manipulator, $r = 2$ with no friction effects for simplicity.In the control affine, state-space model of the manipulator, the state accounts for $x = [q_1, q_2, \dot{q}_1, \dot{q}_2]$. The initial state is $x_{\text{initial}} = [\pi/2, 0, 0, 0]$ and the desired state is $x_{\text{desired}} = [0, 0, 0, 0]$. In Fig. 1, the norm of the distance between the state of the system and the desired state is depicted. The simulation shows that the system (25), under a NMPC strategy, using the triggering condition (24a)-(24b), converges to the final state in the nominal case. In the perturbed case the system converges to a bounded set around the origin.

The next Fig. 2, depicts the triggering moments, during the NMPC strategy. It can be witnessed that using the event-triggered policy, the inter-calculation times are strictly larger

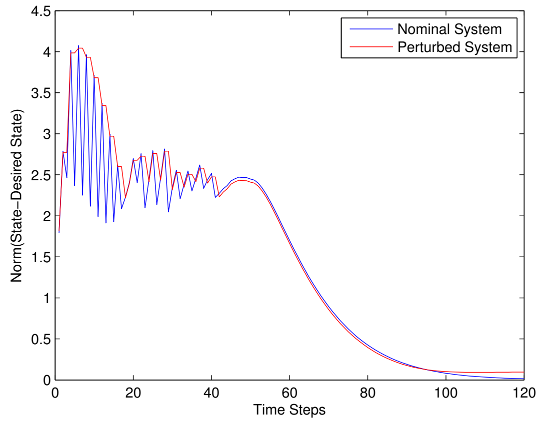

Fig. 1. The norm of the distance between the state of the system (25) and the desired state, i.e. dist $= ||x - x_{\text{desired}}||$. The blue line represents the distance of the nominal system, while the red line represents the distance of the perturbed system, under an additive disturbance.

than one when the system is far away from the equilibrium, until about the 80th time step. After the 80th time step, the system has practically converged to the desired equilibrium.

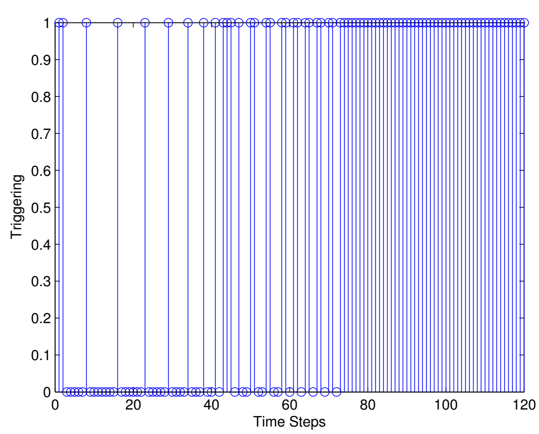

Fig. 2. Triggering instants. When the triggering axis has the value 1, the NMPC algorithm is triggered. For value 0, the NMPC law is implemented on the system in an open-loop fashion.

VI. Conclusions

In this paper, event-triggered strategies for control of both continuous and discrete-time systems under NMPC controllers, were proposed and analyzed. In both cases, uncertain nonlinear systems with additive disturbances, were considered. The main idea behind the event-triggered framework is to trigger the solution of the optimal control problem of the NMPC, only when it is needed. During the inter-event period the control law provided from the previous triggering event, is utilized in an open-loop fashion. This event-based approach is favorable in numerous occasions, because it is possible to reduce the number of times the control law should be computed, thus it can result to the alleviation of the energy consumption, or in the case of networks, it can result to amelioration of the network traffic. The results were illustrated through a simulated example. Future work involves finding the triggering condition in a cooperative control problem of a system of distributed agents which operate in a common environment.

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